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Multilevel interpolation of divergence-free vector fields

Patricio Farrell[1], Kathryn Gillow[2], Holger Wendland[3]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
email: Patricio.Farrell@wias-berlin.de

[2] University of Oxford
Mathematical Institute
University of Oxford
Woodstock Road, Oxford, OX2 6GG, England
email: kathryn.gillow@maths.ox.ac.uk

[3] University of Bayreuth
Department of Mathematics
95440 Bayreuth
Germany
email: holger.wendland@uni-bayreuth.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract:
We introduce a multilevel technique for interpolating scattered data of divergence-free vector fields with the help of matrix-valued compactly supported kernels. The support radius at a given level is linked to the mesh norm of the data set at that level. There are at least three advantages of this method: no grid structure is necessary for the implementation, the multilevel approach is computationally cheaper than solving a large one-shot system and the interpolant is guaranteed to be analytically divergence-free. Furthermore, though we will not pursue this here, our multiscale approach is able to represent multiple scales in the data if present. We will prove convergence of the scheme, stability estimates and give a numerical example.

1. Introduction

A vector field $\mathbf{v}\colon\, \Omega \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is called *divergence-free* if it satisfies

$$\operatorname{div}(\mathbf{v}) = \nabla \cdot \mathbf{v} = 0. \tag{1.1}$$

In fluid dynamics this condition is commonly used to model incompressible fluids. It is of great importance that this constraint is satisfied analytically. For example, when solving the magnetohydrodynamic equations, small numerical errors in the divergence constraint of the magnetic field ($\operatorname{div}(\mathbf{B}) = 0$) lead to unstable and unphysical numerical solutions, see Brackbill & Barnes (1980); McNally (2011).

Therefore, it is crucial to have discretisation techniques which provide analytically divergence-free approximations. One such technique is based upon matrix-valued kernels which have been investigated in Narcowich & Ward (1994); Lowitzsch (2005b,c,a); Fuselier (2008c,b); Narcowich *et al.* (2007); Fuselier *et al.* (2009). These kernels have the additional advantage that they can approximate data at scattered nodes without needing to generate a mesh which becomes, especially for larger dimensions, prohibitively expensive.

Unfortunately, for a growing number of nodes one has to cope with increasing condition numbers and computational costs. Hence, in this paper we combine and analyse for the first time this matrix-valued approximation scheme with a recently investigated multilevel strategy (see for example Le Gia *et al.* (2010); Townsend & Wendland (2013); Wendland (2010)), which reduces the computational cost significantly. The technique has the additional advantage to capture different scales in the data if present.

This paper is organised as follows. In the next section, we introduce the necessary notation and background theory. In the third section we present the analytically divergence-free approximation algorithm, prove its convergence and analyse the condition numbers of the involved matrices. In the final section, we discuss a numerical example.

Though we concentrate on analytically divergence-free approximation spaces, our approach, including convergence and stability proofs, immediately carries over to analytically curl-free approximation spaces if designed similarly, see Fuselier (2008c).

1.1 *Notation*

For non-negative integer k and $\Omega \subseteq \mathbb{R}^d$ let $H^k(\Omega)$ denote the Sobolev space with differentiability order k and integrability power $p = 2$. Define for $u \in H^k(\Omega)$ the Sobolev norms

$$\|u\|^2_{H^k(\Omega)} := \sum_{|\boldsymbol{\alpha}|\leqslant k} \|D^{\boldsymbol{\alpha}} u\|^2_{L_2(\Omega)}. \tag{1.2}$$

For $\Omega = \mathbb{R}^d$ there is another way to characterise and generalise Sobolev spaces via Fourier transforms by defining

$$H^\sigma(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \mid \widehat{f}(\cdot)(1 + \|\cdot\|_2^2)^{\sigma/2} \in L_2(\mathbb{R}^d) \right\},$$

where $0 \leqslant \sigma < \infty$ can now also denote a fractional positive number (Evans, 1998, Chapter 5). The norm on this space is naturally defined by

$$\|f\|^2_{H^\sigma(\mathbb{R}^d)} := (2\pi)^{-d/2} \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\sigma d\boldsymbol{\omega},$$

which is equivalent to the norm (1.2) in the case of $\sigma = k \in \mathbb{N}_0$. Here, the Fourier transform of an integrable function $f \in L_1(\mathbb{R}^d)$ is defined to be

$$\widehat{f}(\boldsymbol{\omega}) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i\mathbf{x}^T\boldsymbol{\omega}} d\mathbf{x}, \qquad \boldsymbol{\omega} \in \mathbb{R}^d$$

and then extended to $L_2(\mathbb{R}^d)$-functions in the usual way.

Other function spaces will be introduced later on.

2. Positive Definite Matrix-Valued Kernels

A continuous scalar-valued function $\phi \colon \mathbb{R}^d \to \mathbb{R}$ is called *positive semi-definite* on $\mathbb{R}^d$ if for all $N \in \mathbb{N}$, any finite set $X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\} \subseteq \mathbb{R}^d$ of pairwise distinct points, and all $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_N)^T \in \mathbb{R}^N$, the quadratic form

$$\sum_{j=1}^N \sum_{k=1}^N \alpha_j \alpha_k \phi(\mathbf{x}_j - \mathbf{x}_k) \tag{2.1}$$

is non-negative. It is called *positive definite* on $\mathbb{R}^d$ if the quadratic form is positive for all $\boldsymbol{\alpha} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$.

We will be concerned mostly with compactly supported radial basis functions for which the Fourier transform exhibits algebraic decay. The basis function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radial if it is of the form $\phi(\mathbf{x}) = \phi_0(\|\mathbf{x}\|_2)$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. The Fourier transform of an integrable function ϕ decays algebraically, if there are two constants $c_1, c_2 > 0$ and some $\tau > d/2$ such that

$$c_1(1+\|\boldsymbol{\omega}\|_2^2)^{-\tau} \leqslant \widehat{\phi}(\boldsymbol{\omega}) \leqslant c_2(1+\|\boldsymbol{\omega}\|_2^2)^{-\tau}. \tag{2.2}$$

We will mostly be interested in radial basis functions with compact support having a Fourier transform with such a decay. Typical examples are given in Johnson (2012); Wendland (1995, 2005).

In this section, we introduce positive definite matrix-valued kernels, which we will eventually use to construct divergence-free approximants. The theory for positive definite matrix-valued kernels was originally developed by Narcowich and Ward in Narcowich & Ward (1994). We will follow an approach by Fuselier for introducing the theory Fuselier (2008c,b), which mimics some of the ideas from the scalar-valued case Wendland (2005).

Let us start by making a reasonable generalisation of positive definite scalar-valued kernels.

DEFINITION 2.1 A continuous matrix-valued kernel $\boldsymbol{\Phi}\colon \mathbb{R}^d \to \mathbb{R}^{n\times n}$ is called positive definite if it is even, $\boldsymbol{\Phi}(-\mathbf{x}) = \boldsymbol{\Phi}(\mathbf{x})$, symmetric, $\boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{\Phi}(\mathbf{x})^T$, and if

$$\sum_{j,k=1}^{N} \boldsymbol{\alpha}_j^T \boldsymbol{\Phi}(\mathbf{x}_j - \mathbf{x}_k)\boldsymbol{\alpha}_k > 0$$

for all $N \in \mathbb{N}$, all pairwise distinct $\mathbf{x}_j \in \mathbb{R}^d$ and all $\boldsymbol{\alpha}_j \in \mathbb{R}^n$, not all of them vanishing.

In this paper, we will exclusively be interested in one specific example of a matrix-valued positive definite kernel. Let $\phi\colon \mathbb{R}^d \to \mathbb{R}$ be a positive definite function in $C^2(\mathbb{R}^d)$. Then we define

$$\boldsymbol{\Phi}\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}, \qquad \boldsymbol{\Phi} := (-\Delta I + \nabla\nabla^T)\phi \tag{2.3}$$

where Δ is the Laplacian, ∇ the gradient and I the d-dimensional identity matrix. The component-wise Fourier transform is given by

$$\widehat{\boldsymbol{\Phi}}(\boldsymbol{\omega}) = \left(\|\boldsymbol{\omega}\|_2^2 I - \boldsymbol{\omega}\boldsymbol{\omega}^T\right)\widehat{\phi}(\boldsymbol{\omega}).$$

For just two space dimensions the kernel $\boldsymbol{\Phi}$ takes the form

$$\boldsymbol{\Phi} = \begin{pmatrix} -\partial_{22} & \partial_{12} \\ \partial_{21} & -\partial_{11} \end{pmatrix}\phi = \begin{pmatrix} -\frac{x_2^2}{r^2}\phi_0''(r) - \frac{x_1^2}{r^3}\phi_0'(r) & \frac{x_1x_2}{r^2}\phi_0''(r) - \frac{x_1x_2}{r^3}\phi_0'(r) \\ \frac{x_1x_2}{r^2}\phi_0''(r) - \frac{x_1x_2}{r^3}\phi_0'(r) & -\frac{x_1^2}{r^2}\phi_0''(r) - \frac{x_2^2}{r^3}\phi_0'(r) \end{pmatrix},$$

provided ϕ is radial with $\phi(\mathbf{x}) = \phi_0(r)$ where $r = \sqrt{x_1^2 + x_2^2}$. Therefore, in general matrix-valued kernels are not radial. Nevertheless, they are commonly referred to as matrix-valued radial basis functions. By construction the kernel $\boldsymbol{\Phi}$ consists of divergence-free columns and rows. It is possible to show that it is indeed positive definite in the sense of Definition 2.1. The following result essentially follows from the general theory in Narcowich & Ward (1994).

THEOREM 2.2 Let $\phi \in W_1^2(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ be positive definite. Then the kernel $\boldsymbol{\Phi}$ is positive definite.

As usual, the space $W_1^2(\mathbb{R}^d)$ consists of all functions u which are, together with their weak derivatives up to order two, in $L_1(\mathbb{R}^d)$.

2.1 *Native Spaces for Matrix-Valued Kernels*

Next, let us introduce native spaces for positive definite matrix-valued kernels. We will follow here mostly ideas from Fuselier (2008c); Wendland (2005). It is possible to develop a similar Hilbert space theory as in the scalar-valued case. Define the function space

$$\mathbf{F}_{\Phi}(\Omega) := \left\{ \sum_{j=1}^{N} \Phi(\cdot - \mathbf{x}_j)\boldsymbol{\alpha}_j \mid \mathbf{x}_j \in \Omega, \boldsymbol{\alpha}_j \in \mathbb{R}^n \right\}$$

and equip it with the inner product

$$\left(\sum_{j=1}^{N} \Phi(\cdot - \mathbf{x}_j)\boldsymbol{\alpha}_j, \sum_{k=1}^{M} \Phi(\cdot - \mathbf{y}_k)\boldsymbol{\beta}_k \right)_{\Phi} := \sum_{j=1}^{N}\sum_{k=1}^{M} \boldsymbol{\alpha}_j^T \Phi(\mathbf{x}_j - \mathbf{y}_k)\boldsymbol{\beta}_k.$$

Note that the bilinear form is indeed an inner product since Φ is assumed to be symmetric and positive definite.

As in the scalar-valued case, we form the closure with respect to the norm induced by the inner product and denote it with $\mathscr{F}_{\Phi}(\Omega)$, i. e.

$$\mathscr{F}_{\Phi}(\Omega) := \overline{\mathbf{F}_{\Phi}(\Omega)}^{\|\cdot\|_{\Phi}}.$$

Again, it is not obvious what these abstract elements in the completion actually mean. But the scalar-valued case gives a good indication what to do. To interpret these elements as functions we define for an element $\mathbf{f} \in \mathscr{F}_{\Phi}(\Omega)$ the function value

$$f_j(\mathbf{x}) := (\mathbf{f}, \Phi(\cdot - \mathbf{x})\mathbf{e}_j)_{\Phi}$$

where $\mathbf{e}_j$ denotes the jth canonical unit vector. Now we can interpret the abstract elements from the completion as continuous functions by defining a linear mapping $R\colon \mathscr{F}_{\Phi}(\Omega) \to C(\Omega, \mathbb{R}^n)$, whose jth component $(1 \leqslant j \leqslant n)$ is given by

$$R(\mathbf{f})_j(\mathbf{x}) := f_j(\mathbf{x}) = (\mathbf{f}, \Phi(\cdot - \mathbf{x})\mathbf{e}_j)_{\Phi}\,.$$

We make several remarks. Firstly, $R(\mathbf{f})(\mathbf{x})$ actually defines a continuous function because for each component we find

$$\begin{aligned} |R(\mathbf{f})_j(\mathbf{x}) - R(\mathbf{f})_j(\mathbf{y})| &= (\mathbf{f}, (\Phi(\cdot - \mathbf{x}) - \Phi(\cdot - \mathbf{y}))\mathbf{e}_j)_{\Phi} \\ &\leqslant \|\mathbf{f}\|_{\Phi} \|(\Phi(\cdot - \mathbf{x}) - \Phi(\cdot - \mathbf{y}))\mathbf{e}_j\|_{\Phi}, \end{aligned}$$

where

$$\|(\Phi(\cdot - \mathbf{x}) - \Phi(\cdot - \mathbf{y}))\mathbf{e}_j\|_{\Phi}^2 = 2\mathbf{e}_j^T \Phi(\mathbf{0})\mathbf{e}_j - 2\mathbf{e}_j^T \Phi(\mathbf{x} - \mathbf{y})\mathbf{e}_j.$$

Thus as $\mathbf{x}$ tends to $\mathbf{y}$, $R(\mathbf{f})_j(\mathbf{x})$ tends to $R(\mathbf{f})_j(\mathbf{y})$ as Φ is assumed to be continuous. Therefore each component of $R(\mathbf{f})$ is continuous, which implies that $R(\mathbf{f})$ defines a continuous function itself.

Lastly, we note that R is injective. The injectivity means that $R\colon \mathscr{F}_{\Phi}(\Omega) \to R(\mathscr{F}_{\Phi}(\Omega))$ is actually a bijective mapping and we can identify each abstract element from $\mathscr{F}_{\Phi}(\Omega)$ with a continuous function from $R(\mathscr{F}_{\Phi}(\Omega))$. Just as in the scalar-valued case, this motivates us to define the native space as the image of R.

DEFINITION 2.3 The native space of a positive definite matrix-valued kernel Φ is defined by

$$\mathcal{N}_\Phi(\Omega) := R(\mathcal{F}_\Phi(\Omega)).$$

It is equipped with the inner product

$$(\mathbf{f},\mathbf{g})_{\mathcal{N}_\Phi(\Omega)} := \left(R^{-1}\mathbf{f}, R^{-1}\mathbf{g}\right)_\Phi.$$

To simplify notation we will sometimes use $(\cdot,\cdot)_\Phi$ instead of $(\cdot,\cdot)_{\mathcal{N}_\Phi(\Omega)}$ and $\|\cdot\|_\Phi$ instead of $\|\cdot\|_{\mathcal{N}_\Phi(\Omega)}$. As in the scalar-valued case, the native space is unique (Fuselier, 2008b, Proposition 1) and there exists a kernel reproduction property.

LEMMA 2.1 (Kernel Reproduction Property) For $\mathbf{f} \in \mathcal{N}_\Phi(\Omega)$, $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\mathbf{x} \in \Omega$ we have

$$\Phi(\cdot - \mathbf{x})\boldsymbol{\alpha} \in \mathcal{N}_\Phi(\Omega),$$
$$(\mathbf{f}, \Phi(\cdot - \mathbf{x})\boldsymbol{\alpha})_\Phi = \mathbf{f}(\mathbf{x})^T \boldsymbol{\alpha}.$$

Proof. For the second property let $\mathbf{f} \in \mathcal{N}_\Phi(\Omega)$. Then, there exists a unique $\mathbf{g} \in \mathcal{F}_\Phi(\Omega)$ such that $R(\mathbf{g}) = \mathbf{f}$. We compute for $1 \leqslant j \leqslant n$

$$\begin{aligned}
\mathbf{f}(\mathbf{x})^T \mathbf{e}_j &= [R(\mathbf{g})(\mathbf{x})]^T \mathbf{e}_j = (\mathbf{g}, \Phi(\cdot - \mathbf{x})\mathbf{e}_j)_\Phi = \left(R^{-1}\mathbf{f}, \Phi(\cdot - \mathbf{x})\mathbf{e}_j\right)_\Phi \\
&= (\mathbf{f}, \Phi(\cdot - \mathbf{x})\mathbf{e}_j)_{\mathcal{N}_\Phi(\Omega)},
\end{aligned}$$

where in the last step we have used that R (and thus R^{-1} as well) leaves $\Phi(\cdot - \mathbf{x})\mathbf{e}_j$ unaltered. This also immediately implies the first property. □

Thus the native space is a Hilbert space of continuous functions on the domain Ω with reproducing kernel Φ. There is a useful characterisation for native spaces which are defined on the whole Euclidean space.

THEOREM 2.4 (Characterisation of Native Spaces) Suppose the positive definite kernel ϕ lies in $W_1^2(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$. Then the native space of the divergence-free kernel is given by

$$\mathcal{N}_\Phi(\mathbb{R}^d) = \left\{\mathbf{f} \in \mathbf{L}_2(\mathbb{R}^d) \cap \mathbf{C}(\mathbb{R}^d) \mid \|\mathbf{f}\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} < \infty, \operatorname{div}(\mathbf{f}) = 0\right\}$$

where the native space norm is given by

$$\|\mathbf{f}\|^2_{\mathcal{N}_\Phi(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2 \widehat{\phi}(\boldsymbol{\omega})} d\boldsymbol{\omega}...$$

Here $\mathbf{L}_2(\mathbb{R}^d)$ and $\mathbf{C}(\mathbb{R}^d)$ denote vector-valued function spaces with each component in $L_2(\mathbb{R}^d)$ or $C(\mathbb{R}^d)$, respectively. Proofs of this can be found in (Fuselier, 2008c, Theorem 2 and 3) as well as in (Wendland, 2009, Theorem 3.4).

2.2 *Native Spaces as Sobolev Spaces*

As in the scalar-valued case, it is possible to interpret the native space under some assumptions as some type of Sobolev space. For scalar valued kernels it is well-known that the native space of a scalar-valued function is norm-equivalent to a classical Sobolev space if the Fourier transform of the kernel has algebraic decay Wendland (2005).

We define the vector-valued Sobolev space $\mathbf{H}^\sigma(\Omega)$ to consist of those vector functions $\mathbf{u} = (u_1, \ldots, u_d)^T : \Omega \to \mathbb{R}^d$ for which each component lies in the scalar-valued Sobolev space $H^\sigma(\Omega)$. A norm on the vector-valued Sobolev space can be defined with the help of the scalar-valued Sobolev norm in the following way

$$\|\mathbf{u}\|_{\mathbf{H}^\sigma(\Omega)} := \left(\sum_{j=1}^{d} \|u_j\|^2_{H^\sigma(\Omega)} \right)^{1/2}.$$

This means we take the discrete ℓ_2 norm of the Sobolev norms of each component for the vector function $\mathbf{u}$. We introduce divergence-free Sobolev spaces,

$$\mathbf{H}^\sigma(\mathbb{R}^d, \mathrm{div}) := \left\{ \mathbf{f} \in \mathbf{H}^\sigma(\mathbb{R}^d) \mid \mathrm{div}(\mathbf{f}) = 0 \right\}.$$

With the help of these spaces we define subspaces

$$\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d) := \left\{ \mathbf{f} \in \mathbf{H}^\sigma(\mathbb{R}^d) \mid \|\mathbf{f}\|_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)} < \infty \right\},$$
$$\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d, \mathrm{div}) := \left\{ \mathbf{f} \in \mathbf{H}^\sigma(\mathbb{R}^d, \mathrm{div}) \mid \|\mathbf{f}\|_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)} < \infty \right\},$$

where

$$\|\mathbf{f}\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)} := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{f}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\sigma+1} d\boldsymbol{\omega}$$

All the spaces introduced in this section are obviously related. Let us consider a divergence-free velocity $\mathbf{u}$, which lies in the space $\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d, \mathrm{div})$ for some $\sigma > 0$. This space, however, is a subspace of $\mathbf{H}^\sigma(\mathbb{R}^d)$. Therefore, we may sometimes write $\mathbf{u}$ in the $\mathbf{H}^\sigma(\mathbb{R}^d)$ norm as well. Moreover, the following simple lemma holds:

LEMMA 2.2 (Relationship between different Sobolev Norms) For $\mathbf{u} \in \widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)$ we have

$$\|\mathbf{u}\|_{\mathbf{H}^\sigma(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)},$$

showing particularly $\widetilde{\mathbf{H}}^\sigma(\mathbb{R})^d \subseteq \mathbf{H}^\sigma(\mathbb{R}^d)$ and also $\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d, \mathrm{div}) \subseteq \mathbf{H}^\sigma(\mathbb{R}^d, \mathrm{div})$.

Proof. Using the definitions above, we obtain

$$\begin{aligned}
\|\mathbf{u}\|^2_{\mathbf{H}^\sigma(\mathbb{R}^d)} &= \sum_{j=1}^{n} \|u_j\|^2_{H^\sigma(\mathbb{R}^d)} = (2\pi)^{-d/2} \sum_{j=1}^{n} \int_{\mathbb{R}^d} |\widehat{u}_j(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\sigma d\boldsymbol{\omega} \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \sum_{j=1}^{n} |\widehat{u}_j(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\sigma d\boldsymbol{\omega} \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \|\widehat{\mathbf{u}}(\boldsymbol{\omega})\|_2^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\sigma d\boldsymbol{\omega} \\
&\leqslant (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{u}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\boldsymbol{\omega}\|_2^2)^{\sigma+1} d\boldsymbol{\omega} \\
&= \|\mathbf{u}\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)},
\end{aligned}$$

where we have used that $1 \leqslant \frac{1+\|\boldsymbol{\omega}\|_2^2}{\|\boldsymbol{\omega}\|_2^2}$. □ Now we are able to state another characterisation of the native spaces that we have discussed so far. To this end, note that if $\phi \in L_1(\mathbb{R}^d)$ has a Fourier transform satisfying (2.2) with $\tau = \sigma + 1$ and $\sigma > d/2$ then $\phi \in C^2(\mathbb{R}^d)$. This can be seen as follows. Under the assumptions, we can invoke the Fourier inversion formula to write

$$\phi(\mathbf{x}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{\phi}(\boldsymbol{\omega}) e^{i\mathbf{x}^T\boldsymbol{\omega}} d\boldsymbol{\omega}.$$

We can differentiate under the integral

$$D^\alpha \phi(\mathbf{x}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \widehat{\phi}(\boldsymbol{\omega})(i\boldsymbol{\omega})^\alpha e^{i\mathbf{x}^T\boldsymbol{\omega}} d\boldsymbol{\omega},$$

as long as the function $\boldsymbol{\omega} \mapsto \widehat{\phi}(\boldsymbol{\omega})(i\boldsymbol{\omega})^\alpha$ belongs to $L_1(\mathbb{R}^d)$. This function is continuous and for second order derivatives, i.e. $|\alpha| = 2$, it behaves for large $\boldsymbol{\omega}$ like $\|\boldsymbol{\omega}\|_2^{-2\sigma-2+|\alpha|} = \|\boldsymbol{\omega}\|_2^{-2\sigma}$, showing integrability as long as $\sigma > d/2$.

THEOREM 2.5 (Native Spaces as Sobolev Spaces) Let $\sigma > d/2$. Let ϕ generate $H^{\sigma+1}(\mathbb{R}^d)$ as its native space, i.e. $\phi \in L_1(\mathbb{R}^d)$ possesses a Fourier transform satisfying (2.2) with $\tau = \sigma + 1$. Then, the native space of the matrix-valued kernel $\Phi = (-\Delta I + \nabla\nabla^T)\phi$ is given by

$$\mathcal{N}_\Phi(\mathbb{R}^d) = \widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d, \mathrm{div}).$$

The proof follows immediately from Theorem 2.4. For details, see Fuselier (2008c); Wendland (2009).

We finish this section by introducing a vector-valued extension operator since we are mostly interested in bounded domains and therefore need to extend our locally defined Sobolev functions to functions defined on the whole Euclidean space. The following result follows from (Wendland, 2009, Proposition 3.8.) by setting the pressure component to zero.

PROPOSITION 2.6 (Extension Operator) Suppose either $d = 2$ or $d = 3$. Let $\sigma \geqslant 0$ and let $\Omega \subseteq \mathbb{R}^d$ be a simply-connected domain with $C^{k,1}$ boundary, where $k \geqslant \sigma$ is some integer. Then there exists a continuous operator

$$\mathbf{E}_{\mathrm{div}} \colon \mathbf{H}^\tau(\Omega, \mathrm{div}) \to \widetilde{\mathbf{H}}^\tau(\mathbb{R}^d, \mathrm{div}), \qquad 0 \leqslant \tau \leqslant \sigma,$$

such that

1. $(\mathbf{E}_{\mathrm{div}}\mathbf{v})|_\Omega = \mathbf{v}|_\Omega$
2. $\|\mathbf{E}_{\mathrm{div}}\mathbf{v}\|_{\widetilde{\mathbf{H}}^\tau(\mathbb{R}^d,\mathrm{div})} \leqslant C\|\mathbf{v}\|_{\mathbf{H}^\tau(\Omega,\mathrm{div})}$,

for all $\mathbf{v} \in \mathbf{H}^\tau(\Omega, \mathrm{div})$, with some $C = C_\tau > 0$.

Finally, we need a lemma from Chernih & Gia (2013). For the convenience of the reader we include its proof here.

LEMMA 2.3 Assume $\mathbf{u} \in \mathbf{H}^\sigma(\Omega, \mathrm{div})$ with $\sigma > 0$. Then

$$\int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{u}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} d\boldsymbol{\omega} \leqslant C\|\mathbf{u}\|_{\mathbf{L}_2(\Omega)}^2.$$

Proof. To prove this result, we need to discuss the extension operator $\mathbf{E}_{\mathrm{div}}$ in more detail. This operator involves the standard, bounded extension operator $E_S : H^\sigma(\Omega) \to H^\sigma(\mathbb{R}^d)$ as well a specifically constructed, bounded operator $T \colon \mathbf{H}^\sigma(\Omega, \mathrm{div}) \to \mathbf{H}^{\sigma+1}(\Omega)$ which satisfies

$$\mathbf{u} = \nabla \times T\mathbf{u}$$

for all $\mathbf{u} \in \mathbf{H}^\sigma(\Omega, \mathrm{div})$. Then $\mathbf{E}_{\mathrm{div}}$ is defined as

$$\mathbf{E}_{\mathrm{div}}(\mathbf{u}) = \nabla \times E_S T\mathbf{u},$$

where E_S is taken component-wise. For more details on the construction of this operator, we refer to Wendland (2009). Using this definition, we compute

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{u}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} d\boldsymbol{\omega} = \int_{\mathbb{R}^d} \frac{\|\boldsymbol{\omega} \times \widehat{E_S T\mathbf{u}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} d\boldsymbol{\omega} &\leqslant C \int_{\mathbb{R}^d} \|\widehat{E_S T\mathbf{u}}(\boldsymbol{\omega})\|_2^2 d\boldsymbol{\omega} \\ &\leqslant C\|E_S T\mathbf{u}\|^2_{\mathbf{L}_2(\mathbb{R}^d)} \leqslant C\|E_S T\mathbf{u}\|^2_{\mathbf{H}^1(\mathbb{R}^d)} \\ &\leqslant C\|T\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \leqslant C\|\mathbf{u}\|^2_{\mathbf{L}_2(\Omega)}.\end{aligned}$$

□

2.3 *Reconstruction Problem*

The following discussion is well-known from the theory of scalar-valued kernels which can easily be extended to matrix-valued kernels Wendland (2009). We provide it nevertheless to remind us of some of the key results. Suppose we want to reconstruct data $\mathbf{f}_1, \dots, \mathbf{f}_N \in \mathbb{R}^n$ at scattered data points $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \mathbb{R}^d$. For some coefficient vectors $\boldsymbol{\alpha}_j \in \mathbb{R}^n$, we can set up a vector-valued interpolant of the form

$$\mathbf{s}_{\mathbf{f},X} := \sum_{j=1}^{N} \Phi(\cdot - \mathbf{x}_j)\boldsymbol{\alpha}_j \tag{2.4}$$

and apply to it the interpolation conditions

$$\mathbf{s}_{\mathbf{f},X}(\mathbf{x}_k) = \mathbf{f}_k \tag{2.5}$$

for $k = 1, \dots, N$. This means we have to solve the $nN \times nN$ block matrix system

$$\begin{pmatrix} \Phi(\mathbf{x}_1 - \mathbf{x}_1) & \dots & \Phi(\mathbf{x}_1 - \mathbf{x}_N) \\ \vdots & & \vdots \\ \Phi(\mathbf{x}_N - \mathbf{x}_1) & \dots & \Phi(\mathbf{x}_N - \mathbf{x}_N) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_N \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_N \end{pmatrix}$$

or in abbreviated form

$$A_{\Phi,X}\boldsymbol{\alpha} = \mathbf{f}. \tag{2.6}$$

Note that for $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_N)^T \in \mathbb{R}^{nN}$, we have

$$\boldsymbol{\beta}^T A_{\Phi,X}\boldsymbol{\beta} = \sum_{j,k=1}^{N} \boldsymbol{\beta}_j^T \Phi(\mathbf{x}_j - \mathbf{x}_k)\boldsymbol{\beta}_k > 0,$$

since Φ is a positive definite kernel. Therefore, the matrix $A_{\Phi,X}$ is positive definite and the interpolant (2.4) with coefficient vectors determined by (2.6) exists and is unique.

Assuming the data are generated by $\mathbf{f}_j = \mathbf{f}(\mathbf{x}_j)$ with $\mathbf{f} \in \mathcal{N}_\Phi(\mathbb{R}^d)$, the interpolant $\mathbf{s}_{\mathbf{f},X}$ is also the best approximation from

$$\mathbf{V}_X = \left\{ \sum_{j=1}^{N} \Phi(\cdot - \mathbf{x}_j)\boldsymbol{\beta}_j \mid \boldsymbol{\beta}_j \in \mathbb{R}^n \right\},$$

to $\mathbf{f}$ in the native space norm. This is a direct consequence of Lemma 2.1 because for any $\mathbf{g} = \sum_{j=1}^N \Phi(\cdot - \mathbf{x}_j)\boldsymbol{\beta}_j \in \mathbf{V}_X$ we deduce the best approximation property

$$\begin{aligned}\left(\mathbf{f} - \mathbf{s}_{\mathbf{f},X}, \mathbf{g}\right)_\Phi &= \sum_{j=1}^{N} \left(\mathbf{f} - \mathbf{s}_{\mathbf{f},X}, \Phi(\cdot - \mathbf{x}_j)\boldsymbol{\beta}_j\right)_\Phi \\ &= \sum_{j=1}^{N} \left[\mathbf{f}(\mathbf{x}_j) - \mathbf{s}_{\mathbf{f},X}(\mathbf{x}_j)\right]^T \boldsymbol{\beta}_j = \mathbf{0},\end{aligned}$$

where we have used the interpolation condition (2.5) in the last step. This best approximation property immediately implies two stability results. Namely,

$$\|\mathbf{f} - \mathbf{s}_{\mathbf{f},X}\|_\Phi \leqslant \|\mathbf{f}\|_\Phi \qquad \text{and} \qquad \|\mathbf{s}_{\mathbf{f},X}\|_\Phi \leqslant \|\mathbf{f}\|_\Phi.$$

In theory, we could use this idea to interpolate divergence-free fields with a divergence-free kernel. However, just as in the scalar-valued kernel case, the condition number of the linear system (2.6) will become very large when interpolating many nodes. Also, the density of the matrices and the resulting computational cost limits the number of points, particularly, since our system now has dimension $nN \times nN$ instead of $N \times N$ in the scalar-valued case. This motivates our multilevel idea in the next section.

3. Multilevel Theory for Divergence-Free Vector Fields

We define scaled kernels via

$$\Phi_j = \Phi_{\delta_j} = (-\Delta I + \nabla\nabla^T)\phi_{\delta_j}, \tag{3.1}$$

where $\phi_\delta(\mathbf{x}) = \delta^{-d}\phi(\mathbf{x}/\delta)$ and ϕ is a function satisfying the decay condition (2.2) with $\tau = \sigma + 1$ and $\sigma > d/2$. These kernels generate for any $0 < \delta \leqslant \delta_c$ native spaces which are norm equivalent to the same Sobolev space which is determined by the smoothness of the kernel.

LEMMA 3.1 (Norm Equivalence for Scaled Matrix-Valued Kernels) Let $\delta \in (0, \delta_c]$. Let ϕ generate $H^{\sigma+1}(\mathbb{R}^d)$ with $\sigma > d/2$. Then

$$\mathcal{N}_{\Phi_\delta}(\mathbb{R}^d) = \widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d, \mathrm{div})$$

and for every $\mathbf{v} \in \widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d, \mathrm{div})$, we have

$$C_1\|\mathbf{v}\|^2_{\mathcal{N}_{\Phi_\delta}(\mathbb{R}^d)} \leqslant \|\mathbf{v}\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d,\mathrm{div})} \leqslant C_2\delta^{-2(\sigma+1)}\|\mathbf{v}\|^2_{\mathcal{N}_{\Phi_\delta}(\mathbb{R}^d)}$$

for two positive constants C_1, C_2.

Proof.

We first consider the upper bound. If $\delta \leqslant 1$, then

$$(1 + \|\boldsymbol{\omega}\|_2^2)^{\sigma+1} = \delta^{-2(\sigma+1)}(\delta^2 + \|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1} \leqslant \delta^{-2(\sigma+1)}(1 + \|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1}.$$

Moreover, if $1 < \delta \leqslant \delta_c$, we can simply use

$$(1+\|\boldsymbol{\omega}\|_2^2)^{\sigma+1} \leqslant (1+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1} \leqslant \delta^{-2(\sigma+1)}\delta_c^{2(\sigma+1)}(1+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1}.$$

With both estimates we can then deduce for $m = \max\{1, \delta_c^{\sigma+1}\}$

$$\begin{aligned}
\|\mathbf{v}\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d,\mathrm{div})} &= (2\pi)^{-d/2}\int_{\mathbb{R}^d}\frac{\|\widehat{\mathbf{v}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2}(1+\|\boldsymbol{\omega}\|_2^2)^{\sigma+1}d\boldsymbol{\omega}\\
&\leqslant \frac{m^2}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{\|\widehat{\mathbf{v}}(\boldsymbol{\omega})\|_2^2}{\delta^{2(\sigma+1)}\|\boldsymbol{\omega}\|_2^2}(1+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1}d\boldsymbol{\omega}\\
&\leqslant m^2c_2(2\pi)^{-d/2}\int_{\mathbb{R}^d}\delta^{-2(\sigma+1)}\frac{\|\widehat{\mathbf{v}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2\widehat{\phi}(\delta\boldsymbol{\omega})}d\boldsymbol{\omega}\\
&= m^2c_2(2\pi)^{-d/2}\int_{\mathbb{R}^d}\delta^{-2(\sigma+1)}\frac{\|\widehat{\mathbf{v}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2\widehat{\phi_\delta}(\boldsymbol{\omega})}d\boldsymbol{\omega}\\
&= C_2\delta^{-2(\sigma+1)}\|\mathbf{v}\|^2_{\mathcal{N}_{\Phi_\delta}(\mathbb{R}^d)},
\end{aligned}$$

where $C_2 = m^2c_2$. For the lower bound, again, we distinguish between two cases. If $\delta \leqslant 1$, then we have

$$(1+\|\boldsymbol{\omega}\|_2^2)^{\sigma+1} \geqslant (1+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1}.$$

Moreover, for $1 < \delta \leqslant \delta_c$, we have

$$(1+\|\boldsymbol{\omega}\|_2^2)^{\sigma+1} = \delta^{-2(\sigma+1)}(\delta^2+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1} \geqslant \delta_c^{-2(\sigma+1)}(1+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1}.$$

Similarly, for $k = \min\{1, \delta_c^{-\sigma-1}\}$, we then find

$$\begin{aligned}
\|\mathbf{v}\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d,\mathrm{div})} &\geqslant k^2(2\pi)^{-d/2}\int_{\mathbb{R}^d}\frac{\|\widehat{\mathbf{v}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2}(1+\|\delta\boldsymbol{\omega}\|_2^2)^{\sigma+1}d\boldsymbol{\omega}\\
&\geqslant k^2c_1(2\pi)^{-d/2}\int_{\mathbb{R}^d}\frac{\|\widehat{\mathbf{v}}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2\widehat{\phi_\delta}(\boldsymbol{\omega})}d\boldsymbol{\omega}\\
&= C_1\|\mathbf{v}\|^2_{\mathcal{N}_{\Phi_\delta}(\mathbb{R}^d)},
\end{aligned}$$

where $C_1 = k^2c_1$. □

Another ingredient of the multilevel convergence proof consists of a sampling inequality for vector-valued functions. This comes from and is proven in Wendland (2009) and gives an error estimate in terms of the *fill distance* or *mesh norm* of a discrete point set $X \subseteq \Omega$ defined by

$$h_{X,\Omega} = \sup_{\mathbf{x}\in\Omega}\inf_{\mathbf{x}_j\in X}\|\mathbf{x}-\mathbf{x}_j\|_2.$$

LEMMA 3.2 (Vector-Valued Sampling Inequality) Let $\Omega \subseteq \mathbb{R}^d$ be a domain with Lipschitz boundary. Furthermore, let $\sigma, \eta \in \mathbb{R}$ with $\sigma > d/2$ and $0 \leqslant \eta \leqslant \sigma$. Assume that $\mathbf{u} \in \mathbf{H}^\sigma(\Omega)$ satisfies $\mathbf{u}|_X = \mathbf{0}$ on a discrete set $X \subseteq \Omega$ with a sufficiently small fill distance $h_{X,\Omega}$. Then,

$$\|\mathbf{u}\|_{\mathbf{H}^\eta(\Omega)} \leqslant Ch_{X,\Omega}^{\sigma-\eta}\|\mathbf{u}\|_{\mathbf{H}^\sigma(\Omega)}$$

with a constant $C > 0$ independent of $\mathbf{u}$.

3.1 *The Multilevel Interpolation Algorithm*

Suppose we are given a sequence of point sets in Ω, namely $X_1, X_2, \ldots$, with decreasing mesh norms $h_1 := h_{X_1,\Omega}, h_2 := h_{X_2,\Omega}, \ldots$. From these mesh norms we construct support radii so that

$$\delta_j = \beta h_j^{\frac{\sigma}{\sigma+1}} \tag{3.2}$$

with some proportionality constant $\beta > 0$. We define scaled kernels as in (3.1). The vector-valued divergence-free interpolants $\mathbf{s}_j$ come from the space

$$\mathbf{V}_j = \left\{ \sum_{\mathbf{x}\in X_j} \Phi_j(\cdot - \mathbf{x})\boldsymbol{\beta}_{\mathbf{x}} \mid \boldsymbol{\beta}_{\mathbf{x}} \in \mathbb{R}^d \right\}.$$

Now, we can state the multilevel interpolation algorithm for matrix-valued kernels, which reconstructs some divergence-free target function $\mathbf{u}$ by a residual correction scheme.

Algorithm 3.1 (Multilevel Interpolation Algorithm)
Input: Data $\mathbf{u}$ with $\operatorname{div}(\mathbf{u}) = 0$ and number of levels n

Set $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{e}_0 = \mathbf{u}$. For $j = 1, \ldots, n$ do

(i) Determine the local correction $\mathbf{s}_j \in \mathbf{V}_j$ to $\mathbf{e}_{j-1}$ on X_j

$$\mathbf{s}_j(\mathbf{x}) = \mathbf{e}_{j-1}(\mathbf{x}) \qquad \mathbf{x} \in X_j$$

(ii) Update the global approximation and the residuals

$$\mathbf{u}_j = \mathbf{u}_{j-1} + \mathbf{s}_j$$
$$\mathbf{e}_j = \mathbf{e}_{j-1} - \mathbf{s}_j$$

Output: Approximate solution $\mathbf{u}_n$ to $\mathbf{u}$

Note that we only need to know the unknown function $\mathbf{u}$ at the discrete data sites X_j.

The residual in this case is in fact the error between the solution $\mathbf{u}$ and its jth multiscale approximation $\mathbf{u}_j$, i. e.

$$\mathbf{e}_j = \mathbf{u} - \mathbf{u}_j.$$

Our main result of this paper shows that the above algorithm indeed converges under the assumption (3.2) on the relation between the support radii δ_j and the fill distances h_j.

THEOREM 3.2 (Convergence of Multilevel Interpolation Algorithm) Let $\sigma > d/2$. Let $\Omega \subseteq \mathbb{R}^d$ be bounded with a $C^{k,1}$ boundary where $k \geqslant \sigma$. Let $X_1, X_2, \ldots$ be a sequence of point sets in Ω with mesh norms $h_1 = h_{X_1,\Omega}$, $h_2 = h_{X_2,\Omega}, \ldots$. Assume

$$\gamma\mu h_j \leqslant h_{j+1} \leqslant \mu h_j \tag{3.3}$$

for $j = 1, 2, \ldots$ and some fixed $\mu \in (0,1)$ and $\gamma \in (0,1)$. Suppose that $\phi \in L_1(\mathbb{R}^d)$ satisfies (2.2) with $\tau = \sigma + 1$, i.e. that its native space is $H^{\sigma+1}(\mathbb{R}^d)$. Define

$$\delta_j = \left(\frac{h_j}{\mu}\right)^{\frac{\sigma}{\sigma+1}} \quad \text{as well as} \quad \Phi_j = \Phi_{\delta_j} = (-\Delta I + \nabla\nabla^T)\phi_{\delta_j}$$

with $\phi_\delta = \delta^{-d}\phi(\cdot/\delta)$. Let $h_1 \leqslant \mu$ be sufficiently small. Finally, let $\mathbf{u} \in \mathbf{H}^\sigma(\Omega, \mathrm{div})$. Then, there exists a positive constant C independent of μ, j and $\mathbf{u}$ such that

$$\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_j\|_{\Phi_{j+1}} \leqslant \alpha \|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}\|_{\Phi_j}, \tag{3.4}$$

for $j = 1, 2, \ldots$ with $\alpha = C\mu^{\sigma-1}$. Moreover, we have the estimates

$$\|\mathbf{u} - \mathbf{u}_k\|_{\mathbf{L}_2(\Omega)} \leqslant \widetilde{C}\alpha^k \|\mathbf{u}\|_{\mathbf{H}^\sigma(\Omega)}$$

for $k = 1, 2, \ldots$ with another constant $\widetilde{C} > 0$ independent of k, α and $\mathbf{u}$. Hence, the multiscale approximation $\mathbf{u}_k$ converges to $\mathbf{u}$ in the L_2 norm if we choose μ so small that $\alpha < 1$.

Proof. It is easy to verify that the support radii are monotonically decreasing. We compute

$$\begin{aligned}
\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_j\|^2_{\Phi_{j+1}} &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{e}_j}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2 \widehat{\phi_{j+1}}(\boldsymbol{\omega})} d\boldsymbol{\omega} \\
&\leqslant \frac{1}{c_1}(2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{e}_j}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\delta_{j+1}\boldsymbol{\omega}\|_2^2)^{\sigma+1} d\boldsymbol{\omega} \\
&=: \frac{1}{c_1}(2\pi)^{-d/2}(I_1 + I_2),
\end{aligned}$$

where we introduced the notation

$$\begin{aligned}
I_1 &= \int\limits_{\|\boldsymbol{\omega}\|_2 \leqslant \frac{1}{\delta_{j+1}}} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{e}_j}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\delta_{j+1}\boldsymbol{\omega}\|_2^2)^{\sigma+1} d\boldsymbol{\omega} \\
I_2 &= \int\limits_{\|\boldsymbol{\omega}\|_2 \geqslant \frac{1}{\delta_{j+1}}} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{e}_j}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2} (1 + \|\delta_{j+1}\boldsymbol{\omega}\|_2^2)^{\sigma+1} d\boldsymbol{\omega}.
\end{aligned}$$

To bound these two integrals we start with the following observation:

$$\begin{aligned}
\|\mathbf{e}_j\|^2_{\mathbf{H}^\sigma(\Omega)} &= \|\mathbf{e}_{j-1} - \mathbf{s}_{\mathbf{e}_{j-1}}\|^2_{\mathbf{H}^\sigma(\Omega)} \\
&= \|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1} - \mathbf{s}_{\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}}\|^2_{\mathbf{H}^\sigma(\Omega)} \\
&\leqslant \|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1} - \mathbf{s}_{\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}}\|^2_{\mathbf{H}^\sigma(\mathbb{R}^d)} \\
&\leqslant \|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1} - \mathbf{s}_{\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}}\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)} \\
&\leqslant C\delta_j^{-2(\sigma+1)} \|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1} - \mathbf{s}_{\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}}\|^2_{\Phi_j} \\
&\leqslant C\delta_j^{-2(\sigma+1)} \|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}\|^2_{\Phi_j}.
\end{aligned} \tag{3.5}$$

Here we have used the definition of $\mathbf{e}_j$, the facts that $\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1} = \mathbf{e}_{j-1}$ on Ω and that the interpolant $\mathbf{s}_{\mathbf{e}_{j-1}}$ equals $\mathbf{s}_{\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}}$ as well as Lemma 2.2, Lemma 3.1 and the best approximation property of the interpolant.

For the first integral, I_1, we have $\delta_{j+1}\|\boldsymbol{\omega}\|_2 \leqslant 1$ and thus by Lemma 2.3, the sampling inequality (Lemma 3.2) and (3.5), it follows that

$$\begin{aligned}
I_1 &\leqslant 2^{\sigma+1}\int_{\|\boldsymbol{\omega}\|_2\leqslant\frac{1}{\delta_{j+1}}} \frac{\|\widehat{\mathbf{E}_{\mathrm{div}}\mathbf{e}_j}(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2}d\boldsymbol{\omega} \\
&\leqslant C\|\mathbf{e}_j\|^2_{\mathbf{L}_2(\Omega)} \\
&\leqslant Ch_j^{2\sigma}\|\mathbf{e}_j\|^2_{\mathbf{H}^\sigma(\Omega)} \\
&\leqslant Ch_j^{2\sigma}\delta_j^{-2(\sigma+1)}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}\|^2_{\Phi_j} \\
&= C_1\mu^{2\sigma}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}\|^2_{\Phi_j},
\end{aligned}$$

using also that by definition we have

$$\frac{h_j^\sigma}{\delta_j^{\sigma+1}} = \mu^\sigma.$$

Next we turn to the second integral I_2. Since $\delta_{j+1}\|\boldsymbol{\omega}\|_2 \geqslant 1$, we find

$$(1+\delta_{j+1}^2\|\boldsymbol{\omega}\|_2^2)^{\sigma+1} \leqslant (2\delta_{j+1}^2\|\boldsymbol{\omega}\|_2^2)^{\sigma+1} \leqslant 2^{\sigma+1}\delta_{j+1}^{2(\sigma+1)}(1+\|\boldsymbol{\omega}\|_2^2)^{\sigma+1}$$

and consequently, using Proposition 2.6 and (3.5) once again, gives

$$\begin{aligned}
I_2 &\leqslant 2^{\sigma+1}\delta_{j+1}^{2(\sigma+1)}\int_{\mathbb{R}^d}\frac{\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_j(\boldsymbol{\omega})\|_2^2}{\|\boldsymbol{\omega}\|_2^2}(1+\|\boldsymbol{\omega}\|_2^2)^{\sigma+1}d\boldsymbol{\omega} \\
&= 2^{\sigma+1}\delta_{j+1}^{2(\sigma+1)}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_j\|^2_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d,\mathrm{div})} \\
&\leqslant C\delta_{j+1}^{2(\sigma+1)}\|\mathbf{e}_j\|^2_{\mathbf{H}^\sigma(\Omega)} \\
&\leqslant C(\delta_{j+1}/\delta_j)^{2(\sigma+1)}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}\|^2_{\Phi_j} \\
&\leqslant C_2\mu^{2\sigma}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_{j-1}\|^2_{\Phi_j}.
\end{aligned}$$

In the last step we have used

$$\left(\frac{\delta_{j+1}}{\delta_j}\right)^{\sigma+1} = \left(\frac{h_{j+1}}{\mu}\right)^\sigma\left(\frac{\mu}{h_j}\right)^\sigma = \left(\frac{h_{j+1}}{h_j}\right)^\sigma \leqslant \mu^\sigma.$$

Combining both estimates now yields (3.4) with

$$\alpha = \sqrt{\frac{1}{c_1}(2\pi)^{-d/2}}(C_1+C_2)^{1/2}\mu^\sigma = C\mu^\sigma.$$

It follows by the sampling inequality, Lemma 2.2 and Lemma 3.1 that

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_k\|^2_{\mathbf{L}_2(\Omega)} &= \|\mathbf{e}_k\|^2_{\mathbf{L}_2(\Omega)} \leqslant Ch_k^{2\sigma}\|\mathbf{e}_k\|^2_{\mathbf{H}^\sigma(\Omega)} \\
&\leqslant Ch_k^{2\sigma}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_k\|_{\widetilde{\mathbf{H}}^\sigma(\mathbb{R}^d)} \\
&\leqslant Ch_k^{2\sigma}\delta_{k+1}^{-2(\sigma+1)}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_k\|^2_{\Phi_{k+1}} \\
&= C\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_k\|^2_{\Phi_{k+1}},
\end{aligned}$$

where we have used the fact that

$$\frac{h_k^\sigma}{\delta_{k+1}^{\sigma+1}} = \mu^\sigma \left(\frac{h_k}{h_{k+1}}\right)^\sigma \leqslant \mu^\sigma \left(\frac{1}{\mu\gamma}\right)^\sigma = \gamma^{-\sigma}.$$

Applying (3.4), n times we can conclude

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_k\|_{\mathbf{L}_2(\Omega)}^2 &\leqslant \widetilde{C}\alpha^{2n}\|\mathbf{E}_{\mathrm{div}}\mathbf{e}_0\|_{\Phi_1}^2 = \widetilde{C}\alpha^{2n}\|\mathbf{E}_{\mathrm{div}}\mathbf{u}\|_{\Phi_1}^2 \\ &\leqslant \widetilde{C}\alpha^{2n}\|\mathbf{u}\|_{\mathbf{H}^\sigma(\Omega)}^2 \end{aligned}$$

with a constant $\widetilde{C} = \widetilde{C}(\gamma) > 0$ independent of α, k and $\mathbf{u}$. □

In contrast to the multilevel interpolation with scalar-valued kernels Wendland (2010), the multilevel interpolation of divergence-free fields with matrix-valued kernels is shown to converge only for a (mildly) non-proportional relationship between mesh norm and support radius. In fact, as we will see in the next section, a proportional relationship does not lead to convergence. The non-proportionality is introduced via the native space norm for matrix-valued kernels.

3.2 *Stability*

In this section, we will discuss how the interpolation matrices $A_{\Phi_\delta,X}$ in the multilevel algorithm depend on the support radii and the separation distance

$$q_X := \min_{j\neq k}\|\mathbf{x}_j - \mathbf{x}_k\|$$

for some data set $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$. We will extend the one-shot results proven by Fuselier Fuselier (2008b,a). Earlier work is due to Lowitzsch (2005c); Narcowich & Ward (1994). Since $A_{\Phi_\delta,X}$ is symmetric and positive definite, we only need to find bounds on the smallest and largest eigenvalues.

For the smallest eigenvalue we will use the following general result by Fuselier Fuselier (2008b,a).

THEOREM 3.3 Let ϕ be an even positive definite function, which possesses a positive Fourier transform $\widehat{\phi} \in C(\mathbb{R}^d/\{\mathbf{0}\})$. Let $\Phi = (-\Delta I + \nabla\nabla^T)\phi$. Define the function

$$M(s) := \inf_{\|\boldsymbol{\omega}\|_2\leqslant s} \widehat{\phi}(\boldsymbol{\omega}).$$

A lower bound on the smallest eigenvalue of the interpolation matrix $A_{\Phi,X}$ is given by

$$\lambda_{\min}(A_{\Phi,X}) \geqslant \left(\frac{s^2}{16\pi}\right)^{(d+2)/2} \frac{M(s)\pi}{(4\pi)^d\Gamma((d+2)/2)}$$

for any $s > 0$ satisfying

$$s \geqslant \widetilde{c}/q_X$$

where $\widetilde{c}$ is a constant independent of ϕ and X.

A direct consequence of this result is the following one. Suppose $\phi \in L_1(\mathbb{R}^d)$ has a Fourier transform satisfying the bound (2.2) with $\tau = \sigma + 1$, meaning in particular $\widehat{\phi}(\boldsymbol{\omega}) \geqslant c_1(1+\|\boldsymbol{\omega}\|_2^2)^{-\sigma-1}$. Then, obviously, $M(s)$ can be bounded from below by $M(s) \geqslant c_1(1+s^2)^{-\sigma-1} \geqslant cs^{-2\sigma-2}$, where the last estimate holds for example for $s \geqslant 1$. Hence, taking $s = c/q_X$, Theorem 3.3 yields in this situation,

$$\lambda_{\min}(A_{\Phi,X}) \geqslant cq_X^{-(d+2)-2\sigma-2} = cq_X^{2\sigma-d}. \tag{3.6}$$

Here, however, we are interested in scaled versions of the basis function, i. e. we are interested in lower bounds for $\lambda_{\min}(A_{\Phi_\delta,X})$ with $\Phi_\delta = (-\Delta I + \nabla\nabla^T)\phi_\delta$, where $\phi_\delta = \delta^{-d}\phi(\cdot/\delta)$. We can derive such lower bounds in two ways. We can either proceed as above using the fact that $\widehat{\phi_\delta} = \widehat{\phi}(\delta\cdot)$ and hence $M(c/q_X) \geqslant c(q_X/\delta)^{2(\sigma+1)}$ yielding

$$\lambda_{\min}(A_{\Phi_\delta,X}) \geqslant c q_X^{-(d+2)}\left(\frac{q_X}{\delta}\right)^{2(\sigma+1)} = c q_X^{2\sigma-d}\delta^{-2\sigma-2} \tag{3.7}$$

for $\delta/q_X \geqslant 1$. Alternatively, we can realise that

$$\Phi_\delta = \delta^{-d-2}[(-\Delta I + \nabla\nabla^T)\phi](\cdot/\delta) = \delta^{-d-2}\Phi(\cdot/\delta)$$

and that $X/\delta := \{\mathbf{x}/\delta \mid \mathbf{x} \in X\}$ has obviously separation distance $q_{X/\delta} = q_X/\delta$. Hence, apart from the scaling factor δ^{-d-2} interpolating with Φ_δ in X is the same as interpolating with Φ in X/δ. Thus, (3.6) yields

$$\lambda_{\min}(A_{\Phi_\delta,X}) \geqslant c\delta^{-d-2}\left(\frac{q_X}{\delta}\right)^{2\sigma-d} = c q_X^{2\sigma-d}\delta^{-2\sigma-2}$$

which is obviously the same as (3.7).

THEOREM 3.4 Suppose ϕ is positive definite and compactly supported, with a Fourier transform satisfying (2.2) with $\tau = \sigma + 1$ and $\sigma > d/2$, i. e. generating $H^{\sigma+1}(\mathbb{R}^d)$. Then,

$$\operatorname{cond}(A_{\Phi_\delta,X}) \leqslant C\left(1 + \frac{2\delta}{q_X}\right)^d \left(\frac{\delta}{q_X}\right)^{2\sigma-d}$$

with a constant $C > 0$ independent of q_X and δ.

Proof. It remains to bound the largest eigenvalue $\lambda_{\max} = \lambda_{\max}(A_{\Phi_\delta,X})$. For this, we follow ideas from Farrell & Wendland (2013). Due to the Gershgorin theorem there is an index $j \in \{1,\ldots,dN\}$ with corresponding data site $\mathbf{x}_{\widetilde{j}}$ with $1 \leqslant \widetilde{j} \leqslant N$ such that

$$|\lambda_{\max}| \leqslant \sum_{k=1}^{dN} |a_{jk}^\delta|$$

with $A_{\Phi_\delta,X} = (a_{jk}^\delta) \in \mathbb{R}^{dN\times dN}$. The entries a_{jk}^δ have the form

$$a_{jk}^\delta = \delta^{-d-2}(D^\alpha\phi)((\mathbf{x}_{\widetilde{j}} - \mathbf{x}_{\widetilde{k}})/\delta)$$

with certain multi-indices $\alpha \in \mathbb{N}_0^d$ of length $|\alpha| = 2$. Note that the data site $\mathbf{x}_{\widetilde{k}}$ corresponds to d indices k. By assumption, ϕ belongs to $C^2(\mathbb{R}^d)$ and has compact support. This means that there is a constant $c_\phi > 0$ such that

$$|a_{jk}^\delta| \leqslant \delta^{-d-2}c_\phi, \qquad 1 \leqslant j,k \leqslant dN$$

Since Φ_δ has compact support in the ball $B(\mathbf{0},\delta)$, only those terms a_{jk}^δ, $1 \leqslant k \leqslant dN$, are nonzero which belong to the index set

$$I_{\widetilde{j}} := \{\widetilde{k} \in \{1,\ldots,N\} \mid \|\mathbf{x}_{\widetilde{j}} - \mathbf{x}_{\widetilde{k}}\| < \delta\}.$$

That is, there are $dn_{\tilde{j}}$ non-zero summands with

$$n_{\tilde{j}} := \left|\{\widetilde{k} \in \{1,\ldots,N\} \mid \|\mathbf{x}_{\tilde{j}} - \mathbf{x}_{\tilde{k}}\|_2 < \delta\}\right|.$$

In Farrell & Wendland (2013) it is shown that we have the estimate

$$n_{\tilde{j}} \leqslant \left(\frac{\delta + q_X/2}{q_X/2}\right)^2 = \left(1 + \frac{2\delta}{q_X}\right)^d.$$

Now we can bound the largest eigenvalue by

$$|\lambda_{\max}| \leqslant \sum_{k=1}^{dN} |a_{jk}^{\delta}| \leqslant dn_{\tilde{j}} \max_{1\leqslant k\leqslant N} |a_{jk}^{\delta}| \leqslant c_{\phi} d\delta^{-d-2}\left(1 + \frac{2\delta}{q_X}\right)^d.$$

Combining this with the lower bound (3.7) on the smallest eigenvalue yields

$$\operatorname{cond}(A_{\Phi_\delta,X}) \leqslant C\left(1 + \frac{2\delta}{q_X}\right)^d \left(\frac{\delta}{q_X}\right)^{2\sigma-d}$$

with a constant $C > 0$ independent of X and δ.

□

Since we can assume that $2\delta \geqslant q_X$, the bound further simplifies to

$$\operatorname{cond}(A_{\Phi_\delta,X}) \leqslant C\left(\frac{\delta}{q_X}\right)^{2\sigma} \tag{3.8}$$

COROLLARY 3.1 Under the assumption $\delta = cq_X$, the condition number is constant and independent of q_X and δ. If we assume that the data sets are quasi-uniform, i.e. satisfy $q_X \approx h_X$, the convergence theorem requires $\delta = cq_X^{\sigma/(\sigma+1)}$. In this case, the condition number behaves like

$$\operatorname{cond}(A_{\Phi_\delta,X}) \leqslant Cq_X^{-2\sigma/(\sigma+1)}.$$

Proof. The statement for the case $\delta = cq_X$ immediately follows from (3.8). Since $\delta = cq_X^{\sigma/(\sigma+1)}$ gives

$$\left(\frac{\delta}{q_X}\right)^{\sigma} = c\delta^{-1} = cq_X^{-\sigma/(\sigma+1)}, \tag{3.9}$$

the second statement also follows from (3.8). □

4. Numerical Example

We run a numerical example for a divergence-free vector field of the form

$$\mathbf{u}(x,y) = \begin{pmatrix} -2x^3y \\ 3x^2y^2 \end{pmatrix} \tag{4.1}$$

on the closed unit square $\Omega = [0,1]^2$. Note that the vector field is indeed divergence-free. We choose regular nested grids as data sites. That is, q_X and $h = h_{X,\Omega}$ are comparable. As basis function we

N	H	$\|\mathbf{e}\|_{\mathbf{L}_2(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{H}^1(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{L}_\infty(\Omega)}$	cond(A)	ratio [%]
9	2^{-1}	1.83e-1	1.53e0	6.11e-1	1.3e2	72.22
25	2^{-2}	3.35e-2	5.00e-1	2.19e-1	9.2e2	74.96
81	2^{-3}	5.62e-3	1.65e-1	6.19e-2	5.2e3	52.44
289	2^{-4}	1.02e-3	6.13e-2	1.75e-2	2.4e4	24.75
1089	2^{-5}	1.91e-4	2.46e-2	4.80e-3	9.1e4	10.19
4225	2^{-6}	3.37e-5	1.02e-2	1.17e-3	2.9e5	4.31
16641	2^{-7}	5.29e-6	4.47e-3	2.81e-4	8.3e5	1.54

Table 1: Convergence study for multilevel interpolation of divergence-free vector field (4.1) with basis function $\phi_{2,3}$ and support radii (4.2) where $\nu = 2.5$.

N	H	$\|\mathbf{e}\|_{\mathbf{L}_2(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{H}^1(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{L}_\infty(\Omega)}$	cond(A)
25	2^{-2}	2.45	1.61	1.48	-2.83
81	2^{-3}	2.57	1.60	1.83	-2.50
289	2^{-4}	2.46	1.43	1.83	-2.20
1089	2^{-5}	2.42	1.31	1.86	-1.93
4225	2^{-6}	2.50	1.27	2.04	-1.69
16641	2^{-7}	2.67	1.20	2.05	-1.50

Table 2: Orders for multilevel interpolation of divergence-free vector field (4.1) with basis function $\phi_{2,3}$ and support radii (4.2) where $\nu = 2.5$.

employ $\phi_{2,3}(r) = (1-r)_+^8(32r^3+25r^2+8r+1)$, i. e. $\sigma+1 = 4.5$. Furthermore, we use a proportionality constant $\nu = 2.5$ and support radii of the form

$$\delta = \nu h^{\frac{\sigma}{\sigma+1}} = \nu h^{\frac{7}{9}}. \tag{4.2}$$

As the numerical results in Tables 1 and 2 show, the numerical solution converges. Note that the number N is actually referring to the amount of data points. The systems that need to be solved are actually twice as big, since $\mathbf{u}$ consists of two spatial components. The parameter H denotes the distance between two horizontal (or vertical) nearest data points of the uniform grid. Thus, this parameter is proportional to the mesh norm h. We point out that the condition numbers are estimated values only since the matrices become quite large. The final column shows the ratio of non-zeros to total numbers of entries in the matrix (in percent). The multilevel interpolant and the errors at the last level are depicted in Figures 1 and 2. Due to Corollary 3.1, we expect the condition number to grow asymptotically not faster than $q_X^{-1.\overline{55}}$.

Tables 3 and 4 show the results for the stationary case, i. e.

$$\delta = \nu h. \tag{4.3}$$

In this case, the algorithm eventually stagnates. According to Corollary 3.1, we expect the condition number to become asymptotically independent of q_X. The numerics seem to corroborate this.

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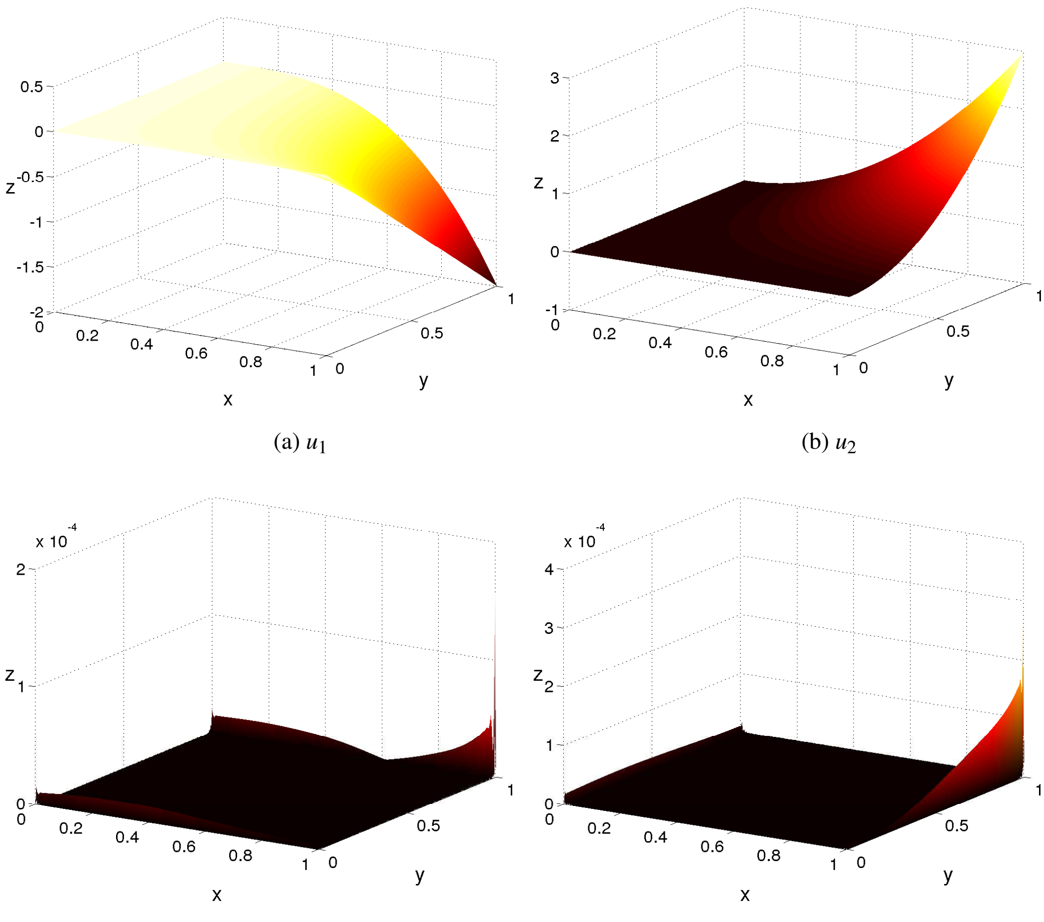


(a) u_1 (b) u_2

(c) e_1 (d) e_2

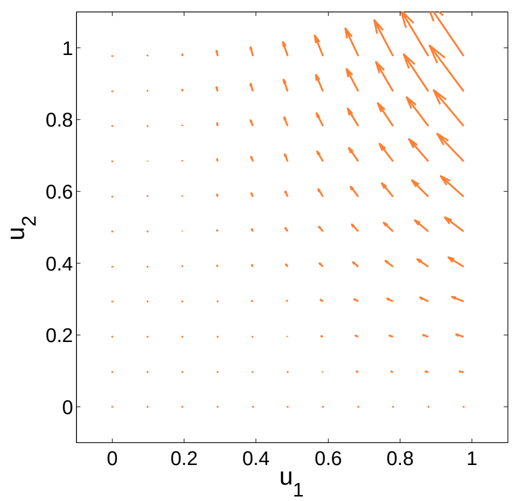
FIG. 1: Both components of multilevel approximation and corresponding errors of divergence-free vector field (4.1) after seven levels, using a basis function $\phi_{2,3}$, $\mu = 0.5$ and $\nu = 2.5$.

N	H	$\|\mathbf{e}\|_{\mathbf{L}_2(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{H}^1(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{L}_\infty(\Omega)}$	cond(A)	ratio [%]
9	2^{-1}	2.00e-1	1.70e0	6.18e-1	9.4e1	72.22
25	2^{-2}	4.10e-2	6.11e-1	2.72e-1	2.3e2	58.00
81	2^{-4}	7.88e-3	2.28e-1	8.69e-2	3.9e2	25.95
289	2^{-4}	1.68e-3	9.77e-2	2.56e-2	5.2e2	8.75
1089	2^{-5}	5.44e-4	6.22e-2	7.00e-3	6.1e2	2.55
4225	2^{-6}	3.90e-4	8.54e-2	1.62e-3	6.7e2	0.69
16641	2^{-7}	4.15e-4	1.66e-2	9.85e-4	6.6e2	0.18

Table 3: Convergence study for multilevel interpolation of divergence-free vector field (4.1) with basis function $\phi_{2,3}$ and support radii (4.3) where $\nu = 2.5$.

N	H	$\|\mathbf{e}\|_{\mathbf{L}_2(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{H}^1(\Omega)}$	$\|\mathbf{e}\|_{\mathbf{L}_\infty(\Omega)}$	cond(A)
25	2^{-2}	2.28	1.47	1.18	-1.29
81	2^{-3}	2.38	1.42	1.65	-0.78
289	2^{-4}	2.23	1.22	1.76	-0.41
1089	2^{-5}	1.63	0.65	1.87	-0.23
4225	2^{-6}	0.48	-0.45	2.11	-0.12
16641	2^{-7}	-0.09	2.37	0.72	0.02

Table 4: Orders for multilevel interpolation of divergence-free vector field (4.1) with basis function $\phi_{2,3}$ and support radii (4.3) where $\nu = 2.5$.



(a) Vector field $\mathbf{u}$

FIG. 2: The divergence-free vector field (4.1) after seven levels, using a basis function $\phi_{2,3}$, $\mu = 0.5$ and $\nu = 2.5$.

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