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Center manifold reduction approach for the lubrication equation

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Abstract

The goal of this study is the reduction of the lubrication equation, modelling thin film dynamics, onto an approximate invariant manifold. The reduction is derived for the physical situation of the late phase evolution of a dewetting thin liquid film, where arrays of droplets connected by an ultrathin film of thickness ε undergo a slow-time coarsening dynamics. With this situation in mind, we construct an asymptotic approximation of the corresponding invariant manifold, that is parametrized by a family of droplet pressures and positions, in the limit when $\varepsilon \to 0$.

The approach is inspired by the paper by Mielke and Zelik [Mem. Amer. Math. Soc., Vol. 198, 2009], where the center manifold reduction was carried out for a class of semilinear systems. In this study this approach is considered for quasilinear degenerate parabolic PDE's such as lubrication equations.

While it has previously been shown by Glasner and Witelski [Phys. Rev. E, Vol. 67, 2003], that the system of ODEs governing the coarsening dynamics, can be obtained via formal asymptotic methods, the center manifold reduction approach presented here, pursues the rigorous justification of this asymptotic limit.

1 Introduction

For many processes in nature and technology the development of complicated or even chaotic patterns of spatially localized structures can be observed during their long-time evolution. Examples range from applications in biology, such as morphogenesis, which can be described by reaction-diffusion systems such as the Gierer-Meinhardt model, to applications in the material sciences such as Ostwald ripening of phase separated patterns in binary alloys, which is governed by the Cahn-Hillard equation. To better understand the evolving patterns, one focus of research is concerned with the possibility to develop so-called *reduced models*, that capture this long-time spatial behaviour, Out of the already large body of literature we mention just a few examples such as [1, 2] or [3, 4].

Much less common though, are studies on the rigorous justification of the corresponding reduced models. However, recently in the work by [5] a center manifold reduction theorem for a class of semilinear parabolic equations, describing a variety of dissipative processes that possess so-called *multi-pulse* solutions, was proved.

Here, we focus on a, from mathematical point of view, more complicated PDE's of quasilinear degenerate parabolic type, such as the lubrication equation that governs the dynamics of thin film flow and is initiated by the recent research on dewetting thin liquid films. These studies show that spatially localized (we will call here) *multi-droplet* structures exist and undergo a slow-time dynamics of coarsening, that has also been observed experimentally in [6, 7]. The driving forces of the dewetting process are the intermolecular potentials between the liquid film and the solid substrate, typically consisting of a long-range attractive van der Waals and short-range Born repulsive potential [8]. The combined potential reduces the unstable film to an ultra-thin layer, that connects the evolving patterns and is given by the minimum ε of the intermolecular potential, i.e. the film settles into an energetically more favorable state, see [9].

The last stage of this dewetting process, namely the long-time coarsening process originates in the breaking up of the evolving thin film patterns. The evolution of this process is commonly described by lubrication models, such as

$$\partial_t h = -\partial_x \Big(h^3 \partial_x \left(\partial_{xx} h - \Pi_\varepsilon(h) \right) \Big) , \tag{1.1}$$

which we will consider in this study. It describes the evolution of the height profile $h(x,t)$ for the free surface of the two-dimensional film, see e.g. [10] for a review. The high order is a result of the contribution from surface tension at the free boundary, reflected by the linearized curvature term $\partial_{xx} h$. A further contribution to the pressure is denoted by $\Pi_\varepsilon(h)$ and represents one from the intermolecular forces. A commonly used expression is given by

$$\Pi_\varepsilon(h) = \frac{\varepsilon^2}{h^3} - \frac{\varepsilon^3}{h^4}, \tag{1.2}$$

It can be written as a derivative of the potential function $U_\varepsilon(h)$,

$$U_\varepsilon(h) = -\frac{\varepsilon^2}{2\,h^2} + \frac{\varepsilon^3}{3\,h^3}, \tag{1.3}$$

where parameter $0 < \varepsilon \ll 1$ is the global minimum of the latter function and gives to the leading order thickness of the ultra-thin layer (see Figure 1).

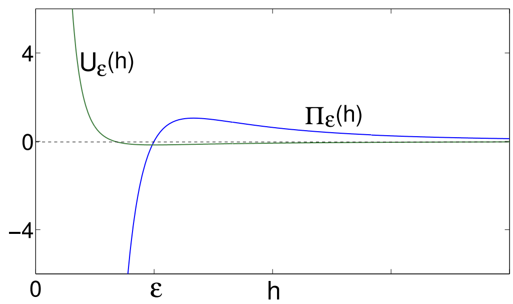


Figure 1: Plots of intermolecular pressure $\Pi_\varepsilon(h)$ and potential function $U_\varepsilon(h)$ for $\varepsilon = 0.1$

In this study we consider (1.1) on a bounded interval $(-L,\, L)$ with boundary conditions

$$\partial_{xxx} h = 0, \quad \text{and} \quad \partial_x h = 0 \quad \text{at} \quad x = \pm L, \tag{1.4}$$

which incorporate zero flux at the boundary and as a consequence imply the conservation of mass law

$$h_c = \frac{1}{2L}\int_{-L}^{L} h(x,t)\,dx \equiv \text{const},\ \forall t > 0. \tag{1.5}$$

It has been shown in [9] that the model (1.1) with boundary conditions (1.4) and initial data $h_0(x)$ has a unique strong positive solution, provided that $h_0(x) \in H^1(-L, L)$, positive for all $x \in (-L,\ L)$ and

$$\int_{-L}^{L} \frac{1}{2}|\partial_x h_0(x)|^2 + U_\varepsilon(h_0(x))\,dx < \infty.$$

Additionally stationary solutions to (1.1) were described in [9]. There, it was shown that (1.1), considered on the whole real line $\mathbb{R}$ for a fixed $\varepsilon > 0$ possess a family of positive nonconstant steady state solutions $\hat{h}_\varepsilon(x-\xi, P)$ parametrized by two constants ξ and $P > 0$. Asymptotically $\hat{h}_\varepsilon(x-\xi, P)$ looks like a *droplet* (see the plot on the right in Figure 3) which has core region where it is well approximated by a parabolic profile and an outer region where to the leading order it is given by ε. In this case ξ is the position of the droplet center and P corresponds to the constant hydrodynamic pressure inside it.

Within the context of thin liquid films one of the first studies of the coarsening dynamics can be found in [11] and [12]. There the authors argued that during the coarsening process each droplet stays very close to a stationary solution $\hat{h}_\varepsilon(x-\xi(t),\ P(t))$ with corresponding position and pressure evolving slowly in time. Therefore, the whole array of coarsening droplets can be considered as a metastable system and well characterized by evolution of droplet pressures and positions (see Figure 2).

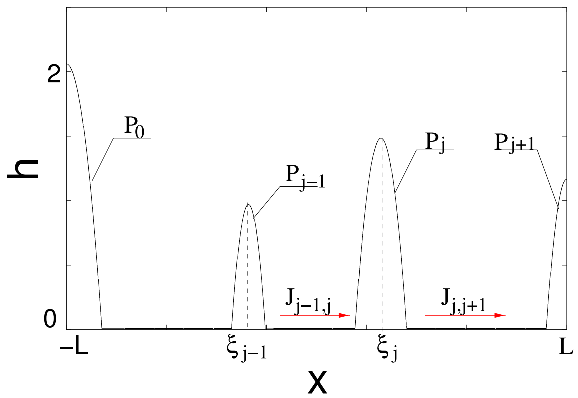


Figure 2: Geometric sketch for an array of several droplets

Using asymptotic methods they then formally derived a reduced ODE model of (1.1) with (1.4) for an array of $N+1$ droplets on a bounded interval $[-L,\ L]$, where the coarsening process is governed by the evolution of pressures $P_j(t)$ and positions $\xi_j(t)$ of the droplets given by

$$\frac{dP_j}{dt} = C_{P,j}(J_{j,j+1} - J_{j-1,j}), \quad \frac{d\xi_j}{dt} = -C_{\xi,j}(J_{j,j+1} + J_{j-1,j}), \quad j = 0, ...N. \tag{1.6}$$

At each time t the right-hand side of (1.6) is a function of the stationary solution $\hat{h}_\varepsilon(x-\xi(t),\, P(t))$ with a so called pressure $C_{P,j}$ and mobility coefficients $C_{\xi,j}$ defined for $j=1,...,N-1$ as

$$C_{P,j} = -\left(\int_{\xi_j-\tilde{L}}^{\xi_j+\tilde{L}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial P}\, dx\right)^{-1},\ C_{\xi,j} = \frac{\displaystyle\int_{\xi_j-\tilde{L}}^{\xi_j+\tilde{L}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx}{\displaystyle 2\int_{\xi_j-\tilde{L}}^{\xi_j+\tilde{L}} \frac{\left(\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)\right)^2}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx}, \tag{1.7a}$$

and for $j=0,N$ as

$$C_{P,0} = -\left(2\int_{-L}^{-L+\tilde{L}} \frac{\partial \hat{h}_\varepsilon(x+L,\, P_0)}{\partial P}\, dx\right)^{-1},\ C_{P,N} = -\left(2\int_{L-\tilde{L}}^{L} \frac{\partial \hat{h}_\varepsilon(x-L,\, P_N)}{\partial P}\, dx\right)^{-1} \tag{1.7b}$$

where

$$\hat{h}_\varepsilon^-(P) = \min_{x\in\mathbb{R}} \hat{h}_\varepsilon(x-\xi,\, P) \tag{1.8}$$

and $\tilde{L}$ was defined as a characteristic length of the support of one droplet. The right-hand side of (1.6) depends also on the fluxes $J_{j+1,\,j}$ between neighboring $j+1$ and j droplets. In [11] the leading order asymptotic approximation for it was found to be

$$J_{j,j+1} = -\frac{V(\hat{h}_\varepsilon^-(P_{j+1})) - V(\hat{h}_\varepsilon^-(P_j))}{[\xi_{j+1} - A/P_{j+1}] - [\xi_j + A/P_j]},\, j=0,...,N-1 \tag{1.9}$$

where $A = 1/\sqrt{3}$ denotes the constant contact angle and

$$V(h) = -3\varepsilon^2 \log(h) - \frac{4\varepsilon^3}{h}.$$

Due to the boundary conditions (1.4) the positions of the first and the last droplet in the array are fixed for all t at the points $x=-L$ and $x=L$, respectively. Therefore, in order to complete the description of (1.6) one defines

$$J_{-1,\,0} = -J_{0,\,1},\quad J_{N,\,N+1} = -J_{N-1,\,N}.$$

In [11] the reduced ODE system of $2N-1$ equations (1.6) with (1.7a)–(1.9) was solved numerically and showed good agreement of the evolution of the droplet pressures P_j and positions ξ_j with those obtained by the direct numerical solution of the lubrication model (1.1) with boundary conditions (1.4). Recently, in [13] the derivation of reduced ODE models was extended the two-dimensional lubrication equations, further extensions for other lubrication type models were developed in [14]. In parallel, in [15] an alternative derivation of reduced ODE models for both one-dimensional and two dimensional cases based on the gradient flow type structure of equation (1.1) was developed. Nevertheless, the full rigorous justification of reduced ODE models retains still an open challenging problem.

Inspired by [5] we pursue a new approach for the derivation of the reduced ODE model corresponding to above no-slip lubrication equation (1.1) considered with boundary conditions (1.4) as an alternative to the one first derived by [11] and by that, make an additional step towards the rigorous justification of the limiting reduced model. It was shown both numerically and asymptotically [13, 16] that the solutions to (1.1)–(1.4) in long time regime are in some sense very close for all times to combinations of finite or even infinite number of stationary solutions (so called pulses) parameterized by a discrete set of parameters. In the case of the lubrication equation one can interpret the stationary solution $\hat{h}_\varepsilon(x,\, P)$ on $\mathbb{R}$ as such a pulse.

Our approach proceeds with the following steps. In section 2 we first summarize some results of [9, 11] about positive nonconstant stationary solutions to (1.1) and prove some new results concerning their asymptotics as $\varepsilon \to 0$. In section 3 we construct an 'approximate invariant' manifold $P_{\mathbf{m}}$ parameterized by a set of positions and pressures in a droplet array. We show that when $\varepsilon > 0$ is sufficiently small every point $\mathbf{m}$ of it is 'almost stationary' with respect to the evolution governed by the lubrication equation and define a special projection operator $P_{\mathbf{m}}$ on $\mathbb{P}_\varepsilon$. In section 4 we prove that in a neighborhood of the 'approximate invariant' manifold every solution $h(\cdot, t)$ of (1.1) can be decomposed into the sum of some point $\mathbf{m}\,(t)$ on the manifold and a remainder function $v(t)$, which is 'orthogonal' to the manifold, i.e $P_{\mathbf{m}}\, v(t) = 0$ for $t > 0$. Next, we decompose (1.1) into a system of two equations: an ODE which describes an evolution on the 'approximate invariant' manifold for $\mathbf{m}\,(t)$ and a quasilinear equation for the remainder $v(t)$. Up to this moment we proceed rigorously. Further, in section 5 we make a formal assumption on the smallness of remainder function $v(t)$ and obtain by this a leading order equation for $\mathbf{m}\,(t)$ on $\mathbb{P}_\varepsilon$, which can be written in the form of the reduced ODE model. Finally, in the conclusion section we compare it with the one derived by [11] and find a good agreement between them.

The invariant manifold based approach applied to the lubrication equation (1.1) is quite different to both approaches of [11, 13] and [15] and provides a nice geometric interpretation for the reduced dynamics. However, a rigorous justification of a center-manifold reduction in the case of the lubrication equation is a more complicated problem than those described by [5], because (1.1) is a quasilinear equation, which additionally degenerates as $h \to 0$. Therefore, in the conclusion section we discuss the main nontrivial open questions arising in our approach that need to be solved.

2 Asymptotics of the stationary solutions

Let us rewrite equation (1.1) in the form

$$\partial_t h + \mathbb{F}_\varepsilon\,(h) = 0 \tag{2.1}$$

and define the corresponding quasilinear elliptic operator as

$$\mathbb{F}_\varepsilon\,(h) = \partial_x \Big(h^3 \partial_x\,(\partial_{xx} h - \Pi_\varepsilon(h))\,\Big). \tag{2.2}$$

As before, we consider (1.1) on the interval $(-L,\, L)$ with boundary conditions (1.4). The following theorem summarizes results by Bertozzi et al. [9] and Glasner and Witelski [12] on the properties of a stationary solution $\hat{h}_\varepsilon(x, P)$ of (2.1) on $\mathbb{R}$.

Theorem 2.1. *Equation (1.1) considered on the whole real line $\mathbb{R}$ has a family of positive nonconstant steady state solutions $\hat{h}_\varepsilon(x, P)$ parameterized by a constant (a so called pressure) $P \in (0,\, P_{max}(\varepsilon))$, where*

$$P_{max}(\varepsilon) = \frac{27}{256\varepsilon}, \tag{2.3}$$

which satisfy

$$\partial_{xx}\hat{h}_\varepsilon(x,\, P) = \Pi_\varepsilon(\hat{h}_\varepsilon(x,\, P)) - P, \tag{2.4a}$$

$$\hat{h}_\varepsilon(x,\, P) = \hat{h}_\varepsilon(-x,\, P), \tag{2.4b}$$

$$\partial_x\hat{h}_\varepsilon(0,\, P) = 0 \quad \text{and} \quad \partial_x\hat{h}_\varepsilon(x,\, P) < 0 \quad \text{for} \quad x > 0. \tag{2.4c}$$

For any numbers $P^* > P_* > 0$ the following asymptotics holds for all $P \in (P_*,\, P^*)$ as $\varepsilon \to 0$:

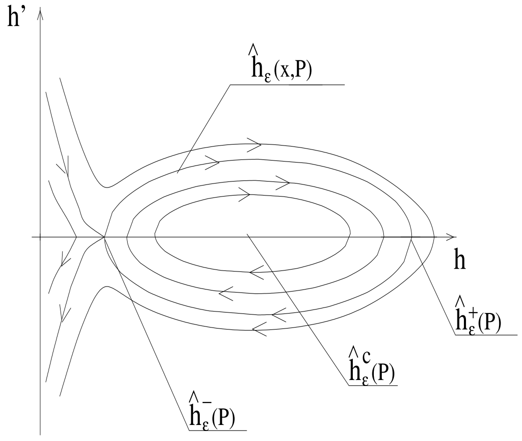
$$\hat{h}_\varepsilon^-(P) = \min_{x\in\mathbb{R}} \hat{h}_\varepsilon(x,\, P) = \epsilon + \epsilon^2 P + O(\epsilon^3). \tag{2.5a}$$

$$\hat{h}_\varepsilon^+(P) = \max_{x\in\mathbb{R}} \hat{h}_\varepsilon(x,\, P) = \frac{1}{6P} + O(\epsilon). \tag{2.5b}$$

Proof: For each $\varepsilon > 0$ it is simple to deduce that any solution to equation

$$h'' = \Pi_\varepsilon\,(h) - P, \tag{2.6}$$

with P being a number, gives a stationary solution to (1.1) on $\mathbb{R}$. The rest of the proof can be done via a phase plane analysis for equation (2.6) as described in [9].(see also Figure 3).



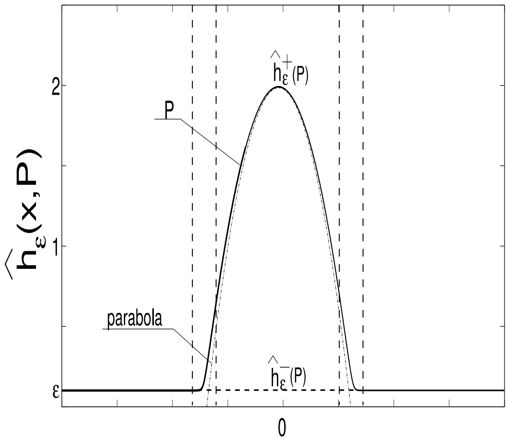


Figure 3: Phase plane portrait for the equation (2.6) (left) and plot of stationary solution $\hat{h}_\varepsilon(x,\, P)$ (right).

It shows that for any fixed $P \in (0,\, P_{max}(\varepsilon))$ there exists a homoclinic loop $\hat{h}_\varepsilon(x,\, P)$ for equation (2.6). The value (2.3) for $P_{max}(\varepsilon)$ is given by the global maximum of $\Pi_\varepsilon(h)$, which is attained at $h_{max} = 4/3\varepsilon$. Moreover, there exists a phase shift such that $\hat{h}_\varepsilon(x,\, P)$ satisfy also (2.4b)–(2.4c). The asymptotics (2.5a)–(2.5b) were derived in [12]. The smallest real root of the algebraic equation $\Pi_\varepsilon(h) = P$ is a saddle-point to equation (2.6) and gives us $\hat{h}^-_\varepsilon(P)$. Expanding the identity $\Pi_\varepsilon(\hat{h}^-_\varepsilon(P)) = P$ in ε one obtains (2.5a). An elliptic center point $\hat{h}^c_\varepsilon(P)$ of equation (2.6) is the other real root of $\Pi_\varepsilon(h) = P$ and has asymptotics

$$\hat{h}^c_\varepsilon(P) = \epsilon(\epsilon P + o(\epsilon))^{-1/3}. \tag{2.7}$$

Once $\hat{h}^-_\varepsilon(P)$ is determined, the first integral to equation (2.4a) can be written as

$$\frac{1}{2}\left(\partial_x \hat{h}_\varepsilon(x,\, P)\right)^2 + \mathcal{U}_\varepsilon(\hat{h}_\varepsilon(x,\, P),\, P) = 0, \tag{2.8}$$

where

$$\mathcal{U}_\varepsilon(h,\, P) = -U_\varepsilon(h) + U_\varepsilon(\hat{h}^-_\varepsilon(P)) + P(h - \hat{h}^-_\varepsilon(P))\,. \tag{2.9}$$

By (2.4b)–(2.4c) $\hat{h}_\varepsilon(x,\, P)$ attains its maximum at $x = 0$, and therefore $\hat{h}^+_\varepsilon(P)$ is determined by the condition $\mathcal{U}_\varepsilon\left(\hat{h}^+_\varepsilon(P),\, P\right) = 0$. Again, after expansion of the last identity in ε one obtains (2.5b). ■

Note that the shifted function $\hat{h}_\varepsilon(x-\xi, P)$ is also a solution to (2.1) on $\mathbb{R}$ for every $\xi \in \mathbb{R}$. Now, using Theorem 2.1 we can prove the following estimates for the stationary solutions:

Proposition 2.2. *For any numbers* $P^* > P_* > 0$ *there exist positive constants* $d, C_k,\ k = 0, 1$ *and* ε_0 *such that for all* $|x| > d, P \in (P_*, P^*)$ *and* $\epsilon \in (0, \epsilon_0)$ *one has*

$$\left|\hat{h}_\varepsilon(x,\, P) - \hat{h}^-_\varepsilon(P)\right| \le C_0\, \exp\left(\frac{d-x}{\sqrt{2}\varepsilon}\right), \tag{2.10a}$$

$$\left|\frac{\partial^k \hat{h}_\varepsilon(x,\, P)}{\partial x^k}\right| \le \frac{C_0}{\varepsilon^k}\, \exp\left(\frac{d-x}{\sqrt{2}\varepsilon}\right) \quad \textit{for } k = 1, 2, 3, 4, \tag{2.10b}$$

$$\frac{\partial \hat{h}_\varepsilon(x,\, P)}{\partial P} \le C_1\, \varepsilon\, (x - d). \tag{2.10c}$$

Proof: Let us define a function

$$F(v) = -\mathcal{U}_\varepsilon(v + \hat{h}^-_\varepsilon(P),\, P),$$

where $\mathcal{U}_\varepsilon(h,\, P)$ is defined by (2.9). From the proof of Theorem 2.1 it follows that $\Pi(\hat{h}^-_\varepsilon(P)) - P = 0$. Using this and (2.9) one obtains

$$\begin{aligned} &F(0) = -\mathcal{U}_\varepsilon(\hat{h}^-_\varepsilon(P),\, P) = 0,\\ &F'(0) = 0,\\ &F''(v) = \Pi'_\varepsilon(v + \hat{h}^-_\varepsilon(P)). \end{aligned} \tag{2.11}$$

Therefore, applying Newton-Leibniz formula to $F(v)$ and integrating once by parts one gets

$$F(v) = \int_0^1 (1-t)\Pi'_\varepsilon(t(v + \hat{h}^-_\varepsilon(P))\,dt\,v^2$$

Substituting in the last expression $v_\varepsilon(x,\,P) = \hat{h}_\varepsilon(x,\,P) - \hat{h}^-_\varepsilon(P)$ and using (2.8), (2.4c) one obtains that

$$\frac{\partial_x v_\varepsilon(x,\,P)}{v_\varepsilon(x,\,P)} = -\sqrt{2\left(\int_0^1 (1-t)\Pi'_\varepsilon\left(t\hat{h}_\varepsilon(x,\,P)\right)\,dt\right)} \text{ for } x > 0. \tag{2.12}$$

By (1.2) and (2.5a) the function $\Pi'_\varepsilon(h)$ decays monotonically to zero on $[\hat{h}^-_\varepsilon(P),\,4/3\,\varepsilon]$ and

$$\Pi'_\varepsilon(\hat{h}^-_\varepsilon(P)) \sim 1/\varepsilon^2. \tag{2.13}$$

Using this and (2.4c) let us define a unique $\nu_\varepsilon(P) > 0$ such that

$$\Pi'_\varepsilon\left(\hat{h}_\varepsilon(\nu_\varepsilon(P),\,P)\right) = \frac{1}{2\,\varepsilon^2}. \tag{2.14}$$

Next, we fix some positive numbers $P^* > P_* > 0$ and show using a contradiction argument, that there exists a number $d > 0$ such that $d > \nu_\varepsilon(P)$ for all sufficiently small $\varepsilon > 0$ and $P \in (P_*,\,P^*)$. Suppose on the contrary, that there exist sequences $\{P_n\},\,\{\varepsilon_n\}$ with $P_n \in (P_*,\,P^*)$ for all $n \in \mathbb{N}$ and $\varepsilon_n \to 0$ such that $\nu_{\varepsilon_n}(P_n) \to +\infty$ as $n \to +\infty$. Then using the asymptotics (2.5a), (2.7) and (2.4c) one concludes that there exists a positive number $\tilde{\varepsilon}$ such that

$$\hat{h}_\varepsilon(x,\,P) - \hat{h}^-_\varepsilon(P) \to 0 \text{ as } x \to \infty \text{ uniformly in } \varepsilon \in (0,\,\tilde{\varepsilon}) \text{ and } P \in (P_*,\,P^*),$$

and hence using (2.13) one concludes

$$\frac{\Pi'_\varepsilon\left(\hat{h}_{\varepsilon_n}(\nu_{\varepsilon_n}(P_n),\,P_n)\right)}{1/\varepsilon_n^2} \to 1 \text{ as } n \to \infty.$$

But the last expression gives a contradiction to definition (2.14). Therefore, the constant $d > 0$ with above properties exists.

Let us now fix $x > d$. Using monotonicity of $\Pi'_\varepsilon(h)$, (2.4c), (2.5a) and the definition of d one obtains

$$\frac{1}{2\varepsilon^2} < \Pi'_\varepsilon(\hat{h}_\varepsilon(d,\,P)) \le \Pi'_\varepsilon(\hat{h}_\varepsilon(x,\,P)) < \Pi'_\varepsilon(\hat{h}^-_\varepsilon) \le \frac{1}{\varepsilon^2} \tag{2.15}$$

for sufficiently small $\varepsilon > 0$ and $P \in (P_*,\,P^*)$. Integrating (2.12) on $(\nu_\varepsilon(P),\,x)$ and using (2.15) one estimates

$$\frac{v_\varepsilon(x,\,P)}{v_\varepsilon(\nu_\varepsilon(P),\,P)} = \exp\left(-\int_{\nu_\varepsilon(P)}^x \sqrt{2\int_0^1 (1-t)\Pi'_\varepsilon(t\,\hat{h}_\varepsilon(x,\,P))\,dt\,dx}\right)$$

$$\le \exp\left[\frac{d-x}{\sqrt{2}\varepsilon}\right]. \tag{2.16}$$

From (2.14) and definition of $v_\varepsilon(x,\,P)$ it follows that

$$v_\varepsilon(\nu_\varepsilon(P),\,P) \le \hat{h}_\varepsilon(\nu_\varepsilon(P),P) \le C_0,$$

where constant C_0 does not depend on ε and P, and therefore

$$\left|\hat{h}_\varepsilon(x,\,P) - \hat{h}^-_\varepsilon(P)\right| \le C_0 \exp\left[\frac{d-x}{\sqrt{2}\varepsilon}\right]. \tag{2.17}$$

Next, by (2.12) and (2.15) one obtains

$$\left|\partial_x \hat{h}_\varepsilon(x,\,P)\right| \le \frac{1}{\varepsilon}\left|\hat{h}_\varepsilon(x,\,P) - \hat{h}^-_\varepsilon(P)\right| \le \frac{C_0}{\varepsilon}\exp\left[\frac{d-x}{\sqrt{2}\varepsilon}\right]. \tag{2.18}$$

For the second derivative using (2.4c) and Peano formula one obtains

$$\left|\partial_{xx}\hat{h}_\varepsilon(x,\,P)\right| = \left|\Pi_\varepsilon(\hat{h}_\varepsilon(x,\,P)) - P\right| \le \left|\Pi'_\varepsilon\left(\theta_\varepsilon(P)\right)\left(\hat{h}_\varepsilon(x) - \hat{h}^-_\varepsilon(P)\right)\right|,$$

where $\theta_\varepsilon(P)$ is a point in interval $\left(\hat{h}^-_\varepsilon,\,\hat{h}_\varepsilon(x,\,P)\right)$. Therefore, using again (2.15) one arrives at

$$|\partial_{xx}\hat{h}_\varepsilon(x,\,P)| \le \frac{C_0}{\varepsilon^2}\exp\left[\frac{d-x}{\sqrt{2}\varepsilon}\right].$$

Analogously, one can derive estimates for $|\partial^k_x\hat{h}_\varepsilon(x,\,P)|$ with $k = 3,\,4$. This together with (2.17)–(2.18) implies (2.10a)–(2.10b) in the case $x > d$.

Next, integrating the first integral (2.8) on a interval $(\eta,\,x)$ with $0 < \eta < x$ one obtains

$$x - \eta = \int_{\hat{h}_\varepsilon(x,\,P)}^{\hat{h}_\varepsilon(\eta,\,P)} \frac{dh}{\sqrt{-2\mathcal{U}_\varepsilon(h,\,P)}}.$$

Differentiation of the last expression with respect to P, using (2.8) and subsequently taking $\eta = x^c_\varepsilon(P)$, where a point $x^c_\varepsilon(P)$ is defined by

$$\hat{h}_\varepsilon(x^c_\varepsilon(P),\,P) := \hat{h}^c_\varepsilon(P),$$

yields

$$\partial_P\hat{h}_\varepsilon(x,\,P) = \frac{\partial_P\hat{h}_\varepsilon(x^c_\varepsilon(P),\,P)}{\partial_x\hat{h}_\varepsilon(x^c_\varepsilon(P),\,P)}\partial_x\hat{h}_\varepsilon(x,\,P) + \partial_x\hat{h}_\varepsilon(x,\,P)\int_{\hat{h}_\varepsilon(x,\,P)}^{\hat{h}^c_\varepsilon(P)} \frac{(h - \hat{h}^-_\varepsilon(P))\,dh}{\sqrt{(-2\mathcal{U}_\varepsilon(h,\,P))^3}}. \tag{2.19}$$

Using that $\mathcal{U}_\varepsilon(h,\,P)$ decreases for fixed $\varepsilon,\,P$ on $(\hat{h}^-_\varepsilon(P),\,\hat{h}^c_\varepsilon(P))$ and again (2.15) one estimates

$$\begin{aligned}
\left|\partial_x\hat{h}_\varepsilon(x,\,P)\int_{\hat{h}_\varepsilon(x,\,P)}^{\hat{h}^c_\varepsilon(P)} \frac{(h - \hat{h}^-_\varepsilon(P))\,dh}{\sqrt{(-2\mathcal{U}_\varepsilon(h,\,P))^3}}\right| &= \int_{\hat{h}_\varepsilon(x,\,P)}^{\hat{h}^c_\varepsilon(P)} \frac{(h - \hat{h}^-_\varepsilon(P))}{-2\mathcal{U}_\varepsilon(h,\,P)}\sqrt{\frac{\mathcal{U}_\varepsilon(\hat{h}_\varepsilon(x,\,P),\,P)}{\mathcal{U}_\varepsilon(h,\,P)}}\,dh \\
\le \int_{\hat{h}_\varepsilon(x,\,P)}^{\hat{h}^c_\varepsilon(P)} \frac{dh}{2\Pi'_\varepsilon\left(\theta_\varepsilon(P)\right)\left(h - \hat{h}^-_\varepsilon(P)\right)} &\le \varepsilon^2 \ln\left(\frac{\hat{h}^c_\varepsilon(P) - \hat{h}^-_\varepsilon(P)}{\hat{h}_\varepsilon(x,\,P) - \hat{h}^-_\varepsilon(P)}\right) \\
\le -\varepsilon^2 \ln\left(\hat{h}_\varepsilon(x,\,P) - \hat{h}^-_\varepsilon(P)\right) &\le C_2\,\varepsilon(x-d),
\end{aligned} \tag{2.20}$$

where the constant C_2 does not depend on ε and P. In the last expression we also used the asymptotic representations (2.5a), (2.7) and the estimate (2.17). Next, using $\Pi_\varepsilon(\hat{h}^c_\varepsilon(P)) - P = 0$ one obtains

$$\Big|\frac{\partial_P \hat{h}_\varepsilon(x^c_\varepsilon(P),\, P)}{\partial_x \hat{h}_\varepsilon(x^c_\varepsilon(P),\, P)}\Big| \le C_3$$

where constant C_3 does not depend on ε and P. Therefore, using (2.18) one obtains

$$\Big|\frac{\partial_P \hat{h}_\varepsilon(x^c_\varepsilon(P),\, P)}{\partial_x \hat{h}_\varepsilon(x^c_\varepsilon(P),\, P)} \partial_x \hat{h}_\varepsilon(x,\, P)\Big| \le \frac{C_3\, C_0}{\varepsilon}\, \exp\left[\frac{d-x}{\sqrt{2}\varepsilon}\right].$$

The last three estimate imply (2.10c) in the case $x > d$. The case $x < -d$ for (2.10a)–(2.10c) can be shown analogously using that $\hat{h}_\varepsilon(x,\, P)$ and $\partial_P \hat{h}_\varepsilon(x,\, P)$ are odd functions in x. ∎

3 The approximating manifold: Definitions, estimates and properties

3.1 The multi-droplet structure

Let us define a set $\mathbb{B}_\varepsilon \subset \mathbb{R}^{2N}$ as

$$\mathbb{B}_\varepsilon = \big\{\mathbf{s} = (P_0,\ P_1, ...,\ P_N, \xi_1, \xi_2..., \xi_{N-1}) \in \mathbb{R}^{2N} : P_j \in (P_*,\ P^*),\ j = 0, ..., N; \\ -L < \xi_1 < ... < \xi_{N-1} < L;\ \xi_i - \xi_{i-1} - 4\,d > 2\sqrt{\varepsilon},\ i = 1, ..., N\big\}, \qquad (3.1)$$

where we assume $\xi_0 = -L$ and $\xi_N = L$.

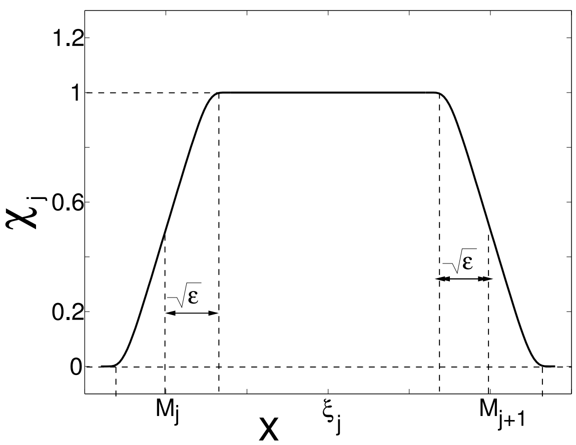


Figure 4: Plot of function $\chi_j(\mathbf{s})(x)$.

Throughout the whole section we fix positive numbers ε_1 and L so that for all $\varepsilon \in (0,\, \varepsilon_1)$ the set $\mathbb{B}_\varepsilon$ is not empty. The boundary of the open set $\mathbb{B}_\varepsilon$ in $\mathbb{R}^{2N}$ topology is given by

$$\begin{aligned}\partial\mathbb{B}_\varepsilon &= \left\{\mathbf{s}\in\mathbb{R}^{2N}:\ \exists j\in\{0,...,N\}: P_j = P_*\right\}\\ &\cup\left\{\mathbf{s}\in\mathbb{R}^{2N}:\ \exists j\in\{0,...,N\}: P_j = P^*\right\}\\ &\cup\left\{\mathbf{s}\in\mathbb{R}^{2N}:\ \exists i\in\{1,...,N\},\ \xi_i-\xi_{i-1}-4\,d = 2\sqrt{\varepsilon}\right\}.\end{aligned} \tag{3.2}$$

Let us define for each $\mathbf{s}\in\mathbb{B}_\varepsilon$ and $j\in\{1,...,N\}$ the points

$$M_j = \frac{\xi_j+\xi_{j-1}}{2} \tag{3.3}$$

and functions $\chi, \chi_j(\mathbf{s})\in C^\infty(\mathbb{R})$ (see Figure 4) as

$$\chi(x) = \begin{cases} 0, & x\le -\sqrt{\varepsilon}\\ \dfrac{1}{2}\left(1+\tanh\left(\tan\left(\dfrac{\pi}{2\sqrt{\varepsilon}}\,x\right)\right)\right), & -\sqrt{\varepsilon}<x<\sqrt{\varepsilon}\\ 1, & x\ge\sqrt{\varepsilon}\end{cases}; \tag{3.4}$$

for $j = 1, ..., N-1$

$$\begin{aligned}\chi_j(\mathbf{s})(x) &= \chi_j(\xi_{j-1},\ \xi_j,\ \xi_{j+1},\ P_{j-1},\ P_j,\ P_{j+1},\ x)\\ &= \begin{cases}\chi(x-M_j), & x<M_j+\sqrt{\varepsilon}\\ 1, & M_j+\sqrt{\varepsilon}\le x\le M_{j+1}-\sqrt{\varepsilon}\\ 1-\chi(x-M_{j+1}), & x>M_{j+1}-\sqrt{\varepsilon}\end{cases};\end{aligned}$$

and for $j = 0, N$

$$\begin{aligned}\chi_0(\mathbf{s})(x) &= \chi_0(\xi_1,\ P_0,\ P_1,\ x) = \begin{cases}1, & 0\le x\le M_1-\sqrt{\varepsilon}\\ 1-\chi(x-M_1), & x>M_1-\sqrt{\varepsilon}\end{cases},\\ \chi_N(\mathbf{s})(x) &= \chi_0(\xi_{N-1},\ P_{N-1},\ P_N,\ x) = \begin{cases}\chi(x-M_N), & x\le M_N+\sqrt{\varepsilon}\\ 1, & x>M_N+\sqrt{\varepsilon}\end{cases}.\end{aligned} \tag{3.5}$$

One can see that for all $x\in[0,\ L]$ and $\mathbf{s}\in\mathbb{B}_\varepsilon$ it holds that $\sum_{j=0}^N \chi_j(\mathbf{s})(x)\equiv 1$.

Define next a mapping $\mathbf{m}_\varepsilon:\ \mathbb{B}_\varepsilon\to L^\infty(-L,\ L)$, which maps a point $\mathbf{s}\in\mathbb{B}_\varepsilon$ to a function $\mathbf{m}_\varepsilon(\mathbf{s})\in C^\infty(-L,\ L)$ satisfying boundary conditions (1.4) as follows:

$$\forall\mathbf{s}\in\mathbb{B}_\varepsilon\ \ \mathbf{m}_\varepsilon(\mathbf{s})(x) = \sum_{j=0}^N \chi_j(\mathbf{s})(x)\hat{h}_\varepsilon(x-\xi_j,\ P_j), \tag{3.6}$$

where again $\xi_0=-L,\ \xi_N = L$. The image of $\mathbf{m}_\varepsilon$ defines a smooth $2N$-dimensional submanifold in L^∞, which we denote as $\mathbb{P}_\varepsilon$. Like in [5] we define a boundary of $\mathbb{P}_\varepsilon$ as $\partial\mathbb{P}_\varepsilon = \mathbf{m}_\varepsilon(\partial\mathbb{B}_\varepsilon)$. From (3.6) it follows that every point $\mathbf{m}(\mathbf{s})\in\mathbb{P}_\varepsilon$ is a composition of $N+1$ stationary solutions to the lubrication equation (2.1). Following to [5] we call such a composition as a *multi-droplet* or a *multi-pulse structure* (see the example in Figure 5). We note, that the mapping $\mathbf{m}_\varepsilon$ is a diffeomorphism between $\mathbb{B}_\varepsilon$ and $\mathbb{P}_\varepsilon$ and therefore, below in this section, we associate with each $\mathbf{m}\in\mathbb{P}_\varepsilon$ a unique $\mathbf{s}\in\mathbb{B}_\varepsilon$ such that $\mathbf{m}_\varepsilon(\mathbf{s}) := \mathbf{m}$.

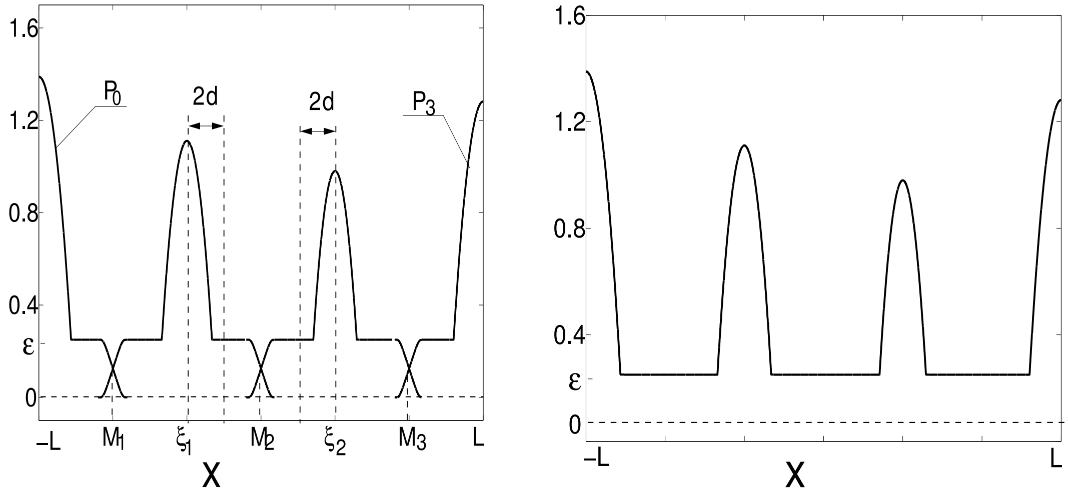


Figure 5: Example of a multi-droplet structure, four truncated pulses $\chi_j(\mathbf{s})(x)\hat{h}_\varepsilon(x - \xi_j,\, P_j),\, j = 0, 1, 2, 3$ (left) and their sum $\mathbf{m}_\varepsilon(\mathbf{s})(x)$ (right).

The tangent space $\mathbb{T}_{\mathbf{m}}\mathbb{P}_\varepsilon$ of manifold $\mathbb{P}_\varepsilon$ at a point $\mathbf{m} \in \mathbb{P}_\varepsilon$ is given by span of functions $\{\phi_0(\mathbf{s}), \phi_1(\mathbf{s}), ..., \phi_{2N-1}(\mathbf{s})\}$, where $\phi_j(\mathbf{s}) \in C_c^\infty(-L,\, L)$ are defined as follows:

$$\phi_j(\mathbf{s}) \;=\; \frac{\partial \mathbf{m}_\varepsilon(\mathbf{s})}{\partial P_j} \quad \text{for} \quad j = 0, ..., N, \tag{3.7a}$$

$$\phi_{N+j}(\mathbf{s}) \;=\; \frac{\partial \mathbf{m}_\varepsilon(\mathbf{s})}{\partial \xi_j} \quad \text{for} \quad j = 1, ..., N-1. \tag{3.7b}$$

Using definitions (3.3)–(3.6) one can see that for $j \in \{1, ..., N-1\}$ functions $\phi_j(\mathbf{s})(x)$ and $\phi_{N+j}(\mathbf{s})(x)$ have a compact support on an interval

$$I_j = (M_j - \sqrt{\varepsilon},\; M_{j+1} + \sqrt{\varepsilon}) \tag{3.8}$$

and can be represented as:

$$\phi_j(\mathbf{s})(x) = \begin{cases} \dfrac{\partial \hat{h}_\varepsilon(x - \xi_j,\, P_j)}{\partial P}\chi(x - M_j)\,, & x < M_j + \sqrt{\varepsilon} \\ \\ \dfrac{\partial \hat{h}_\varepsilon(x - \xi_j,\, P_j)}{\partial P}\,, & x \in [M_j + \sqrt{\varepsilon},\, M_{j+1} - \sqrt{\varepsilon}]\;, \\ \dfrac{\partial \hat{h}_\varepsilon(x - \xi_j,\, P_j)}{\partial P}(1 - \chi(x - M_{j+1}))\,, & x > M_{j+1} - \sqrt{\varepsilon} \end{cases} \tag{3.9}$$

$$
\phi_{N+j}(\mathbf{s})(x) = \begin{cases} \frac{1}{2}\chi'(x-M_j)\Big(\hat{h}_\varepsilon(x-\xi_{j-1},\, P_{j-1}) - \hat{h}_\varepsilon(x-\xi_j,\, P_j)\Big) \\ -\dfrac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial x}\chi(x-M_j), & x < M_j + \sqrt{\varepsilon} \\ \\ -\dfrac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial x}, & x \in \left[M_j + \sqrt{\varepsilon},\, M_{j+1} - \sqrt{\varepsilon}\right] \\ \\ \frac{1}{2}\chi'(x-M_{j+1})\Big(\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon(x-\xi_{j+1},\, P_{j+1})\Big) \\ -\dfrac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial x}(1-\chi(x-M_{j+1})), & x > M_{j+1} - \sqrt{\varepsilon} \end{cases} \tag{3.10}
$$

The remaining two functions have a representation:

$$
\phi_0(\mathbf{s})(x) = \begin{cases} \dfrac{\partial \hat{h}_\varepsilon(x+L, P_0)}{\partial P}, & x \in [0,\; M_1 - \sqrt{\varepsilon}] \\ \dfrac{\partial h_\varepsilon(x+L,\, P_0)}{\partial P}(1-\chi(x-M_1)), & x > M_1 - \sqrt{\varepsilon} \end{cases}
$$

$$
\phi_N(\mathbf{s})(x) = \begin{cases} \dfrac{\partial \hat{h}_\varepsilon(x-L,\, P_N)}{\partial P}\chi(x-M_N), & x < M_N + \sqrt{\varepsilon} \\ \\ \dfrac{\partial \hat{h}_\varepsilon(x-L,\, P_N)}{\partial P}, & x \in [M_N + \sqrt{\varepsilon},\; L] \end{cases}
$$

The next proposition shows that the right-hand side (2.2) of the lubrication equation (2.1) is small on the manifold $\mathbb{P}_\varepsilon$ (due to the fact that it is formed by compositions of stationary solutions). For this reason we refer to $\mathbb{P}_\varepsilon$ as *'approximate stationary'* or *'approximate invariant'*.

Proposition 3.1. *For every $\mathbf{m} \in \mathbb{P}_\varepsilon$ and sufficiently small $\varepsilon > 0$ one has*

$$
\Big|\Big|\mathbb{F}_\varepsilon(\mathbf{m})\Big|\Big|_{L^\infty(-L,\,L)} \leq K\varepsilon^{3/2},
$$

where the constant $K > 0$ does not depend on $\mathbf{m}$ or ε.

Proof: Let us fix below any $\mathbf{m} \in \mathbb{P}_\varepsilon$ corresponding to some $\mathbf{s} \in \mathbb{B}_\varepsilon$. Due to the definitions (3.5)–(3.6) for all $x \in [0,\; M_1 - \sqrt{\varepsilon}] \cup [M_N + \sqrt{\varepsilon},\; L]$ one has $\mathbb{F}_\varepsilon(\mathbf{m})(x) \equiv 0$. Let us estimate $\mathbb{F}_\varepsilon(\mathbf{m})(x)$ on the interval I_j from (3.8) for every $j \in \{1, ..., N-1\}$.

Due to (3.5)–(3.6) one has a representation:

$$\mathbb{F}_\varepsilon(\mathbf{m})(x) = \begin{cases} \mathbb{F}_\varepsilon\Big((1-\chi(x-M_j))\hat{h}_\varepsilon(x-\xi_{j-1},\, P_{j-1}) + \\ +\chi(x-M_j)\hat{h}_\varepsilon(x-\xi_j,\, P_j)\Big), & x\in[M_j-\sqrt{\varepsilon},\; M_j+\sqrt{\varepsilon}] \\ \\ 0, & x\in[M_j+\sqrt{\varepsilon},\; M_{j+1}-\sqrt{\varepsilon}] \\ \\ \mathbb{F}_\varepsilon\Big((1-\chi(x-M_{j+1}))\hat{h}_\varepsilon(x-\xi_j,\, P_j) + \\ +\chi(x-M_{j+1})\hat{h}_\varepsilon(x-\xi_{j+1},\, P_{j+1})\Big), & x\in[M_{j+1}-\sqrt{\varepsilon},\; M_{j+1}+\sqrt{\varepsilon}] \end{cases} \tag{3.11}$$

Let us first estimate $\mathbb{F}_\varepsilon(\mathbf{m})(x)$ for $x\in[M_j-\sqrt{\varepsilon},\; M_j+\sqrt{\varepsilon}]$. Due to the asymptotic representation (2.5a) and the definitions (3.4), (3.6), we conclude that for sufficiently small $\varepsilon>0$ it holds that

$$\begin{aligned} \varepsilon &\le \min\Big\{\hat{h}_\varepsilon(x-\xi_j,\, P_j),\; \hat{h}_\varepsilon(x-\xi_{j-1},\, P_{j-1})\Big\} \le |\mathbf{m}(x)| \\ &\le \max\Big\{\hat{h}_\varepsilon(x-\xi_j,\, P_j),\; \hat{h}_\varepsilon(x-\xi_{j-1},\, P_{j-1})\Big\} \le \varepsilon + 2\,P^*\varepsilon^2, \end{aligned} \tag{3.12}$$

where $\min$ and $\max$ are taken in $x\in[M_j-\sqrt{\varepsilon},\; M_j+\sqrt{\varepsilon}]$. Therefore, for such x and sufficiently small $\varepsilon>0$ one obtains

$$|\Pi_\varepsilon(\mathbf{m}(x))| = \Big|\varepsilon^{-1}\left(\left(\frac{\varepsilon}{\mathbf{m}}\right)^3 - \left(\frac{\varepsilon}{\mathbf{m}}\right)^4\right)\Big| \le \Big|\varepsilon^{-1}\left(1-\left(\frac{1}{1+2\,P^*\varepsilon}\right)^4\right)\Big| \le K_0. \tag{3.13}$$

In the same manner one obtains

$$|\Pi'_\varepsilon(\mathbf{m}(x))| \le K_1/\varepsilon^2 \quad \text{and} \quad |\Pi''_\varepsilon(\mathbf{m}(x))| \le K_2/\varepsilon^3,$$

where constants the K_i, $i=0,1,2$ do not depend on $\mathbf{m}\in\mathbb{P}_\varepsilon$, $\varepsilon>0$ and $x\in[M_j-\sqrt{\varepsilon},\; M_j+\sqrt{\varepsilon}]$. Using definition (3.4) one obtains that

$$\Big|\frac{d^k\chi}{dx^k}\Big| \le \left(\frac{\pi}{2\sqrt{\varepsilon}}\right)^k, \quad \text{for} \quad k\in\mathbb{N}_0 \quad \text{uniformly in} \quad x\in\mathbb{R}. \tag{3.14}$$

By the estimate (2.10b) and definition (3.1)

$$\Big|\frac{\partial^k\hat{h}_\varepsilon(x-\xi_j,P_j)}{\partial x^k}\Big| \le \frac{C_0}{\varepsilon^k}\exp\left(-\frac{d}{\sqrt{2}\varepsilon}\right)$$

for all $\mathbf{s}\in\mathbb{B}_\varepsilon$, $j=0,...,N$, $x\in[M_j-\sqrt{\varepsilon},\; M_j+\sqrt{\varepsilon}]$ and $k=1,...,4$. Therefore, using also (3.12) one obtains

$$\Big|\frac{\partial^k\mathbf{m}(x)}{\partial x^k}\Big| \le K_3\,\varepsilon^{1-k/2}, \quad k=0,\;1,...,4,$$

where constant the $K_3 > 0$ does not depend on $\mathbf{m} \in \mathbb{P}_\varepsilon$, $\varepsilon > 0$ and $x \in [M_j - \sqrt{\varepsilon},\ M_j + \sqrt{\varepsilon}]$. Finally, using the last five estimates one obtains for all $x \in [M_j - \sqrt{\varepsilon},\ M_j + \sqrt{\varepsilon}]$

$$\begin{aligned} |\mathbb{F}_\varepsilon(\mathbf{m})(x)| \ \leq\ & |\mathbf{m}^3 \mathbf{m}_{xxxx}| + |\mathbf{m}^3 \Pi'_\varepsilon(\mathbf{m})\mathbf{m}_{xx}| + |\mathbf{m}^3 \Pi''_\varepsilon(\mathbf{m})\mathbf{m}_x^2| \\ & + |3\mathbf{m}^2 \mathbf{m}_x \mathbf{m}_{xxx}| + |3\mathbf{m}^2 \mathbf{m}_x \Pi'_\varepsilon(\mathbf{m})\mathbf{m}_x| \leq K_5\, \varepsilon^{3/2}. \end{aligned} \quad (3.15)$$

In the very same manner an analogous estimate on $|\mathbb{F}_\varepsilon(\mathbf{m})|$ can be obtained for $x \in [M_{j+1} - \sqrt{\varepsilon},\ M_{j+1} + \sqrt{\varepsilon}]$, and therefore using (3.11) one ends up with

$$\Big|\Big|\mathbb{F}_\varepsilon(\mathbf{m})\Big|\Big|_{L^\infty(I_j)} \leq K\, \varepsilon^{3/2} \quad \text{for every} \quad j \in \{1, ..., N-1\}.$$

■

3.2 Projection on the tangent space $\mathbb{T}_\mathbf{m}\mathbb{P}_\varepsilon$

Next, we define for each $\mathbf{m} \in \mathbb{P}_\varepsilon$ the orthogonal $L^2(-L, L)$-projection on $\mathbb{T}_\mathbf{m}\mathbb{P}_\varepsilon$ acting in $L^\infty(-L,\ L)$ using so called "adjoint function" $\psi_j(\mathbf{s}) \in C_c^\infty(-L,\ L),\ j = 0, ..., 2N-1$. Namely, we define

$$\begin{aligned} &\psi_j(\mathbf{s})(x) = C_j(\mathbf{s})\chi_j(\mathbf{s})(x),\ j = 0, ..., N \\ &\psi_{N+j}(\mathbf{s})(x) = C_{N+j}(\mathbf{s})\chi_j(\mathbf{s})(x) \int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s - \xi_j,\ P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s - \xi_j,\ P_j)^3}\, ds,\ j = 1, ..., N-1, \end{aligned} \quad (3.16)$$

where we denote

$$\begin{aligned} C_j(\mathbf{s}) &= \left(\int_{M_j + \sqrt{\varepsilon}}^{M_{j+1} - \sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x - \xi_j,\ P_j)}{\partial P}\, dx \right)^{-1}, \\ C_{N+j}(\mathbf{s}) &= \left(\int_{M_j + \sqrt{\varepsilon}}^{M_{j+1} - \sqrt{\varepsilon}} \frac{\left(\hat{h}_\varepsilon(x - \xi_j,\ P_j) - \hat{h}_\varepsilon^-(P_j)\right)^2}{\hat{h}_\varepsilon(x - \xi_j,\ P_j)^3}\, dx \right)^{-1}, \text{ for } j = 1, ..., N-1 \end{aligned} \quad (3.17)$$

and

$$C_0(\mathbf{s}) = \left(\int_{-L}^{M_1 - \sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x + L,\ P_0)}{\partial P}\, dx \right)^{-1},\ C_N(\mathbf{s}) = \left(\int_{M_N + \sqrt{\varepsilon}}^{L} \frac{\partial \hat{h}_\varepsilon(x - L,\ P_N)}{\partial P}\, dx \right)^{-1}. \quad (3.18)$$

Again for $j \in \{1, ..., N-1\}$ the functions $\psi_j(\mathbf{s})(x)$ and $\psi_{N+j}(\mathbf{s})(x)$ have compact support on I_j given by (3.8).

Remark 3.2. Recall that the formal adjoint $\mathbb{F}_\varepsilon{}'(\mathbf{m})^*$ to the operator, obtained by differentiation of $\mathbb{F}_\varepsilon(\mathbf{m})$ at a point $\mathbf{m} \in \mathbb{P}_\varepsilon$, and due to the definitions (2.2), (3.6), acts as:

$$\mathbb{F}_\varepsilon{}'(\mathbf{m})^*[\psi(\mathbf{s})](x) = \Big(\Pi'_\varepsilon\Big(\hat{h}_\varepsilon(x-\xi_j,\, P_j)\Big) - \partial_{xx}\Big)\Big[\partial_x\Big(\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3\psi(x)\Big)\Big]$$

for $x \in [M_j + \sqrt{\varepsilon},\; M_{j+1} - \sqrt{\varepsilon}]$. From this it follows that $\mathbb{F}_\varepsilon{}'(\mathbf{m})^*[\psi_j(\mathbf{s})](x) \equiv 0$ for $x \in [M_j + \sqrt{\varepsilon},\; M_{j+1} - \sqrt{\varepsilon}]$. This justifies the name 'adjoint' for the functions $\psi_j(\mathbf{s})$.

Before defining a projection on $\mathbb{T}_{\mathbf{m}}\mathbb{P}_\varepsilon$ we prove two helpful propositions.

Proposition 3.3. *There exists a positive number $K > 0$ such that for all $\mathbf{m} \in \mathbb{P}_\varepsilon$, sufficiently small $\varepsilon > 0$ and $j,\, k \in \{0, ..., 2N-1\}$ one has*

$$\Big|(\psi_j(\mathbf{s}),\; \phi_k(\mathbf{s})) - \delta_{j,k}\Big| \le K\,\varepsilon^{3/2}, \tag{3.19}$$

where $(\cdot,\;\cdot)$ denotes the standard inner product in $L^2(-L,\, L)$.

Proof:
a) Let us first consider $(\psi_j(\mathbf{s}),\; \phi_j(\mathbf{s}))$ for $j \in \{1, ..., N-1\}$. By definitions (3.7b) and (3.16) one has:

$$\begin{aligned}\frac{(\psi_j, \phi_j)}{C_j(\mathbf{s})} \;=\; & \int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial P}\, dx \\ & + \int_{M_j-\sqrt{\varepsilon}}^{M_j+\sqrt{\varepsilon}} \Big(\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial P}\chi(x-M_j)\Big)\chi(x-M_j)\, dx \\ & + \int_{M_{j+1}-\sqrt{\varepsilon}}^{M_{j+1}+\sqrt{\varepsilon}} \Big(\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial P}\chi(x-M_{j+1})\Big)(1-\chi(x-M_{j+1}))\, dx.\end{aligned} \tag{3.20}$$

By (2.10c) and (3.1) there exists a positive number K_0 such that for all $\mathbf{s} \in \mathbb{B}_\varepsilon$,
$x \in [\xi_{j-1} + 2d,\; \xi_j - 2d]$ with $j = 1, ..., N$ and sufficiently small $\varepsilon > 0$ it holds

$$\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial P} \le K_0\,\varepsilon. \tag{3.21}$$

From the last two expressions and estimate (3.14) one obtains:

$$|(\psi_j(\mathbf{s}),\; \phi_j(\mathbf{s})) - 1| \le K_1\,\varepsilon^{3/2} C_j(\mathbf{s}).$$

By definition (3.17) $C_j(\mathbf{s})$ is bounded uniformly in $\mathbf{s} \in \mathbb{B}_\varepsilon$ and $j = 0, ..., N$. Therefore, the estimate (3.19) for this case follows.

b) Let us consider $(\psi_j(\mathbf{s}),\ \phi_{N+j}(\mathbf{s}))$ for $j \in \{1, ..., N-1\}$. By definitions (3.3), (3.4) and (3.7b), (3.16) one has:

$$\frac{(\psi_j,\ \phi_{N+j})}{C_j(\mathbf{s})} = -\int_{\xi_j-2d}^{\xi_j+2d} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial x}$$
$$+\int_{M_j-\sqrt{\varepsilon}}^{\xi_j-2d} \Big(\frac{1}{2}\chi'(x-M_j)\left(\hat{h}_\varepsilon(x-\xi_{j-1},\ P_{j-1}) - \hat{h}_\varepsilon(x-\xi_j,\ P_j)\right)$$
$$-\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial x}\chi(x-M_j)\Big)\chi(x-M_j)\,dx$$
$$+\int_{\xi_j+2d}^{M_{j+1}+\sqrt{\varepsilon}} \Big(\frac{1}{2}\chi'(x-M_{j+1})\left(\hat{h}_\varepsilon(x-\xi_j,\ P_j) - \hat{h}_\varepsilon(x-\xi_{j+1},\ P_{j+1})\right)$$
$$-\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial x}\chi(x-M_{j+1})\Big)(1-\chi(x-M_{j+1}))\,dx. \qquad (3.22)$$

The first integral in the last expression is identically zero because of (2.4b). By (2.10a) and (2.5a) for $x \in [\xi_{j-1}+d,\ \xi_j-d]$ one has

$$|\hat{h}_\varepsilon(x-\xi_{j-1}, P_{j-1}) - \hat{h}_\varepsilon(x-\xi_j, P_j)| \le (P^* - P_*)\varepsilon^2 + O(\varepsilon^3). \qquad (3.23)$$

Using this, (3.14) and (2.10b) one obtains

$$|(\psi_j(\mathbf{s}),\ \phi_{N+j}(\mathbf{s}))| \le K_2\,\varepsilon^{3/2},$$

where constant the $K_2 > 0$ does not depend on $\mathbf{s} \in \mathbb{B}_\varepsilon$, $j = 1, ..., N-1$ and ε.

c) Let us estimate the inner products for 'neighbors' $(\psi_j(\mathbf{s}),\ \phi_{j-1}(\mathbf{s}))$ for $j \in \{1, ..., N-1\}$. By definitions (3.7b) and (3.16) one has

$$\Big|\frac{(\psi_j(\mathbf{s}),\ \phi_{j-1}(s))}{C_j(\mathbf{s})}\Big| = \Big|\int_{M_j-\sqrt{\varepsilon}}^{M_j+\sqrt{\varepsilon}} \Big((1-\chi(x-M_j))\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial P}\Big)\chi(x-M_j)\,dx\Big| \le K_3\,\varepsilon^{3/2},$$

where we used again estimates (3.14), (3.21).

d) Let us estimate $(\psi_{j+N}(\mathbf{s}),\ \phi_{j+N}(\mathbf{s}))$ for $j \in \{1, ..., N-1\}$.

$$\frac{(\psi_{j+N},\ \phi_{j+N})}{C_{j+N}(\mathbf{s})} = -\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial x}\int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\ P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\ P_j)^3}\,ds\,dx$$
$$+\int_{M_{j+1}-\sqrt{\varepsilon}}^{M_{j+1}+\sqrt{\varepsilon}} \Big(\frac{1}{2}\chi'(x-M_{j+1})\left(\hat{h}_\varepsilon(x-\xi_j,\ P_j) - \hat{h}_\varepsilon(x-\xi_{j+1},\ P_{j+1})\right)$$
$$-\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial x}(1-\chi(x-M_{j+1}))\Big)\int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\ P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\ P_j)^3}\,ds\,dx$$
$$+\int_{M_j-\sqrt{\varepsilon}}^{M_j+\sqrt{\varepsilon}} \Big(\frac{1}{2}\chi'(x-M_j)\left(\hat{h}_\varepsilon(x-\xi_{j-1},\ P_{j-1}) - \hat{h}_\varepsilon(x-\xi_j,\ P_j)\right)$$
$$-\frac{\partial \hat{h}_\varepsilon(x-\xi_j,\ P_j)}{\partial x}\chi(x-M_j)\Big)\int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\ P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\ P_j)^3}\,ds\,dx. \qquad (3.24)$$

Let us integrate by parts the first integral at the right-hand side of (3.24):

$$
\begin{aligned}
&-\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial x} \int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\, P_j)-\hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\, P_j)^3}\, ds\, dx = \frac{1}{C_{N+j}(\mathbf{s}\,)} \\
&-\left(\hat{h}_\varepsilon(M_{j+1}-\sqrt{\varepsilon}-\xi_j,\, P_j)-\hat{h}_\varepsilon^-(P_j)\right)\int_{\xi_j}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j)-\hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx \\
&+\left(\hat{h}_\varepsilon(M_j+\sqrt{\varepsilon}-\xi_j,\, P_j)-\hat{h}_\varepsilon^-(P_j)\right)\int_{\xi_j}^{M_j-\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j)-\hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx.
\end{aligned}
\tag{3.25}
$$

Using this and estimates (2.10a), (2.5a)–(2.5b) one obtains

$$
\begin{aligned}
&\Big|\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial x} \int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\, P_j)-\hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\, P_j)^3}\, ds\, dx + 1/C_{N+j}(\mathbf{s}\,)\Big| \\
&\qquad\qquad \le \frac{K_4}{\varepsilon^3}\exp\left(-\frac{d}{\sqrt{2}\varepsilon}\right).
\end{aligned}
\tag{3.26}
$$

As done in paragraph **b)**, the remaining terms in (3.24) can be estimated by $O(\varepsilon^{3/2})$, using (3.14), (3.23) and (2.10b). Therefore, from (3.24) one ends up with

$$
|(\psi_{j+N}(\mathbf{s}\,),\ \phi_{j+N}(\mathbf{s}\,))-1| \le K_5\, \varepsilon^{3/2}.
$$

e) The rest of inner products $(\psi_j(\mathbf{s}\,),\ \phi_j(\mathbf{s}\,))$, which were not considered yet, can be estimated in a similar way as in in **b)** - **c)**. ■

Proposition 3.4. *For every* $\mathrm{s}\, \in \mathbb{B}_\varepsilon$ *there exist functions*

$$
\bar{\psi}_0(\mathbf{s}\,),\ \bar{\psi}_1(\mathbf{s}\,), ..., \bar{\psi}_{2N-1}(\mathbf{s}\,) \in C^\infty(-L,\, L),
$$

such that for all sufficiently small $\varepsilon > 0$ *and every* $j,\ k \in \{0, ...2N-1\}$ *one has*

$$
\left(\bar{\psi}_j(\mathbf{s}\,),\ \phi_k(\mathbf{s}\,)\right) = \delta_{j,k}. \tag{3.27}
$$

Moreover, there exists a positive number K *not depending on* $\mathrm{s}\,, \varepsilon, j$ *such that*

$$
\left|\left|\psi_j(\mathbf{s}\,)-\bar{\psi}_j(\mathbf{s}\,)\right|\right|_{L^\infty(-L,\, L)} \le K\, \varepsilon^{3/2}. \tag{3.28}
$$

Proof: Let us search $\bar{\psi}_j(\mathbf{s}\,)$ in the form

$$
\bar{\psi}_j(\mathbf{s}\,) = \sum_{i=0}^{2N-1} B_i^j(\mathbf{s}\,)\psi_i(\mathbf{s}\,)
$$

From (3.27) it necessarily follows that the vector $\left[B_0^j,\ B_1^j, ...,\ B_{2N-1}^j\right]^T$ is the solution to a linear system of $2N$ algebraic equations given as

$$
\begin{pmatrix}
(\psi_0,\ \phi_0) & (\psi_1,\ \phi_0) & ... & (\psi_{2N-1},\ \phi_0) \\
... & ... & ... & ... \\
(\psi_0,\ \phi_j) & (\psi_1,\ \phi_j) & ... & (\psi_{2N-1},\ \phi_j) \\
... & ... & ... & ... \\
(\psi_0,\ \phi_{2N-1}) & (\psi_1,\ \phi_{2N-1}) & ... & (\psi_{2N-1},\ \phi_{2N-1})
\end{pmatrix}
\begin{bmatrix} B_0^j \\ ... \\ B_j^j \\ ... \\ B_{2N-1}^j \end{bmatrix}
=
\begin{bmatrix} 0 \\ ... \\ 1 \\ ... \\ 0 \end{bmatrix}. \tag{3.29}
$$

By Proposition 2.3 for sufficiently small $\varepsilon > 0$ the matrix $A(\mathbf{s})$ of the system (3.29) has a diagonal dominance and is therefore invertible. Hence, there exists a unique solution to (3.29) and existence of $\bar{\psi}_j(\mathbf{s})$ satisfying (3.27) is proved. Moreover, one has $A(\mathbf{s}) = \mathrm{Id} + D(\mathbf{s})$, where by Proposition 2.3 $||D(\mathbf{s})||_2 \leq \mathrm{const}\, \varepsilon^{3/2}$ uniformly in $\mathbf{s}$. Expanding the inverse to A_ε as a Neumann series

$$A^{-1} = \sum_{k=0}^{\infty}(-1)^k D^k,$$

one obtains from (3.29) that

$$|B_i^j(\mathbf{s}) - \delta_{i,j}| \leq \mathrm{const}\, \varepsilon^{3/2}.$$

Estimate (3.28) follows from this and the uniform bounds in $\mathbf{s} \in \mathbb{B}_\varepsilon$ by definition (3.16) for $||\psi_j(\mathbf{s})||_{L^\infty(-L,\,L)}$, $j = 0, ..., 2N-1$. ■

Finally, we define for every $\mathbf{m} \in \mathbb{P}_\varepsilon$ a linear operator $P_\mathbf{m}$ acting on $v \in L^\infty(-L,\, L)$ by

$$P_\mathbf{m}\, v = \sum_{j=0}^{2N-1} \left(\bar{\psi}_j(\mathbf{s}),\; v\right) \phi_j(\mathbf{s}). \tag{3.30}$$

From this definition it is clear that the image of $P_\mathbf{m}$ belongs to $\mathbb{T}_\mathbf{m}\mathbb{P}_\varepsilon$ and from the orthogonality conditions (3.27) it follows that $P_\mathbf{m}{}^2 = P_\mathbf{m}$. Thus, $P_\mathbf{m}$ are indeed projections on the tangent space $\mathbb{T}_\mathbf{m}\mathbb{P}_\varepsilon$. From definitions (3.7b), (3.16) one can deduce that $||P_\mathbf{m}||_{\mathcal{L}(L^\infty(-L,\,L),\; L^\infty(-L,\,L))}$ is bounded uniformly in $\mathbf{m} \in \mathbb{P}_\varepsilon$, and $P_\mathbf{m}$ is Fréchet differentiable with respect to $\mathbf{m}$.

4 Decomposition in a neighborhood of the manifold

We start this section by showing that in a sufficiently small $L^\infty(-L,\, L)$ neighborhood of the 'approximate invariant' manifold $\mathbb{P}_\varepsilon$ every function $h(x)$ can be decomposed into the sum of some point $\mathbf{m} \in \mathbb{P}_\varepsilon$ and the remainder function v such that $P_\mathbf{m}\, v = 0$. Everywhere below $\mathcal{O}_\delta(\mathbb{P}_\varepsilon)$ and $\mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon)$ denote L^∞ neighborhoods with diameters δ and δ_1 of $\mathbb{P}_\varepsilon$ and its boundary $\partial\mathbb{P}_\varepsilon$, respectively.

Theorem 4.1. *There exist positive constants ε_1 and δ, δ_1 such that for all $\varepsilon \in (0,\, \varepsilon_1)$ there exist a nonlinear differentiable function $\pi_\varepsilon : \mathcal{O}_\delta(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon) \to \mathbb{P}_\varepsilon$, which satisfies*

$$P_{\pi_\varepsilon(h)}(h - \pi_\varepsilon(h)) \equiv 0, \quad \textit{for all} \quad h \in \mathcal{O}_\delta(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon).$$

Proof: Let us first show, that the required projector can be constructed locally for $\mathcal{O}_\delta(\mathbf{m}_0)$ a neighborhood of each point $\mathbf{m}_0 \in \mathbb{P}_\varepsilon$. If $h \in \mathcal{O}_\delta(\mathbf{m}_0)$ then $h = \mathbf{m}_0 + w$ with $||w||_{L^\infty} \leq \delta$ and the required $\mathbf{m} = \pi_\varepsilon(h)$ is found from equation

$$P_\mathbf{m}(\mathbf{m}_0 + w - \mathbf{m}) = 0. \tag{4.1}$$

Recall, that $\mathbb{P}_\varepsilon = \mathbf{m}_\varepsilon(\mathbb{B}_\varepsilon)$, where $\mathbf{m}_\varepsilon$ is a diffeomorphism between the open set $\mathbb{B}_\varepsilon$ given by (3.1) and $\mathbb{P}_\varepsilon$. Therefore, there exist points $\mathbf{s}$, $\mathbf{s}_0 \in \mathbb{B}_\varepsilon$, such that $\mathbf{m}_0 = \mathbf{m}_\varepsilon(\mathbf{s}_0)$ and $\mathbf{m} = \mathbf{m}_\varepsilon(\mathbf{s})$. Moreover, using definition (3.30) one can rewrite (4.1) as:

$$\big(\mathbf{m}_\varepsilon(\mathbf{s}_0) + w - \mathbf{m}_\varepsilon(\mathbf{s}),\ \bar{\psi}_j(\mathbf{s})\big) = 0, \quad \text{for} \quad j = 0, ..., 2N-1. \tag{4.2}$$

Define now a function $F_\varepsilon : \mathbb{R}^{2N} \times L^\infty(-L,\, L) \to \mathbb{R}^{2N}$ as

$$F_\varepsilon(\mathbf{s}, w)_j = \big(\mathbf{m}_\varepsilon(\mathbf{s}_0) + w - \mathbf{m}_\varepsilon(\mathbf{s}),\ \bar{\psi}_j(\mathbf{s})\big)\,, \quad j = 0, ..., 2N-1.$$

Then one has $F_\varepsilon(\mathbf{s}_0,\ 0) = 0$ and

$$(\partial_{\mathbf{s}} F_\varepsilon(\mathbf{s}_0,\ 0)\,\delta\mathbf{s})_j = -(\mathbf{m}'_\varepsilon(\mathbf{s})\,\delta\mathbf{s}, \bar{\psi}_j(\mathbf{s}_0)) = -\sum_{i=0}^{i=2N-1} \big(\phi_i(\mathbf{s})\,\delta s_i, \bar{\psi}_j(\mathbf{s}_0)\big) = -\delta s_j, \tag{4.3}$$

where we denoted $\delta\mathbf{s} = [\delta s_0, \delta s_1, ..., \delta s_{2N-1}]$ and used the orthogonality relations (3.27), which hold for sufficiently small $\varepsilon > 0$. From this it follows that $D_{\mathbf{s}} F_\varepsilon(\mathbf{s}_0, 0) = -\mathrm{Id}$, and therefore by the implicit function theorem, that there exist a constant $\delta > 0$ not depending on ε, such that for all w with $||w||_{L^\infty} \leq \delta$, equation $F_\varepsilon(\mathbf{s}, w) = 0$ has a unique solution $\mathbf{s} = \tilde{\mathbf{s}}_\varepsilon(w)$ or, what is the same, there exists a unique $\mathbf{m} = \tilde{\mathbf{m}}_\varepsilon(w)$ satisfying equation (4.2). Moreover, again due to (4.3) there exists a positive constant δ_1 not depending on ε such that one can choose δ not depending on the choice of $\mathbf{m}_0 \in \mathbb{P}_\varepsilon \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon)$. Therefore, one can construct the required projector π_ε globally on $\mathcal{O}_\delta(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon)$. The differentiability of $\pi_\varepsilon(h)$ also follows from the implicit function theorem. ■

In [9] it was shown that for every positive initial data $h_0 \in H^1(-L,\, L)$ with

$$\int_{-L}^{L} \frac{1}{2}|\partial_x h_0|^2 + U(h_0)\, dx < \infty,$$

for all $t > 0$, there exists a unique positive smooth solution $h(x,t)$ to (2.1) with boundary conditions (1.4) such that $h(x, 0) = h_0(x)$. We restrict ourselves to consider (2.1) in a small neighborhood $\mathcal{O}_\delta(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon)$ of the 'approximate invariant' manifold $\mathbb{P}_\varepsilon$. Taking δ sufficiently small and using definition of $\mathbb{P}_\varepsilon$ one obtains that any $h_0 \in \mathcal{O}_\delta(\mathbb{P}_\varepsilon) \cap H^1(-L,\, L)$ is positive, and therefore any solution to (2.1) with (1.4) such that $h(x, 0) = h_0(x)$ exists for all $t > 0$. According to Theorem 4.1 such a solution can be uniquely decomposed as follows:

$$h(t) = \mathbf{m}(t) + v(t),\ \mathbf{m}(t) \in \mathbb{P}_\varepsilon \tag{4.4a}$$

$$P_{\mathbf{m}(t)}\, v(t) \equiv 0. \tag{4.4b}$$

Inserting this into equation (2.1) one can write it in an equivalent form

$$\partial_t v + \mathbb{F}_\varepsilon{}'(\mathbf{m}(t))v(t) = -\mathbb{F}_\varepsilon(\mathbf{m}(t)) - \mathbb{F}_\varepsilon(v(t), \mathbf{m}(t)) - \mathbf{m}'(t), \tag{4.5}$$

where $\mathbb{F}_\varepsilon(v,\mathbf{m}) = \mathbb{F}_\varepsilon(v+\mathbf{m}) - \mathbb{F}_\varepsilon(\mathbf{m}) - \mathbb{F}_\varepsilon{}'(\mathbf{m})v$ and the differential $\mathbb{F}_\varepsilon{}'(\mathbf{m}(t))$ is taken in an appropriate space, e.g. $W^{4,2}_{bc}(-L,L)$ where

$$W^{4,2}_{bc}(-L,L) = \{u \in W^{4,2}_{bc}(-L,L) |\; u \text{ satisfies (1.4)}\}.$$

Let us now differentiate (4.4b) with respect to time

$$P'_{\mathbf{m}(t)}[\mathbf{m}'(t)]v(t) + P_{\mathbf{m}(t)}\,\partial_t v(t) \equiv 0.$$

Applying the projection $P_{\mathbf{m}(t)}$ to (4.5) and noting the last expression yields a differential equation for $\mathbf{m}(t)$ on the manifold $\mathbb{P}_\varepsilon$ in the following form

$$(\mathrm{Id} - \mathbb{D}(\mathbf{m}(t))[\cdot]v(t))\mathbf{m}'(t) = P_{\mathbf{m}(t)}\left(-\mathbb{F}_\varepsilon(\mathbf{m}(t)) - \mathbb{F}_\varepsilon(v(t),\mathbf{m}(t))\right) - \mathbb{S}(\mathbf{m}(t))v(t), \tag{4.6}$$

where

$$\mathbb{D}(\mathbf{m})[\delta\mathbf{m}]v = P'_{\mathbf{m}}[\delta\mathbf{m}]v, \tag{4.7a}$$

$$\mathbb{S}(\mathbf{m})v = P_{\mathbf{m}}(\mathbb{F}_\varepsilon{}'(\mathbf{m})v). \tag{4.7b}$$

Let us also denote

$$\mathbb{M}(\mathbf{m},v)w = (\mathrm{Id} - \mathbb{D}(\mathbf{m})[\cdot]v)^{-1}w \tag{4.8}$$

Then for each $v \in L^\infty(-L,\,L)$ such that $||v||_{L^\infty}$ is sufficiently small and each $\mathbf{m} \in \mathbb{P}_\varepsilon$ the operator $\mathbb{M}(\mathbf{m},\,v) :\; L^\infty(-L,\,L) \to \mathbb{P}_\varepsilon$ is well defined and can be represented by a Neumann series

$$\mathbb{M}(\mathbf{m},v) = \sum_{i=0}^{\infty} (\mathbb{D}(\mathbf{m})[\cdot]v)^i\,.$$

Thus, equation (4.6) can be written in the following more convenient form

$$\mathbf{m}'(t) = f(\mathbf{m}(t),\; v(t)), \tag{4.9}$$

where

$$f(\mathbf{m},\; v) := \mathbb{M}(\mathbf{m},\; v)\left(-P_{\mathbf{m}}(\mathbb{F}_\varepsilon(\mathbf{m}) + \mathbb{F}_\varepsilon(v,\;\mathbf{m})) - \mathbb{S}(\mathbf{m})v\right). \tag{4.10}$$

We conclude that if $h(t)$ solves (2.1) and $h(t) \in \mathcal{O}_\delta(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon)$ on $[0,\;T]$ then the associated functions $\mathbf{m}(t)$ and $v(t)$ satisfy on $[0,\;T]$ the following system

$$\begin{cases} \partial_t v + \mathbb{F}_\varepsilon{}'(\mathbf{m}(t))v(t) = h(\mathbf{m}(t),\; v(t),\; \mathbf{m}'(t)) \\ \mathbf{m}'(t) = f(\mathbf{m},v) \end{cases}, \tag{4.11}$$

where we denoted $h(\mathbf{m},\; v,\; w) = -\mathbb{F}_\varepsilon(\mathbf{m}) - \mathbb{F}_\varepsilon(v,\mathbf{m}) - w$. Vice versa, any solution $(\mathbf{m}(t),\; v(t))$ to (4.11) on $[0,T]$ with sufficiently small $v(t)$ satisfying $\mathbb{P}_{\mathbf{m}(0)}v(0) = 0$ generates a unique solution $u(t) = \mathbf{m}(t) + v(t)$ to equation (2.1). Therefore, instead of the initial lubrication equation (2.1) we can consider the associated system (4.11).

5 Equations on the manifold

In this section we derive the reduced ODE system via a reduction onto the approximate invariant manifold. We assume that for any solution to (2.1) having at $t = 0$ an initial value in the neighborhood $\mathcal{O}_\delta(\mathbb{P}_\varepsilon) \setminus \mathcal{O}_{\delta_1}(\partial\mathbb{P}_\varepsilon)$ defined in Theorem 4.1, the norm of remainder $v(t)$, obtained via decomposition (4.4a), is negligibly small for all $t > 0$ in comparison with that of $\mathbf{m}(t) \in \mathbb{P}_\varepsilon$. Then putting $v(t) \equiv 0$ for $t > 0$ we obtain to leading order

$$\mathbf{m}'(t) = f(\mathbf{m}(t), 0) \tag{5.1}$$

for equation (4.9), which describes the evolution of $\mathbf{m}(t)$ on the manifold $\mathbb{P}_\varepsilon$. Below we transform (5.1) to an ODE system describing the evolution of pressures $P_j(t)$ and positions $\xi_j(t)$ of the multi-droplet structure $\mathbf{m}(t)$.

Putting $v(t) \equiv 0$ into definitions (4.7b)–(4.8), (4.10) one writes (5.1) as

$$\mathbf{m}'(t) = -P_{\mathbf{m}}\mathbb{F}_\varepsilon(\mathbf{m}).$$

Let us rewrite the last equation in a coordinate form on manifold $\mathbb{P}_\varepsilon$. Denoting as before

$$\mathbf{s} = (s_0, ..., s_{2N-1}) := (P_0, ..., P_N, \xi_1, ..., \xi_{N-1})$$

such that $\mathbf{m} = \mathbf{m}_\varepsilon(\mathbf{s})$ and taking the standard scalar product in $L^2(-L, L)$ of $\mathbf{m}'(t)$ with $\bar{\psi}_j(\mathbf{s})$ for $j = 0, ..., 2N-1$ one gets

$$(\mathbf{m}'(t), \bar{\psi}_j(\mathbf{s})) = \sum_{i=0}^{i=2N-1} \left(\phi_i(\mathbf{s})\frac{ds_i}{dt}, \bar{\psi}_j(\mathbf{s})\right) = \frac{ds_j}{dt}, \tag{5.2}$$

where we used definition (3.7b) and the orthogonality conditions (3.27). On the other hand

$$(P_{\mathbf{m}}\mathbb{F}_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})), \bar{\psi}_j(\mathbf{s})) = (\mathbb{F}_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})), \bar{\psi}_j(\mathbf{s}))$$

By Proposition 2.4 one has for all $\mathbf{s} \in \mathbb{B}_\varepsilon$ and $j = 0, ..., 2N-1$

$$(\mathbb{F}_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})), \bar{\psi}_j(\mathbf{s})) \sim (\mathbb{F}_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})), \psi_j(\mathbf{s})) \quad \text{as} \quad \varepsilon \to 0. \tag{5.3}$$

Next, by definition (3.16) and representation (3.11) one has for $j = 1, ..., N-1$

$$\begin{aligned}
(\mathbb{F}_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})), \psi_j(\mathbf{s})) = C_j(\mathbf{s})\Big(&\int_{M_j-\sqrt{\varepsilon}}^{M_j+\sqrt{\varepsilon}} \chi(x-M_j)\mathbb{F}_\varepsilon(\mathbf{m})(x)\,dx \\
&+ \int_{M_{j+1}-\sqrt{\varepsilon}}^{M_{j+1}+\sqrt{\varepsilon}} (1-\chi(x-M_{j+1}))\mathbb{F}_\varepsilon(\mathbf{m})(x)\,dx\Big) \\
&= C_j(\mathbf{s})\left(\int_{\theta_j}^{M_j+\sqrt{\varepsilon}} \mathbb{F}_\varepsilon(\mathbf{m})(x)\,dx + \int_{M_{j+1}-\sqrt{\varepsilon}}^{\theta_{j+1}} \mathbb{F}_\varepsilon(\mathbf{m})(x)\,dx\right) \\
&= \frac{J(\mathbf{s})(\theta_{j+1}) - J(\mathbf{s})(\theta_j)}{\displaystyle\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j, P_j)}{\partial P}\,dx},
\end{aligned} \tag{5.4}$$

where we used the second mean-value theorem for integration (see paragraph 1.13 in [17]) with θ_j being some point in $(M_j - \sqrt{\varepsilon},\, M_j + \sqrt{\varepsilon})$ and introduced for each s $\in \mathbb{B}_\varepsilon$ a flux function $J(\mathbf{s}) \in C^\infty(-L,\, L)$ as

$$J(\mathbf{s}) = \mathbf{m}_\varepsilon(\mathbf{s})^3 \partial_x \left(-\Pi_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})) + \partial_{xx}\mathbf{m}_\varepsilon(\mathbf{s})\right). \tag{5.5}$$

Analogously, by definition (3.16) for $j = 1, ..., N-1$:

$$\begin{aligned}
&(\mathbb{F}_\varepsilon(\mathbf{m}_\varepsilon(\mathbf{s})),\, \psi_{j+N}(\mathbf{s})) \\
&= C_{N+j}(\mathbf{s})\Big(\int_{M_j-\sqrt{\varepsilon}}^{M_j+\sqrt{\varepsilon}} \chi(x-M_j) \int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\, P_j)^3}\, ds\, \mathbb{F}_\varepsilon(\mathbf{m})(x)\, dx \\
&+ \int_{M_{j+1}-\sqrt{\varepsilon}}^{M_{j+1}+\sqrt{\varepsilon}} (1-\chi(x-M_{j+1})) \int_{\xi_j}^{x} \frac{\hat{h}_\varepsilon(s-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(s-\xi_j,\, P_j)^3}\, ds\, \mathbb{F}_\varepsilon(\mathbf{m})(x)\, dx\Big) \\
&= C_{N+j}(\mathbf{s})\Big(\int_{\xi_j}^{M_j+\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx \int_{\theta_j}^{M_j+\sqrt{\varepsilon}} \mathbb{F}_\varepsilon(\mathbf{m})(x)\, dx \\
&+ \int_{\xi_j}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx \int_{M_{j+1}-\sqrt{\varepsilon}}^{\theta_{j+1}} \mathbb{F}_\varepsilon(\mathbf{m})(x)\Big)\, dx \\
&= \frac{\displaystyle\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx}{\displaystyle 2\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{(\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j))^2}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx} (J(\mathbf{s})(\theta_{j+1}) + J(\mathbf{s})(\theta_j)).
\end{aligned} \tag{5.6}$$

Finally, denoting

$$\begin{aligned}
J_{j-1,\,j} &= J(\mathbf{s})(\theta_j),\ j = 1, ..., N-1 \\
J_{-1,\,0} &= -J_{0,\,1},\ \ J_{N,\,N+1} = -J_{N-1,\,N}
\end{aligned} \tag{5.7}$$

and combining (5.2), (5.3), (5.6) one obtains the following coordinate form for the leading order equation of (5.1) as $\varepsilon \to 0$

$$\begin{aligned}
\frac{dP_j}{dt} &= C_{P,j} \cdot (J_{j,\,j+1} - J_{j-1,\,j}), \\
\frac{d\xi_j}{dt} &= -C_{\xi,j} \cdot (J_{j,\,j+1} + J_{j-1,\,j}),\ j = 0, ..., N
\end{aligned} \tag{5.8}$$

where for $j = 1, ..., N-1$

$$C_{P,j} = -\left(\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\partial \hat{h}_\varepsilon(x-\xi_j,\, P_j)}{\partial P}\, dx \right)^{-1}, \tag{5.9}$$

$$C_{\xi,j} = \frac{\displaystyle\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx}{\displaystyle 2\int_{M_j+\sqrt{\varepsilon}}^{M_{j+1}-\sqrt{\varepsilon}} \frac{\left(\hat{h}_\varepsilon(x-\xi_j,\, P_j) - \hat{h}_\varepsilon^-(P_j)\right)^2}{\hat{h}_\varepsilon(x-\xi_j,\, P_j)^3}\, dx}; \tag{5.10}$$

and

$$
\begin{aligned}
C_{P,0} &= -\left(2\int_{-L}^{M_1-\sqrt{\varepsilon}} \frac{\partial h_0}{\partial P}\,dx\right)^{-1}, \\
C_{P,N} &= -\left(2\int_{M_N+\sqrt{\varepsilon}}^{L} \frac{\partial h_N}{\partial P}\,dx\right)^{-1}.
\end{aligned} \tag{5.11}
$$

We conclude that (5.8)–(5.11) gives us a new reduced ODE model describing the evolution of pressures and positions in multi-droplet structures governed by the no-slip lubrication equation (1.1) considered with boundary conditions (1.4).

Conclusions and discussion

Comparing (5.8)–(5.11) with the ODE model (1.6), obtained via formal asymptotic methods in [11], together with coefficients (1.7a)–(1.7b) one observes that formally they do have the same structure. The differences between them appear in the formulas for the coefficients $C_{P,j},\ C_{\xi,j}$ and the fluxes $J_{j-1,\,j}$. In the definitions (5.11) the interval of integration $[M_j+\sqrt{\varepsilon},\ M_{j+1}-\sqrt{\varepsilon}]$ represents the support of the j-th droplet, while in (1.7a)–(1.7b) this corresponds to $[-\tilde{L},\ \tilde{L}]$. While in the latter case, for given positions $\xi_j,\ j=1,...,N-1$ of the droplets in the array of $N+1$, the singular integrals are estimated asymptotically, in the former case we can calculate the interval $[M_j+\sqrt{\varepsilon},\ M_{j+1}-\sqrt{\varepsilon}]$ explicitly using formula (3.3). In the case of system (1.6) the fluxes $J_{j-1,\,j}$ between neighboring droplets are derived asymptotically and given by (1.9), while for the system (5.8) derived here, we obtained the formulas (5.5), (5.7), and $\theta_j\in(M_j-\sqrt{\varepsilon},\ M_j+\sqrt{\varepsilon})$ in definition (5.7) arise after application of a mean-value theorem.

Both reduced ODE models give us in some sense a leading order approximation for the late phase evolution of solutions to the no-slip lubrication equation. The 'approximate invariant' approach developed here for the derivation of the system (5.8) gives us possibilities for analytical error estimates of the obtained approximations. Namely, using Propositions 3.3, 3.4 and formula (5.3) one can now estimate that passing from equation (5.1) to system (5.8) terms of $O(\varepsilon^3)$ are neglected. In addition we note that having an estimate of the remainder function $v(t)$ from the decomposition (4.4a) one could similarly estimate the order of magnitude of terms that are neglected when passing from the exact equation (4.9) on the 'approximate invariant' manifold $\mathbb{P}_\varepsilon$ to its leading order (5.1).

Finally, we summarize that the analysis of this paper together with the results of [13, 16] all suggest that the reduced ODE models (5.8) (or (1.6)) are valid in the limit $\varepsilon\to 0$. The next step towards a rigorous justification of our approach could be the proof of a corresponding center manifold existence theorem. In turn this theorem concerns the spectrum of the operator (2.2) linearized at a point $\mathbf{m}\in\mathbb{P}_\varepsilon$ as $\varepsilon\to 0$.

In contrast to the center manifold existence theorem for the class of semilinear parabolic equations and the main assumptions on the spectrum used in the approach of [5] (see assumptions (2.20) and (2.25) there), it turns out that here the corresponding linearized eigenvalue problem has a more complex structure. This initiated our analysis of the asymptotics of the spectrum for (2.2) linearized at the stationary solution $h_{0,\varepsilon}$, which describes physically a droplet on a bounded interval. The results of this analysis are contained in a companion paper [18], where we show that the corresponding linearized eigenvalue problem is a singular perturbed one and the spectrum of it tends to zero as $\varepsilon \to 0$.

While this by itself is an interesting problem, we moreover expect that the construction of the center manifold will be a singularly perturbed problem as well. However, as of now there are only a few studies in the literature that rigorously show existence of a singularly perturbed center manifold, see e.g. [19] and [20].

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