ASYMPTOTICALLY OPTIMAL WEIGHTED NUMERICAL INTEGRATION

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ABSTRACT. We study numerical integration of Hölder–type functions with respect to weights on the real line. Our study extends previous work by F. Curbera, [2] and relies on a connection between this problem and the approximation of distribution functions by empirical ones. The analysis is based on a lemma which is important within the theory of optimal designs for approximating stochastic processes.

As an application we reproduce a variant of the well known result for weighted integration of Brownian paths, see e.g., [8].

1. Introduction, Problem Formulation

The present study is initiated by work of F. Curbera [2], which was devoted to asymptotically optimal numerical quadrature of Lipschitz functions with respect to a Gaussian weight. We generalize this to a broad class of possible weight functions.

To be precise, given $1 \leq p < \infty$ let $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ be an *integrable* function which moreover satisfies:

(I_p)
(i) φ is a non vanishing bounded continuous function.
(ii) $\int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx < \infty$.

Remark 1. The integrability of the weight function is certainly necessary to study integration, since otherwise constant functions would not be integrable. Requirement (i) implies that we concentrate on weights, which are regular on bounded intervals, hence no singularities are allowed there. All we are interested in is the behavior for $|x| \to \infty$, which is controlled by requirement (ii). In view of Hölder's Inequality this is certainly fulfilled for weights possessing certain moments, i.e., for which there is $\varepsilon > 0$, such that $\int_{\mathbb{R}} |x|^{\frac{1+\varepsilon}{p}} \varphi(x) dx < \infty$.

For a given real function $f\colon \mathbb{R} \to \mathbb{R}$ we let

$$I_\varphi(f) := \int_{\mathbb{R}} f(x)\varphi(x)dx.$$

We aim at approximating $I_\varphi(f)$ by using a quadrature formula

$$\mathrm{u}(f) := \sum_{j=1}^{n} c_j f(x_j),$$

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where n is a number of knots, while $c_j,\quad j=1,\dots,n$ and $x_j,\quad j=1,\dots,n$ are weights and knots, respectively. The error of such quadrature rule u at a function f is given by

$$e(f,\mathrm{u}) := |I_\varphi(f) - \mathrm{u}(f)|.$$

We shall apply quadrature rules to the integration of Hölder functions f with constant bounded by L, i.e., for $1 < q < \infty$ we denote

$$\mathcal{F}_q(L) := \{f\colon \mathbb{R} \to \mathbb{R} \quad \text{abs. cont. with derivative } f' \text{ and } \|f'\|_q \le L\}.$$

In case $q = \infty$ this is identified with the space of Lipschitz functions f satisfying $|f(y) - f(x)| \le L|y - x|,\quad x, y \in \mathbb{R}$. Thus we are interested in the overall error of a quadrature rule u given by

$$e(\mathcal{F}_q(L),\mathrm{u}) := \sup_{f\in\mathcal{F}_q(L)} e(f,\mathrm{u}).$$

The important quantity under consideration is

$$e_n(\mathcal{F}_q(L),\varphi) := \inf_{\mathrm{u}\in\mathcal{Q}_n} e(\mathcal{F}_q(L),\mathrm{u}),$$

where the infimum is taken over all quadrature rules u using at most n knots.

Without loss of generality we shall assume $\int_{\mathbb{R}} \varphi(x)dx = 1$ throughout, although the formulation of the results does not differ for other normalization.

We shall prove the following

Theorem 1. *Given $1 < q \le \infty$, let the weight function φ satisfy (I_p) for $\frac{1}{p} = 1 - \frac{1}{q}$. Then we have*

$$\lim_{n\to\infty} n e_n(\mathcal{F}_q(L),\varphi) = \frac{L}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}}dx\right)^{(p+1)/p}. \tag{1}$$

A sequence of asymptotically optimal quadrature rules is provided by

$$\mathrm{u}_n(f) := \sum_{j=1}^n c_{j,n} f(x_{j,n}),\quad f \in \mathcal{F}_q(L),$$

with knots determined by

$$\int_{x_{j,n}}^{x_{j+1,n}} \varphi(x)^{\frac{p}{p+1}}dx = \frac{1}{n+1}\int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}}dx,\quad j = 0,\dots,n, \tag{2}$$

(where $x_{0,n} := -\infty$). Asymptotically optimal weights are given by

$$c_{1,n} := \int_{x_{0,n}}^{x_{2,n}} \varphi(x)dx - \frac{1}{2}\int_{x_{1,n}}^{x_{2,n}} \varphi(x)dx,$$

$$c_{j,n} := \frac{1}{2}\int_{x_{j-1,n}}^{x_{j+1,n}} \varphi(x)dx,\quad j = 2,\dots,n-1,$$

$$c_{n,n} := \int_{x_{n-1,n}}^{1} \varphi(x)dx - \frac{1}{2}\int_{x_{n-1,n}}^{x_{n,n}} \varphi(x)dx.$$

Remark 2. The above asymptotically optimal weights may hardly be calculated in most cases. But we have the following approximation.

$$c_{j,n} = \frac{\varphi(\xi_{j,n})(x_{j+1,n} - x_{j-1,n})}{2}, \quad j = 2, \ldots, n-1,$$

where the $\xi_{j,n}$ were obtained using the Mean Value Theorem. Further we have

$$\int_{x_{j-1,n}}^{x_{j+1,n}} \varphi(x)^{\frac{p}{p+1}} dx = \frac{2}{n+1} \int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx,$$

which leads to

$$(x_{j+1,n} - x_{j-1,n})\, \varphi(\eta_{j,n})^{\frac{p}{p+1}} = \frac{2}{n+1} \int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx,$$

again by the Mean Value Theorem, such that we derived the asymptotic expression

$$c_{j,n} \asymp \frac{\varphi(x_{j,n})^{\frac{1}{p+1}}}{n+1} \int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx, \quad j = 1, \ldots, n.$$

Thus in practice the optimal weights may be replaced by

$$\tilde{c}_{j,n} := \frac{\varphi(x_{j,n})^{\frac{1}{p+1}}}{n+1} \int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx, \quad j = 1, \ldots, n.$$

Although formally these weights do not obey the necessary condition $\sum_{j=1}^{n} \tilde{c}_{j,n} = 1$, they work well in many cases as reported in [2].

Example. As an example we exhibit the result proven in [2]. We let

$$\varphi_\sigma(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

for some $\sigma > 0$. Theorem 1 with $p = 1$ yields

$$\lim_{n\to\infty} n e_n(\mathcal{F}_q(L), \varphi_\sigma) = L\sigma\sqrt{\frac{\pi}{2}},$$

which corresponds to [2, page 16], by noting that n there corresponds to $2n+1$ here. We also obtain asymptotically optimal quadrature rules. As a special instance we recover the asymptotic quadrature rule provided in [2, Thm. 3]. However we do not pay attention to results concerning additional properties, although the regular sequence of knots described above in (2) will be distributed symmetrically for odd n.

For a recent publication concerning rigorous results on existence and uniqueness of optimal knots we refer to [1].

The proof of this theorem will follow from a result on optimal approximation of probability distribution functions by empirical ones. This problem of convergence of probability distributions is made precise now.

Suppose we are given two distribution functions F and G on the real line possessing pth absolute moments. In this case the distance between these distributions can be measured in the L_p–sense, see [6, Ch. 3.2] for more details, by letting

$$\theta_p(F,G) := \left(\int_{\mathbb{R}} |F(x)-G(x)|^p dx\right)^{1/p}.$$

So we may ask for approximating a given distribution function F by an empirical one, i.e., a step function

$$Q(x) := \sum_{j=1}^{n} c_j \chi_{(-\infty,x)}(x_j), \quad x \in \mathbb{R},$$

for a finite number n of steps. Thus we ask for

$$e_n(F,p) := \inf\left\{\theta_p(F,Q), \quad Q \text{ has at most } n \text{ steps}\right\}. \tag{3}$$

Though there is vast literature concerning probability metrics, see [6] for further references, this specific type of questions does not seem to be settled. We shall prove the following

Theorem 2. *Let F be a probability distribution function possessing a density function satisfying (I_p). Then we have*

$$\lim_{n\to\infty} n e_n(F,p) = \frac{1}{2}\left(\frac{1}{p+1}\right)^{1/p} \left(\int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx\right)^{(p+1)/p}. \tag{4}$$

An asymptotically optimal sequence $x_{j,n}$, $j = 1,\dots,n$ and corresponding jump heights $c_{j,n}$ are provided as stated in Theorem 1.

The proof of Theorem 2 relies on a general lemma, see the Basic Lemma below, which turned out to be important within a different area, the *optimal design problem* for the approximation of stochastic processes, studied by [7] and many others. We refer to [4, 5] for further information. Most of the arguments required to prove the Basic Lemma can be found there.

Suppose we were able to prove Theorem 2. Then it is an easy task to complete the

Proof of Theorem 1. We first observe that any quadrature rule $\mathrm{u} = \sum_{j=1}^{n} c_j \delta_{x_j}$ with finite error has to integrate constant functions exactly, which amounts to $\sum_{j=1}^{n} c_j = 1$. Thus every quadrature rule can be assigned a distribution function Q via

$$Q(x) := \sum_{j=1}^{n} c_j \chi_{(-\infty,x)}(x_j), \quad x \in \mathbb{R}.$$

Moreover we may rewrite for any function $f \in \mathcal{F}_q(L)$ and quadrature rule u the respective error by

$$e(f,\mathrm{u}) = \left|\int_{\mathbb{R}} f dF - \int_{\mathbb{R}} f dQ\right|,$$

where F is the distribution function corresponding to the weight φ. This yields, using integration by parts,

$$\sup_{f\in\mathcal{F}_q(L)} e(f,\mathrm{u}) = L \sup_{f\in\mathcal{F}_q(1)} e(f,\mathrm{u}) = L\theta_p(F,Q),$$

for the last equality see [6, Ex. 4.3.2]. This means $e_n(\mathcal{F}_q(L), \varphi) = Le_n(F, p)$, completing the proof of Theorem 1. □

Below we are first going to discuss the Basic Lemma and proceed by proving Theorem 2.

Finally we establish a further (equivalent for p=2) problem, the approximate computation of weighted integrals of paths of stochastic processes, as initiated in [7], see the compound discussion in [8].

2. The Basic Lemma

Notion and notation within this section is close to [5]. Let $\alpha\colon (0,1) \to \mathbb{R}^+$ be continuous satisfying

$$(II_p) \qquad \int_0^1 \alpha(t)^{1/(p+1)} dt < \infty \quad \text{and} \quad \int_0^1 \frac{1}{\alpha(t)^{p/(p+1)}} dt < \infty.$$

Each design $(t_1, \dots, t_n)$ of knots in $(0,1)$ with $t_j \le t_{j+1}, \quad j = 0, \dots, n$, where $t_0 := 0$ and $t_{n+1} = 1$, we assign a partition

$$\Pi := \{\Delta_j = [t_j, t_{j+1}), \quad j = 0, \dots, n\}$$

of $(0,1)$. A sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ is said to be *uniformly fine* if for every $0 < a < b < 1$

$$\inf_{n\in\mathbb{N}} \max_{0\le j\le n} |\Delta_{j,n} \cap [a,b]| = 0.$$

The aim of this section is to prove the following

Basic Lemma. (i) *For any continuous function α satisfying (II_p), uniformly fine sequence Π_n of partitions and choice of $\xi_{j,n} \in \Delta_{j,n}$ we have*

$$\liminf_{n\to\infty} n^p \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} \ge \left(\int_0^1 \alpha(t)^{1/(p+1)} dt \right)^{p+1}.$$

(ii) *Moreover, if partitions Π_n are chosen such that*

$$(5) \qquad n \max_{0\le j\le n} \int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} dt \longrightarrow \int_0^1 \alpha(t)^{1/(p+1)} dt,$$

then we even have

$$(6) \qquad \lim_{n\to\infty} n^p \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} = \left(\int_0^1 \alpha(t)^{1/(p+1)} dt \right)^{p+1}.$$

Proof. As mentioned above the statements of the Basic Lemma are essentially contained in [5, Lemma 3 and Thm. 1]. We briefly sketch the arguments for the convenience of the reader.

First notice, that (i) is certainly true for subsequences, say $n_k, \quad k \in \mathbb{N}$ along which $\max_{0\le j\le n_k} |\Delta_{j,n} \cap [a,b]| > 0$. Hence we assume that $\lim_{n\to\infty} |\Delta_{j,n} \cap [a,b]| = 0$.

To prove (i) fix any interval $[a,b] \subset (0,1)$. For $n \in \mathbb{N}$ let $I_n := \{j, \quad \Delta_{j,n} \subset [a,b]\}$. Hölder's Inequality yields

$$n^p \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} \geq \left(\sum_{j \in I_n} \alpha(\xi_{j,n})^{1/(p+1)} |\Delta_{j,n}| \right)^{p+1}.$$

The right–hand side sum is the Darboux sum for the integral $\int_a^b \alpha(t)^{1/(p+1)} dt$, which is the only possible limit. Thus

$$\liminf_{n \to \infty} n^p \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} \geq \left(\int_a^b \alpha(t)^{1/(p+1)} dt \right)^{p+1}.$$

Since this is valid for every choice of $[a,b]$ the proof of (i) is complete.

The proof of (ii) is also based on Hölder's Inequality, which provides under assumption (II_p) for any interval $\Delta \subset [0,1]$

$$|\Delta|^{p+1} \leq \left(\int_\Delta \alpha(t)^{1/(p+1)} dt \right)^p \left(\int_\Delta \frac{1}{\alpha(t)^{p/(p+1)}} dt \right).$$

We use this to derive

$$\begin{aligned} n^p \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} &\leq \sum_{j=0}^{n} \left(n \int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} dt \right)^p \left(\int_{\Delta_{j,n}} \frac{\alpha(\xi_{j,n})}{\alpha(t)^{p/(p+1)}} dt \right) \\ &\leq \left(n \max_{0 \leq j \leq n} \int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} dt \right)^p \left(\int_0^1 \sum_{j=0}^{n} \frac{\alpha(\xi_{j,n})}{\alpha(t)^{p/(p+1)}} \chi_{\Delta_{j,n}}(t) dt \right). \end{aligned} \tag{7}$$

The right–hand side integral in (7) converges to $\int_0^1 \alpha(t)^{1/(p+1)} dt$ by our assumption (II_p). Moreover the assumption (5) finally ensures

$$\limsup_{n \to \infty} n^p \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} \leq \left(\int_0^1 \alpha(t)^{1/(p+1)} dt \right)^{p+1}.$$

Together with (i) this completes the proof of the lemma. □

3. Proof of Theorem 2

In this section we turn to the proof of Theorem 2.

Let $Q(x) := \sum_{j=1}^n c_j \chi_{(-\infty,x)}(x_j), \quad x \in \mathbb{R}$ be any empirical distribution function. We substitute $t := F(x)$ and rewrite

$$\theta_p(F,Q)^p = \int_0^1 |t - Q(F^{-1}(t))|^p \frac{dt}{\varphi(F^{-1}(t))}.$$

Letting

$$\alpha(t) := \left(\varphi(F^{-1}(t)) \right)^{-1}, \quad t \in (0,1) \tag{8}$$

and observing that $R(t) := Q(F^{-1}(t)), \quad t \in (0,1)$ is also an empirical distribution function (for the uniform distribution on $(0,1)$) this transfers to

$$\theta_p(F,Q)^p = \int_0^1 |t - R(t)|^p \alpha(t) dt. \tag{9}$$

We let $t_j := F(x_j), \quad j = 1, \ldots, n$ and $t_0 := 0, \quad t_{n+1} := 1$ and $\Delta_j := [t_j, t_{j+1})$ and write

$$R(t) = \sum_{j=1}^{n} c_j \chi_{(0,t_j)}(t), \quad t \in (0,1).$$

First we are going to prove the following

Lemma 1. *For any sequence Q_n of empirical distribution functions with $\theta_p(F,Q_n) \to 0$ the corresponding sequence Π_n of partitions $\Pi_n := \{\Delta_{j,n}, \quad j = 0, \ldots, n\}$ must be uniformly fine.*

Proof. Suppose not. Then there exists an interval $[a,b] \subset (0,1)$, a constant $0 < c < \frac{1}{4}$, where we assume $a \le c \le 1 - c \le b$ for technical reasons, and a sequence $\Delta_{j(n),n}$ with $|\Delta_{j(n),n} \cap [a,b]| \ge c$. Let $\bar{\Delta}_{j(n),n}$ denote the corresponding intervals $\bar{\Delta}_{j(n),n} := [x_{j(n),n}, x_{j(n)+1,n})$.

First suppose that $j(n) \in \{0, n\}$ infinitely often. Identifying the corresponding subsequence with $j(n)$ again and assuming without loss of generality that $j(n) = 0, \quad n \in \mathbb{N}$, we conclude that $x_{1,n} \ge F^{-1}(c)$. Since there are no knots before $x_{1,n}$ we obtain

$$\theta_p(F,Q_n)^p \ge \int_{-\infty}^{F^{-1}(c)} F(x)^p dx > 0,$$

such that this is not converging to 0. Consequently, possible limit points of $j(n)$ can only be in $\{1, \ldots, n-1\}$. In this case $\bar{\Delta}_{j(n),n} \subset [F^{-1}(a), F^{-1}(b)]$ and we infer

$$\begin{aligned}
\theta_p(F,Q_n)^p &\ge \min_c \int_{x_{j(n),n}}^{x_{j(n)+1,n}} |F(x) - c|^p dx \\
&\ge \frac{|\bar{\Delta}_{j(n),n}|^{p+1}}{2^p(p+1)} \min_{x \in [F^{-1}(a), F^{-1}(b)]} \varphi(x) \\
&\ge \frac{c^{p+1}}{2^p(p+1)} \min_{x \in [F^{-1}(a), F^{-1}(b)]} \varphi(x) > 0,
\end{aligned}$$

which also contradicts $\theta_p(F,Q_n) \to 0$. Thus the sequence Π_n must be uniformly fine. ☐

We next check that the function α satisfies (II_p), if the weight function φ was to satisfy (I_p). Indeed, employing representation (8) we confirm that

$$\int_0^1 \alpha(t)^{1/(p+1)} dt = \int_{\mathbb{R}} \frac{\varphi(x)}{\varphi(x)^{1/(p+1)}} dx = \int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} < \infty$$

and also

$$\int_0^1 \frac{1}{\alpha(t)^{p/(p+1)}} dt = \int_{\mathbb{R}} \varphi(x)^{1+p/(p+1)} dx \le \max_{x \in \mathbb{R}} \varphi(x)^{\frac{p}{p+1}} < \infty.$$

Now, let Q_n be any sequence of empirical distribution functions and R_n the respective transformed one. Returning to the representation (9) and applying the Mean Value Theorem we find a sequence $\xi_{j,n} \in \Delta_{j,n}, \quad j = 0, \ldots, n, \quad n \in \mathbb{N}$ for which

$$\begin{aligned}\theta_p(F, Q_n)^p &\geq \sum_{j=0}^{n} \alpha(\xi_{j,n}) \min_c \int_{\Delta_{j,n}} |t - c|^p dt \\ &\geq \frac{1}{2^p(p+1)} \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1}.\end{aligned}$$

By Lemma 1 and the Basic Lemma we conclude that

$$\liminf_{n\to\infty} n^p e_n^p(F, p) \geq \frac{1}{2^p(p+1)} \left(\int_0^1 \alpha(t)^{1/(p+1)} dt \right)^{p+1},$$

establishing the right hand side in (4) as a lower bound.

Next suppose the sequence Q_n of empirical distributions be chosen with knots and weights as given in Theorem 1, especially the knots $x_{j,n}$ satisfy (2). Arguing as above and remembering $t_{j,n} = F(x_{j,n})$, this transfers to

$$\int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} dt = \frac{1}{n+1} \int_0^1 \alpha(t)^{1/(p+1)} dt, \quad j = 0, \ldots, n.$$

Next we use the representation of the corresponding R_n to derive

$$\int_0^1 |t - R_n(t)|^p \alpha(t) dt = \int_0^{t_{1,n}} t^p \alpha(t) dt + \sum_{j=1}^{n-1} \alpha(\xi_{j,n}) \frac{|\Delta_{j,n}|^{p+1}}{2^p(p+1)} + \int_{t_{n,n}}^1 |1 - t|^p \alpha(t) dt.$$

Since the first and last summands above tend to 0 faster than n^{-p}, as can be seen from calculations similar to the ones in (7), they may be replaced by

$$\alpha(\xi_{0,n}) \frac{t_{1,n}^{p+1}}{2^p(p+1)} \quad \text{and} \quad \alpha(\xi_{n,n}) \frac{|1 - t_{n,n}|^{p+1}}{2^p(p+1)}, \text{ resp.}$$

without spoiling the asymptotics. We conclude

$$\lim_{n\to\infty} n^p \int_0^1 |t - R_n(t)|^p \alpha(t) dt = \lim_{n\to\infty} n^p \frac{1}{2^p(p+1)} \sum_{j=0}^{n} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1}.$$

An application of the Basic Lemma yields

$$\lim_{n\to\infty} n e_n(F, p) = \lim_{n\to\infty} n e(F, Q_n) = \frac{1}{2} \left(\frac{1}{p+1} \right)^{1/p} \left(\int_{\mathbb{R}} \varphi(x)^{\frac{p}{p+1}} dx \right)^{(p+1)/p},$$

which completes the proof of Theorem 2.

Remark 3. The previous results extend in a natural way to weights which live on bounded or one–sided intervals in $\mathbb{R}$, which means that they have to satisfy appropriate versions of (I_p).

The situation of weighted integration on a finite interval has (implicitly) been treated in [8]. There the authors indicate a correspondence between the integration problem and the approximate computation of stochastic integrals, which we also stress below in Section 4.

Remark 4. Also one might include additional weights $g\colon \mathbb{R} \to \mathbb{R}^+$ which satisfy (i) in (I_p) and consider

$$\theta_p(F, G, g) := \left(\int_{\mathbb{R}} |F(x) - G(x)|^p g(x) dx \right)^{1/p}.$$

In this case the functions φ in the statements of the results have to be replaced by $\frac{\varphi}{g}$.

4. Application to weighted integration of Brownian paths

Below we are going to exploit a general principle relating the worst case error of integration to an average case one, which probably goes back to [8]. We will not use much details and refer the reader to [3], where further information as well as references are given.

Suppose we are given a Brownian motion $X := (X_t)_{t \geq 0}$, $\quad X_0 = 0$, on a probability space $(\Omega, \mathcal{F}, P)$, which has almost surely continuous paths and has covariance kernel

$$\mathbf{E}_P X_s X_t = \min \{s, t\}, \quad s, t \geq 0.$$

Given a weight as introduced above we aim at approximating

$$I_\varphi(X(\omega)) := \int_0^\infty X_t(\omega) \varphi(t) dt, \quad \omega \in \Omega,$$

by a quadrature formula

$$\mathrm{u}(X(\omega)) := \sum_{j=1}^n c_j X_{t_j}(\omega), \quad \omega \in \Omega.$$

Observe that both $I_\varphi(X)$ as well as $\mathrm{u}(X)$ are real random variables. The corresponding error is measured in mean square sense, hence

$$e^{avg}(I_\varphi, \mathrm{u}) := \left(\mathbf{E}_P |I_\varphi(X) - \mathrm{u}(X)|^2 \right)^{1/2}$$

and we let

$$e_n^{avg}(I_\varphi) := \inf_{\mathrm{u} \in \mathcal{Q}_n} e^{avg}(I_\varphi, \mathrm{u})$$

denote the nth minimal error on the average (with respect to the Wiener measure).

As before we denote by F and Q the distribution functions corresponding to the weight φ and the quadrature formula u, respectively. The main observation is as follows. For any Borel measure, say μ, on $[0, \infty)$ we let $\langle X, \mu \rangle_\omega := \int_0^\infty X_t(\omega) d\mu(t)$.

Then we obtain the following equalities.

$$
\begin{aligned}
\mathbf{E}_P|\langle X,\mu\rangle|^2 &= \mathbf{E}_P \int_0^\infty\int_0^\infty X_s(\omega)X_t(\omega)d\mu(t)d\mu(s)\\
&= \int_0^\infty\int_0^\infty \min\{s,t\}\, d\mu(t)d\mu(s)\\
&= \int_0^\infty |\mu([s,\infty))|^2 ds\\
&= \int_0^\infty |\mu([0,s))-\mu([0,\infty))|^2 ds. \qquad (10)
\end{aligned}
$$

Within our context $\mu([0,s)) = F(s) - Q(s)$. First note that $1 = F(\infty) = Q(\infty)$ is required to make (10) finite, which amounts to

$$\mathbf{E}_P|I_\varphi(X) - \mathrm{u}(X)|^2 = \int_0^\infty |F(s)-Q(s)|^2 ds.$$

Thus Theorem 2 immediately implies

Corollary 1. *If the weight obeys* (I_2) *(with integral extending from 0 to* ∞*) then*

$$\lim_{n\to\infty} n e_n^{avg}(I_\varphi) = \frac{1}{\sqrt{12}}\left(\int_0^\infty \varphi(x)^{2/3}dx\right)^{3/2}.$$

Corresponding sequences of asymptotically optimal knots and weights are given as in Theorem 1 (for $p=2$*).*

Remark 5. As mentioned above such result (on a bounded interval) is discussed in the running example in [8], see e.g., equation (3.16) there. As indicated there the condition on the weight function φ can be relaxed. The authors also establish the relation between average case integration error for a measure with given covariance and the worst case integration error over functions from the unit ball in the reproducing kernel Hilbert space. Here this relation of worst and average case errors is provided by relating Theorems 1, 2 and Corollary 1 and reads

$$e_n(\mathcal{F}_2(1),\varphi) = e_n^{avg}(I_\varphi),$$

after mentioning that $\mathcal{F}_2(1)$ is the unit ball of the reproducing kernel Hilbert space W_2^1 of the Brownian motion X.

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