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Existence of weak solutions for a PDE system describing phase separation and damage processes including inertial effects

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Abstract

In this paper, we consider a coupled PDE system describing phase separation and damage phenomena in elastically stressed alloys in the presence of inertial effects. The material is considered on a bounded Lipschitz domain with mixed boundary conditions for the displacement variable. The main aim of this work is to establish existence of weak solutions for the introduced hyperbolic-parabolic system. To this end, we first adopt the notion of weak solutions introduced in [HK11]. Then we prove existence of weak solutions by means of regularization, time-discretization and different variational techniques.

1 Introduction

In micro-electronic materials such as solder alloys, different physical processes are shaping the micro-structure. For a realistic description of these structures, phase separation, coarsening and elasticity as well as damage phenomena have to be taken into account. A fully coupled system has been originally studied in [HK11] and further developed in [HK13b] allowing, for instance, inhomogeneous elastic energy densities. The corresponding degenerating case has been analyzed in [HK12]. To the authors' best knowledge, before these works, phase separation and damage processes have only been investigated independently of each other in the mathematical literature.

Phase separation and coarsening phenomena are usually described by phase–field models of Cahn–Hilliard type. The evolution is modeled by a parabolic diffusion equation for the phase fractions. To include elastic effects, resulting from stresses caused by different elastic properties of the phases, Cahn-Hilliard systems are coupled with an elliptic equation in the case of a quasi-static balance of forces. Such coupled Cahn-Hilliard systems with elasticity are also called Cahn-Larché systems. Since in general the mobility, stiffness and surface tension coefficients depend on the phases (see for instance [BDM07] and [BDDM07] for the explicit structure deduced by the embedded atom method), the mathematical analysis of the coupled problem is very complex. Existence results were derived for special cases in [CMP00, Gar00, BP05] (constant mobility, stiffness and surface tension coefficients), in [BCD$^+$02] (concentration dependent mobility, two space dimensions), [SP13b, SP13a] (concentration dependent surface tension and nonlinear diffusion) and in [PZ08] in an abstract measure-valued setting (concentration dependent mobility and surface tension tensors).

Damage behavior, however, originates from breaking atomic links in the material from a microscopic point of view whereas a macroscopic theory may specify damage in the isotropic case by a scalar-valued variable related to the proportion of damaged bonds in the micro-structure of the material with respect to the undamaged ones. According to the latter perspective, phase-field models are quite common to model smooth transitions between damaged and undamaged material states. Such phase-field models have been mainly investigated for incomplete damage which means that damaged material cannot loose all its elastic energy.

Existence and uniqueness results for damage models of viscoelastic materials are proven in [BSS05] for scalar-valued displacements. Higher dimensional damage models are analytically investigated in [BS04, MR06, MT10, KRZ13, RR12] and, there, existence and regularity properties are shown. A coupled system describing incomplete damage, linear elasticity and phase separation appeared in [HK11, HK13b]. There, existence of weak solutions has been proven under mild assumptions, where, for instance, the stiffness tensor may be material-dependent and the chemical free energy may be of polynomial or logarithmic type. All these works are based on the gradient-of-damage model proposed by Frémond and Nedjar [FN96] (see also [Fré02]) which describes damage as a result from microscopic movements in the solid. The distinction between a balance law for the microscopic forces and constitutive relations of the material yield a satisfying derivation of an evolution law for the damage propagation from the physical point of view. In particular, the gradient of the damage variable enters the resulting equation and serves as a regularization term for the mathematical analysis as well as it ensures the structural size effect. Internal constraints are ensured by

the presence of non-smooth operators (subdifferential operators) in the evolution system. Hence, in the case that the evolution of the damage is assumed to be uni-directional, i.e. the damage process is irreversible, the microforce balance law becomes a doubly-nonlinear differential inclusion.

The main aim of this paper is to generalize the results for hyperbolic-parabolic damage systems introduced in [HK13a] to coupled phase-field systems describing *phase separation* and *damage processes* in the presence of inertial terms with mixed boundary conditions on *non-smooth (Lipschitz) domains*. The novelty of this contribution is to obtain existence results for *phase separation* with elasticity including *inertial effects* and damage processes on Lipschitz domains. We first utilize and adjust the notion of weak solutions introduced in [HK11]. Then, we prove existence of weak solutions by means of regularization, time-discretization and different variational techniques. To this end, an energy estimate has, for instance, to be established and several convergence properties are shown.

1.1 Energies and evolutionary equations

Here, we qualify our model formally and postpone a rigorous treatment to Section 4. The presented model is based on two functionals, i.e. a generalized Ginzburg-Landau free energy functional $\mathcal{E}$ and a damage pseudo-dissipation potential $\mathcal{R}$ (in the sense by Moreau). The free energy density φ of the system is given by

$$\varphi(\varepsilon(u), c, \nabla c, z, \nabla z) := \frac{1}{p}|\nabla z|^p + \frac{1}{2}|\nabla c|^2 + W(c, \varepsilon(u), z) + f(z) + \Psi(c), \tag{1}$$

where the gradient terms penalize spatial changes of the variables c and z. W denotes the elastically stored energy density accounting for elastic deformations and damage effects, f is the damage dependent potential and Ψ stands for the chemical energy density.

The overall free energy $\mathcal{E}$ of Ginzburg-Landau type has the following structure:

$$\mathcal{E}(u, c, z) := \int_\Omega \Big(\varphi(\varepsilon(u), c, \nabla c, z, \nabla z) + I_{[0,\infty)}(z) \Big) \, \mathrm{d}x. \tag{2}$$

In this context, $I_{[0,\infty)}$ signifies the indicator function of the subset $[0,\infty) \subseteq \mathbb{R}$, i.e. $I_{[0,\infty)}(x) = 0$ for $x \in [0,\infty)$ and $I_{[0,\infty)}(x) = \infty$ for $x < 0$. We assume that the energy dissipation for the damage process is triggered by a rate-dependent dissipation potential $\mathcal{R}$ of the form

$$\mathcal{R}(\dot{z}) := \int_\Omega \Big(\frac{1}{2}|\dot{z}|^2 + I_{(-\infty,0]}(\dot{z}) \Big) \, \mathrm{d}x. \tag{3}$$

The governing evolutionary equations for a system state $q = (u, c, z)$ can be expressed by virtue of the functionals (2) and (3). More precisely, the evolution is driven by the following hyperbolic-parabolic system of differential equations and differential inclusions:

$$\begin{aligned}
\textit{diffusion:} \qquad & c_t = \mathrm{div}(m(c,z)\nabla\mu), & \text{(4a)} \\
& \mu = -\Delta c + W_{,c}(c, \epsilon(u), z) + \Psi'(c), & \text{(4b)} \\
\textit{balance of forces:} \qquad & u_{tt} - \mathrm{div}\left(W_{,e}(c, \epsilon(u), z)\right) = l, & \text{(4c)} \\
\textit{damage evolution:} \qquad & 0 \in \partial_z \mathcal{E}(u, c, z) + \partial_{\dot{z}} \mathcal{R}(\partial_t z) \quad \textit{or equivalently} & \text{(4d)} \\
& z_t = -\Delta_p z + W_{,z}(c, \epsilon(u), z) + f'(z) + \xi + \varphi = 0, & \text{(4e)} \\
& \xi \in \partial I_{[0,\infty)}(z), & \text{(4f)} \\
& \varphi \in \partial I_{(-\infty,0]}(z_t). & \text{(4g)}
\end{aligned}$$

The Cahn-Hilliard system (4a)-(4b) describes phase separation phenomena in alloys, the hyperbolic equation (4c) formulates the balance of forces including inertial effects and the inclusion (4d)-(4g) is an evolution law for the damage processes. The sub-gradients correspond to the constraints that the damage is non-negative and irreversible. Let us note that linear contributions in f model damage activation thresholds.

We choose Dirichlet conditions for the displacements u on a subset Γ of the boundary $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$. Let $b : [0,T] \times \Gamma \to \mathbb{R}^n$ be a function which prescribes the displacements on Γ for a fixed chosen time interval $[0,T]$. The imposed boundary and initial conditions and constraints are as follows:

$$\textit{boundary displacements}: \qquad u = b \text{ on } \Gamma_D \times (0,T), \tag{5a}$$

$$\textit{initial concentration}: \qquad c(0) = c^0 \text{ in } \Omega, \tag{5b}$$

$$\textit{initial displacements}: \qquad u(0) = u^0, \ u_t(0) = v^0 \text{ in } \Omega, \tag{5c}$$

$$\textit{initial damage}: \qquad z(0) = z^0 \text{ in } \Omega. \tag{5d}$$

Moreover, we use natural boundary conditions for the remaining variables on (parts of) the boundary:

$$W_{,e}(c, \epsilon(u), z) \cdot \nu = 0 \qquad\qquad \text{on } \Gamma_N \times (0,T), \tag{6a}$$

$$\nabla c \cdot \nu = \nabla z \cdot \nu = m(c,z)\nabla\mu \cdot \nu = 0 \qquad\qquad \text{on } \partial\Omega, \tag{6b}$$

$$\tag{6c}$$

where ν stands for the outer unit normal to $\partial\Omega$.

We like to mention that mass conservation of the system follows from the diffusion equation (4a) and (6b), i.e.

$$\int_\Omega c(t) - c^0 \, \mathrm{d}x = 0 \text{ for all } t \in [0,T].$$

In the next section, we state the precise assumptions that are needed for a rigorous analysis. Section 3 presents the main results. We give a notion of weak solutions evolved from [HK13a] and state the existence theorem in Subsection 3.1. Since the proof is based on regularization techniques, we also give the weak notion and the associated existence result for the regularized system in Subsection 3.2. In the main part, Section 4, the existence proof is carried out first for the regularized case and then for the limiting case.

2 Notation and assumptions

Throughout this work, let $p > n$ be a constant and let $\Omega \subseteq \mathbb{R}^n$ ($n = 1, 2, 3$) be a bounded Lipschitz domain. For the Dirichlet boundary Γ_D and the Neumann boundary Γ_N of $\partial\Omega$, we adopt the assumptions from [Ber11], i.e., Γ_D and Γ_N are non-empty and relatively open sets in $\partial\Omega$ with finitely many path-connected components such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$.

The considered time interval is denoted by $[0,T]$ and $\Omega_t := \Omega \times [0,t]$ for $t \in [0,T]$. The partial derivative of a function h with respect to a variable s is abbreviated by $h_{,s}$. The set $\{v > 0\}$ for a function $v \in W^{1,p}(\Omega)$ has to be read as $\{x \in \overline{\Omega} \,|\, v(x) > 0\}$ by employing the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ (because $p > n$).

The elastic energy density W is assumed to be of the form

$$W(c, e, z) = \frac{1}{2}\mathbf{C}(z)(e - e^*(c)) : (e - e^*(c)), \tag{7}$$

where e^* denotes the eigenstrain and $\mathbf{C}$ the material stiffness tensor which depends on the damage variable. For e^*, we assume the linear relation $e^*(c) = c\hat{e}$ with $\hat{e} \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ (Vegard's law). We choose the stiffness tensor function $\mathbf{C} \in \mathcal{C}^1([0,1]; \mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{n \times n}))$, where $\mathcal{L}_{\mathrm{sym}}(\mathbb{R}^{n \times n})$ denotes the linear mappings from $\mathbb{R}^{n \times n}$ into $\mathbb{R}^{n \times n}$ which are symmetric. We also assume the properties

$$\mathbf{C}(z)e : e \geq \eta|e|^2, \qquad \mathbf{C}'(z)e : e \geq 0 \tag{8}$$

for all $e \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$, $z \in [0,1]$ and a constant $\eta > 0$ independent of e and z.

Furthermore, we choose the mobility $m \in C(\mathbb{R} \times [0,1]; \mathbb{R}^+)$ and suppose that the chemical energy density $\Psi \in \mathcal{C}^1(\mathbb{R})$ can be decomposed into

$$\Psi(c) = \Psi_1(c) + \Psi_2(c) \quad \text{for } c \in \mathbb{R},$$

where $\Psi_1, \Psi_2 \in \mathcal{C}^1(\mathbb{R})$ with Ψ_1 convex and $\Psi_1 \geq 0$.

In addition, we assume the following growth conditions:

$$|\Psi'(c)| \leq C(1 + |c|^{2^\star/2}), \tag{9a}$$
$$|\Psi_2'(c)| \leq C(|c| + 1) \tag{9b}$$

for all $c \in \mathbb{R}$. Moreover, the mobility function should satisfy

$$C_1 \leq m(c, z) \leq C_2 \tag{10}$$

for all $c \in \mathbb{R}$, $z \in [0, 1]$. Here, $C_1, C_2 > 0$ denote constants independent of c and z, and $2^\star$ is the Sobolev critical exponent.

The damage dependent potential f entering equation (4d) is assumed to be a function of $\mathcal{C}^1([0, 1]; \mathbb{R}^+)$.

3   Main results

3.1   Notion of weak solutions and existence results

In what follows we define for $k \geq 1$ the spaces

$$W_+^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) \,|\, u \geq 0 \text{ a.e. in } \Omega \right\},$$
$$W_-^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) \,|\, u \leq 0 \text{ a.e. in } \Omega \right\},$$
$$H_{\Gamma_\mathrm{D}}^k(\Omega) := \left\{ u \in H^k(\Omega) \,|\, u = 0 \text{ on } \Gamma_\mathrm{D} \text{ in the sense of traces} \right\}.$$

Let the following initial-boundary data and volume forces be given:

boundary data:	$b \in H^1(0, T; H^2(\Omega; \mathbb{R}^n)) \cap W^{2,1}(0, T; L^2(\Omega; \mathbb{R}^n))$,
initial values:	$c^0 \in H^1(\Omega)$, $u^0 \in H^1(\Omega; \mathbb{R}^n)$, $v^0 \in L^2(\Omega; \mathbb{R}^n)$,
	$z^0 \in W^{1,p}(\Omega)$ with $0 \leq z^0 \leq 1$ a.e. in Ω,
external volume forces:	$l \in L^2(0, T; L^2(\Omega; \mathbb{R}^n))$.

A weak formulation of system (4)-(6) is given in the following definition.

Definition 3.1 (Weak solution) *A weak solution of the PDE system* (4)-(6) *for the data* $(l, b, c^0, u^0, v^0, z^0)$ *is a 5-tuple* (c, u, z, μ, ξ) *satisfying the following properties:*

- *spaces:*

  $c \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))^*)$,
  $u \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^n)) \cap W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^n)) \cap H^2(0, T; (H_{\Gamma_\mathrm{D}}^1(\Omega; \mathbb{R}^n))^*)$
    with $u = b$ *on* $\Gamma_\mathrm{D} \times (0, T)$, $u(0) = u^0$ *a.e. in* Ω, $\partial_t u(0) = v^0$ *a.e. in* Ω,
  $z \in L^\infty(0, T; W^{1,p}(\Omega)) \cap H^1(0, T; L^2(\Omega))$
    with $z(0) = z^0$ *in* Ω, $z \geq 0$ *a.e. in* Ω_T, $\partial_t z \leq 0$ *a.e. in* Ω_T,
  $\mu \in L^2(0, T; H^1(\Omega))$,
  $\xi \in L^\infty(0, T; L^1(\Omega))$.

- *for all* $\zeta \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ *with* $\zeta(T) = 0$:

$$\int_{\Omega_T} (c - c^0)\partial_t \zeta \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} m(c, z)\nabla\mu \cdot \nabla\zeta \, \mathrm{d}x \, \mathrm{d}t \tag{11}$$

- *for all $\zeta \in L^2(0, T; H^1(\Omega))$ and for a.e. $t \in (0, T)$:*

$$\int_\Omega \mu\, \zeta\, \mathrm{d}x = \int_\Omega \left(\nabla c \cdot \nabla \zeta + W_{,c}(c, \epsilon(u), z)\zeta + \Psi'(c)\zeta\right) \mathrm{d}x \tag{12}$$

- *for all $\zeta \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^n)$ and for a.e. $t \in (0, T)$:*

$$\langle \partial_{tt} u, \zeta \rangle_{H^1} + \int_\Omega W_{,e}(c, \epsilon(u), z) : \epsilon(\zeta)\, \mathrm{d}x = \int_\Omega l \cdot \zeta\, \mathrm{d}x \tag{13}$$

- *for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in (0, T)$:*

$$0 \leq \int_\Omega \left(|\nabla z|^{p-2}\nabla z \cdot \nabla \zeta + (W_{,z}(c, \epsilon(u), z) + f'(z) + \partial_t z + \xi)\zeta\right) \mathrm{d}x \tag{14}$$

- *for all $\zeta \in L^\infty_+(\Omega)$ and for a.e. $t \in (0, T)$:*

$$0 \geq \int_\Omega \xi(\zeta - z)\, \mathrm{d}x \tag{15}$$

- *total energy inequality for a.e. $t \in (0, T)$:*

$$\mathcal{E}(t) + \mathcal{K}(t) + \mathcal{D}(0, t) \leq \mathcal{E}(0) + \mathcal{K}(0) + \mathcal{W}_{\mathrm{ext}}(0, t) \tag{16}$$

with

$$
\begin{aligned}
\text{free energy:} \quad & \mathcal{E}(t) := \int_\Omega \left(\frac{1}{p}|\nabla z(t)|^p + \frac{1}{2}|\nabla c(t)|^2 + W(c(t), \epsilon(u(t)), z(t))\right) \mathrm{d}x \\
& \qquad\qquad + \int_\Omega \big(f(z(t)) + \Psi(c(t))\big) \mathrm{d}x, \\
\text{kinetic energy:} \quad & \mathcal{K}(t) := \int_\Omega \frac{1}{2}|\partial_t u(t)|^2\, \mathrm{d}x, \\
\text{dissipation:} \quad & \mathcal{D}(0, t) := \int_{\Omega_t} \big(|\partial_t z|^2 + m(c, z)|\nabla \mu|^2\big) \mathrm{d}x\, \mathrm{d}s, \\
\text{external work:} \quad & \mathcal{W}_{\mathrm{ext}}(0, t) := \int_{\Omega_t} W_{,e}(c, \epsilon(u), z) : \epsilon(\partial_t b)\, \mathrm{d}x\, \mathrm{d}s \\
& \qquad\qquad - \int_{\Omega_t} \partial_t u \cdot \partial_{tt} b\, \mathrm{d}x\, \mathrm{d}s + \int_{\Omega_t} l \cdot (\partial_t u - \partial_t b)\, \mathrm{d}x\, \mathrm{d}s \\
& \qquad\qquad - \int_\Omega v^0 \cdot \partial_t b^0\, \mathrm{d}x + \int_\Omega \partial_t u(t) \cdot \partial_t b(t)\, \mathrm{d}x.
\end{aligned}
$$

Remark 3.2 *Let (c, u, z, μ, ξ) be a weak solution. Furthermore, if additionally*

$$c \in H^1(0, T; H^1(\Omega)), \quad u \in H^1(0, T; H^1(\Omega; \mathbb{R}^n)), \quad z \in H^1(0, T; W^{1,p}(\Omega)),$$

then for a.e. $t \in (0, T)$

$$
\begin{aligned}
& z_t - \Delta_p z + W_{,z}(c, \epsilon(u), z) + f'(z) + \xi + \varphi = 0 \ \ in \ \big(W^{1,p}(\Omega)\big)^*, \\
& \xi \in \partial I_{W^{1,p}_+(\Omega)}(z), \\
& \varphi \in \partial I_{W^{1,p}_-(\Omega)}(\partial_t z).
\end{aligned}
$$

Moreover, the energy inequality (16) becomes an energy balance.

The main aim of this work is to prove existence of weak solutions in the sense above.

Theorem 3.3 *Let the assumptions in Section 2 be satisfied. To the given data l, b, c^0, u^0, v^0, z^0, there exists a weak solution of system (4)-(6) in the sense of Definition 3.1.*

3.2 Notion of weak solutions for a regularized system and existence results

We will first study a regularized version of our phase separation-damage model. The passage to the limit is performed in Section 4.2. The regularization is needed in the existence proof in the first instance to pass from the time-discrete to the time-continuous system.

The regularized PDE system for $\delta > 0$ is given by

$$
\begin{aligned}
c_t &= \operatorname{div}(m(c,z)\nabla\mu), \\
\mu &= -\Delta c + W_{,c}(c, \epsilon(u), z) + \Psi'(c) + \delta c_t, \\
u_{tt} &- \operatorname{div}\left(W_{,e}(c, \epsilon(u), z)\right) + \delta Au = l, \\
z_t &- \Delta_p z + W_{,z}(c, \epsilon(u), z) + f'(z) + \xi + \varphi = 0, \\
\xi &\in \partial I_{[0,\infty)}(z), \\
\varphi &\in \partial I_{(-\infty,0]}(z_t),
\end{aligned}
$$

where the linear operator $A : H^2(\Omega; \mathbb{R}^n) \to (H^2(\Omega; \mathbb{R}^n))^*$ is defined as

$$
\langle Au, v \rangle_{H^2} := \int_\Omega \langle \nabla(\nabla u), \nabla(\nabla v) \rangle_{\mathbb{R}^{n \times n \times n}} \, \mathrm{d}x := \sum_{1 \le i,j,k \le n} \int_\Omega \frac{\mathrm{d}^2 u_k}{\mathrm{d}x_i \mathrm{d}x_j} \frac{\mathrm{d}^2 v_k}{\mathrm{d}x_i \mathrm{d}x_j} \, \mathrm{d}x.
$$

A weak formulation of the regularized system such as in Definition 3.1 can be obtained with the corresponding modifications including the δ-terms.

Definition 3.4 (Weak solution of the regularized system) *A weak solution of the regularized PDE system for the data $(l, b, c^0, u^0, v^0, z^0)$ is a 5-tuple (c, u, z, μ, ξ) satisfying the following properties:*

- *spaces:*

$$
\begin{aligned}
&c \in L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)), \\
&\quad \text{with } c(0) = c^0 \text{ a.e. in } \Omega, \\
&u \in L^\infty(0,T; H^2(\Omega; \mathbb{R}^n)) \cap W^{1,\infty}(0,T; L^2(\Omega; \mathbb{R}^n)) \cap H^2(0,T; (H^2_{\Gamma_D}(\Omega; \mathbb{R}^n))^*) \\
&\quad \text{with } u = b \text{ on } \Gamma_D \times (0,T), \ u(0) = u^0 \text{ a.e. in } \Omega, \ \partial_t u(0) = v^0 \text{ a.e. in } \Omega, \\
&z \in L^\infty(0,T; W^{1,p}(\Omega)) \cap H^1(0,T; L^2(\Omega)) \\
&\quad \text{with } z(0) = z^0 \text{ in } \Omega, \ z \ge 0 \text{ a.e. in } \Omega_T, \ \partial_t z \le 0 \text{ a.e. in } \Omega_T, \\
&\mu \in L^2(0,T; H^1(\Omega)), \\
&\xi \in L^\infty(0,T; L^1(\Omega)).
\end{aligned}
$$

- *for all $\zeta \in H^1(\Omega)$ and for a.e. $t \in (0,T)$:*

$$
\int_{\Omega_T} (\partial_t c)\, \zeta \, \mathrm{d}x \, \mathrm{d}t = -\int_{\Omega_T} m(c,z) \nabla\mu \cdot \nabla\zeta \, \mathrm{d}x \, \mathrm{d}t \tag{17}
$$

- *for all $\zeta \in H^1(\Omega)$ and for a.e. $t \in (0,T)$:*

$$
\int_\Omega \mu\, \zeta \, \mathrm{d}x = \int_\Omega \left(\nabla c \cdot \nabla\zeta + W_{,c}(c, \epsilon(u), z)\zeta + \Psi'(c)\zeta + \delta\, (\partial_t c)\, \zeta \right) \mathrm{d}x \tag{18}
$$

- *for all $\zeta \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^n)$ and for a.e. $t \in (0,T)$:*

$$
\langle \partial_{tt} u, \zeta \rangle_{H^1} + \int_\Omega W_{,e}(c, \epsilon(u), z) : \epsilon(\zeta) \, \mathrm{d}x + \delta \langle Au, \zeta \rangle_{H^2} = \int_\Omega l \cdot \zeta \, \mathrm{d}x \tag{19}
$$

- *for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in (0,T)$:*

$$
0 \le \int_\Omega \left(|\nabla z|^{p-2} \nabla z \cdot \nabla\zeta + (W_{,z}(c, \epsilon(u), z) + f'(z) + \partial_t z + \xi)\zeta \right) \mathrm{d}x \tag{20}
$$

- *for all $\zeta \in L_+^\infty(\Omega)$ and for a.e. $t \in (0, T)$:*

$$0 \geq \int_\Omega \xi(\zeta - z)\,\mathrm{d}x \tag{21}$$

- *total energy inequality for a.e. $t \in (0, T)$:*

$$\mathcal{E}(t) + \mathcal{K}(t) + \mathcal{D}(0, t) \leq \mathcal{E}(0) + \mathcal{K}(0) + \mathcal{W}_{\mathrm{ext}}(0, t) \tag{22}$$

with

free energy:
$$\mathcal{E}(t) := \int_\Omega \left(\frac{1}{p}|\nabla z(t)|^p + \frac{1}{2}|\nabla c(t)|^2 + W(c(t), \epsilon(u(t)), z(t)) \right)\mathrm{d}x$$
$$+ \int_\Omega \big(f(z(t)) + \Psi(c(t)) \big)\,\mathrm{d}x + \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2},$$

kinetic energy:
$$\mathcal{K}(t) := \int_\Omega \frac{1}{2}|\partial_t u(t)|^2\,\mathrm{d}x,$$

dissipation:
$$\mathcal{D}(0, t) := \int_{\Omega_t} \big(|\partial_t z|^2 + \delta|\partial_t c|^2 + m(c, z)|\nabla\mu|^2 \big)\,\mathrm{d}x\,\mathrm{d}s,$$

external work:
$$\mathcal{W}_{\mathrm{ext}}(0, t) := \int_{\Omega_t} W_{,e}(c, \epsilon(u), z) : \epsilon(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \delta \int_0^t \langle Au(s), \partial_t b(s)\rangle_{H^2}\,\mathrm{d}s$$
$$- \int_{\Omega_t} \partial_t u \cdot \partial_{tt} b\,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t} l \cdot (\partial_t u - \partial_t b)\,\mathrm{d}x\,\mathrm{d}s$$
$$- \int_\Omega v^0 \cdot \partial_t b^0\,\mathrm{d}x + \int_\Omega \partial_t u(t) \cdot \partial_t b(t)\,\mathrm{d}x.$$

The proof of the main result, see Theorem 3.3, is based on the existence of weak solutions for the regularized system.

Theorem 3.5 *Let the assumptions in Section 2 be satisfied. To the given data l, b, c^0, u^0, v^0, z^0, there exists a weak solution of the regularized system in the sense of Definition 3.4.*

4 Proof of the existence theorems

4.1 Existence proof for the regularized system

For the existence proof of the regularized system, we will use a semi-implicit Euler scheme solved by a recursive minimization procedure.

Let $\tau > 0$ denote the discretization fineness and let $M_\tau := \lfloor T/\tau \rfloor$ be the number of discrete time points. We fix a $k \in 1, \ldots, M_\tau$ and define the functional $\mathcal{F}_\tau^k : H^1(\Omega) \times H^2(\Omega; \mathbb{R}^n) \times W^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}_\tau^k(c, u, z) := \int_\Omega \left(\frac{1}{p}|\nabla z|^p + \frac{1}{2}|\nabla c|^2 + W(c, \epsilon(u), z) + f(z) + \Psi(c) - l(k\tau) \cdot u \right)\mathrm{d}x$$
$$+ \frac{\delta}{2}\langle A^{k-1}u, u\rangle_{H^2} + \frac{\tau}{2}\left\| \frac{z - z_\tau^{k-1}}{\tau} \right\|_{L^2}^2 + \frac{\tau^2}{2}\left\| \frac{u - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2} \right\|_{L^2}^2$$
$$+ \frac{1}{2\tau}\left\| \frac{c - c_\tau^{k-1}}{\tau} \right\|_{V_0}^2 + \frac{\delta}{2\tau}\left\| \frac{c - c_\tau^{k-1}}{\tau} \right\|_{L^2}^2,$$

where $V_0 = \{\zeta \in (H^1(\Omega))^* | \langle \zeta, \mathbf{1}\rangle_{(H^1)^* \times H^1} = 0\}$. Note that the inverse operator $A^{k-1,-1} : V_0 \to U_0 := \{\zeta \in (H^1(\Omega)) | \int_\Omega \zeta\,\mathrm{d}x = 0\}$ of the operator $A^{k-1} : U_0 \to V_0$ given by

$$u \mapsto \langle \nabla u, m(c_\tau^{k-1}, z_\tau^{k-1})\nabla\cdot \rangle_{L^2}$$

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is well defined. The space V_0 is endowed with the scalar product

$$\langle u, v \rangle_{V_0} := \langle \nabla(A^{-1}u), m(c_\tau^{k-1}, z_\tau^{k-1})\nabla(A^{-1}v)\rangle_{L^2}.$$

We refer to [Gar00] for details.

A minimizer of $\mathcal{F}_\tau^k$ in the subspace

$$\left\{ c \in H^1(\Omega) \mid \int_\Omega (c - c^0)\, \mathrm{d}x = 0\, \mathrm{d}x \right\} \times \left\{ u \in H^2(\Omega; \mathbb{R}^n) \mid u|_{\Gamma_\mathrm{D}} = b(\tau k)|_{\Gamma_\mathrm{D}} \right\}$$
$$\times \left\{ z \in W^{1,p}(\Omega) \mid 0 \le z \le z_\tau^{k-1} \right\} \quad (23)$$

obtained by the direct method in the calculus of variations is denoted by $(c_\tau^k, u_\tau^k, z_\tau^k)$. More precisely, by a recursive minimization procedure starting from the initial values (c^0, u^0, z^0) and $u^{-1} := u^0 - \tau v^0$, we obtain functions $(c_\tau^k, u_\tau^k, z_\tau^k)$ for $k = 0, \ldots, M_\tau$. The velocity field v_τ^k is set to $(u_\tau^k - u_\tau^{k-1})/\tau$ and b_τ^k and l_τ^k are given by $b(\tau k)$ and $l(\tau k)$.

Let $w_\tau^k \in \{l_\tau^k, b_\tau^k, c_\tau^k, u_\tau^k, v_\tau^k, z_\tau^k, \mu_\tau^k\}$, we introduce the piecewise constant interpolations w_τ, w_τ^- and the linear interpolation $\widehat{w}_\tau$ with respect to time as

$$w_\tau(t) := w_\tau^k \qquad\qquad \text{with } k = \lceil t/\tau \rceil,$$
$$w_\tau^-(t) := w_\tau^{\max\{0, k-1\}} \qquad\qquad \text{with } k = \lceil t/\tau \rceil,$$
$$\widehat{w}_\tau(t) := \beta w_\tau^k + (1-\beta)w_\tau^{\max\{0, k-1\}} \qquad\qquad \text{with } k = \lceil t/\tau \rceil,\ \beta = \frac{t - (k-1)\tau}{\tau}$$

and the piecewise constant functions t_τ and t_τ^- as

$$t_\tau := \lceil t/\tau \rceil \tau = \min\{k\tau \mid k \in \mathbb{N}_0 \text{ and } k\tau \ge t\},$$
$$t_\tau^- := \max\{0, t_\tau - \tau\}.$$

We would like to remark that, by definition, $w_\tau(t) = w_\tau(t_\tau)$ for all $t \in [0, T]$ and

$$\partial_t \widehat{v}_\tau(t) = \frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2}$$

for $t \in \lceil t/\tau \rceil$.

Since the functions $(c_\tau^k, u_\tau^k, z_\tau^k)$ are minimizers, we obtain the following necessary conditions (Euler-Lagrange equations) by direct methods in the calculus of variations, cf. [HK11, HK12, HK13a]:

Lemma 4.1 *There exists a time-discrete weak solution in the following sense:*

■ *spaces:*

$$c_\tau, c_\tau^- \in L^\infty(0, T; H^1(\Omega)), \qquad\qquad \widehat{c}_\tau \in W^{1,\infty}(0, T; H^1(\Omega)),$$
$$u_\tau, v_\tau \in L^\infty(0, T; H^2(\Omega; \mathbb{R}^n)), \qquad\qquad \widehat{u}_\tau, \widehat{v}_\tau \in W^{1,\infty}(0, T; H^2(\Omega; \mathbb{R}^n)),$$
$$z_\tau, z_\tau^- \in L^\infty(0, T; W^{1,p}(\Omega)), \qquad\qquad \widehat{z}_\tau \in W^{1,\infty}(0, T; W^{1,p}(\Omega)),$$
$$\mu_\tau \in L^\infty(0, T; H^1(\Omega)),$$

with

$$c_\tau(0) = c^0 \text{ a.e. in } \Omega,\ u_\tau(0) = u^0 \text{ a.e. in } \Omega,\ z_\tau(0) = z^0 \text{ in } \Omega,\ v_\tau(0) = v^0 \text{ a.e. in } \Omega,$$
$$u_\tau = b_\tau \text{ on } \Gamma_\mathrm{D} \times (0, T),\ z_\tau \ge 0 \text{ a.e. in } \Omega_T,\ \partial_t \widehat{z}_\tau \le 0 \text{ a.e. in } \Omega_T,$$

■ *for all $\zeta \in L^2(0, T; H^1(\Omega))$:*

$$\int_{\Omega_T} (\partial_t \widehat{c}_\tau)\zeta\, \mathrm{d}x\, \mathrm{d}t = -\int_{\Omega_T} m(c_\tau^-, z_\tau^-)\nabla\mu_\tau \cdot \nabla\zeta\, \mathrm{d}x\, \mathrm{d}t, \quad (24)$$

■ *for all $\zeta \in H^1(\Omega)$ and for a.e. $t \in (0, T)$:*

$$\int_\Omega \mu_\tau \zeta \, \mathrm{d}x = \int_\Omega \left(\nabla c_\tau \cdot \nabla \zeta + W_{,c}(c_\tau, \epsilon(u_\tau), z_\tau) \zeta + \Psi'(c_\tau) \zeta + \delta(\partial_t \widehat{c}_\tau) \zeta \right) \mathrm{d}x, \tag{25}$$

■ *for all $\zeta \in H^2_{\Gamma_{\mathrm{D}}}(\Omega; \mathbb{R}^n)$ and for a.e. $t \in (0, T)$:*

$$\int_\Omega \partial_t \widehat{v}_\tau \cdot \zeta \, \mathrm{d}x + \int_\Omega W_{,e}(c_\tau, \epsilon(u_\tau), z_\tau) : \epsilon(\zeta) \, \mathrm{d}x + \delta \langle A u_\tau, \zeta \rangle_{H^2} = \int_\Omega l_\tau \cdot \zeta \, \mathrm{d}x, \tag{26}$$

■ *for a.e. $t \in (0, T)$ and for all $\zeta \in W^{1,p}(\Omega)$ with $0 \le \zeta + z_\tau(t) \le z_\tau^-(t)$:*

$$0 \le \int_\Omega \left(|\nabla z_\tau|^{p-2} \nabla z_\tau \cdot \nabla \zeta + (W_{,z}(c_\tau, \epsilon(u_\tau), z_\tau) + f'(z_\tau) + \partial_t \widehat{z}_\tau) \zeta \right) \mathrm{d}x. \tag{27}$$

Lemma 4.2 (A priori estimates) *There exists a constant $C > 0$ independent of δ such that*

(i) $\|\nabla c_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} < C, \quad \|\partial_t \widehat{c}_\tau\|_{L^2(0,T;L^2(\Omega))} < C,$

(ii) $\|u_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} < C, \quad \|v_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} < C,$

$\|\widehat{u}_\tau\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} < C,$

$\|\widehat{v}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^1(0,T;(H^2_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^n))^*)} < C,$

(iii) $\|\nabla z_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))} < C, \quad \|\partial_t \widehat{z}_\tau\|_{L^2(0,T;L^2(\Omega))} < C,$

(iv) $\|\nabla \mu_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} < C, \quad \|m(c_\tau^-, z_\tau^-)^{1/2} \nabla \mu_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} < C.$

Proof. We split the proof into two steps. We first prove the a priori estimates (i), (ii) and (iv) and then we deduce estimate (iii).

First a priori estimates. Testing (24) with $\tau \mu_\tau$, testing (25) with $c_\tau - c_\tau^-$, testing (26) with $u_\tau - u_\tau^- - (b_\tau - b_\tau^-)$, and adding everything, yield

$$T_1(t) + T_2(t) + T_3(t) + T_4(t) + T_5(t) \le 0$$

with

$$T_1(t) := \int_\Omega \nabla c_\tau(t) \cdot \nabla(c_\tau(t) - c_\tau^-(t)) \, \mathrm{d}x + \int_\Omega \delta \langle \nabla(\nabla u_\tau(t)), \nabla(\nabla(u_\tau(t) - u_\tau^-(t))) \rangle \, \mathrm{d}x$$

$$+ \int_\Omega \partial_t \widehat{v}_\tau(t) \cdot (u_\tau(t) - u_\tau^-(t)) \, \mathrm{d}x,$$

$$T_2(t) := \tau \int_\Omega m(c_\tau(t), z_\tau(t)) |\nabla \mu_\tau(t)|^2 \, \mathrm{d}x + \tau \int_\Omega \delta |\partial_t \widehat{c}_\tau(t)|^2 \, \mathrm{d}x,$$

$$T_3(t) := \int_\Omega W_{,c}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t))(c_\tau(t) - c_\tau^-(t)) \, \mathrm{d}x$$

$$+ \int_\Omega W_{,e}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(u_\tau(t) - u_\tau^-(t)) \, \mathrm{d}x$$

$$T_4(t) := \tau \int_\Omega \Psi'(c_\tau(t)) \partial_t \widehat{c}_\tau(t) \, \mathrm{d}x - \tau \int_\Omega l_\tau(t) \cdot \partial_t \widehat{u}_\tau(t) \, \mathrm{d}x$$

$$T_5(t) := -\tau \int_\Omega \left(\partial_t \widehat{v}_\tau(t) \cdot \partial_t \widehat{b}_\tau(t) + W_{,e}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(\partial_t \widehat{b}_\tau(t)) \right) \mathrm{d}x,$$

$$- \tau \int_\Omega \left(\delta \langle \nabla(\nabla u_\tau(t)), \nabla(\nabla(\partial_t \widehat{b}_\tau(t))) \rangle_{\mathbb{R}^{n \times n \times n}} - l_\tau(t) \cdot \partial_t \widehat{b}_\tau(t) \right) \mathrm{d}x.$$

These terms are estimated in the following.

- Convexity estimates yield

$$T_1(t) \geq \frac{1}{2}\|\nabla c_\tau(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla c_\tau^-(t)\|^2_{L^2(\Omega)}$$
$$+ \frac{\delta}{2}\|\nabla(\nabla u_\tau(t))\|^2_{L^2(\Omega;\mathbb{R}^{n\times n\times n})} - \frac{\delta}{2}\|\nabla(\nabla u_\tau^-(t))\|^2_{L^2(\Omega;\mathbb{R}^{n\times n\times n})}$$
$$+ \frac{1}{2}\|v_\tau(t)\|^2_{L^2(\Omega;\mathbb{R}^n)} - \frac{1}{2}\|v_\tau^-(t)\|^2_{L^2(\Omega;\mathbb{R}^n)}.$$

- We obtain for small $\eta > 0$:

$$T_2(t) \geq \eta \int_{t_\tau^-}^{t_\tau} \left(\|\nabla \mu_\tau(s)\|^2_{L^2(\Omega;\mathbb{R}^n)} + \delta\|\partial_t \widehat{c}_\tau(s)\|^2_{L^2(\Omega)} \right) \mathrm{d}s.$$

- By the convexity argument and by $z_\tau \leq z_\tau^-$, we gain

$$W_{,e}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(u_\tau(t) - u_\tau^-(t))$$
$$\geq W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau(t))$$
$$\geq W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t))$$
$$+ \int_{t_\tau^-}^{t_\tau} W_{,c}(\widehat{c}_\tau(s), \epsilon(u_\tau^-(s)), z_\tau(s))\partial_t \widehat{c}_\tau(s) \, \mathrm{d}s,$$

and conclude ($\eta > 0$ is chosen as small as necessary)

$$T_3(t) \geq \int_\Omega \left(W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) \right) \mathrm{d}x$$
$$+ \int_{t_\tau^-}^{t_\tau} \int_\Omega W_{,c}(c_\tau(s), \epsilon(u_\tau(s)), z_\tau(s))\partial_t \widehat{c}_\tau(s) \, \mathrm{d}x \, \mathrm{d}s$$
$$+ \int_{t_\tau^-}^{t_\tau} \int_\Omega W_{,c}(\widehat{c}_\tau(s), \epsilon(u_\tau^-(s)), z_\tau(s))\partial_t \widehat{c}_\tau(s) \, \mathrm{d}x \, \mathrm{d}s$$
$$\geq \int_\Omega \left(W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) \right) \mathrm{d}x$$
$$- C_\eta \int_{t_\tau^-}^{t_\tau} \left(\|W_{,c}(c_\tau(s), \epsilon(u_\tau(s)), z_\tau(s))\|^2_{L^2(\Omega)} + \|W_{,c}(\widehat{c}_\tau(s), \epsilon(u_\tau^-(s)), z_\tau(s))\|^2_{L^2(\Omega)} \right) \mathrm{d}s$$
$$- \eta \int_{t_\tau^-}^{t_\tau} \|\partial_t \widehat{c}_\tau(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s$$
$$\geq \int_\Omega \left(W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) \right) \mathrm{d}x$$
$$- \widehat{C}_\eta \int_{t_\tau^-}^{t_\tau} \left(\|c_\tau(s)\|^2_{L^2(\Omega)} + \|c_\tau^-(s)\|^2_{L^2(\Omega)} + \|\epsilon(u_\tau(s))\|^2_{L^2(\Omega)} + \|\epsilon(u_\tau^-(s))\|^2_{L^2(\Omega)} \right) \mathrm{d}s$$
$$- \eta \int_{t_\tau^-}^{t_\tau} \|\partial_t \widehat{c}_\tau(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s.$$

- Convexity of Ψ_1 combined with growth condition (9b) and Young's inequality show

$$T_4(t) \geq \int_\Omega \Psi_1(c_\tau(t)) \, \mathrm{d}x - \int_\Omega \Psi_1(c_\tau^-(t)) \, \mathrm{d}x - \eta \int_{t_\tau^-}^{t_\tau} \left(\|\partial_t \widehat{c}_\tau(s)\|^2_{L^2(\Omega)} + \|l_\tau(s)\|^2_{L^2(\Omega;\mathbb{R}^n)} \right) \mathrm{d}s$$
$$- C_\eta \int_{t_\tau^-}^{t_\tau} \left(\|\Psi_2'(c_\tau(s))\|^2_{L^2(\Omega)} + \|v_\tau(s)\|^2_{L^2(\Omega;\mathbb{R}^n)} \right) \mathrm{d}s$$
$$\geq \int_\Omega \Psi_1(c_\tau(t)) \, \mathrm{d}x - \int_\Omega \Psi_1(c_\tau^-(t)) \, \mathrm{d}x - \eta \int_{t_\tau^-}^{t_\tau} \left(\|\partial_t \widehat{c}_\tau(s)\|^2_{L^2(\Omega)} + \|l_\tau(s)\|^2_{L^2(\Omega;\mathbb{R}^n)} \right) \mathrm{d}s$$
$$- C_\eta \int_{t_\tau^-}^{t_\tau} \left(\|c_\tau(s)\|^2_{L^2(\Omega)} + \|v_\tau(s)\|^2_{L^2(\Omega;\mathbb{R}^n)} \right) \mathrm{d}s.$$

■ By using the discrete integration by parts formula, i.e.,

$$\int_{t_\tau^-}^{t_\tau} \int_\Omega \partial_t \widehat{v}_\tau \cdot \partial_t \widehat{b}_\tau \, dx \, ds = \int_\Omega v_\tau(t) \cdot \partial_t \widehat{b}_\tau(t) \, dx - \int_\Omega v_\tau^-(t) \cdot \partial_t \widehat{b}_\tau(t - \tau) \, dx$$
$$- \int_{t_\tau^-}^{t_\tau} \int_\Omega v_\tau^-(s) \cdot \frac{\partial_t \widehat{b}_\tau(s) - \partial_t \widehat{b}_\tau(s - \tau)}{\tau} \, dx \, ds, \tag{28}$$

we obtain

$$T_5(t) \geq -\int_\Omega v_\tau(t) \cdot \partial_t \widehat{b}_\tau(t) \, dx + \int_\Omega v_\tau^-(t) \cdot \partial_t \widehat{b}_\tau(t - \tau) \, dx$$
$$- \int_{t_\tau^-}^{t_\tau} \left(\eta \| v_\tau^-(s) \|_{L^2(\Omega)}^2 + C_\eta \left\| \frac{\partial_t \widehat{b}_\tau(s) - \partial_t \widehat{b}_\tau(s - \tau)}{\tau} \right\|_{L^2(\Omega)}^2 \right) ds$$
$$- \int_{t_\tau^-}^{t_\tau} \left(\eta \| c_\tau(s) \|_{L^2(\Omega)}^2 + \eta \| \epsilon(u_\tau(s)) \|_{L^2(\Omega;\mathbb{R}^{n \times n})}^2 + C_\eta \| \epsilon(\partial_t \widehat{b}_\tau(s)) \|_{L^2(\Omega;\mathbb{R}^{n \times n})}^2 \right) ds$$
$$- \int_{t_\tau^-}^{t_\tau} \left(\eta \| \nabla(\nabla u_\tau(s)) \|_{L^2(\Omega;\mathbb{R}^{n \times n \times n})}^2 + C_\eta \| \nabla(\nabla \partial_t \widehat{b}_\tau(s)) \|_{L^2(\Omega;\mathbb{R}^{n \times n \times n})}^2 \right) ds$$
$$- \int_{t_\tau^-}^{t_\tau} \left(\eta \| l_\tau(s) \|_{L^2(\Omega;\mathbb{R}^n)}^2 + C_\eta \| \partial_t \widehat{b}_\tau(s) \|_{L^2(\Omega;\mathbb{R}^n)}^2 \right) ds.$$

Summing over the discrete time points $t_\tau = 0, \tau, \ldots, k\tau$ for an arbitrary but fixed chosen $k \in \mathbb{N}$, we can apply Gronwall's inequality and obtain the following boundedness properties:

$$\| \nabla c_\tau \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} < C, \tag{29}$$
$$\| \partial_t \widehat{c}_\tau \|_{L^2(0,T;L^2(\Omega))} < C, \tag{30}$$
$$\| \nabla(\nabla u_\tau) \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n \times n \times n}))} < C, \tag{31}$$
$$\| \epsilon(u_\tau) \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{n \times n}))} < C, \tag{32}$$
$$\| v_\tau \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n))} < C, \tag{33}$$
$$\| \nabla \mu_\tau \|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} < C, \tag{34}$$

where $C > 0$ is independent of τ. Combining estimates (31)-(33) with Korn's inequality, we obtain

$$\| u_\tau \|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} < C.$$

Consequently, by noticing $v_\tau = \partial_t \widehat{u}_\tau$,

$$\| \widehat{u}_\tau \|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} < C.$$

A comparison argument in (26) also gives

$$\| \widehat{v}_\tau \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^1(0,T;(H^2_{\Gamma_D}(\Omega;\mathbb{R}^n))^*)} < C.$$

Second a priori estimates. Testing (27) with $z_\tau^- - z_\tau$, yields

$$\int_\Omega |\nabla z_\tau(t)|^{p-2} \nabla z_\tau(t) \cdot \nabla(z_\tau(t) - z_\tau^-(t)) \, dx + \frac{1}{2} \tau \| \partial_t \widehat{z}_\tau(t) \|_{L^2(\Omega)}^2$$
$$\leq -\tau \int_\Omega \left(W_{,z}(c(\tau(t)), \epsilon(u_\tau(t)), z_\tau(t)) \partial_t \widehat{z}_\tau(t) + f'(z_\tau(t)) \partial_t \widehat{z}_\tau(t) \right) dx.$$

Now we apply a convexity estimate and get

$$\frac{1}{p} \| \nabla z_\tau(t) \|_{L^p(\Omega;\mathbb{R}^n)}^p - \frac{1}{p} \| \nabla z_\tau^-(t) \|_{L^p(\Omega;\mathbb{R}^n)}^p + \frac{1}{2} \tau \| \partial_t \widehat{z}_\tau(t) \|_{L^2(\Omega)}^2$$
$$\leq \tau \eta \| \partial_t \widehat{z}_\tau(t) \|_{L^2(\Omega)}^2 + \tau C_\eta \left(1 + \| c_\tau(t) \|_{L^4(\Omega)}^4 + \| \epsilon(u_\tau(t)) \|_{L^4(\Omega;\mathbb{R}^{n \times n})}^4 \right).$$

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We end up with

$$\|\nabla z_\tau\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))} < C,$$
$$\|\partial_t \widehat{z}_\tau\|_{L^2(0,T;L^2(\Omega))} < C,$$

where $C > 0$ is independent of τ. □

By applying Poincaré's inequality, standard weak and weakly-star compactness results to the above a priori estimates, we obtain the following convergence properties.

Lemma 4.3 (Convergence properties) *There exist functions*

$$c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$
$$u \in L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap W^{2,\infty}(0,T;(H^2_{\Gamma_D}(\Omega;\mathbb{R}^n))^*),$$
$$z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$
$$\mu \in L^2(0,T;H^1(\Omega))$$

and subsequences (omitting the subscript) such that for all $r \geq 1$ and $s < 2^*$:

$$
\begin{align}
c_\tau, c_\tau^- \to c & \quad \textit{weakly-star in } L^\infty(0,T;H^1(\Omega)), & (35) \\
& \quad \textit{strongly in } L^r(0,T;L^s(\Omega)), \textit{ a.e. in } \Omega_T, & (36) \\
\widehat{c}_\tau \to c & \quad \textit{weakly-star in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), & (37) \\
u_\tau, u_\tau^- \to u & \quad \textit{weakly-star in } L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)), & (38) \\
& \quad \textit{strongly in } L^r(0,T;H^1(\Omega;\mathbb{R}^n)), \textit{ a.e. in } \Omega_T, & (39) \\
\widehat{u}_\tau \to u & \quad \textit{weakly-star in } L^\infty(0,T;H^2(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)), & (40) \\
v_\tau, v_\tau^- \to \partial_t u & \quad \textit{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)), & (41) \\
\widehat{v}_\tau \to \partial_t u & \quad \textit{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^1(0,T;(H^2_{\Gamma_D}(\Omega;\mathbb{R}^n))^*), & (42) \\
z_\tau, z_\tau^- \to z & \quad \textit{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)), & \\
& \quad \textit{strongly in } L^r(0,T;L^r(\Omega;\mathbb{R}^n)), \textit{ a.e. in } \Omega_T, & (43) \\
\widehat{z}_\tau \to z & \quad \textit{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), & (44) \\
\mu_\tau \to \mu & \quad \textit{weakly in } L^2(0,T;H^1(\Omega)), & (45) \\
m(c_\tau^-, z_\tau^-)^{\frac{1}{2}} \nabla \mu_\tau \to m(c,z)^{\frac{1}{2}} \nabla \mu & \quad \textit{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^n)) & (46)
\end{align}
$$

as $\tau \searrow 0$.

Strong convergence of a subsequence of $\{\nabla z_\tau\}$ in $L^p(\Omega_T;\mathbb{R}^n)$ can be shown as in [HK13a] by a tricky approximation argument.

Lemma 4.4 (cf. [HK13a]) *There exists a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ such that $z_{\tau_k} \to z$ in $L^p(0,T;W^{1,p}(\Omega))$ as $\tau_k \searrow 0$.*

For a time discrete solution of the regularized system, we can prove the validity of an energy inequality of type (16) except the additional discretization error terms $e_\tau^1, \ldots, e_\tau^4$ which will turn out to converge to 0 in a certain sense as $\tau \searrow 0$.

Lemma 4.5 (Discrete energy inequality) *Let a time-discrete weak solution be given as in Lemma 4.1. Then the following energy estimate is satisfied for a.e. $t \in (0,T)$:*

$$
\mathcal{E}_\tau(t) + \mathcal{K}_\tau(t) + \mathcal{D}_\tau(0,t) + \int_0^{t_\tau} \left(e_\tau^1(s) + e_\tau^2(s) + e_\tau^3(s) + e_\tau^4(s) \right) \mathrm{d}s
$$
$$
\leq \mathcal{E}_\tau(0) + \mathcal{K}_\tau(0) + \mathcal{W}_{\tau,\text{ext}}(0,t) \tag{47}
$$

with the discrete energies

$$\mathcal{E}_\tau(t) := \int_\Omega \left(\frac{1}{p}|\nabla z_\tau(t)|^p + \frac{1}{2}|\nabla c_\tau(t)|^2 + W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) + f(z_\tau(t)) + \Psi(c_\tau(t)) \right) \mathrm{d}x$$
$$+ \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2},$$

$$\mathcal{K}_\tau(t) := \int_\Omega \frac{1}{2}|v_\tau(t)|^2 \, \mathrm{d}x,$$

$$\mathcal{D}_\tau(0,t) := \int_0^{t_\tau} \int_\Omega \left(|\partial_t \widehat{z}_\tau|^2 + \delta|\partial_t \widehat{c}_\tau|^2 + m(c_\tau^-, z_\tau^-)\nabla\mu_\tau \cdot \nabla\mu_\tau \right) \mathrm{d}x \, \mathrm{d}s,$$

$$\mathcal{W}_{\tau,\mathrm{ext}}(0,t) := \int_0^{t_\tau} \int_\Omega W_{,e}(c_\tau, \epsilon(u_\tau), z_\tau) : \epsilon(\partial_t \widehat{b}_\tau) \, \mathrm{d}x \, \mathrm{d}s + \delta \int_0^{t_\tau} \langle Au_\tau(s), \partial_t \widehat{b}_\tau(s)\rangle_{H^2} \, \mathrm{d}s$$
$$+ \int_0^{t_\tau} \int_\Omega l_\tau \cdot \left(\partial_t \widehat{u}_\tau - \partial_t \widehat{b}_\tau \right) \mathrm{d}x \, \mathrm{d}s - \int_\Omega v^0 \cdot \partial_t \widehat{b}_\tau(0) \, \mathrm{d}x + \int_\Omega v_\tau(t) \cdot \partial_t \widehat{b}_\tau(t) \, \mathrm{d}x$$
$$- \int_0^{t_\tau} \int_\Omega v_\tau^-(s) \cdot \frac{\partial_t \widehat{b}_\tau(s) - \partial_t \widehat{b}_\tau(s-\tau)}{\tau} \, \mathrm{d}x \, \mathrm{d}s,$$

and the error terms

$$e_\tau^1(t) := \int_\Omega \left(\frac{W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) - W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau(t))}{\tau} \right) \mathrm{d}x$$
$$+ \int_\Omega W_{,z}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) \, \partial_t \widehat{z}_\tau(t) \, \mathrm{d}x,$$

$$e_\tau^2(t) := \int_\Omega \left(\frac{W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) - W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau^-(t))}{\tau} \right) \mathrm{d}x$$
$$+ \int_\Omega W_{,c}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) \, \partial_t \widehat{c}_\tau(t) \, \mathrm{d}x,$$

$$e_\tau^3(t) := \int_\Omega \frac{\Psi(c_\tau^-(t)) - \Psi(c_\tau(t))}{\tau} \, \mathrm{d}x + \int_\Omega \Psi'(c_\tau(t)) \, \partial_t \widehat{c}_\tau(t) \, \mathrm{d}x,$$

$$e_\tau^4(t) := \int_\Omega \frac{f(z_\tau^-(t)) - f(z_\tau(t))}{\tau} \, \mathrm{d}x + \int_\Omega f'(z_\tau(t)) \, \partial_t \widehat{z}_\tau(t) \, \mathrm{d}x.$$

Proof. We compute by using convexity of W with respect to e:

$$\int_\Omega W_{,e}(c_\tau, \epsilon(u_\tau), z_\tau) : \epsilon(u_\tau - u_\tau^-) \, \mathrm{d}x$$
$$\geq \int_\Omega \left(W(c_\tau, \epsilon(u_\tau), z_\tau) - W(c_\tau^-, \epsilon(u_\tau^-), z_\tau^-) \right) \mathrm{d}x$$
$$+ \int_\Omega \left(W(c_\tau, \epsilon(u_\tau^-), z_\tau^-) - W(c_\tau, \epsilon(u_\tau^-), z_\tau) \right) \mathrm{d}x$$
$$+ \int_\Omega \left(W(c_\tau^-, \epsilon(u_\tau^-), z_\tau^-) - W(c_\tau, \epsilon(u_\tau^-), z_\tau^-) \right) \mathrm{d}x. \tag{48}$$

We test (26) with $u_\tau - u_\tau^- - (b_\tau - b_\tau^-)$, apply (48), use further convexity arguments and end up with

$$\frac{1}{2}\left\|v_\tau(t)\right\|_{L^2}^2 - \frac{1}{2}\left\|v_\tau^-(t)\right\|_{L^2}^2 + \frac{\delta}{2}\langle Au_\tau(t), u_\tau(t)\rangle_{H^2} - \frac{\delta}{2}\langle Au_\tau^-(t), u_\tau^-(t)\rangle_{H^2}$$
$$+ \int_\Omega \left(W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) \right) \mathrm{d}x - \int_\Omega \partial_t \widehat{v}_\tau(t) \cdot \left(b_\tau(t) - b_\tau^-(t) \right) \mathrm{d}x$$
$$+ \int_\Omega \left(W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) - W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau(t)) \right) \mathrm{d}x$$
$$+ \int_\Omega \left(W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) - W(c_\tau(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) \right) \mathrm{d}x$$

$$\leq \int_\Omega l_\tau(t) \cdot \left(u_\tau(t) - u_\tau^-(t) - (b_\tau(t) - b_\tau^-(t)) \right) \mathrm{d}x$$

$$+ \int_\Omega W_{,e}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(b_\tau(t) - b_\tau^-(t)) \,\mathrm{d}x + \delta \langle Au_\tau(t), b_\tau(t) - b_\tau^-(t) \rangle_{H^2}. \tag{49}$$

Using the convexity estimate

$$\int_\Omega |\nabla z_\tau|^{p-2} \nabla z_\tau \cdot \nabla(z_\tau - z_\tau^-) \,\mathrm{d}x \geq \frac{1}{p} \|\nabla z_\tau\|_{L^p}^p - \frac{1}{p} \|\nabla z_\tau^-\|_{L^p}^p$$

and testing (27) with $z_\tau^- - z_\tau$ yield

$$\frac{1}{p} \|\nabla z_\tau(t)\|_{L^p}^p - \frac{1}{p} \|\nabla z_\tau^-(t)\|_{L^p}^p + \tau \|\partial_t \widehat{z}_\tau(t)\|_{L^2}^2$$

$$\leq \int_\Omega \left(W_{,z}(\epsilon(u_\tau(t)), z_\tau(t)) + f'(z_\tau(t)) \right) \left(z_\tau^-(t) - z_\tau(t) \right) \mathrm{d}x. \tag{50}$$

Next we test equation (24) with $\tau \mu_\tau$ and (25) with $(c_\tau - c_\tau^-)$ and add the two derived equations. We obtain by means of the convexity property

$$\int_\Omega \nabla c_\tau \cdot \nabla(c_\tau - c_\tau^-) \,\mathrm{d}x \geq \frac{1}{2} \|\nabla c_\tau\|_{L^2}^2 - \frac{1}{2} \|\nabla c_\tau^-\|_{L^2}^2$$

the estimate

$$\frac{1}{2} \|\nabla c_\tau(t)\|_{L^2}^2 - \frac{1}{2} \|\nabla c_\tau^-(t)\|_{L^2}^2 + \int_\Omega \Big(W_{,c}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t))(c_\tau(t) - c_\tau^-(t))$$

$$+ \Psi'(c_\tau(t))(c_\tau(t) - c_\tau^-(t)) + \tau \, m(c_\tau^-(t), z_\tau^-(t)) \nabla \mu_\tau(t) \cdot \nabla \mu_\tau(t) \Big) \mathrm{d}x + \delta \tau \|\partial_t \widehat{c}_\tau(t)\|_{L^2}^2 \leq 0. \tag{51}$$

Adding the estimates (49)–(51), we end up with

$$\frac{1}{2} \|v_\tau(t)\|_{L^2}^2 - \frac{1}{2} \|v_\tau^-(t)\|_{L^2}^2 + \frac{\delta}{2} \langle Au_\tau(t), u_\tau(t) \rangle_{H^2} - \frac{\delta}{2} \langle Au_\tau^-(t), u_\tau^-(t) \rangle_{H^2}$$

$$+ \frac{1}{2} \|\nabla c_\tau(t)\|_{L^2}^2 - \frac{1}{2} \|\nabla c_\tau^-(t)\|_{L^2}^2 + \frac{1}{p} \|\nabla z_\tau(t)\|_{L^p}^p - \frac{1}{p} \|\nabla z_\tau^-(t)\|_{L^p}^p$$

$$+ \tau \Big(\|\partial_t \widehat{z}_\tau(t)\|_{L^2}^2 + \delta \|\partial_t \widehat{c}_\tau(t)\|_{L^2}^2 + \int_\Omega m(c_\tau^-(t), z_\tau^-(t)) \nabla \mu_\tau(t) \cdot \nabla \mu_\tau(t) \,\mathrm{d}x \Big)$$

$$- \int_\Omega \partial_t \widehat{v}_\tau(t) \cdot \left(b_\tau(t) - b_\tau^-(t) \right) \mathrm{d}x + \int_\Omega \left(W(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) - W(c_\tau^-(t), \epsilon(u_\tau^-(t)), z_\tau^-(t)) \right) \mathrm{d}x$$

$$+ \int_\Omega f(z_\tau(t)) \,\mathrm{d}x - \int_\Omega f(z_\tau^-(t)) \,\mathrm{d}x + \int_\Omega \Psi(c_\tau(t)) \,\mathrm{d}x$$

$$- \int_\Omega \Psi(c_\tau^-(t)) \,\mathrm{d}x + \tau \left(e_\tau^1(t) + e_\tau^2(t) + e_\tau^3(t) + e_\tau^4(t) \right)$$

$$\leq \int_\Omega l_\tau(t) \cdot \left(u_\tau(t) - u_\tau^-(t) - (b_\tau(t) - b_\tau^-(t)) \right) \mathrm{d}x$$

$$+ \int_\Omega W_{,e}(c_\tau(t), \epsilon(u_\tau(t)), z_\tau(t)) : \epsilon(b_\tau(t) - b_\tau^-(t)) \,\mathrm{d}x + \delta \langle Au_\tau(t), b_\tau(t) - b_\tau^-(t) \rangle_{H^2}$$

with the error terms $e_\tau^1(t)$, $e_\tau^2(t)$, $e_\tau^3(t)$ and $e_\tau^4(t)$. Summing over the discrete time points and taking into account the discrete integration by parts formula (28), we finally obtain the claim. $\qquad \square$

Proof of Theorem 3.5 We are going to establish the equalities and inequalities of the weak formulation (17)-(22).

- ■ (Cahn-Hilliard equation)
  Because of the convergence properties (37), (36), (43) and (45) we may pass to the limit in (24) and obtain (17).

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To establish (18), we first integrate (25) over time from $t = 0$ to $t = T$. The growth condition (9a) and the convergence properties (45), (35), (36), (39), (43) and (37) allow us to pass to the limit in the integrated version of (25) which shows (18).

■ (Balance equation of forces)
By using the canonical embedding $L^2(\Omega; \mathbb{R}^n) \hookrightarrow (H^2_{\Gamma_D}(\Omega; \mathbb{R}^n))^*$, it follows for all $\zeta \in H^2_{\Gamma_D}(\Omega; \mathbb{R}^n)$

$$\int_\Omega \partial_t \widehat{v}_\tau(t) \cdot \zeta \, \mathrm{d}x = \langle \partial_t \widehat{v}_\tau(t), \zeta \rangle_{H^2}.$$

Keeping this identity in mind, integrating (26) from $t = 0$ to $t = T$ and passing to the limit $\tau \searrow 0$ by using (42) and (36), (39), (43) and (38), we obtain (19).

■ (Variational inequality for z)
To obtain the variational inequalities (20) and (21), we can proceed as in [HK13a]. In particular, (20) is valid for the subgradient

$$\xi = -\chi_{\{z=0\}} \max\Big\{0, W_{,z}(c, \epsilon(u), z) + f'(z)\Big\}, \tag{52}$$

which satisfies (21), where $\chi_{\{z=0\}}$ is the characteristic function of the set $\{z = 0\}$.

■ (Energy inequality)
To treat the energy inequality (47), we set

$$
\begin{aligned}
A_\tau(t) &:= \int_\Omega \Big(\frac{1}{p}|\nabla z_\tau(t)|^p + \frac{1}{2}|\nabla c_\tau(t)|^2 + W(c(t), \epsilon(u_\tau(t)), z_\tau(t)) + f(z_\tau(t)) + \Psi(c_\tau(t))\Big) \, \mathrm{d}x \\
&\quad - \int_\Omega \Big(\frac{1}{p}|\nabla z^0|^p + \frac{1}{2}|\nabla c^0|^2 + W(c^0, \epsilon(u^0), z^0) + f(z^0) + \Psi(c^0)\Big) \, \mathrm{d}x \\
&\quad + \int_\Omega \frac{1}{2}|v_\tau(t)|^2 \, \mathrm{d}x - \int_\Omega \frac{1}{2}|v^0|^2 \, \mathrm{d}x + \frac{\delta}{2}\langle A u_\tau(t), u_\tau(t)\rangle_{H^2} - \frac{\delta}{2}\langle A u^0, u^0\rangle_{H^2} \\
&\quad - \int_\Omega v_\tau(t) \cdot \partial_t \widehat{b}_\tau(t) \, \mathrm{d}x + \int_\Omega v^0 \cdot \partial_t \widehat{b}_\tau(0) \, \mathrm{d}x \\
B_\tau(t) &:= \int_0^{t_\tau} \int_\Omega \Big(|\partial_t \widehat{z}_\tau|^2 + \delta|\partial_t \widehat{c}_\tau|^2 + m(c_\tau, z_\tau)\nabla\mu_\tau \cdot \nabla\mu_\tau\Big) \, \mathrm{d}x \, \mathrm{d}s \\
&\quad - \int_0^{t_\tau} \int_\Omega l_\tau \cdot \Big(\partial_t \widehat{u}_\tau - \partial_t \widehat{b}_\tau\Big) \, \mathrm{d}x \, \mathrm{d}s \\
&\quad - \int_0^{t_\tau} \int_\Omega W_{,e}(c_\tau, \epsilon(u_\tau), z_\tau) : \epsilon(\partial_t \widehat{b}_\tau) \, \mathrm{d}x \, \mathrm{d}s - \delta \int_0^{t_\tau} \langle A u_\tau(s), \partial_t \widehat{b}_\tau(s)\rangle_{H^2} \, \mathrm{d}s \\
&\quad + \int_0^{t_\tau} \int_\Omega v_\tau^-(s) \cdot \frac{\partial_t \widehat{b}_\tau(s) - \partial_t \widehat{b}_\tau(s - \tau)}{\tau} \, \mathrm{d}x \, \mathrm{d}s, \\
E_\tau^1(t) &:= \int_0^{t_\tau} e_\tau^1(s) \, \mathrm{d}s, \quad E_\tau^2(t) := \int_0^{t_\tau} e_\tau^2(s) \, \mathrm{d}s, \\
E_\tau^3(t) &:= \int_0^{t_\tau} e_\tau^3(s) \, \mathrm{d}s, \quad E_\tau^4(t) := \int_0^{t_\tau} e_\tau^4(s) \, \mathrm{d}s.
\end{aligned}
$$

Then, (47) is equivalent to

$$A_\tau(t) + B_\tau(t) + E_\tau^1(t) + E_\tau^2(t) + E_\tau^3(t) + E_\tau^4(t) \leq 0. \tag{53}$$

Furthermore, by the a priori estimates, we observe that

$$|A_\tau(t)| + |B_\tau(t)| + |E_\tau^1(t)| + |E_\tau^2(t)| + E_\tau^3(t) + |E_\tau^4(t)| < C \tag{54}$$

for all $t \in [0, T]$ and for all $\tau > 0$ (along a subsequence τ_k). Next, we consider the $\liminf_{\tau \searrow 0}$ of each term in (53) separately.

- By the already proven convergence properties and by lower semi-continuity arguments, we obtain

$$\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} A_\tau(t) \, \mathrm{d}t \geq \int_{t_1}^{t_2} A(t) \, \mathrm{d}t \text{ for all } 0 \leq t_1 \leq t_2 \leq T, \tag{55}$$

where A is defined as A_τ but c_τ, u_τ, z_τ, v_τ and $\widehat{b}_\tau$ are substituted by their continuous limits. Note that this lim inf–estimate does not necessarily hold pointwise a.e. in t because, for instance, we do not know $v_\tau(t) \to v(t)$ weakly in $L^2(\Omega; \mathbb{R}^n)$ for a.e. t (see (41)).

- Let $0 \leq t_1 \leq t_2 \leq T$ be arbitrary. By Fatou's lemma, by (44) and by a lower semi-continuity argument, we obtain

$$\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} \int_0^{t_\tau} \int_\Omega |\partial_t \widehat{z}_\tau(s)|^2 \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t \geq \int_{t_1}^{t_2} \left(\liminf_{\tau \searrow 0} \int_0^{t_\tau} \int_\Omega |\partial_t \widehat{z}_\tau(s)|^2 \, \mathrm{d}x \, \mathrm{d}s \right) \mathrm{d}t$$
$$\geq \int_{t_1}^{t_2} \int_0^t \int_\Omega |\partial_t z(s)|^2 \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t. \tag{56}$$

Analogously,

$$\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} \int_0^{t_\tau} \int_\Omega \delta |\partial_t \widehat{c}_\tau(s)|^2 \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t \geq \int_{t_1}^{t_2} \int_0^t \int_\Omega \delta |\partial_t c(s)|^2 \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t \tag{57}$$

and, by (46),

$$\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} \int_0^{t_\tau} \int_\Omega m(c_\tau^-(s), z_\tau^-(s)) \nabla \mu_\tau(s) \cdot \nabla \mu_\tau(s) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t$$
$$\geq \int_{t_1}^{t_2} \int_0^t \int_\Omega m(c(s), z(s)) \nabla \mu(s) \cdot \nabla \mu(s) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t. \tag{58}$$

Taking also (54) and the already known convergence properties into account, we obtain

$$\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} B_\tau(t) \, \mathrm{d}t \geq \int_{t_1}^{t_2} B(t) \, \mathrm{d}t, \tag{59}$$

where B is defined as B_τ but c_τ, $\widehat{c}_\tau$, u_τ, $\widehat{u}_\tau$, v_τ^-, z_τ, $\widehat{z}_\tau$, $\widehat{\mu}_\tau$ and $\widehat{b}_\tau$ are substituted by their continuous counterparts and $\frac{\partial_t \widehat{b}_\tau(t) - \partial_t \widehat{b}_\tau(t-\tau)}{\tau}$ by $\partial_{tt} b(t)$.

- Due to the differentiability of $\mathbf{C}$ we have

$$\mathbf{C}(z_\tau^-) = \mathbf{C}(z_\tau) + \mathbf{C}'(z_\tau)(z_\tau^- - z_\tau) + r(z_\tau^- - z_\tau), \; \frac{r(\eta)}{\eta} \to 0 \text{ as } \eta \to 0. \tag{60}$$

Hence, we obtain

$$\int_0^t \int_\Omega \frac{1}{2} \frac{\mathbf{C}(z_\tau^-) - \mathbf{C}(z_\tau)}{\tau} \big(\epsilon(u_\tau^-) - e^*(c)\big) : \big(\epsilon(u_\tau^-) - e^*(c)\big) \, \mathrm{d}x \, \mathrm{d}s$$
$$= \int_0^t \int_{\{z_\tau^-(s) \neq z_\tau(s)\}} \frac{1}{2} \left(\mathbf{C}'(z_\tau) \frac{z_\tau^- - z_\tau}{\tau} + \frac{r(z_\tau^- - z_\tau)}{z_\tau^- - z_\tau} \frac{z_\tau^- - z_\tau}{\tau} \right) \big(\epsilon(u_\tau^-) - e^*(c)\big)$$
$$: \big(\epsilon(u_\tau^-) - e^*(c)\big) \, \mathrm{d}x \, \mathrm{d}s \tag{61}$$

Because of

$$\left\| \frac{r(z_\tau^- - z_\tau)}{z_\tau^- - z_\tau} \right\|_{L^\infty(\{z_\tau^- \neq z_\tau\})} \leq \left\| \frac{\mathbf{C}(z_\tau^-) - \mathbf{C}(z_\tau)}{z_\tau^- - z_\tau} \right\|_{L^\infty(\{z_\tau^- \neq z_\tau\})}$$
$$+ \left\| \mathbf{C}'(z_\tau) \frac{z_\tau^- - z_\tau}{z_\tau^- - z_\tau} \right\|_{L^\infty(\{z_\tau^- \neq z_\tau\})} < C,$$

and $\frac{r(z_\tau^- - z_\tau)}{|z_\tau^- - z_\tau|} \to 0$ a.e. in Ω_T as $\tau \searrow 0$ we conclude by Lebesgue's generalized convergence theorem

$$\left\| \frac{r(z_\tau^- - z_\tau)}{z_\tau^- - z_\tau} \right\|_{L^q(\{z_\tau^- \neq z_\tau\})} \to 0 \text{ for every } q \geq 1.$$

Using this and the already known convergence properties, we end up with

$$\text{left hand side of } (61) \to \int_{\Omega_t} W_{,z}(c, \epsilon(u), z) \partial_t z \, \mathrm{d}x \, \mathrm{d}s$$

and, consequently, $E_\tau^1(t) \to 0$ as $\tau \searrow 0$. Together with the uniform boundedness (54), this implies

$$\int_{t_1}^{t_2} E_\tau^1(t) \, \mathrm{d}t \to 0 \text{ as } \tau \searrow 0 \text{ for all } 0 \leq t_1 \leq t_2 \leq T. \tag{62}$$

The convergence

$$\int_{t_1}^{t_2} E_\tau^4(t) \, \mathrm{d}s \to 0 \text{ as } \tau \searrow 0 \text{ for all } 0 \leq t_1 \leq t_2 \leq T, \tag{63}$$

can be shown as above.

■ Noticing the linearity of e^*, a short calculation yields

$$\int_0^{t_\tau} \int_\Omega \frac{W(c_\tau^-, \epsilon(u_\tau^-), z_\tau^-) - W(c_\tau, \epsilon(u_\tau^-), z_\tau^-)}{\tau} \, \mathrm{d}x$$
$$= \int_0^{t_\tau} \int_\Omega \mathbf{C}(z_\tau^-)\left(\epsilon(u_\tau^-) - e^*\left(\frac{c_\tau^- + c_\tau}{2}\right)\right) : e^*(\partial_t \widehat{c}_\tau) \, \mathrm{d}x \, \mathrm{d}s.$$

Due to the already known convergence properties, we obtain

$$\int_0^{t_\tau} \int_\Omega \frac{W(c^-, \epsilon(u^-), z^-) - W(c, \epsilon(u^-), z^-)}{\tau} \, \mathrm{d}x \, \mathrm{d}s \to -\int_{\Omega_t} W_{,c}(c, \epsilon(u), z) \partial_t c \, \mathrm{d}x \, \mathrm{d}s$$

and, consequently, $E_\tau^2(t) \to 0$ as $\tau \searrow 0$. Together with the uniform boundedness (54), this implies

$$\int_{t_1}^{t_2} E_\tau^2(t) \, \mathrm{d}t \to 0 \text{ as } \tau \searrow 0 \text{ for all } 0 \leq t_1 \leq t_2 \leq T. \tag{64}$$

■ The claim

$$\liminf_{\tau \searrow 0} \int_{t_1}^{t_2} E_\tau^3(t) \, \mathrm{d}t \geq 0 \text{ for all } 0 \leq t_1 \leq t_2 \leq T \tag{65}$$

can be shown by the following arguments: On the one hand, convexity of Ψ_1 yields

$$\frac{\Psi_1(c_\tau^-) - \Psi_1(c_\tau)}{\tau} + \Psi_1'(c_\tau) \partial_t \widehat{c}_\tau \geq 0.$$

On the other hand, by using the differentiability property of Ψ_2, we obtain (cf. (60))

$$\frac{\Psi_2(c_\tau^-) - \Psi_2(c_\tau)}{\tau} + \Psi_2'(c_\tau) \partial_t \widehat{c}_\tau = \frac{r(c_\tau^- - c_\tau)}{\tau} \text{ with } \frac{r(\eta)}{\eta} \to 0 \text{ as } \eta \to 0.$$

In the non-trivial case $c_\tau^- - c_\tau \neq 0$, we can argue as follows. Since $\frac{r(c_\tau^- - c_\tau)}{\tau} = \frac{r(c_\tau^- - c_\tau)}{c_\tau^- - c_\tau} \frac{c_\tau^- - c_\tau}{\tau}$ and since $\frac{c_\tau^- - c_\tau}{\tau}$ is bounded in $L^2(\Omega_T)$, it remains to show

$$\frac{r(c_\tau^- - c_\tau)}{c_\tau^- - c_\tau} \to 0 \text{ in } L^2(\Omega_T) \text{ as } \tau \searrow 0. \tag{66}$$

Indeed, it converges pointwise to 0 a.e. in Ω_T and applying the mean value theorem yields (here $\xi \in [\min\{c_\tau^-, c_\tau\}, \max\{c_\tau^-, c_\tau\}]$)

$$\left| \frac{r(c_\tau^- - c_\tau)}{c_\tau^- - c_\tau} \right| = \left| \frac{\Psi_2(c_\tau^-) - \Psi_2(c_\tau)}{c_\tau^- - c_\tau} - \Psi_2'(c_\tau) \right|$$
$$\leq |\Psi_2'(\xi)| + |\Psi_2'(c_\tau)|$$
$$\leq C(1 + |\xi| + |c_\tau|)$$
$$\leq C(1 + |c_\tau^-| + 2|c_\tau|).$$

Therefore, the left hand side is bounded in $L^\infty(0, T; L^{2^*}(\Omega))$. Lebesgue's generalized convergence theorem yields (66). We end up with $\liminf_{\tau \searrow 0} E_\tau^3(t) \geq 0$ as $\tau \searrow 0$. Fatou's lemma shows the claim.

If we combine (55), (59), (62), (64), (63) and (65), we finally obtain

$$0 \geq \liminf_{\tau \searrow 0} \int_{t_1}^{t_2} \left(A_\tau(t) + B_\tau(t) + E_\tau^1(t) + E_\tau^2(t) + E_\tau^3(t) + E_\tau^4(t) \right) \mathrm{d}t$$
$$\geq \int_{t_1}^{t_2} \left(A(t) + B(t) \right) \mathrm{d}t.$$

for all $0 \leq t_1 \leq t_2 \leq T$. Thus, $A(t) + B(t) \leq 0$ for a.e. $t \in (0, T)$ which is the desired energy inequality (14).

Hence, we obtain existence of weak solutions in the sense of Definition 3.4. $\qquad\square$

4.2 Existence proof for the limit system

We now study the limit $\delta \searrow 0$. For each $\delta > 0$, we obtain a weak solution $(c_\delta, u_\delta, z_\delta, \mu_\delta, \xi_\delta)$ in the sense of Definition 3.4.

Lemma 4.6 (A priori estimates) *There exists a constant $C > 0$ independent of δ such that*

(i) $\|c_\delta\|_{L^\infty(0,T;H^1(\Omega))} < C$, $\quad \sqrt{\delta}\|\partial_t c_\delta\|_{L^2(0,T;L^2(\Omega))} < C$,
$\|\partial_t c_\delta\|_{L^2(0,T;(H^1(\Omega))^*)} < C$,

(ii) $\|u_\delta\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n))} < C$, $\quad \sqrt{\delta}\|u_\delta\|_{L^\infty(0,T;H^2(\Omega;\mathbb{R}^n))} < C$,
$\|u_\delta\|_{H^2(0,T;(H_{\Gamma_D}^2(\Omega;\mathbb{R}^n))^*)} < C$,

(iii) $\|z_\delta\|_{L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))} < C$,

(iv) $\|\mu_\delta\|_{L^2(0,T;H^1(\Omega))} < C$, $\quad \|m(c_\delta, z_\delta)^{1/2} \nabla \mu_\delta\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} < C$.

Proof. From the energy inequality (22), we infer the second inequality of (i), the first two inequalities of (ii), (iii) and the second inequality of (iv). By considering (19), we get

$$\langle \partial_{tt} u_\delta(t), \zeta \rangle_{H^2} \leq C(\|\epsilon(u_\delta(t))\|_{L^2} + \|c_\delta(t)\|_{L^2})\|\epsilon(\zeta)\|_{L^2} + \delta \|\nabla(\nabla u_\delta(t))\|_{L^2} \|\nabla(\nabla\zeta)\|_{L^2}$$
$$+ \|l\|_{L^2} \|\zeta\|_{L^2}$$

and, therefore,

$$\|u_\delta\|_{H^2(0,T;(H_{\Gamma_D}^2(\Omega;\mathbb{R}^n))^*)} < C.$$

Due to $\int_\Omega c_\delta(t)\,\mathrm{d}x = const.$ and the boundedness of $\|\nabla c_\delta(t)\|_{L^2(\Omega)}$, we derive by Poincaré's inequality the first inequality of (i).

From (17) and (18) we obtain boundedness of $\int_\Omega \mu_\delta(t)\,dx$. Since $\|\nabla\mu_\delta(t)\|_{L^2(\Omega_T)}$ is also bounded, Poincaré's inequality yields the first inequality of (iv).

Finally, we know from the boundedness of $\{\nabla\mu_\delta\}$ in $L^2(\Omega_T;\mathbb{R}^n)$ that $\{\partial_t c_\delta\}$ is also bounded in $L^2(0,T;(H^1(\Omega))^*)$ with respect to δ by applying equation (17). Hence, the third inequality of (i) is satisfied. $\qquad\square$

Lemma 4.7 (Convergence properties) *There exist functions*

$$c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*),$$
$$u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;(H^2_{\Gamma_D}(\Omega;\mathbb{R}^n))^*),$$
$$z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$
$$\mu \in L^2(0,T;H^1(\Omega))$$

and subsequences (omitting the subscript) such that for all $r \geq 1$ and $s < 2^*$:

$$\begin{array}{llr}
c_\delta \to c & \text{weakly-star in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*), & (67) \\
& \text{strongly in } L^2(\Omega_T), \text{ a.e. in } \Omega_T, & (68) \\
u_\delta \to u & \text{weakly-star in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)), & (69) \\
& \text{weakly-star in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)), & (70) \\
z_\delta \to z & \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), & (71) \\
& \text{strongly in } L^r(0,T;L^r(\Omega;\mathbb{R}^n)), \text{ a.e. in } \Omega_T, & (72) \\
& \text{strongly in } L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^n)), & (73) \\
& \text{strongly in } C(\overline{\Omega_T}), & (74) \\
\mu_\delta \to \mu & \text{weakly in } L^2(0,T;H^1(\Omega)), & (75) \\
m(c_\delta,z_\delta)^{1/2}\nabla\mu_\delta \to m(c,z)^{1/2}\nabla\mu & \text{weakly in } L^2(0,T;L^2(\Omega,\mathbb{R}^n)) & (76)
\end{array}$$

as $\delta \searrow 0$.

Proof. Lemma 4.6 reveals the existence of functions

$$c \in L^\infty(0,T;H^1(\Omega))$$
$$u \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)) \cap H^2(0,T;(H^2_{\Gamma_D}(\Omega;\mathbb{R}^n))^*),$$
$$z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$
$$\mu \in L^2(0,T;H^1(\Omega)),$$
$$m(c,z)^{1/2}\nabla\mu \in L^2(0,T;L^2(\Omega,\mathbb{R}^n))$$

and subsequences indexed by δ_k such that

$$\begin{array}{llr}
c_{\delta_k} \to c & \text{weakly-star in } L^\infty(0,T;H^1(\Omega)), & (77) \\
u_{\delta_k} \to u & \text{weakly-star in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^n)), & (78) \\
z_{\delta_k} \to z & \text{weakly-star in } L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)), & (79) \\
\mu_{\delta_k} \to \mu & \text{weakly in } L^2(0,T;H^1(\Omega)), & (80) \\
m(c_{\delta_k},z_{\delta_k})^{1/2}\nabla\mu_{\delta_k} \to w & \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^n)). & (81)
\end{array}$$

Due to the strong convergence properties of $\{c_{\delta_k}\}$, $\{z_{\delta_k}\}$ and the growth assumptions on the mobility function m, we infer

$$w = m(c,z)^{1/2}\nabla\mu.$$

In the following, we omit the subscript k. Furthermore, property (i) of Lemma 4.6 shows that $\{c_\delta\}$ converges strongly to an element c in $L^2(\Omega_T)$ as $\delta \searrow 0$ for a subsequence by a compactness result due to Aubin and Lions ([Sim86]). By choosing a further subsequence we also obtain pointwise almost everywhere convergence.

By applying the same technique as for Lemma 4.4, strong convergence of ∇z_δ in $L^p(\Omega_T; \mathbb{R}^n)$ can be obtained. Note that we need the assumption $\mathbf{C}'(\cdot) \geq 0$, see (8). We conclude that

$$z_\delta \to z \qquad \text{strongly in } L^p(0, T; W^{1,p}(\Omega)).$$

Furthermore, by Lemma 4.6 (iii), we find

$$z_\delta \to z \qquad \text{strongly in } \mathcal{C}(\overline{\Omega_T})$$

for a subsequence by an Aubin-Lions type compactness result (cf. [Sim86]). $\qquad\square$

Next, we will proof our main result.

Proof of Theorem 3.3

■ (Cahn-Hilliard equation)
Writing (17) in the form

$$\int_{\Omega_T} (c_\delta - c^0) \partial_t \zeta \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} m(c_\delta, z_\delta) \nabla \mu_\delta \cdot \nabla \zeta \, \mathrm{d}x \, \mathrm{d}t,$$

by only allowing test-functions $\zeta \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with $\zeta(T) = 0$ we may pass to the limit by means of the convergence properties (68), (73) and (75) and receive (11).

Equation (12) can be obtained by integrating (18) over time and taking advantage of the convergence properties (75), (67), (68), (69), (73) and Lemma 4.6 (i).

■ (Balance equation of forces)
Integrating (19) from 0 to T and using (68), (69), (74) and the convergence $\int_0^T \delta \langle Au_\delta, \zeta \rangle_{H^2} \, \mathrm{d}t \to 0$ due to Lemma 4.6 (ii) we conclude

$$\int_0^T \langle \partial_{tt} u, \zeta \rangle_{H^2} \, \mathrm{d}t + \int_{\Omega_T} W_{,e}(c, \epsilon(u), z) : \epsilon(\zeta) \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} l \cdot \zeta \, \mathrm{d}x \, \mathrm{d}t \tag{82}$$

for all $\zeta \in L^\infty(0, T; H^2_{\Gamma_D}(\Omega; \mathbb{R}^n))$. Therefore, (13) is true for all $\zeta \in H^2_{\Gamma_D}(\Omega; \mathbb{R}^n)$ and a.e. $t \in (0, T)$. Using the density of the set $H^2_{\Gamma_D}(\Omega; \mathbb{R}^n)$ in $H^1_{\Gamma_D}(\Omega; \mathbb{R}^n)$ (here we need the assumption that the boundary parts Γ_D and Γ_N have finitely many path-connected components, see [Ber11]), we can identify $\partial_{tt} u(t) \in (H^1_{\Gamma_D}(\Omega; \mathbb{R}^n))^*$ and (13) is true for all $\zeta \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^n)$ and a.e. $t \in (0, T)$. Furthermore, $\partial_{tt} u \in L^\infty(0, T; (H^1_{\Gamma_D}(\Omega; \mathbb{R}^n))^*)$.

■ (Variational inequality for z)
The variational inequality can be shown as in [HK13a]. We choose the following cluster points with respect to a subsequence:

$$\chi_\delta := \chi_{\{z_\delta > 0\}} \to \chi \qquad \text{weakly-star in } L^\infty(\Omega_T), \tag{83}$$

$$\eta_\delta := \chi_{\{z_\delta = 0\} \cap \{W_{,z}(c_\delta, \epsilon(u_\delta), z_\delta) + f'(z_\delta) \leq 0\}} \to \eta \qquad \text{weakly-star in } L^\infty(\Omega_T), \tag{84}$$

$$F_\delta := \chi_{\{z_\delta > 0\}} \sqrt{\frac{\mathbf{C}'(z_\delta)}{2}} (\epsilon(u_\delta) - e^*(c_\delta)) \to F \qquad \text{weakly in } L^2(\Omega_T; \mathbb{R}^{n \times n}), \tag{85}$$

$$G_\delta := \chi_{\{z_\delta = 0\} \cap \{W_{,z}(c_\delta, \epsilon(u_\delta), z_\delta) + f'(z_\delta) \leq 0\}} \times \tag{86}$$

$$\times \sqrt{\frac{\mathbf{C}'(z_\delta)}{2}} (\epsilon(u_\delta) - e^*(c_\delta)) \to G \qquad \text{weakly in } L^2(\Omega_T; \mathbb{R}^{n \times n}). \tag{87}$$

Note that since $\mathbf{C}'(\cdot)$ is symmetric and positive definite matrix, its square root exists. By (69) and (74), we obtain for a.e. $x \in \{z > 0\}$

$$\chi(x) = 1, \quad \eta(x) = 0, \quad F(x) = \sqrt{\frac{\mathbf{C}'(z(x))}{2}} (\epsilon(u)(x) - e^*(c(x))), \quad G(x) = 0 \tag{88}$$

because of the following arguments:

Let $\zeta \in L^2(\Omega_T; \mathbb{R}^{n \times n})$ with supp$(\zeta) \subseteq \{z > 0\}$. Then, by (74), we obtain supp$(\zeta) \subseteq \{z_\delta > 0\}$ for all sufficiently small $\delta > 0$. By (85), we find

$$\int_{\Omega_T} F_\delta : \zeta \, \mathrm{d}x \, \mathrm{d}t \to \int_{\Omega_T} F : \zeta \, \mathrm{d}x \, \mathrm{d}t.$$

On the other hand, by (69), (note that δ can be chosen arbitrarily small)

$$\int_{\Omega_T} F_\delta : \zeta \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} \sqrt{\frac{\mathbf{C}'(z_\delta)}{2}} (\epsilon(u_\delta) - e^*(c_\delta)) : \zeta \, \mathrm{d}x \, \mathrm{d}t$$

$$\to \int_{\Omega_T} \sqrt{\frac{\mathbf{C}'(z)}{2}} (\epsilon(u) - e^*(c)) : \zeta \, \mathrm{d}x \, \mathrm{d}t$$

Thus,

$$\int_{\Omega_T} \sqrt{\frac{\mathbf{C}'(z)}{2}} (\epsilon(u) - e^*(c)) : \zeta \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} F : \zeta \, \mathrm{d}x \, \mathrm{d}t$$

The other identities in (88) follow analogously.

Now let $\zeta \in L^\infty(0, T; W_-^{1,p}(\Omega))$. Taking (52) into account, inequality (12) becomes by integration over time

$$0 \le \int_{\Omega_T} \left(|\nabla z_\delta|^{p-2} \nabla z_\delta \cdot \nabla \zeta + \partial_t z_\delta \zeta \right) \mathrm{d}x \, \mathrm{d}t + \int_{\{z_\delta > 0\}} (W_{,z}(c_\delta, \epsilon(u_\delta), z_\delta) + f'(z_\delta)) \, \zeta \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\{z_\delta = 0\} \cap \{W_{,z}(c_\delta, \epsilon(u_\delta), z_\delta) + f'(z_\delta) \le 0\}} (W_{,z}(c_\delta, \epsilon(u_\delta), z_\delta) + f'(z_\delta)) \, \zeta \, \mathrm{d}x \, \mathrm{d}t.$$

Applying $\limsup_{\delta \searrow 0}$ on both sides and multiplying by -1 yield

$$0 \ge \lim_{\delta \searrow 0} \int_{\Omega_T} \left(|\nabla z_\delta|^{p-2} \nabla z_\delta \cdot \nabla(-\zeta) + \partial_t z_\delta (-\zeta) \right) \mathrm{d}x \, \mathrm{d}t$$

$$+ \liminf_{\delta \searrow 0} \int_{\Omega_T} (F_\delta)^2 (-\zeta) \, \mathrm{d}x \, \mathrm{d}t + \liminf_{\delta \searrow 0} \int_{\Omega_T} \chi_\delta \, f'(z_\delta)(-\zeta) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \liminf_{\delta \searrow 0} \int_{\Omega_T} (G_\delta)^2 (-\zeta) \, \mathrm{d}x \, \mathrm{d}t + \liminf_{\delta \searrow 0} \int_{\Omega_T} \eta_\delta \, f'(z_\delta)(-\zeta) \, \mathrm{d}x \, \mathrm{d}t.$$

Weakly lower semicontinuous arguments, the uniformly convergence property (74) and the properties listed in (88) give

$$0 \ge \int_{\Omega_T} \left(|\nabla z|^{p-2} \nabla z \cdot \nabla(-\zeta) + \partial_t z(-\zeta) \right) \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\{z > 0\}} (W_{,z}(c, \epsilon(u), z) + f'(z)) \, (-\zeta) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\{z = 0\}} \left((F^2 + G^2) + (\chi + \eta) f'(z) \right) (-\zeta) \, \mathrm{d}x \, \mathrm{d}t.$$

This inequality may also be written in the following form:

$$0 \le \int_{\Omega_T} \left(|\nabla z|^{p-2} \nabla z \cdot \nabla \zeta + (W_{,z}(c, \epsilon(u), z) + f'(z) + \partial_t z) \zeta \right) \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\{z = 0\}} \left((F^2 + G^2) + (\chi + \eta) f'(z) - W_{,z}(c, \epsilon(u), z) - f'(z) \right) \zeta \, \mathrm{d}x \, \mathrm{d}t.$$

Therefore,

$$0 \le \int_{\Omega_T} \left(|\nabla z|^{p-2} \nabla z \cdot \nabla \zeta + (W_{,z}(c, \epsilon(u), z) + f'(z) + \partial_t z + \xi) \zeta \right) \mathrm{d}x \, \mathrm{d}t$$

with
$$\xi := \chi_{\{z=0\}} \min\Big\{0, (F^2 + G^2) + (\chi + \eta - 1)f'(z) - W_{,z}(c, \epsilon(u), z)\Big\}.$$
This proves (14) and (15).

- (Energy inequality)
  To prove the energy inequality (16), we can proceed as in the proof of Theorem 3.5. Integrating (22) with respect to time on $[t_1, t_2]$ yields ($0 \le t_1 \le t_2 \le T$)

$$\int_{t_1}^{t_2} (A_\delta(t) + B_\delta(t) + C_\delta(t)) \, \mathrm{d}t \le 0 \tag{89}$$

with

$$
\begin{aligned}
A_\delta(t) := & \int_\Omega \left(\frac{1}{p} |\nabla z_\delta(t)|^p + \frac{1}{2} |\nabla c_\delta(t)|^2 + W(c_\delta, \epsilon(u_\delta(t)), z_\delta(t)) + f(z_\delta(t)) + \Psi(c_\delta(t)) \right) \mathrm{d}x \\
& - \int_\Omega \left(\frac{1}{p} |\nabla z^0|^p + \frac{1}{2} |\nabla c^0|^2 + W(c^0, \epsilon(u^0), z^0) + f(z^0) + \Psi(c^0) \right) \mathrm{d}x \\
& + \int_\Omega \frac{1}{2} |\partial_t u_\delta(t)|^2 \, \mathrm{d}x - \int_\Omega \frac{1}{2} |v^0|^2 \, \mathrm{d}x - \int_\Omega \partial_t u_\delta(t) \cdot \partial_t b(t) \, \mathrm{d}x + \int_\Omega v^0 \cdot \partial_t b^0 \, \mathrm{d}x, \\
B_\delta(t) := & \int_{\Omega_t} \left(|\partial_t z_\delta|^2 + \delta |\partial_t c_\delta|^2 + m(c_\delta, z_\delta) \nabla \mu_\delta \cdot \nabla \mu_\delta \right) \mathrm{d}x \, \mathrm{d}s \\
& - \int_{\Omega_t} W_{,e}(c_\delta, \epsilon(u_\delta), z_\delta) : \epsilon(\partial_t b) \, \mathrm{d}x \, \mathrm{d}s + \int_{\Omega_t} \partial_t u_\delta \cdot \partial_{tt} b \, \mathrm{d}x \, \mathrm{d}s \\
& - \int_{\Omega_t} l \cdot (\partial_t u_\delta - \partial_t b) \, \mathrm{d}x \, \mathrm{d}s, \\
C_\delta(t) := & \frac{\delta}{2} \langle A u_\delta(t), u_\delta(t) \rangle_{H^2} - \frac{\delta}{2} \langle A u^0, u^0 \rangle_{H^2} - \delta \int_0^t \langle A u_\delta(t), \partial_t b(t) \rangle_{H^2} \, \mathrm{d}t.
\end{aligned}
$$

Let A be the corresponding integral expression to A_δ, where c_δ, u_δ and z_δ are replaced by c, u and z, respectively. Furthermore, let

$$
\begin{aligned}
B(t) := & \int_{\Omega_t} \left(|\partial_t z|^2 + m(c, z) \nabla \mu \cdot \nabla \mu \right) \mathrm{d}x \, \mathrm{d}s - \int_{\Omega_t} W_{,e}(c, \epsilon(u), z) : \epsilon(\partial_t b) \, \mathrm{d}x \, \mathrm{d}s \\
& + \int_{\Omega_t} \partial_t u \cdot \partial_{tt} b \, \mathrm{d}x \, \mathrm{d}s - \int_{\Omega_t} l \cdot (\partial_t u - \partial_t b) \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}
$$

The limit passage in (89) can be performed as follows.

- Weakly lower semi-continuity arguments show
$$\liminf_{\delta \searrow 0} \int_{t_1}^{t_2} A_\delta(t) \, \mathrm{d}t \ge \int_{t_1}^{t_2} A(t) \, \mathrm{d}t.$$

- Fatou's lemma and weakly lower semicontinuous arguments for $\nabla \mu_\delta$ as well as the convergence property for c_δ, u_δ, z_δ (see (67), (68), (69), (71), (72)) show (cf. (56)-(58))
$$\liminf_{\delta \searrow 0} \int_{t_1}^{t_2} B_\delta(t) \, \mathrm{d}t \ge \int_{t_1}^{t_2} B(t) \, \mathrm{d}t.$$

- We have
$$C_\delta(t) \ge -\frac{\delta}{2} \langle A u^0, u^0 \rangle_{H^2} - \delta \|u_\delta(t)\|_{H^2(\Omega; \mathbb{R}^n)} \|\partial_t b(t)\|_{H^2(\Omega; \mathbb{R}^n)}.$$
  By Lemma 4.6 (ii), we obtain
$$\liminf_{\delta \searrow 0} \int_{t_1}^{t_2} C_\delta(t) \, \mathrm{d}t \ge 0.$$

We end up with $\int_{t_1}^{t_2} A(t) + B(t) \, \mathrm{d}t \le 0$ for all $0 \le t_1 \le t_2 \le T$. This proves (16).

Putting all steps together, Theorem 3.3 is proven. $\qquad\square$

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