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Equilibrium figures of viscous fluids governed by external forces and surface tension

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Abstract

We reconsider the problem of determining equilibrium figures of an isolated drop of an incompressible viscous liquid. The fluid body is subject to an external force density, and, along the free boundary, to surface tension. Here the term "equilibrium figure" means that the whole configuration is assumed to be stationary with respect to a uniformly rotating reference frame. Moreover, the pressure outside the fluid body is assumed to be close to a constant, and the fluid body itself is assumed to be close to the unit ball. The existence of such configurations is proved by applying successive approximations, under certain smallness and symmetry conditions on the external and inertial forces. The smallness assumptions are in some sense stronger, while the symmetry assumptions are weaker compared to previous results.

In case surface tension is no longer present and is (or is not) replaced by self-gravitation, the perturbation problem degenerates. The mathematical difficulties are sketched, along with a proposal of how to overcome these difficulties. Details will be presented in forthcoming papers.

1 Introduction

An equilibrium figure is a fluid flow, which is stationary with respect to a uniformly rotating reference frame, the domain occupied by the fluid - henceforth called "fluid body" - being governed by forces of various type. The present paper is the first in a three-piece series of papers dealing with equilibrium figures of viscous fluids.[1] In any case there will be some externally applied force density, not depending on the particular shape of the fluid body. Concerning the forces depending on the fluid body itself we will distinguish three cases:

a) surface tension (and possibly self-gravitation),

b) self-gravitation,

c) no compensating force at all.

[1]This series will represent a reorganized and slightly extended version of parts of the author's doctoral dissertation, written under the direction of Professor Matthias Günther at the University of Leipzig.

Here we shall give an overall introduction to the subject, with special emphasis on the case a). The remaining two parts of the series will be devoted to the cases b) and c), respectively.

To start with, let us recall the following problem: Determine the possible equilibrium figures of a force-free incompressible and inviscid liquid droplet, the boundary Σ of which being held together by surface tension. Denoting by $K(x)$ the mean curvature of Σ and by $r^2(x)$ the squared distance from the axis of rotation, both computed at some generic point $x \in \Sigma$, the governing equation of the problem reads as follows

$$K(x) + \frac{1}{2}\omega^2 r^2(x) = c\,, \quad x \in \Sigma\,, \tag{1.1}$$

cf. [10, Kap. 3]. Here $\omega \in \mathbb{R}$ is given, and $c \in \mathbb{R}$ is to be determined as well. For later purposes we note that, letting $n : \Sigma \longrightarrow S^2$ denote the unit outward normal along Σ, and letting div denote the surface divergence operator associated with Σ, one has (up to a constant positive factor)

$$K = -\operatorname{div} n\,. \tag{1.2}$$

In what follows the axis of rotation is always assumed to be the x_3-axis of some cartesian coordinate system, say. Then

$$r^2(x) = x_1^2 + x_2^2\,, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3\,.$$

E. Hölder was the first to prove local existence and uniqueness of solutions of (1.1), if ω is choosen sufficiently small, cf. [5]. A global existence proof via variational methods has been given in [1]. The classical model has been extended to a number of directions. The interested reader is referred to [12] for an introductory overview.

Now, if we take non-vanishing viscosity and at the same time an applied force density $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ into account, we are led to the following free boundary problem for a system of equations of Navier-Stokes type:

$$\begin{aligned}
-\Delta v + \nabla p + \alpha\big(\operatorname{div}(v \otimes v) - \omega \mathcal{C}_1 v - \omega^2 \mathcal{C}_2\big) &= f && \text{in } \Omega\,, && (1.3)\\
\operatorname{div} v &= 0 && \text{in } \Omega\,, && (1.4)\\
T(p,v).n &= Kn && \text{along } \Sigma\,, && (1.5)\\
v.n &= 0 && \text{along } \Sigma\,. && (1.6)
\end{aligned}$$

The equations (1.3), (1.4) are the Navier-Stokes equations for an incompressible fluid, reformulated in a frame of reference rotating with an angular velocity of constant amount ω about some fixed axis. The expressions $\omega \mathcal{C}_1 v$ and $\omega^2 \mathcal{C}_2$ represent the coriolis and centrifugal forces, respectively. With the same notation as above and the same convention concerning the axis of rotation we have

$$\mathcal{C}_1 v = 2\, v \wedge e_3\,, \quad \mathcal{C}_2 = \frac{1}{2}\nabla r^2\,, \tag{1.7}$$

where $e_3 := (0,0,1)$ and the symbol "$\wedge$" is used to denote the cross product in $\mathbb{R}^3$. In (1.5), $T(p,v)$ denotes the stress tensor, the cartesian coordinates of which being defined by

$$e_i.T(p,v).e_j = -p\delta_{ij} + \partial_i v^j + \partial_j v^i \ . \tag{1.8}$$

The equations (1.5), (1.6) represent the boundary conditions according to our assumption that the boundary Σ of the domain Ω occupied by the drop is governed by surface tension. The exterior pressure is assumed to be constant throughout the entire space, cf. Remark 3.9 below. Note that the whole configuration Ω, v, p, f is assumed to be stationary with respect to the rotating reference frame. In case $\omega \neq 0$ the given force density f can only be thought of as being fixed, if it is axisymmetric with respect to the axis of rotation.

The first attempt to prove existence of solutions of (1.3)–(1.6) has been made by J. Bemelmans, cf. [2]. He studied the local perturbation problem for (1.3)–(1.6) near the static sphere, under the additional assumption $\omega = 0$, and he proves local existence and uniqueness under certain symmetry assumptions on the applied force density f. In [3] regularity of solutions is investigated, and analyticity of the solution for analytic data is shown. However, in [2] there remain a number of questions open, a few of which will be addressed here. For instance, from [2, Satz 1] it is not clear, for which kind of forces equilibrium figures exist.

The paper is organized as follows. In the next section we derive a rigorous formulation of the problem, introducing the main ingredients of what is sometimes called "domain-perturbation method". The corresponding techniques become more and more standard, but we shall explain them in detail. Roughly speaking, the unknown free boundary Σ is written as a graph of some unknown function $\zeta : \Sigma_0 \longrightarrow \mathbb{R}$, where Σ_0 is the unit sphere. By resolving (1.3)–(1.5) for $v = v(\zeta, \omega, f)$, and by inserting this solution into (1.6), the problem is reduced to a scalar nonlinear operator equation, the nonlinear operator acting between Sobolev spaces. However, since the Neumann problem (1.3)–(1.5) admits eigensolutions we will have to pass to a slightly modified version of the perturbation problem. In section 3 this modified problem is solved by successive approximations. Mainly symmetry assumptions on the data are used in order to eliminate some branching equations, and in order to pass from the modified to the actual free boundary problem.

In case the mean curvature K is removed from the right hand side of equation (1.5) and is (or is not) replaced by another, in some sense "lower order" compensating force such as self-gravitation, the corresponding perturbation problem degenerates. As already mentioned above, both these cases will be discussed separately in two forthcoming papers. As a precursor, the specific mathematical difficulties at hand are explained rather briefly in section 4.

2 Setting of the problem

Let $\Omega_0 := \{x \in \mathbb{R}^3 : |x| < 1\}$ be the unit ball, and let $\Sigma_0 := \partial\Omega_0$ be the unit sphere. The basic function spaces we shall use are the Sobolev spaces $H^s(\Omega_0)$ and $H^s(\Sigma_0)$ of equivalence classes of real-valued functions having generalized derivatives up to order $s \geqslant 0$ in $L^2(\Omega_0)$ or $L^2(\Sigma_0)$, respectively. Proceeding from the definition of $H^s(\mathbb{R}^n)$ by Fourier transformation, the norms in the latter spaces are defined by partitions of unity and local immersions into $\mathbb{R}^n$, $n = 2, 3$. For more precise definitions and fundamental properties such as trace and extension theorems, the reader is referred to [6, App. B]. We only mention here that the spaces $H^s(\Omega_0)$ and $H^s(\Sigma_0)$ coincide with the spaces $\bar{H}^{\mathrm{loc}}_{(s)}(\Omega_0)$ and $H_{(s)}(\Sigma_0)$, respectively, introduced there. The fact that these spaces form Banach algebras when $s > \frac{3}{2}$ or $s > 1$, respectively, will be used frequently and without explicit reference. As usual, we shall work here with functions rather than with equivalence classes, choosing the representatives as smooth as possible. Spaces of vector-valued functions are defined in the usual manner.

In what follows, the symbol U^s refers to a zero neighbourhood of the Sobolev space $H^s := H^s(\Sigma_0)$. The diameter of the set U^s is always assumed to be choosen sufficiently small, if necessary. Given some "boundary perturbation" $\zeta \in U^s$, $s > 2$, we introduce a corresponding C^1-diffeomorphism $\theta_\zeta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by the definition

$$\theta_\zeta(x) := \big(1 + \widetilde{\zeta}(x)\big)x \ , \tag{2.1}$$

where $\zeta \longmapsto \widetilde{\zeta} \in \mathcal{L}\left(H^s(\Sigma_0), H^{s+1/2}(\mathbb{R}^3)\right)$ is a fixed linear extension operator. Without loss of generality we may assume that

$$\widetilde{\zeta}(x) = 0 \ , \quad \text{if} \quad \big||x| - 1\big| \geqslant \frac{1}{2} \ , \tag{2.2}$$

and, denoting by n_0 the unit outward normal along Σ_0, that

$$\frac{\partial\widetilde{\zeta}}{\partial n_0} = 0 \quad \text{along } \Sigma_0 \ . \tag{2.3}$$

Using θ_ζ we define corresponding perturbed domains:

$$\Omega_\zeta := \theta_\zeta(\Omega_0) \ , \quad \Sigma_\zeta := \theta_\zeta(\Sigma_0) \ . \tag{2.4}$$

Obviously, there holds $\partial\Omega_\zeta = \Sigma_\zeta$. Furthermore we introduce the metric quantities $g_{jk}(\zeta)$, $j, k = 1, 2, 3$, by

$$g_{jk}(\zeta) := \partial_j\theta_\zeta.\partial_k\theta_\zeta \ . \tag{2.5}$$

Let

$$g(\zeta) := \det\big(g_{jk}(\zeta)\big) \ , \tag{2.6}$$

and let $g^{jk}(\zeta)$, $a_{jk}(\zeta)$, $j, k = 1, 2, 3$, be defined implicitely by the relations

$$g^{jk}(\zeta)g_{kl}(\zeta) = \delta_{jl} \ , \quad a_{jk}(\zeta)a_{jl}(\zeta) = g(\zeta)g^{kl}(\zeta) \ . \tag{2.7}$$

Finally, since $g(\zeta)$ is positive and is uniformly bounded away from 0, we can set

$$\sqrt{g}(\zeta) := \sqrt{g(\zeta)}\,. \tag{2.8}$$

From these definitions it is not that difficult to see that these mappings are analytic, more precisely

$$g_{jk}, g^{jk}, g, \sqrt{g}, a_{jk} \in C^\omega\left(U^s, H^{s-1/2}_{\mathrm{loc}}(\mathbb{R}^3)\right)\,. \tag{2.9}$$

The respective linearizations, that is the Fréchet derivatives about the reference configuration $\zeta = 0$ are readily computed:

$$\begin{aligned}
D_\zeta g_{jk}(0)\eta &= \partial_j(x^k\widetilde{\eta}) + \partial_k(x^i\widetilde{\eta})\,, && (2.10)\\
D_\zeta g^{jk}(0) &= -D_\zeta g_{jk}(0)\,, && (2.11)\\
D_\zeta g(0)\eta &= 2\partial_l\left(x^l\widetilde{\eta}\right)\,, && (2.12)\\
D_\zeta\sqrt{g}(0) &= \frac{1}{2}D_\zeta g(0)\,, && (2.13)\\
D_\zeta a_{jk}(0)\eta &= \delta_{jk}\partial_l(x^l\widetilde{\eta}) - \partial_j(x^k\widetilde{\eta})\,. && (2.14)
\end{aligned}$$

Next, we wish to transform the differential operators occuring in the system of equations (1.3)–(1.6) onto the fixed domain Ω_0 or it's boundary Σ_0, respectively. To this end we define, denoting by X^k, $k = 1, 2, 3$, the cartesian components of $X \in \mathbb{R}^3$,

$$\left(\Delta_\zeta v\right)^l := \frac{1}{\sqrt{g(\zeta)}}\partial_j\left(\sqrt{g(\zeta)}g^{jk}(\zeta)\partial_k v^l\right)\,, \tag{2.15}$$

$$\left(\nabla_\zeta p\right)^l := \frac{1}{\sqrt{g(\zeta)}}a_{lk}(\zeta)\partial_k p\,, \tag{2.16}$$

$$\operatorname{div}_\zeta v := \frac{1}{\sqrt{g(\zeta)}}\partial_j\left(\sqrt{g(\zeta)}v^j\right)\,, \tag{2.17}$$

$$\left(\operatorname{div}_\zeta(v\otimes w)\right)^l := \operatorname{div}_\zeta\left(vw^l\right)\,, \tag{2.18}$$

$$\left(T_\zeta(p,v)\right)^{kl} := -p\delta_{kl} + \left(\nabla_\zeta v^k\right)^l + \left(\nabla_\zeta v^l\right)^k\,. \tag{2.19}$$

Note that these operators depend analytically on the boundary perturbation ζ in the sense that

$$\zeta \longmapsto \Delta_\zeta \;\in C^\omega\Big(U^s, \mathcal{L}\big(H^{s+1/2}(\Omega_0,\mathbb{R}^3), H^{s-3/2}(\Omega_0,\mathbb{R}^3)\big)\Big)\,,\quad s > 3\,, \tag{2.20}$$

$$\zeta \longmapsto \nabla_\zeta \;\in C^\omega\Big(U^s, \mathcal{L}\big(H^{s+1/2}(\Omega_0), H^{s-1/2}(\Omega_0,\mathbb{R}^3)\big)\Big)\,,\quad s > 2\,, \tag{2.21}$$

$$\zeta \longmapsto \operatorname{div}_\zeta \in C^\omega\Big(U^s, \mathcal{L}\big(H^{s-1/2}(\Omega_0,\mathbb{R}^3), H^{s-3/2}(\Omega_0)\big)\Big)\,,\quad s > 3\,. \tag{2.22}$$

The operator Δ_ζ has been constructed just to ensure that $(\Delta v)\circ\theta_\zeta = \Delta_\zeta(v\circ\theta_\zeta)$ for sufficiently smooth v, ζ.

Proposition 2.1. *Assume we are given C^1-mappings $v = v(\zeta)$ and $f = f(\zeta)$, say. More precisely, let $v \in C^1\big(U^s, H^{s+1/2}(\Omega_0, \mathbb{R}^3)\big)$, $f \in C^1\big(U^s, H^{s-3/2}(\Omega_0, \mathbb{R}^3)\big)$, $s > 3$, such that $\Delta_\zeta v(\zeta) = f(\zeta)$. If we split*

$$D_\zeta v(0)\eta = \big(x.\nabla v(0)\big)\widetilde{\eta} + \dot{v} \ , \tag{2.23}$$

where $\dot{v} \in H^{s+1/2}(\Omega_0, \mathbb{R}^3)$ depends linear and bounded on $\eta \in H^s$, then we have

$$D_\zeta f(0)\eta = \big(x.\nabla f(0)\big)\widetilde{\eta} + \Delta\dot{v} \ . \tag{2.24}$$

Proof. Indeed, if we write just for a moment $\Delta_\zeta = \Delta(\zeta)$, $\Delta = \Delta(0)$, we get from (2.15) and (2.10)–(2.13)

$$\begin{aligned} D_\zeta \Delta(0)\eta &= -\partial_l(x^l\widetilde{\eta})\Delta + \partial_j\left(\partial_l(x^l\widetilde{\eta})\partial_j\right) + \partial_j\left\{\left[-\partial_j(x^k\widetilde{\eta}) - \partial_k(x^j\widetilde{\eta})\right]\partial_k\right\} \\ &= -\Delta(x^k\widetilde{\eta})\partial_k - \left\{\partial_j(x^k\widetilde{\eta}) + \partial_k(x^j\widetilde{\eta})\right\}\partial_{jk} \\ &= -\Delta(x^k\widetilde{\eta})\partial_k - 2\partial_j(x^k\widetilde{\eta})\partial_{jk} \ , \end{aligned}$$

and

$$\begin{aligned} D_\zeta\left\{\Delta_\zeta v(\zeta)\right\}|_{\zeta=0}\eta &= \Delta\big(D_\zeta v(0)\eta\big) - \Delta(x^k\widetilde{\eta})\partial_k v(0) - 2\partial_j(x^k\widetilde{\eta})\partial_{jk}v(0) \\ &= \Delta\big(\widetilde{\eta}x^l\partial_l v(0)\big) + \Delta\dot{v} - \Delta(x^k\widetilde{\eta})\partial_k v(0) - 2\partial_j(x^k\widetilde{\eta})\partial_{jk}v(0) \\ &= \big(x.\nabla f(0)\big)\widetilde{\eta} + \Delta\dot{v} \ , \end{aligned}$$

which proves the assertion. □

Remark 2.2. Similar relations are of course valid for the other differential operators defined above. Note that (2.23), (2.24) could have been also obtained, at least formally, by linearizing $(\Delta\overline{v}(\zeta))\circ\theta_\zeta = \overline{f}(\zeta)\circ\theta_\zeta$, where $\overline{v}(\zeta)\circ\theta_\zeta = v(\zeta)$ and $\overline{f}(\zeta)\circ\theta_\zeta = f(\zeta)$. This fact will be exploited later on to compute the linearization of mappings being defined in terms of solutions of partial differential equations, the coefficients of which depend on ζ. □

Now, proceeding in the same spirit as before we set

$$n_\zeta := n_{\Sigma_\zeta} \circ \theta_\zeta \ , \tag{2.25}$$

where n_{Σ_ζ} denotes the unit outward normal of Ω_ζ along Σ_ζ. Making use of the metric quantities introduced above, cf. (2.5), (2.7), we find that

$$n_\zeta^j = \mathrm{tr}_{\Sigma_0}\left[\big(a_{lm}(\zeta)a_{nm}(\zeta)x^l x^n\big)^{-1} a_{jk}(\zeta)x^k\right] \ . \tag{2.26}$$

Setting $n(\zeta) := n_\zeta$ one has

$$n \in C^\omega(U^s, H^{s-1}) \ ,$$

cf. (2.9), and

$$D_\zeta n(0)\eta = -\,\mathrm{tr}_{\Sigma_0}[\nabla\widetilde{\eta}] = -\nabla_{\Sigma_0}\eta \ , \tag{2.27}$$

cf. (2.14), (2.3). Henceforth we shall write $\nabla\eta$ instead of $\nabla_{\Sigma_0}\eta$ for the surface gradient of η .

The mean curvature operator is defined by

$$K_\zeta := -\operatorname{div}_\zeta n_\zeta \ . \tag{2.28}$$

By setting $K(\zeta) := K_\zeta$ we have

$$K \in C^\omega(U^s, H^{s-2}) \ , \tag{2.29}$$

and a straightforward computation yields

$$D_\zeta K(0)\eta = -\operatorname{div}_{\Sigma_0}(-\nabla_{\Sigma_0}\eta) + 2\eta = (\Delta_{\Sigma_0} + 2I)\,\eta \ ,$$

with Δ_{Σ_0} being the Laplace-Beltrami operator.

We are now in a position to formulate a perturbation problem corresponding to (1.3)–(1.6). But before we carry this out, let us note that we will actually discuss a slightly modified problem. The reason can be found in the occurence of eigensolutions of the Neumann boundary value problem for the Stokes operator. Now, recall the following Gauss-Green formula for the Stokes operator:

$$\begin{aligned}\int\limits_\Omega \Big(u.(\Delta v - \nabla p) - v.(\Delta u - \nabla q)\Big)\,\mathrm{d}x + \int\limits_\Omega (q\operatorname{div} v - p\operatorname{div} u)\,\mathrm{d}x \ &= \\ = \int\limits_\Sigma \Big(u.T(p,v).n - v.T(q,u).n\Big)\,\mathrm{d}o_x \ .\end{aligned} \tag{2.30}$$

The formula (2.30) is valid for $u, v \in H^2(\Omega, \mathbb{R}^3)$ and $p, q \in H^1(\Omega)$, provided the domain Ω is sufficiently smooth, and it can be directly derived from Gauss' theorem. By multiplying the right hand side of equation (1.3) with $u(x) = e_k$ and $u(x) = e_k \wedge x$, respectively, and by integrating the resulting equation over Ω, we obtain from (2.30) the following six integrability conditions on f:

$$\int\limits_\Omega f(x)\,\mathrm{d}x = 0 \ , \quad \int\limits_\Omega f(x) \wedge x\,\mathrm{d}x = 0 \ . \tag{2.31}$$

These conditions express the fact, that in equilibrium both the total force and the total torque exerted by f vanish. Obviously, these conditions can not be satisfied without paying attention to the solution itself. In view of this, we will introduce additional expressions which guarantee that the conditions (2.31) are always satisfied. In order to make this more precise, and in order to simplify the notation, we introduce functions φ_j and ψ_j, $j = 1, 2, 3$, by

$$\varphi_j(x) := e_j \ , \quad \psi_j(x) := e_j \wedge x \ . \tag{2.32}$$

Moreover, we set

$$\begin{aligned} V_T &:= \mathcal{L}\big(\{\varphi_j; j=1,2,3\}\big) , \\ V_R &:= \mathcal{L}\big(\{\psi_j; j=1,2,3\}\big) , \\ V_S &:= V_T \oplus V_R . \end{aligned}$$

These sets shall be interpreted as subspaces of $H^s(\Omega_0,\mathbb{R}^3)$, $s \geqslant 0$. Setting

$$\begin{aligned} H^s(\Omega_0,\mathbb{R}^3) \ominus V_T &:= \left\{ v \in H^s(\Omega_0,\mathbb{R}^3); \int_{\Omega_0} v(x)\,\mathrm{d}x = 0 \right\} , \\ H^s(\Omega_0,\mathbb{R}^3) \ominus V_R &:= \left\{ v \in H^s(\Omega_0,\mathbb{R}^3); \int_{\Omega_0} \operatorname{rot} v(x)\,\mathrm{d}x = 0 \right\} , \\ H^s(\Omega_0,\mathbb{R}^3) \ominus V_S &:= \big(H^s(\Omega_0,\mathbb{R}^3) \ominus V_T\big) \cap \big(H^s(\Omega_0,\mathbb{R}^3) \ominus V_R\big) \end{aligned}$$

we have for $Q = T,R,S$ the topological direct decompositions

$$H^s(\Omega_0,\mathbb{R}^3) = V_Q \oplus \big(H^s(\Omega_0,\mathbb{R}^3) \ominus V_Q\big) .$$

We denote by P_Q the projector onto V_Q. In view of $\operatorname{rot}(e_k \wedge x) = 2e_k$ we get

$$\begin{aligned} P_T v &= \frac{1}{|\Omega_0|} \left\{ \int_{\Omega_0} \big(v(x)\big)^j \,\mathrm{d}x \right\} \varphi_j , \\ P_R v &= \frac{1}{2|\Omega_0|} \left\{ \int_{\Omega_0} \big(\operatorname{rot} v(x)\big)^j \,\mathrm{d}x \right\} \psi_j , \\ P_S v &= P_T v + P_R v . \end{aligned}$$

With these preparations, our perturbation problem can be formulated as follows: Given $\alpha \geqslant 0$, $\omega \in \mathbb{R}$, $w \in V_S$ and some force density $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, we are looking for solutions $\zeta : \Sigma_0 \longrightarrow \mathbb{R}$, $v : \Omega_0 \longrightarrow \mathbb{R}^3$, $p : \Omega_0 \longrightarrow \mathbb{R}$ and $\Pi^{(1)}, \Pi^{(2)} \in \mathbb{R}^3$ of the following system of equations:

$$-\Delta_\zeta v + \nabla_\zeta p + \alpha\left(\operatorname{div}_\zeta (v \otimes v) - \omega \mathcal{C}_1 v - \omega^2 \mathcal{C}_{2,\zeta}\right) + \Pi^{(1)} = \operatorname{rstr}_{\Omega_0}\!\big[f \circ \theta_\zeta\big] , \tag{2.33}$$

$$\operatorname{div}_\zeta v = 0 , \tag{2.34}$$

$$T_\zeta(p,v).n_\zeta + \Pi^{(2)} \wedge n_\zeta = K_\zeta n_\zeta , \tag{2.35}$$

$$P_S v = w , \tag{2.36}$$

$$\operatorname{tr}_{\Sigma_0}[v]\,.n_\zeta = 0 , \tag{2.37}$$

where $\mathcal{C}_1$ and $\mathcal{C}_{2,\zeta}$ are defined by

$$\mathcal{C}_1 v = 2\, v \wedge e_3 , \quad \mathcal{C}_{2,\zeta} = \frac{1}{2}\left(\nabla r^2\right) \circ \theta_\zeta , \tag{2.38}$$

respectively. Note that all expressions occuring in (2.33)–(2.37) can be interpreted in the classical sense, provided that $s > 4$. For the time being one may assume that f is a smooth vector field.

As a first step towards the solution of (2.33)–(2.37) we shall now study the boundary value problem (2.33)–(2.36). It's well-posedness follows by a perturbation argument. What we need to carry out the proof can be found in the subsequent two Propositions.

Proposition 2.3. *The mapping* $F : C^\infty(\Sigma_0) \times C^\infty(\mathbb{R}^3, \mathbb{R}^3) \longrightarrow C^\infty(\Omega_0)$ *defined by*

$$F(\zeta, f) := \mathrm{rstr}_{\Omega_0}\big[f \circ \theta_\zeta\big]$$

extends to a C^k*-mapping*

$$F : U^s \times H^{t+k}(\mathbb{R}^3) \longrightarrow H^t(\Omega_0) \ ,$$

provided that $s > 3$, $k \in \mathbb{N}_0$, $t \in \mathbb{R}$, *and* $0 \leqslant t \leqslant s - \frac{1}{2}$. *In case* $k \geqslant 1$ *there holds*

$$D_\zeta F(0, f)\eta = \mathrm{rstr}_{\Omega_0}\big[(x.\nabla f)\,\widetilde{\eta}\big] \ .$$

Proof. In case $s + \frac{1}{2} \in \mathbb{N}$, $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the assertion is just a special case of [8, Theorem 3.4]. In case $s + \frac{1}{2} \in \mathbb{N}$ and t not necessarily integer, the assertion can be derived by an interpolation argument, cf. [6, Corollary B.1.6]. In the general case we choose $n \in \mathbb{N}$ such that $s - \frac{1}{2} \leqslant n \leqslant s + \frac{1}{2}$. Now, let $0 \leqslant t \leqslant s - \frac{1}{2}$. Then $0 \leqslant t \leqslant s - \frac{1}{2} \leqslant n$. Since $n > 2$ the assertion follows by the steps carried out before. □

We shall now discuss the following Neumann boundary value problem for the Stokes operator:

$$-\Delta v + \nabla p + \Pi^{(1)} = f \ , \tag{2.39}$$

$$\mathrm{div}\, v = g \ , \tag{2.40}$$

$$T(p, v).n_0 + \Pi^{(2)} \wedge n_0 = h \ , \tag{2.41}$$

$$P_S v = w \ . \tag{2.42}$$

Proposition 2.4. *Let* $s \geqslant 2$. *Assume we are given* $f \in H^{s-2}(\Omega_0, \mathbb{R}^3)$, $g \in H^{s-1}(\Omega_0)$, $h \in H^{s-\frac{3}{2}}(\Sigma_0, \mathbb{R}^3)$ *and* $w \in V_S$. *Then there is a unique solution* $v \in H^s(\Omega_0, \mathbb{R}^3)$, $p \in H^{s-1}(\Omega_0)$, $\Pi^{(j)} \in \mathbb{R}^3$, $j = 1, 2$, *of* (2.39)–(2.42) *depending linearly and bounded on the data.*

Proof. If we define

$$\Pi^{(1)} := \frac{1}{|\Omega_0|}\left\{ \int_{\Omega_0} (f - \nabla g)\,\mathrm{d}x + \int_{\Sigma_0} h\,\mathrm{d}o_x \right\} \ , \tag{2.43}$$

$$\Pi^{(2)} := -\frac{1}{2|\Omega_0|}\left\{ \int_{\Omega_0} (f - \nabla g) \wedge x\,\mathrm{d}x + \int_{\Sigma_0} h \wedge x\,\mathrm{d}o_x \right\} \ , \tag{2.44}$$

and

$$
\begin{aligned}
\overline{f} &:= f - \Pi^{(1)} , \\
\overline{g} &:= g , \\
\overline{h} &:= h - \Pi^{(2)} \wedge n_0 ,
\end{aligned}
$$

then

$$
\int\limits_{\Omega_0} \left(\overline{f} - \nabla\overline{g}\right).w \,\mathrm{d}x + \int\limits_{\Sigma_0} \overline{h}.w \,\mathrm{d}o_x = 0 , \quad w \in V_S . \tag{2.45}
$$

It therefore remains to prove that, under the additional assumption that (2.45) is satisfied,

$$
-\Delta v + \nabla p = \overline{f} , \tag{2.46}
$$

$$
\operatorname{div} v = \overline{g} , \tag{2.47}
$$

$$
T(p,v).n_0 = \overline{h} \tag{2.48}
$$

has a unique solution $v \in H^s(\Omega_0, \mathbb{R}^3) \ominus V_R$, $p \in H^{s-1}(\Omega_0)$. To this end we define bounded mappings $a : H^1(\Omega_0,\mathbb{R}^3) \times H^1(\Omega_0,\mathbb{R}^3) \longrightarrow \mathbb{R}$ and $l : H^1(\Omega_0,\mathbb{R}^3) \longrightarrow \mathbb{R}$ by

$$
\begin{aligned}
a(u,v) &:= \frac{1}{2}\int\limits_{\Omega_0} \left(\partial_i u^j + \partial_j u^i\right)\left(\partial_i v^j + \partial_j v^i\right) \mathrm{d}x , \\
l(\varphi) &:= \int\limits_{\Omega_0} \left(\overline{f} - \nabla\overline{g}\right).\varphi \,\mathrm{d}x + \int\limits_{\Sigma_0} \overline{h}.\varphi \,\mathrm{d}o_x .
\end{aligned}
$$

Given $v \in H^2(\Omega_0,\mathbb{R}^3)$, $p \in H^1(\Omega_0)$ satisfying (2.46)–(2.48) we find

$$
\begin{aligned}
\int\limits_{\Omega_0} \overline{f}.\varphi \,\mathrm{d}x &= \int\limits_{\Omega_0} \left(-\partial_{ii} v^j + \partial_j p\right)\varphi^j \,\mathrm{d}x \\
&= -\int\limits_{\Sigma_0} n_0^i \partial_i v^j \varphi^j \,\mathrm{d}o_x + \int\limits_{\Omega_0} \partial_i v^j \partial_i \varphi^j \,\mathrm{d}x + \int\limits_{\Sigma_0} n_0^j p \varphi^j \,\mathrm{d}o_x - \int\limits_{\Omega_0} p \operatorname{div}\varphi \,\mathrm{d}x \\
&= -\int\limits_{\Sigma_0} n_0^i (\partial_i v^j + \partial_j v^i)\varphi^j \,\mathrm{d}o_x + \int\limits_{\Sigma_0} n_0^j p \varphi^j \,\mathrm{d}o_x + \\
&\qquad + \int\limits_{\Sigma_0} n_0^i \partial_j v^i \varphi^j \,\mathrm{d}o_x + \int\limits_{\Omega_0} \partial_i v^j \partial_i \varphi^j \,\mathrm{d}x - \int\limits_{\Omega_0} p \operatorname{div}\varphi \,\mathrm{d}x \\
&= a(v,\varphi) - \int\limits_{\Omega_0} p \operatorname{div}\varphi \,\mathrm{d}x + \int\limits_{\Omega_0} \nabla\overline{g}.\varphi \,\mathrm{d}x - \int\limits_{\Sigma_0} \overline{h}.\varphi \,\mathrm{d}o_x ,
\end{aligned}
$$

provided that $\varphi \in H^1(\Omega_0,\mathbb{R}^3)$. Given $\overline{f} \in L^2(\Omega_0,\mathbb{R}^3)$, $\overline{g} \in H^1(\Omega_0)$ and $\overline{h} \in H^{\frac{1}{2}}(\Sigma_0,\mathbb{R}^3)$, such that (2.45) is satisfied, we call $(v,p) \in H^1(\Omega_0,\mathbb{R}^3) \times L^2(\Omega_0)$ a weak solution of (2.46)–(2.48), iff

$$
\begin{aligned}
a(v,\varphi) - \int\limits_{\Omega_0} p \operatorname{div} \varphi \, \mathrm{d}x &= l(\varphi) \, , \quad \varphi \in H^1(\Omega_0, \mathbb{R}^3) \, , \\
\operatorname{div} v &= g \, .
\end{aligned}
$$

In a next step, we proof the following assertion: Let $v \in H^1(\Omega_0, \mathbb{R}^3)$. There exists some $p \in L^2(\Omega_0)$, such that (v,p) is a weak solution, iff

$$
\begin{aligned}
a(v,v) - 2l(v) &= \operatorname{Min}\big\{ a(u,u) - 2l(u); u \in H^1(\Omega_0, \mathbb{R}^3), \operatorname{div} u = \overline{g} \big\} \, , && (2.49) \\
\operatorname{div} v &= \overline{g} \, . && (2.50)
\end{aligned}
$$

Let (v,p) be a weak solution, and let $u \in H^1(\Omega_0, \mathbb{R}^3)$ be given, such that $\operatorname{div} u = \overline{g}$. Then $a(v,v) - \int_{\Omega_0} p\overline{g} \, \mathrm{d}x = l(v)$ and

$$
\begin{aligned}
0 &\leqslant a(v-u, v-u) \\
&= l(v-u) - a(u,v) + a(u,u) \\
&= l(v-u) - l(u) - \int_{\Omega_0} p\overline{g} \, \mathrm{d}x + a(u,u) \, ,
\end{aligned}
$$

hence (2.49). In order to prove the other direction we consider the linear operator $A : H^1(\Omega_0, \mathbb{R}^3) \longrightarrow L^2(\Omega_0)$ defined by $A\varphi := \operatorname{div} \varphi$. Given $\varphi \in H^1(\Omega_0, \mathbb{R}^3)$ we let $\psi \in H^2(\Omega_0)$ be the uniquely determined solution of the boundary value problem

$$
\begin{aligned}
\Delta \psi &= \operatorname{div} \varphi \quad \text{in } \Omega_0 \, , \\
\psi &= 0 \quad \text{along } \Sigma_0 \, .
\end{aligned}
$$

Then $h := \varphi - \nabla \psi \in \ker A$, and from standard elliptic theory we have an estimate

$$
\inf_{a \in \ker A} \|\varphi - a\|_{H^1} \leqslant \|\varphi - h\|_{H^1} \leqslant \|\psi\|_{H^2} \leqslant C \|A\varphi\|_{L^2} \, .
$$

So $\operatorname{img} A$ is closed, cf. [14, Part 2, Theorem 3.E]. If we are now given any solution v of the minimum problem (2.49), (2.50), we see by deriving the associated variational equation, that the funcional $\varphi \longmapsto a(v,\varphi) - l(\varphi)$ vanishes on $\ker A$. From Riesz' representation theorem, cf. [14, Part 2, S. 363], we deduce that there is a $p \in L^2(\Omega_0)$ satisfying $a(v,\varphi) - l(\varphi) = (p, A\varphi)_{L^2}$ for all $\varphi \in H^1(\Omega_0, \mathbb{R}^3)$. The assertion is thus proved.

Now, the solution of the minimum problem (2.49), (2.50) can be achieved in the usual manner, by applying the Lax-Milgram theorem. The strong coercivity of a over the space $H^1(\Omega_0, \mathbb{R}^3) \ominus V_S$ of admissible functions is a consequence of Korn's second inequality

$$
a(v,v) \geqslant C_1 \sum_{ij} \int\limits_{\Omega_0} \big(\partial_i v^j\big)^2 \, \mathrm{d}x \, , \quad v \in H^1(\Omega_0, \mathbb{R}^3) \ominus V_R
$$

and Poincarés inequality

$$\int_{\Omega_0} |\nabla v|^2 \, \mathrm{d}x \geqslant C_2 \|v\|^2_{H^1(\Omega_0)} \ , \quad v \in H^1(\Omega_0) \ , \quad \int_{\Omega_0} v \, \mathrm{d}x = 0 \ .$$

The uniqueness of the solution of (2.46)–(2.48) up to elements in $V_S \times \{0\}$ can be proved as follows. The strong coercivity estimate for a immediately implies that for any solution (v, p) of (2.46)–(2.48) with $\overline{f} = 0$, $\overline{g} = 0$, $\overline{h} = 0$ we must have $v \in V_S$. The pressure satisfies $(p, \operatorname{div} \varphi)_{L^2} = 0$ for all $\varphi \in H^1(\Omega_0, \mathbb{R}^3)$, hence $p = 0$.

Once having established a unique solution, it's regularity can be derived from standard elliptic theory: Consider the constant coefficient operator $P(\partial)$ defined by

$$P(\partial) \begin{pmatrix} v \\ p \end{pmatrix} := \begin{pmatrix} -\Delta v + \nabla p \\ \operatorname{div} v \end{pmatrix} \ .$$

The corresponding principal symbol

$$p(\xi) = \begin{pmatrix} |\xi|^2 & 0 & 0 & i\xi_1 \\ 0 & |\xi|^2 & 0 & i\xi_2 \\ 0 & 0 & |\xi|^2 & i\xi_3 \\ i\xi_1 & i\xi_2 & i\xi_3 & 0 \end{pmatrix}$$

is regular whenever $\xi = (\xi_1, \xi_2, \xi_3) \neq 0$. Moreover, the boundary operator $T(p, v).n_0$ satisfies the complementing boundary condition, and so the system (2.39)–(2.42) is elliptic in the sense of [6, Definition 20.1.1]. The Fredholm property of (2.39)–(2.42) now follows from well-known results on elliptic systems, cf. [6, 20.1]. ☐

Proposition 2.5. *Let $s > 4$, and let $k \in \mathbb{N}_0$. Then there exist open zero neighbourhoods $U^s \subset H^s(\Sigma_0)$, $I \subset \mathbb{R}$, $V \subset V_S$, $W^{s-\frac{5}{2}+k} \subset H^{s-\frac{5}{2}+k}(\mathbb{R}^3, \mathbb{R}^3)$ and C^k-mappings*

$$\begin{aligned} v : \ & U^s \times I \times V \times W^{s-\frac{5}{2}+k} \longrightarrow H^{s-\frac{1}{2}}(\Omega_0, \mathbb{R}^3) \ , \\ p : \ & U^s \times I \times V \times W^{s-\frac{5}{2}+k} \longrightarrow H^{s-\frac{3}{2}}(\Omega_0) \ , \\ \Pi^{(j)} : \ & U^s \times I \times V \times W^{s-\frac{5}{2}+k} \longrightarrow \mathbb{R}^3 \ , \quad j = 1, 2 \ , \end{aligned}$$

such that for $\zeta \in U^s$, $\omega \in I$, $w \in V$, $f \in W^{s-\frac{5}{2}+k}$ the quadrupel $v(\zeta, \omega, w, f)$, $p(\zeta, \omega, w, f)$, $\Pi^{(j)}(\zeta, \omega, w, f)$, $j = 1, 2$, represents the unique solution of the boundary value problem (2.33)–(2.36).

Proof. The proof is an immediate consequence of the preceding statements, cf. Propositions 2.3 and 2.4, and the implicit function theorem. ☐

Definition 2.6. Making use of the notions and notations introduced so far, we define a C^1-mapping $\mathcal{F} : U^s \times I \times V \times W^{s-\frac{3}{2}} \longrightarrow H^{s-1}$ by

$$\mathcal{F}(\zeta, \omega, w, f) := \operatorname{tr}_{\Sigma_0}\big[v(\zeta, \omega, w, f)\big].n_\zeta \ . \tag{2.51}$$

☐

The problem has been reduced to the scalar nonlinear operator equation

$$\mathcal{F}(\zeta,\omega,w,f)=0\ . \tag{2.52}$$

It seems to be natural, and it turns out to be necessary to get a unique solution, to prescribe the mass and the center of gravity of the perturbed domains:

$$\int\limits_{\Omega_\zeta}\mathrm{d}x=\int\limits_{\Omega_0}\mathrm{d}x\ ,\quad \int\limits_{\Omega_\zeta}x\,\mathrm{d}x=\int\limits_{\Omega_0}x\,\mathrm{d}x\ . \tag{2.53}$$

To this end we introduce the vector spaces V_0 and V_1 consisting of traces along Σ_0 of constant or linear functions, respectively. We set $V_{01} := V_0 \oplus V_1$. As an immediate consequence of the implicit function theorem there results a mapping $h \in C^\omega(U^s \cap (H^s \ominus V_{01}), V_{01})$, such that the domain $\Omega_{\zeta+h(\zeta)}$ has the same volume and the same center of gravity as Ω_0. Moreover, there holds

$$h(0)=0\ ,\quad D_\zeta h(0)=0\ . \tag{2.54}$$

For later purposes we denote by $P_\imath$, $\imath = 0,1,01$, the orthogonal projector onto $V_\imath$. Accordingly, we set $P_\imath^\perp := I - P_\imath$.

The modified version of the perturbation problem now reads as

$$\mathcal{F}\big(\zeta+h(\zeta),\omega,w,f\big)=0\ , \tag{2.55}$$

with ω, w, f given, and with $\zeta \in U^s \cap (H^s \ominus V_{01})$ unknown. A solution of (2.55) is a solution of the original free boundary problem too, iff

$$\Pi^{(j)}\big(\zeta+h(\zeta),\omega,w,f\big)=0\ ,\quad j=1,2\ . \tag{2.56}$$

We finish this section by proving two statements, one concerning the symmetry of solutions of (2.33)–(2.36), cf. Proposition 2.7, and another one concerning explicit representation formulae for the mappings $\Pi^{(j)}$, $j=1,2$, cf. Proposition 2.8. Both these results will prove to be useful when solving (2.55) and (2.56).

Proposition 2.7. *Let v, p, Π be the solution of* (2.33)–(2.36) *subject to the data ζ, ω, w, f. Given an orthogonal mapping $\mathcal{O}:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, such that $\omega\,(\mathcal{O}e_3 \mp e_3)=0$, the solution $v_\mathcal{O}$, $p_\mathcal{O}$, $\Pi_\mathcal{O}$ of* (2.33)–(2.36) *subject to the data $\zeta_\mathcal{O} := \zeta\circ\mathcal{O}^{-1}$, $\omega_\mathcal{O} := \pm(\det\mathcal{O})\,\omega$, $w_\mathcal{O} := \mathcal{O}w\circ\mathcal{O}^{-1}$, $f_\mathcal{O} := \mathcal{O}f\circ\mathcal{O}^{-1}$ is*

$$\begin{aligned} v_\mathcal{O} &= \mathcal{O}v\circ\mathcal{O}^{-1}\ ,\\ p_\mathcal{O} &= p\circ\mathcal{O}^{-1}\ ,\\ \Pi^{(1)}_\mathcal{O} &= \mathcal{O}\Pi^{(1)}\ ,\\ \Pi^{(2)}_\mathcal{O} &= (\det\mathcal{O})\,\mathcal{O}\Pi^{(2)}\ . \end{aligned}$$

In particular, from $\zeta_\mathcal{O}=\zeta$, $\omega_\mathcal{O}=\omega$, $w_\mathcal{O}=w$, $f_\mathcal{O}=f$ it follows that $v_\mathcal{O}=v$, $p_\mathcal{O}=p$, $\Pi_\mathcal{O}=\Pi$.

Proof. We drop the explicit dependencies on the boundary perturbation ζ in the notation. We shall instead subscribe the letter $\mathcal{O}$ whenever a quantity refers to the corresponding transformed one. It suffices to prove that

$$\begin{aligned} -\Delta_{\mathcal{O}} v_{\mathcal{O}} + \nabla_{\mathcal{O}} p_{\mathcal{O}} + \alpha \left(\operatorname{div}_{\mathcal{O}}(v_{\mathcal{O}} \otimes v_{\mathcal{O}}) - \omega_{\mathcal{O}} \mathcal{C}_1 v_{\mathcal{O}} - \omega_{\mathcal{O}}^2 \mathcal{C}_{2,\mathcal{O}} v_{\mathcal{O}} \right) + \Pi_{\mathcal{O}}^{(1)} &= f_{\mathcal{O}} \circ \theta_{\mathcal{O}} \ , \\ \operatorname{div}_{\mathcal{O}} v_{\mathcal{O}} &= 0 \ , \\ T_{\mathcal{O}}(p_{\mathcal{O}}, v_{\mathcal{O}}).n_{\mathcal{O}} + \Pi_{\mathcal{O}}^{(2)} \wedge n_{\mathcal{O}} &= K_{\mathcal{O}} n_{\mathcal{O}} \ , \\ P_S v_{\mathcal{O}} &= w_{\mathcal{O}} \ . \end{aligned} \tag{2.57}$$

At first we have $\theta_{\mathcal{O}} = \mathcal{O}\theta \circ \mathcal{O}^{-1}$, and therefore

$$f_{\mathcal{O}} \circ \theta_{\mathcal{O}} = (\mathcal{O} f \circ \mathcal{O}^{-1}) \circ (\mathcal{O}\theta \circ \mathcal{O}^{-1}) = \mathcal{O}(f \circ \theta) \circ \mathcal{O}^{-1} \ . \tag{2.58}$$

By definition

$$\Pi_{\mathcal{O}}^{(1)} = \mathcal{O}\Pi^{(1)} \circ \mathcal{O}^{-1} \ . \tag{2.59}$$

Of course there holds

$$n_{\mathcal{O}} = \mathcal{O} n \circ \mathcal{O}^{-1} \ , \quad K_{\mathcal{O}} = K \circ \mathcal{O}^{-1} \ . \tag{2.60}$$

Moreover, given some matrix $A \in \mathbb{R}^{3\times 3}$ and vectors $a, b \in \mathbb{R}^3$, one has $A^t \left(Aa \wedge Ab \right) = (\det A) \left(a \wedge b \right)$. Hence,

$$\Pi_{\mathcal{O}}^{(2)} \wedge n_{\mathcal{O}} = (\det \mathcal{O}) \mathcal{O} \Pi^{(2)} \wedge \mathcal{O} n \circ \mathcal{O}^{-1} = \mathcal{O}(\Pi^{(2)} \wedge n) \circ \mathcal{O}^{-1} \ . \tag{2.61}$$

In what follows we shall identify the mapping $\mathcal{O}$ with the matrix representing this mapping with respect to the canonical basis in $\mathbb{R}^3$, $\mathcal{O} = (O_{ij})$. We then get the following identities ($\partial\theta$ denotes the Jacobian):

$$\begin{aligned} \partial\theta_{\mathcal{O}} &= \mathcal{O}. \left((\partial\theta) \circ \mathcal{O}^{-1} \right) .\mathcal{O}^t \ , \\ (g_{ij,\mathcal{O}}) &= \mathcal{O}.(g_{ij} \circ \mathcal{O}^{-1}).\mathcal{O}^t \ , \\ (g_{\mathcal{O}}^{ij}) &= \mathcal{O}.(g^{ij} \circ \mathcal{O}^{-1}).\mathcal{O}^t \ , \\ g_{\mathcal{O}} &= g \circ \mathcal{O}^{-1} \ , \\ (a_{ij,\mathcal{O}}) &= \mathcal{O}.(a_{ij} \circ \mathcal{O}^{-1}).\mathcal{O}^t \ . \end{aligned}$$

For the expressions on the left hand side of (2.57) we get

$$
\begin{aligned}
e_i.\left(\Delta_{\mathcal{O}} v_{\mathcal{O}}\right) &= \frac{1}{\sqrt{g_{\mathcal{O}}}}\partial_l\Big(\sqrt{g_{\mathcal{O}}}g^{lk}_{\mathcal{O}}\partial_k\left(O_{ij}v^j\circ\mathcal{O}^{-1}\right)\Big) \\
&= O_{ij}O_{l\mu}O_{k\nu}\frac{1}{\sqrt{g}\circ\mathcal{O}^{-1}}\partial_l\Big(\sqrt{g}\circ\mathcal{O}^{-1}g^{\mu\nu}\circ\mathcal{O}^{-1}\partial_k\left(v^j\circ\mathcal{O}^{-1}\right)\Big) \\
&= O_{ij}O_{l\mu}O_{k\nu}O_{k\kappa}O_{l\tau}\frac{1}{\sqrt{g}\circ\mathcal{O}^{-1}}\partial_\tau\left(\sqrt{g}g^{\mu\nu}(\partial_\kappa v^j)\right)\circ\mathcal{O}^{-1} \\
&= e_i.\left(\mathcal{O}\left(\Delta v\right)\circ\mathcal{O}^{-1}\right)\ ; \qquad (2.62)
\end{aligned}
$$

$$
\begin{aligned}
e_i.\left(\nabla_{\mathcal{O}} p_{\mathcal{O}}\right) &= \frac{1}{\sqrt{g_{\mathcal{O}}}}a_{ij,\mathcal{O}}\partial_j\left(p\circ\mathcal{O}^{-1}\right) \\
&= O_{i\mu}O_{j\nu}O_{j\alpha}\frac{1}{\sqrt{g}\circ\mathcal{O}^{-1}}a_{\mu\nu}\circ\mathcal{O}^{-1}\left(\partial_\alpha p\right)\circ\mathcal{O}^{-1} \qquad (2.63) \\
&= e_i.\left(\mathcal{O}\left(\nabla p\right)\circ\mathcal{O}^{-1}\right)\ ;
\end{aligned}
$$

$$
\begin{aligned}
e_i.\big(\operatorname{div}_{\mathcal{O}}(v_{\mathcal{O}}\otimes v_{\mathcal{O}})\big) &= \frac{1}{\sqrt{g_{\mathcal{O}}}}a_{lk,\mathcal{O}}\partial_k\left(v^l_{\mathcal{O}}v^i_{\mathcal{O}}\right) \\
&= \frac{1}{\sqrt{g}\circ\mathcal{O}^{-1}}O_{l\mu}O_{k\nu}a_{\mu\nu}\circ\mathcal{O}^{-1}\partial_k\left(O_{l\alpha}O_{i\beta}v^\alpha\circ\mathcal{O}^{-1}v^\beta\circ\mathcal{O}^{-1}\right) \\
&= \frac{1}{\sqrt{g}\circ\mathcal{O}^{-1}}O_{l\mu}O_{k\nu}O_{l\alpha}O_{i\beta}O_{k\gamma}a_{\mu\nu}\circ\mathcal{O}^{-1}\partial_\gamma\left(v^\alpha v^\beta\right)\circ\mathcal{O}^{-1} \\
&= e_i.\left(\mathcal{O}\operatorname{div}(v\otimes v)\circ\mathcal{O}^{-1}\right)\ ; \qquad (2.64)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{div}_{\mathcal{O}} v_{\mathcal{O}} &= e_i.\left(\nabla_{\mathcal{O}}(O_{i\nu}v^\nu\circ\mathcal{O}^{-1})\right) \\
&= \delta_{ik}O_{k\mu}O_{i\nu}(\partial_\mu v^\nu)\circ\mathcal{O}^{-1} \qquad (2.65) \\
&= (\operatorname{div} v)\circ\mathcal{O}^{-1}\ ;
\end{aligned}
$$

$$
\begin{aligned}
e_i.T_{\mathcal{O}}(p_{\mathcal{O}},v_{\mathcal{O}}).n_{\mathcal{O}} &= \delta_{ij}\Bigg(-p\circ\mathcal{O}^{-1}\delta_{jk}+\frac{1}{\sqrt{g}\circ\mathcal{O}^{-1}}\times \\
&\quad\times\Big(O_{j\beta}O_{\alpha\gamma}O_{k\delta}O_{\alpha\mu}a_{\beta\gamma}\circ\mathcal{O}^{-1}(\partial_\mu v^\delta)\circ\mathcal{O}^{-1}+ \\
&\quad+O_{k\beta}O_{\alpha\gamma}O_{j\delta}O_{\alpha\mu}a_{\beta\gamma}\circ\mathcal{O}^{-1}(\partial_\mu v^\delta)\circ\mathcal{O}^{-1}\Big)\Bigg)O_{kl}n^l\circ\mathcal{O}^{-1} \\
&= e_i.\left(\mathcal{O}\left(T(p,v).n\right)\circ\mathcal{O}^{-1}\right)\ . \qquad (2.66)
\end{aligned}
$$

If we assume $\omega = 0$ the assertion is proved by summing up (2.58)–(2.66). In case

$\omega \neq 0$, $\omega\,(\mathcal{O}e_3 \mp e_3) = 0$ we have

$$\begin{aligned}
\mathcal{C}_1 v_{\mathcal{O}} &= 2\, v_{\mathcal{O}} \wedge e_3 \\
&= 2\,(\mathcal{O}v \wedge e_3) \circ \mathcal{O}^{-1} \\
&= 2\left(\mathcal{O}v \wedge \mathcal{O}\mathcal{O}^{-1} e_3\right) \circ \mathcal{O}^{-1} \\
&= 2\,(\det \mathcal{O})\, \mathcal{O}\left(v \wedge \mathcal{O}^{-1} e_3\right) \circ \mathcal{O}^{-1} \\
&= \pm\,(\det \mathcal{O})\, \mathcal{O}\,(\mathcal{C}_1 v) \circ \mathcal{O}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_{2,\mathcal{O}} &= \frac{1}{2}\left(\nabla r^2\right) \circ \theta_{\mathcal{O}} \\
&= \frac{1}{2}\left(\nabla r^2\right) \circ \mathcal{O} \circ \theta \circ \mathcal{O}^{-1} \\
&= \frac{1}{2}\mathcal{O}\left(\nabla r^2\right) \circ \theta \circ \mathcal{O}^{-1} \\
&= \mathcal{O}\mathcal{C}_2 \circ \mathcal{O}^{-1} \;,
\end{aligned}$$

from which the assertion follows in general. □

Proposition 2.8. *Let $\zeta \in U^s$, $\omega \in I$, $w \in V$ and $f \in W^{s-\frac{3}{2}}$ be given, $s > 4$. Then*

$$\begin{aligned}
\Pi^{(1)}(\zeta, \omega, w, f) &= \frac{1}{|\Omega_\zeta|} \int_{\Omega_\zeta} f(x)\, \mathrm{d}x \;, \\
\Pi^{(2)}(\zeta, \omega, w, f) &= -\frac{1}{2|\Omega_\zeta|} \int_{\Omega_\zeta} \left(f(x) - \Pi^{(1)}(\zeta, \omega, w, f)\right) \wedge x\, \mathrm{d}x \;.
\end{aligned}$$

Proof. From Proposition 2.5, our assumption $s > 4$ and from the well-known embedding theorems we conclude that there is a classical solution of

$$\begin{aligned}
-\Delta v + \nabla p + \alpha\left(\operatorname{div}(v \otimes v) - \omega \mathcal{C}_1 v - \omega^2 \mathcal{C}_2\right) + \Pi^{(1)} &= f \;, \\
\operatorname{div} v &= 0 \;, \\
T(p, v).n_{\Sigma_\zeta} + \Pi^{(2)} \wedge n_{\Sigma_\zeta} &= K_{\Sigma_\zeta} n_{\Sigma_\zeta} \;.
\end{aligned} \tag{2.67}$$

Here $\Pi^{(j)} = \Pi^{(j)}(\zeta, \omega, w, f)$, $j = 1, 2$, and v, p are of class C^2 and C^1 up to the boundary, respectively. We set $\Omega := \Omega_\zeta$, $\Sigma := \Sigma_\zeta$, $n := n_{\Sigma_\zeta}$, $K := K_{\Sigma_\zeta}$ and extend $n : \Sigma \longrightarrow S^2$ to a mapping $\widetilde{n} : \widetilde{\Sigma} \longrightarrow S^2$ defined in an open neighbourhood $\widetilde{\Sigma}$ of Σ. Then $K = \partial_j \widetilde{n}^j$. Now, let W denote any of the functions e_k or $e_k \wedge x$, $k = 1, 2, 3$.

Then, dropping the explicit dependencies of $\widetilde{n}$ and W on $x \in \Omega$, we have

$$\begin{aligned}\int\limits_{\Sigma} \partial_j \widetilde{n}^j n^k W^k \, \mathrm{d}o_x &= \int\limits_{\Omega} \partial_k \left(\partial_j \widetilde{n}^j W^k\right) \mathrm{d}x \\ &= \int\limits_{\Omega} \partial_{jk} \widetilde{n}^j W^k \, \mathrm{d}x \\ &= \int\limits_{\Sigma} \widetilde{n}^j \partial_k \widetilde{n}^j W^k \, \mathrm{d}o_x - \int\limits_{\Omega} \partial_k \widetilde{n}^j \partial_j W^k \, \mathrm{d}x \, . \end{aligned} \tag{2.68}$$

Since $\widetilde{n}^j \widetilde{n}^j = 1$, the first summand in (2.68) vanishes. Since $\partial_j W^k + \partial_k W^j = 0$ and $\int_\Omega \partial_k \widetilde{n}^j \partial_j W^k \, \mathrm{d}x = \int_\Sigma n^j n^k \partial_j W^k \, \mathrm{d}o_x$, the second expression in (2.68) vanishes as well, and we have thus proved that

$$\int\limits_{\Sigma} K n . W \, \mathrm{d}o_x = 0 \, . \tag{2.69}$$

Thus, multiplying the right hand side of (2.67) by W and integrating about Ω, the proposition is an immediate consequence of Greens formula (2.30). □

3 Equilibrium figures

In order to resolve the equation (2.55) for $\zeta = \zeta(\omega, w, f)$ we wish to apply successive approximations. It is therefore necessary to compute the linearization about some given initial configuration.

Proposition 3.1. *Let $\omega_0 \in I$, $w_0 \in V$, and let $f_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a smooth vector field. Setting $v_0 := v(0, \omega_0, w_0, f_0)$, $p_0 := p(0, \omega_0, w_0, f_0)$, $\Pi_0 := \Pi(0, \omega_0, w_0, f_0)$ we have for $\eta \in H^s$:*

$$D_\zeta \mathcal{F}(0, \omega_0, w_0, f_0)\eta = -\operatorname{tr}_{\Sigma_0}[v_0] . \nabla\eta + \left(\frac{\partial v_0}{\partial n_0} . n_0\right) \eta + \operatorname{tr}_{\Sigma_0}[\dot{v}] . n_0 \, , \tag{3.1}$$

where $\dot{v} \in H^{s-\frac{1}{2}}(\Omega_0, \mathbb{R}^3)$ depends linearly and bounded on $\eta \in H^s(\Sigma_0)$ and is to be determined from the following system of equations:

$$\begin{aligned} -\Delta \dot{v} + \nabla \dot{p} + \dot{\Pi}^{(1)} + A\dot{v} &= 0 \, , \\ \operatorname{div} \dot{v} &= 0 \, , \\ T(\dot{p}, \dot{v}) . n_0 + \dot{\Pi}^{(2)} \wedge n_0 &= B\eta \, , \\ P_S \dot{v} &= 0 \, , \end{aligned}$$

where A, B are given by

$$\begin{aligned} A\dot{v} &= \alpha\big(\operatorname{div}(\dot{v} \otimes v_0) + \operatorname{div}(v_0 \otimes \dot{v}) - \omega_0 \mathcal{C}_1 \dot{v}\big) \, , \\ B\eta &= \big(D_\zeta K(0)\eta\big) n_0 + K(0)\big(-\nabla\eta\big) - \\ &\quad - \left[T(p_0, v_0).(-\nabla\eta) + \left(\frac{\partial T(p_0, v_0)}{\partial n_0} . n_0\right) \eta + \Pi_0^{(2)} \wedge (-\nabla\eta)\right] . \end{aligned}$$

Proof. By (2.51) and (2.27) we have

$$D_\zeta \mathcal{F}(0,\omega_0,w_0,f_0)\eta = \mathrm{tr}_{\Sigma_0}[v_0]\,.\,(-\nabla\eta) + \mathrm{tr}_{\Sigma_0}[D_\zeta v(0,\omega_0,w_0,f_0)\eta]\,.n_0\ .$$

The rest follows by setting

$$\begin{aligned} D_\zeta v(0,\omega_0,w_0,f_0)\eta &= (x.\nabla v_0)\,\widetilde{\eta} + \dot{v}\ ,\\ D_\zeta p(0,\omega_0,w_0,f_0)\eta &= (x.\nabla p_0)\,\widetilde{\eta} + \dot{p}\ ,\end{aligned}$$

and by "computing" - as described in Proposition 2.1 and Remark 2.2 - the boundary value problem satisfied by $\dot{v}$, $\dot{p}$, $\dot{\Pi} := D_\zeta\Pi(0,\omega_0,w_0,f_0)$. □

Proposition 3.2. *Let let*

$$\mathcal{A} := D_\zeta\mathcal{F}(0,0,0,0) \in \mathcal{L}(H^s,H^{s-1})\ . \tag{3.2}$$

Then, one has

$$\ker\mathcal{A} = V_0\oplus V_1\ ,\quad \operatorname{coker}\mathcal{A} = V_0\oplus V_1\ . \tag{3.3}$$

Proof. From Proposition 3.1 we deduce that $\mathcal{A} = \Lambda\circ D_\zeta K(0)$, where Λ is a kind of "scalar" Neumann-to-Dirichlet operator for the Stokes operator: More precisely, $\Lambda\eta = \mathrm{tr}_{\Sigma_0}[\dot{v}]\,.n$, where

$$\begin{aligned} -\Delta\dot{v} + \nabla\dot{p} + \dot{\Pi}^{(1)} &= 0\ ,\\ \operatorname{div}\dot{v} &= 0\ ,\\ T(\dot{p},\dot{v}).n_0 + \dot{\Pi}^{(2)}\wedge n_0 &= \eta\, n_0\ ,\\ P_S\dot{v} &= 0\ .\end{aligned}$$

One has

$$\ker\Lambda = V_0\oplus V_1\ ,\quad \operatorname{coker}\Lambda = V_0\oplus V_1\ . \tag{3.4}$$

On the other hand we know that

$$D_\zeta K(0) = \Delta_{\Sigma_0} + 2I\ , \tag{3.5}$$

with Δ_{Σ_0} the Laplace-Beltrami operator. By expansions into spherical harmonics we get

$$\ker(\Delta_{\Sigma_0}+2I) = V_1\ ,\quad \operatorname{coker}(\Delta_{\Sigma_0}+2I) = V_1\ . \tag{3.6}$$

Putting (3.4) and (3.6) together, the conclusion follows. □

Theorem 3.3. *Let $s>4$. Then there exist positive real numbers δ_j, $j=1,2,3$, and a C^1-mapping $Z=Z(\omega,w,f)$, such that for $|\omega|<\delta_1$, $\|w\|<\delta_2$, $\|f\|_{s-\frac{3}{2}}<\delta_3$ and $\zeta\in U^s$ the equivalence*

$$P_1^\perp\mathcal{F}\big(\zeta+h(\zeta),\omega,w,f\big) = 0 \quad\Longleftrightarrow\quad \zeta = Z(\omega,w,f)$$

is valid.

Proof. The mapping $\widetilde{\mathcal{F}}$ defined by

$$\widetilde{\mathcal{F}}(\zeta,\omega,w,f) := P_{01}^{\perp}\mathcal{F}\big(\zeta+h(\zeta),\omega,w,f\big)$$

is a C^1-mapping. It maps a zero neighbourhood of $(U^s \cap (H^s \ominus V_{01})) \times I \times V \times W^{s-\frac{3}{2}}$ into $H^s \ominus V_{01}$. It's linearization $D_\zeta\widetilde{\mathcal{F}}(0,0,0,0) = P_{01}^{\perp}\mathcal{A}|_{H^s\ominus V_{01}}$ is bijective, cf. (3.2), the second relation in (2.54) and (3.3). The proof of Theorem 3.3 with $P_1^{\perp}$ replaced by $P_{01}^{\perp}$ is now an immediate consequence of the usual implicit function theorem. But the lacking equation $c := P_0\mathcal{F}\big(\zeta+h(\zeta),\omega,w,f\big) = 0$ is automatically satisfied. Indeed: from $c|\Sigma| = \int_\Sigma (v.n)(x)\,\mathrm{d}o_x = \int_\Omega \operatorname{div} v(x)\,\mathrm{d}x = 0$ it follows that $c = 0$. □

With the aid of the last theorem, our local existence problem has been reduced to a finite dimensional system of equations, consisting of (2.56) and of

$$P_1\mathcal{F}\big(\zeta+h(\zeta),\omega,w,f\big) = 0\ . \tag{3.7}$$

Most of these equations can be made automatically satisfied by imposing additional symmetry assumptions on the data. At first we shall look at the case

$$\omega = 0\ .$$

Then we have the following

Proposition 3.4. *(i) Let f, w be reflection symmetric with respect to three linearly independend hyperplanes containing the origin. Then, the equations* (3.7), (2.56) *are automatically satisfied.*

(ii) Let f, w be reflection symmetric with respect to the hyperplane $\{x_3 = 0\}$ and axially symmetric with respect to the axis $\{x_1^2 + x_2^2 = 0\}$. Then, the equations (3.7) *are automatically satisfied, and the system of equations* (2.56) *is equivalent to the equation*

$$e_3.\Pi^{(2)}\big(\zeta+h(\zeta),0,w,f\big) = 0\ . \tag{3.8}$$

Proof. We denote by $\Pi^{(0)}$ the expression on the left hand side of equation (3.7). (i): Let $a_1, a_2, a_3 \in \mathbb{R}^3$ be a system of three linearly independent 3-vectors, such that $E_k := \{x \in \mathbb{R}^3 : a_k.x = 0\}$, $k = 1,2,3$, is a hyperplane with respect to which f and w are reflection symmetric. Denoting by $\mathcal{O}_k$ the reflection about E_k, one has $\mathcal{O}_k a_k = -a_k$ and $\mathcal{O}_k b = b$ for all $b \in \mathbb{R}^3$ satisfying $b.a_k = 0$, $k = 1,2,3$. By assumption we conclude

$$\mathcal{O}_k\Pi^{(0)} = \Pi^{(0)}\ , \quad \mathcal{O}_k\Pi^{(1)} = \Pi^{(1)}\ , \quad \mathcal{O}_k\Pi^{(2)} = (\det\mathcal{O}_k)\,\Pi^{(2)} = -\Pi^{(2)}\ , \tag{3.9}$$

cf Proposition 2.7. By multiplying the first and the second identity by a_k, we get $\Pi^{(j)}.a_k = -\Pi^{(j)}.a_k$, hence $\Pi^{(j)}.a_k = 0$, $j = 0,1$, $k = 1,2,3$. So we have proved that $\Pi^{(j)} = 0$, $j = 0,1$. Now let $b_1, b_2, b_3 \in \mathbb{R}^3$ be a system of linear independent vectors satisfying $a_k.b_k = 0$, $k = 1,2,3$. By multiplying the last relation in (3.9) with b_k we get $\Pi^{(2)}.b_k = -\Pi^{(2)}.b_k = 0$, $k = 1,2,3$.

(ii): As above we conclude that $\Pi^{(j)}.e_3 = 0$, $j = 0, 1$. Denoting by $\mathcal{O}$ an arbitrary orthogonal mapping, which leaves the x_3-axis invariant, one has

$$\mathcal{O}\Pi^{(0)} = \Pi^{(0)} , \quad \mathcal{O}\Pi^{(1)} = \Pi^{(1)} , \quad \mathcal{O}\Pi^{(2)} = (\det \mathcal{O})\, \Pi^{(2)} = \Pi^{(2)} .$$

That implies $\Pi^{(j)}.e_k = 0$, $j = 0, 1, 2$, $k = 1, 2$, which proves all the statements of the Proposition. □

It remains to consider what we refer to the "axially and reflection symmetric case". We set

$$w_1(x) := e_3 \wedge x ,$$

and introduce a small real parameter σ, such that

$$w = w_\sigma = \sigma w_1 .$$

By Proposition 2.8 the remaining equation (3.8) is equivalent to

$$\int\limits_{\Omega_{Z(0,w_\sigma,f)}} f.w_1 \,\mathrm{d}x = \int\limits_{\Omega_0} \big(f.w_1\big) \circ \theta_{Z(0,w_\sigma,f)} \sqrt{g}_{Z(0,w_\sigma,f)} \,\mathrm{d}x = 0 . \tag{3.10}$$

In general, the equation (3.10) shall be intuitively regarded as an equation for determining σ as a function depending on f. We wish to show that this is indeed the case under certain assumptions on f. But of course, one can not hope to give a complete description of the solution manifold. We merely discuss here a few examples.

Example 3.5. Assume

$$\int\limits_{\Omega_0} f.w_1 \,\mathrm{d}x = 0 , \tag{3.11}$$

and assume furthermore that $f.w_1$ vanishes in an open neighbourhood of Σ_0. Then, the relation (3.10) is automatically satisfied, provided that σ is choosen sufficiently small. □

Example 3.6. Let f be the sum of a smooth gradient and a smooth poloidal vector field, viz. $f = f_1 + f_2$, $f_1 = \nabla\psi_1$, $f_2 = \operatorname{rot}\operatorname{rot}(x\psi_2)$. Concerning the notion of poloidal and toroidal vector fields, we refer to [11]. Then, the relation (3.10) is automatically satisfied no matter how large σ is. Indeed, using the same notation as above, one has

$$\int\limits_{\Omega_{Z(0,w_\sigma,f)}} f_1.w_1 \,\mathrm{d}x = \int\limits_{\Sigma_{Z(0,w_\sigma,f)}} \psi_1 n_{Z(0,w_\sigma,f)}.w_1 \,\mathrm{d}o_x - \int\limits_{\Omega_{Z(0,w_\sigma,f)}} \psi_1 \operatorname{div} w_1 \,\mathrm{d}x ,$$

and both integrands on the right hand side of the last equation vanish identically. On the other hand, we have

$$\int\limits_{\Omega_{Z(0,w_\sigma,f)}} f_2.w_1 \,\mathrm{d}x = 0$$

because of the identity

$$\operatorname{rot}\big(\operatorname{rot}(x\psi_2)\big) = \nabla\big(\psi_2 + x.\nabla\psi_2\big) - x\Delta\psi_2\ ,$$

the proof of which is left to the reader. □

Example 3.7. Assume $f = f_\varepsilon = \varepsilon f_1$, where ε is a small real parameter and f_1 is an axially and reflection symmetric toroidal force density, $f_1(x) = \operatorname{rot}(x\psi(x))$, satisfying

$$\int_{\Omega_0} f_1.w_1\, \mathrm{d}x = 0\ . \tag{3.12}$$

Furthermore, we assume that ψ is analytic. As f_1 is toroidal, the velocity component v of the solution of (2.33)–(2.37) is toroidal as well. Note that $\Delta(\operatorname{rot}(x\psi)) = \operatorname{rot}(x\Delta\psi)$, $\Delta(\operatorname{rot}(\operatorname{rot}(x\psi))) = \operatorname{rot}(\operatorname{rot}(x\Delta\psi))$. Therefore, and in view of (3.12), we have for the solution $\zeta(\sigma,\varepsilon) := Z(0, w_\sigma, f_\varepsilon)$ of the modified free boundary problem an expansion of the form

$$\zeta(\sigma,\varepsilon) = \frac{1}{2}\varepsilon^2\zeta_{02} + \varepsilon\sigma\zeta_{11} + \frac{1}{2}\sigma^2\zeta_{20} + o(\varepsilon^2+\sigma^2)\ . \tag{3.13}$$

Then, if

$$\int_{\Sigma_0} f_1.w_1\,\zeta_{20}\ \mathrm{d}o_x \neq 0\ , \tag{3.14}$$

the equation (3.10) can be uniquely resolved for $\sigma = \sigma(\varepsilon)$ in the neighbourhood of any $(\sigma_0, 0)$, $\sigma_0 \neq 0$ small. The condition (3.14) is satisfied, for instance, if $f_1 = w_1$, since ζ_{20} is proportional along Σ_0 to the second order Legendre function $P_2 = P_2(x_3)$, cf. [10, (7.2;1)]. □

Remark 3.8. The case

$$\omega \neq 0$$

can be treated similarly. But note that only in the axially and reflection symmetric case we can prove the existence of solutions of the original free boundary problem with the aid of Proposition 2.7.

Remark 3.9. Recall that the model (1.3)–(1.6) is only suitable, if the pressure p_{ext} outside Ω is equal to a constant. Note that changing p_{ext} by an additive constant does not affect the linearization $\mathcal{A}$. When taking an arbitrary exterior pressure p_{ext} into account we get an additional expression $-p_{\mathrm{ext}}n$ on the right hand side of equation (1.5). Now, if ∇p_{ext} is not small and has the wrong sign (or no sign at all), additional eigensolutions may occur. However, the perturbation analysis carried out allows for small perturbations in $H^{s-3/2}_{\mathrm{loc}}(\mathbb{R}^3)$ of some constant exterior pressure . □

4 Outlook

With an eye on the history of the subject we shall now take self-gravitation as an additional or alternative driving force into account. For the sake of simplicity we confine ourselves to the case of a constant atmospheric pressure outside the fluid body under consideration. The basic equations then read as follows:

$$\begin{aligned}
-\Delta v + \nabla p + \alpha \big(\operatorname{div}(v \otimes v) - \omega \mathcal{C}_1 v - \omega^2 \mathcal{C}_2\big) &= f && \text{in } \Omega\,, && (4.1)\\
\operatorname{div} v &= 0 && \text{in } \Omega\,, && (4.2)\\
T(p,v).n &= (\kappa K + \gamma G)\, n && \text{along } \Sigma\,, && (4.3)\\
v.n &= 0 && \text{along } \Sigma\,. && (4.4)
\end{aligned}$$

Here, κ and γ denote non-negative parameters, not necessarily positive, and G be the Newtonian gravitational potential of the domain Ω, that is

$$G(x) = \int_{\Omega} \frac{\mathrm{d}x'}{|x' - x|}\,.$$

The remaining expressions in (4.1)–(4.4) are defined as before. Note that, as the gravitational force $f_{\mathrm{g}} = \nabla(\gamma G)$ is a gradient, it's potential γG can be absorbed into the pressure, and consequently into the boundary conditions (4.3).

The perturbation problem associated with (4.1)–(4.4) is nothing but (2.33)–(2.37), with K_ζ replaced by $\kappa K_\zeta + \gamma G_\zeta$, where G_ζ is defined by

$$G_\zeta(x) := \int_{\Omega_\zeta} \frac{\mathrm{d}x'}{|x' - \theta_\zeta(x)|}\,. \tag{4.5}$$

It is well known that from (4.5) we get via $G(\zeta) := G_\zeta$ a mapping

$$G \in C^\omega(U^s, H^s)\,, \tag{4.6}$$

with linearization

$$D_\zeta G(0) = \frac{\partial G_0}{\partial n_0} I + S = -\frac{4\pi}{3} I + S\,,$$

where S is the single-layer potential operator along Σ_0, viz.

$$\big(S\eta\big)(x) = \int_{\Sigma_0} \frac{\eta(x')}{|x - x'|}\,\mathrm{d}o_{x'}\,.$$

The reader is referred to [9], [10] for proofs of these statements. In view of (4.6) the new problem is just a smooth and compact perturbation of the previous one, and the Propositions 2.4, 2.7 and 2.8 immediately extend to the new situation. We define $\mathcal{F}$ and $\mathcal{A}$ as before, cf. Definition 2.6 and (3.2). By expansions into spherical harmonics it is easily seen, that (3.3) remains valid, as long as $\kappa > 0$ and $\gamma \geqslant 0$.

Hence, as long as $\kappa > 0$ and $\gamma \geqslant 0$, the corresponding perturbation problem can be treated as before.

However, the mathematical analysis of the model (4.1)–(4.4) carried out so far depends heavily on the nice properties of the mean curvature operator K. So what happens, if K is removed? It turns out that the corresponding perturbation problem is in some sense degenerated, even if $\gamma > 0$. Indeed: The nonlinear operator $\mathcal{F}(\,.\,,\omega,w,f)$ cannot be better than of order -1 with respect to the scale H^s. But if $\kappa = 0$ and $\gamma > 0$ the inverse linearization is of order -1 too, $\mathcal{A}^{-1} \in \mathcal{L}(H^{s-1}, H^{s-2})$, and so in applying the usual successive iteration procedure

$$\zeta_{n+1} = \zeta_n - \mathcal{A}^{-1}\mathcal{F}(\zeta_n,\omega,w,f)\ , \quad n = 0,1,\dots\ ,$$

one looses two derivatives in each step. Hence, the results obtained so far cannot be carried over straightforwardly to the case of vanishing surface tension. In case $\gamma = 0$, the situation seems to be even more complicated, because the linearization about the static ball-shaped solution vanishes identically.

During the past decades, a number of methods have been developed to overcome difficulties of the kind at issue, among them generalized implicit function theorems, cf. [13], and quasilinearization techniques, cf. [7]. In his paper [4], J. Bemelmans used an abstract theorem borrowed from [13] in order to prove some local existence results for (4.1)–(4.4) when $\kappa = 0$ and $\gamma > 0$. But the argumentation in [4] is not satisfying in so far as a crucial property of the so-called approximate inverse linearization could have been only established by a tricky construction, namely by introducing spaces of functions depending on the "intermediate" boundary perturbations ζ_n, $n = 1,2,\dots$. A complete and rigorous discussion of the linearized problem remains still open, and it is unknown, whether this gap can be bridged by the method proposed in [4]. On the other hand, there seems to be no literature at all concerning (4.1)–(4.4) when $\kappa = \gamma = 0$.

An observation, which leads to a partial solution of the problem, is the following: Consider (2.33)–(2.37) with K_ζ replaced by γG_ζ, $\gamma \geqslant 0$. Then the right hand side of equation (2.35) is of order -1 rather than of order -2, hence the mapping $v(\,.\,,\omega,w,f)$ is of order 0 with respect to the scale H^s. Since the mapping $\zeta \longmapsto n_\zeta$ is a first order partial differential operator, the mapping $\mathcal{F}(\,.\,,\omega,w,f)$ is a quasidifferential operator. It turns out, that this fact can be used in order to derive a-priori estimates under certain solvability conditions on the linearized problem, cf. [7]. It should be mentioned, that already in [4] a reference has been made to Kato's paper [7]. However, the question of applicability of these methods to the problem at hand remained open.

Note that a different way to treat the problem as a scalar operator equation would be to define

$$\mathcal{G}(\zeta,f) := n_\zeta . T_\zeta\big(p(\zeta,f), v(\zeta,f)\big) . n_\zeta\ ,$$

where $p(\zeta,f), v(\zeta,f)$ solve a "mixed" Dirichlet-Neumann boundary value problem. We have dropped ω, w and Π for simplicity. Of course, both the equations $\mathcal{F} = 0$

and $\mathcal{G} = 0$ represent the same problem, and as long as $\kappa > 0$ - the regular case - both these equations can be solved via the implicit function theorem. But if we assume $\kappa = 0$ - the degenerate case - the structure of the nonlinear operator $\mathcal{F}$ is by far more favourable.

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