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Weak Solutions to a Stationary Heat Equation with Nonlocal Radiation Boundary Condition and Right-Hand Side in L^p $(p \geq 1)$.

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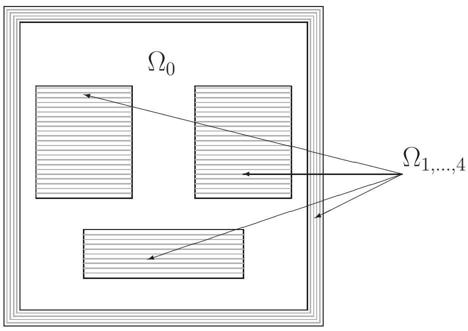
Abstract

Accurate modeling of heat transfer in high-temperatures situations requires to account for the effect of heat radiation. In complex applications such as Czochralski's method for crystal growth, in which the conduction radiation heat transfer problem couples to an induction heating problem and to the melt flow problem, we hardly can expect from the mathematical theory that the heat sources will be in a better space than L^1. In such situations, the results of [LT01] on the unique solvability of the heat conduction problem with surface radiation do not apply, since a right-hand side in L^p with $p < \frac{6}{5}$ no longer belongs to the dual of the Banach space V in which coercivity is obtained. In this paper, we focus on a stationary heat equation with non-local boundary conditions and L^p right-hand side with $1 \leq p \leq \infty$ arbitrary. Essentially, we construct an approximation procedure and, thanks to new coercivity results, we are able to produce energy estimates that involve only the L^p-norm of the heat-sources, and to pass to the limit.

Introduction

Accurate modeling of heat transfer in high-temperatures situations requires to account for the effect of heat radiation. In the field of industrial applications, crystal growth, for example, has motivated a lot of mathematical work on this topic ([Phi03], [KPS04], [KP05], [MPT06], [Mey06], [Voi01]). For this type of applications, situations are relevant in which a transparent medium is *enclosed* by one or several opaque, or diffuse grey bodies, such as in the following 2D-picture:



When heat is supplied to the bodies, each point on the boundary of the transparent cavity, denoted by Ω_0, emits radiation, and at the same time receives radiation emitted at the other parts of the surface that it can see. This effect can be modeled by means of nonlocal radiation boundary conditions for the conductive heat flux (see for example ([Voi01], [KPS04]).

From the point of view of mathematical analysis, an important result was attained in the paper [LT01], in which existence and uniqueness of generalized solutions were proved for the heat equation with radiation boundary conditions and heat sources in the class $[W^{1,2}]^*$.

In the present paper, we want to extend these results to the case that the heat source density might be less regular. In many applications, the heat sources have to be computed from Maxwell's equations (resistive/inductive heating) or from the Navier-Stokes equations (heat conducting fluids). From the viewpoint of the presently available regularity theory, this leads in complex situations (temperature-dependent coefficients, nonsmooth surfaces) to heat source densities that belong only to L^1, or at most to $L^{1+\epsilon}$. The latter observations have motivated research on elliptic problems with L^1 right-hand sides (see for example [BG92], [Rak91]). A L^1-theory is also particularly attractive for the heat equation, in that it leads to natural energy estimates, that is, to estimates in terms of the total heating power, the quantity which is actually controlled in applications.

The mathematical problem. We assume that $\Omega_1, \ldots, \Omega_m$ are disjoint bounded domains in $\mathbb{R}^3$, separated from each other by a transparent medium Ω_0. They represent opaque bodies with different material properties. The bounded domain $\Omega \subset \mathbb{R}^3$ such that

$$\overline{\Omega} = \bigcup_{i=0}^{m} \overline{\Omega_i}\,, \tag{1}$$

is assumed to be connected.

We assume that all materials involved are *grey materials*, see [LT98] or [KPS04]. Therefore, radiation only needs to be considered at the surface of the bodies Ω_i $(i = 1, \ldots, m)$. We define

$$\partial\Omega_{\text{Rad}} := \bigcup_{i=1}^{m} \partial\Omega_i\,,$$

as the surface where heat radiation occurs. Note that the interactions between the parts of the surface that are located in the cavity Ω_0 need to be taken into account.

To this aim, a kernel $w : \partial\Omega_{\text{Rad}} \times \partial\Omega_{\text{Rad}} \longrightarrow \mathbb{R}$, the so-called *view factor*, is introduced by

$$w(z, y) := \begin{cases} \dfrac{\vec{n}(z) \cdot (y - z)\ \vec{n}(y) \cdot (z - y)}{\pi |z - y|^4}\, \Theta(z, y) & \text{if } z \neq y\,, \\ 0 & \text{if } z = y\,, \end{cases} \tag{2}$$

where the *visibility function*

$$\Theta(z, y) = \begin{cases} 1 & \text{if }]z, y[\cap \overline{\Omega \setminus \Omega_0} = \emptyset\,, \\ 0 & \text{else,} \end{cases} \tag{3}$$

is penalizing the nonconvexity of the surface $\partial\Omega_{\text{Rad}}$ as well as the presence of obstacles. In these definitions, the symbol $]z, y[$ is an abbreviation for $\text{conv}\{z, y\} \setminus \{z, y\}$, and $\vec{n}$ is the outward-pointing unit normal to $\partial\Omega_{\text{Rad}}$. Throughout the introduction, we assume that $\partial\Omega_{\text{Rad}}$ has a sufficient regularity for the kernel w to be everywhere well defined.

By straightforward geometrical considerations, one verifies that

$$w(z,y) \geq 0 \;\text{ for all } (z,y) \in \partial\Omega_{\text{Rad}} \times \partial\Omega_{\text{Rad}}\,.$$

We introduce the set of all interacting points $\Sigma_o \subset \partial\Omega_{\text{Rad}}$ by

$$\Sigma_o := \left\{ z \in \partial\Omega_{\text{Rad}} \,\middle|\, \exists\, y \in \partial\Omega_{\text{Rad}} \,:\, w(z,y) > 0 \right\}, \tag{4}$$

i.e. a point $z \in \partial\Omega_{\text{Rad}}$ will belong to Σ_o if it can see at least one other point of $\partial\Omega_{\text{Rad}}$. The splitting $\partial\Omega_{\text{Rad}} = \Gamma \cup \Sigma$, where

$$\Sigma := \overline{\Sigma_o}, \qquad \Gamma := \partial\Omega_{\text{Rad}} \setminus \Sigma\,, \tag{5}$$

gives a disjoint decomposition of the boundary.

Throughout the paper, we will assume that heat-transfer in the transparent medium Ω_0 only occurs by radiation, or at least that the heat conduction taking place in Ω_0 is negligible, i.e. $\kappa = 0$ in Ω_0. We will address the problem of determining the temperature in the opaque components $\Omega_1, \ldots, \Omega_m$ of the domain as (P).

The domain of computation Ω is thus given by $\Omega := \bigcup_{i=1}^m \Omega_i$. As a matter of fact, Ω is *disconnected*. We consider the equations

$$(P) \begin{cases} -\operatorname{div}\,(\kappa(T)\,\nabla T) &= f \qquad \text{in } \Omega\,,\\ -\kappa(T)\dfrac{\partial T}{\partial \vec{n}} &= R - J \quad \text{on } \Sigma\,, \end{cases}$$

where R denotes the radiosity (outgoing radiation), and J denotes the incoming heat radiation J on the surface Σ.

The radiosity R has to be the sum of the radiation emitted according to the Stefan-Boltzmann law, and of the reflected part of the incoming radiation J. Thus,

$$R = \epsilon\,\sigma\,|T|^3\,T + (1-\epsilon)\,J \qquad \text{on } \Sigma. \tag{6}$$

where the emissivity ϵ is a material function that attains values in $[0,1]$, and σ denotes the Stefan-Boltzmann constant. In the chosen model, one has

$$J = K(R) \qquad \text{on } \Sigma\,, \tag{7}$$

where K is the linear integral operator given by

$$(K(f))(z) = \int_\Sigma w(z,y)\,f(y)\,dS_y \qquad \text{for } z \in \Sigma\,, \tag{8}$$

with the kernel (2). We also have to supply a boundary condition on the set Γ. For the sake of generality, we assume the disjoint decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_i is relatively open for $i = 1, 2, 3$. We consider the boundary conditions

$$T = T_0 \text{ on } \Gamma_1\,, \qquad -\,\kappa(T)\frac{\partial T}{\partial \vec{n}} = \alpha\,(T - T_{\rm Ext}) \text{ on } \Gamma_2\,,$$

$$-\,\kappa(T)\frac{\partial T}{\partial \vec{n}} = \epsilon\,\sigma\,(T^4 - T_{\rm Ext}^4) \text{ on } \Gamma_3\,, \tag{9}$$

where the imposed temperature T_0, the external temperature $T_{\rm Ext}$, as well as the coefficient $\alpha > 0$ are given. As above, ϵ is the emissivity on the surface Γ_3, and σ is the Stefan-Boltzmann constant.

Weak formulation of (P). It is possible to immediately eliminate the integral equations (6), (7) on Σ and to derive a weak formulation of (P) that only involves the unknown T. This was shown in [Tii97b]. From the relations (6), (7), it follows that

$$(I - (1-\epsilon)\,K)\,(R) = \epsilon\,\sigma\,|T|^3\,T\,,$$

Note that throughout the paper, occurrence of the functions ϵ and $1-\epsilon$ in connection with an integral operator simply implies multiplication. Assuming that the operator $(I - (1-\epsilon)K)$ is invertible in a suitable Banach space, we then can write

$$R = (I - (1-\epsilon)K)^{-1}\,\epsilon\,(\sigma\,|T|^3\,T)\,,$$

$$J = K(I - (1-\epsilon)K)^{-1}\,\epsilon\,(\sigma\,|T|^3\,T)\,.$$

We introduce the notation

$$G := (I - K)\,(I - (1-\epsilon)K)^{-1}\,\epsilon\,. \tag{10}$$

Then, we can rewrite the conditions on the boundary Σ as follows:

$$-\,\kappa(T)\,\frac{\partial T}{\partial \vec{n}} \;=\; G(\sigma |T|^3\,T) \text{ on } \Sigma\,.$$

If ψ is a smooth function that vanishes on Γ_1, we find the relation

$$\int_\Omega \kappa(T)\,\nabla T\cdot\nabla\psi + \int_{\Gamma_2} \alpha\,T\,\psi + \int_{\Gamma_3} \sigma\,\epsilon\,|T|^3\,T\,\psi + \int_\Sigma G(\sigma\,|T|^3\,T)\,\psi$$

$$= \int_\Omega f\,\psi + \int_{\Gamma_2} \alpha\,T_{\rm Ext}\,\psi + \int_{\Gamma_3} \sigma\,\epsilon\,T_{\rm Ext}^4\,\psi\,. \tag{11}$$

We introduce the notations

$$\langle A\,T\,,\,\psi\rangle := \int_\Omega \kappa(T)\,\nabla T\cdot\nabla\psi + \int_{\Gamma_2} \alpha\,T\,\psi\,, \qquad \tilde{f} := \begin{cases} 0 & \text{on } \Gamma_1\,, \\ \alpha\,T_{\rm Ext} & \text{on } \Gamma_2\,, \\ \sigma\,\epsilon\,T_{\rm Ext}^4 & \text{on } \Gamma_3\,. \end{cases} \tag{12}$$

We rewrite (11) as

$$\langle A\,T\,,\,\psi\rangle + \int_{\Gamma_3} \sigma\,\epsilon\,|T|^3\,T\,\psi + \int_\Sigma G(\sigma\,|T|^3\,T)\,\psi = \int_\Omega f\,\psi + \int_\Gamma \tilde{f}\,\psi\,. \tag{13}$$

Definition 0.1. Let $\Omega \subset \mathbb{R}^3$ have the structure (1), where Ω_i are disjoint domains with $\partial\Omega_i \in \mathcal{C}^{0,1}$. If the decomposition $\partial\Omega_{\text{Rad}} = \Sigma \cup \Gamma$, which is given by (5), leads to $\Sigma\,,\Gamma \in \mathcal{C}^{0,1}$, we set

$$V^{p,q}(\Omega) := \left\{ u \in W^{1,p}(\Omega) \;\middle|\; \gamma(u) \in L^q(\Sigma \cup \Gamma_3) \right\}\,,$$

where γ is the trace operator. The subscript Γ_1 denotes the subspace of functions that vanish on the surface Γ_1.

In order to define a weak solution, we still need assumptions on the coefficients κ, ϵ. Throughout the paper, we assume that κ_i is a continuous function of temperature, and that there exists positive constant κ_l, κ_u such that

$$0 < \kappa_l \leq \kappa_i(s) \leq \kappa_u < \infty \quad \text{for } s \in \mathbb{R}, \quad \text{for each } \Omega_i\,. \tag{14}$$

We assume that ϵ is a measurable function of the position and that there exists a positive number ϵ_l such that

$$0 < \epsilon_l \leq \epsilon(z) \leq 1 \quad \text{for } z \in \Sigma\,. \tag{15}$$

For a real number $p \in]1,\infty[$, we use through the paper the notation $p' = p/p-1$ for the conjugated exponent.

Definition 0.2. We call $T \in V^{p,4}(\Omega)$, $1 < p \leq \infty$ a *weak solution* to (P) if $T = T_0$ almost everywhere on Γ_1, and if T satisfies the integral relation (13) for all $\psi \in V^{p',\infty}(\Omega)$.

Situation and structure of the paper. In the papers [Tii97b], [Tii97a], [LT98] existence and uniqueness of generalized solutions were proved for the problem (P) in enclosure-free systems. In [LT01], the authors then found a more general way to obtain energy estimates, and were able to extend the previous results to general geometries, including enclosures. The two remaining fundamental assumptions of the paper [LT01], necessary to obtain the results, are that the surface Σ belongs to $\mathcal{C}^{1,\alpha}$ piecewise, and that the heat source density f belongs to $[W^{1,2}(\Omega)]^*$.

Our purpose and main focus is to extend these known results to the case that the right-hand side f might be a less regular function. This is motivated by the fact that in concrete applications (resistive / inductive heating, heat conductive fluids), the mathematical theory often only provides L^1, or at most $L^{1+\epsilon}$ regularity for the heat source density.

Our plan is as follows.

In the first section we prove and recall some essential properties of the operators K and G. These properties allow us to derive, in the second section, new coercivity inequalities for the nonlinear form $\langle A\,T\,,\,\psi\rangle + \int_\Sigma G(\sigma\,|T|^3\,T)\,\psi$. The first two sections are extending the available knowledge about the nonlocal radiation operators and they may interest the reader in their own right.

In the next sections, we turn to more specifically study the problem (P). In the third section, the uniqueness issue is briefly treated. The fourth section is devoted to the proof of a general existence result for the problem (P) in the case that the right-hand side f belongs to L^p, with $p > 1$ arbitrary. The case $p = 1$ is treated separately in the fifth section.

Since the regularity of the interface Σ is an important issue for applications in crystal growth, we draw the attention of the reader to another by-product of the discussion of the first two sections: the compactness of the operator K (i.e. the restriction $\partial\Omega \in \mathcal{C}^{1,\alpha}$ piecewise), turns out to be unnecessary to prove the existence of weak solution for the standard case $f \in [W^{1,2}(\Omega)]^*$.

1 The Operators K and G

We use the following notations. If X, Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of the linear continuous mappings from X into Y. We denote by $\mathcal{K}(X, Y)$ the subspace of $\mathcal{L}(X, Y)$ that contains the compact mappings from X into Y. For $B \in \mathcal{L}(X, Y)$ and $x \in X$, we denote by $B(x) \in Y$ the value of B at x. If B, $C \in \mathcal{L}(X, X)$, we denote by $B\,C(x)$ the element $B(C(x))$. As in the introduction, we will use the notation $\partial\Omega_{\text{Rad}} := \bigcup_{i=1}^m \partial\Omega_i$.

We start by studying the operators K and G. Parts of the following results have already been proved in [Tii97b], [Tii97a], [LT98], [LT01]. Throughout this section, we assume that $\partial\Omega_i \in \mathcal{C}^{0,1}$ for $i = 1, \ldots, m$. We denote by S the corresponding surface measure. Under these assumptions, it is well known that the outward-pointing unit normal to $\partial\Omega$, which we denote by $\vec{n}$, is defined almost everywhere in the sense of the measure S on $\partial\Omega$. In order to ensure good measurability properties of the kernel w, we in addition assume that $\vec{n}$ is a $S-$almost everywhere continuous function. This is the case, for instance, if $\partial\Omega$ is a piecewise $\mathcal{C}^1$ surface.

We at first consider the integral operator K. For its study, we need to state a few elementary properties of the kernel w. Under the cited assumptions, the proof of the following Lemma is straighforward.

Lemma 1.1. If $\partial\Omega_{\text{Rad}} \in \mathcal{C}^{k,\alpha}$ with $k \in \mathbb{N}$ and $0 < \alpha \leq 1$, then the set

$$\Sigma_z := \{y \in \partial\Omega_{\text{Rad}} \,:\, w(z, y) \neq 0\}$$

is a $\mathcal{C}^{k,\alpha}$ surface for $S-$almost all $z \in \partial\Omega_{\text{Rad}}$.

Lemma 1.2. The view-factor given by (2) satisfies the conditions

$$\begin{cases} w(z, y) = w(y, z) & \forall (z, y) \in \Sigma \times \Sigma\,, \\ w(z, y) \geq 0 & \forall (z, y) \in \Sigma \times \Sigma\,, \\ \int_\Sigma w(z, y) dS_y \leq 1 & \forall z \in \Sigma\,. \end{cases} \tag{16}$$

Proof. We use the method proposed in [Tii97a]. With the notation of Lemma 1.1, we write

$$\int_\Sigma w(z, y)\, dS_y = \int_{\Sigma_z} \frac{\vec{n}(z) \cdot (y - z)\, \vec{n}(y) \cdot (z - y)}{\pi |y - z|^4}\, dS_y\,.$$

We can also write the integrated function in the following way:

$$\frac{\vec{n}(z)\cdot(y-z)\,\vec{n}(y)\cdot(z-y)}{\pi|y-z|^4} = -\frac{\cos(\phi_y)\ \cos(\phi_z)}{\pi|y-z|^2}\,, \tag{17}$$

where ϕ_y [resp. ϕ_z] is the angle between $\vec{n}(y)$ [resp. $\vec{n}(z)$] and $(z-y)$ [resp. $(y-z)$]. Representation (17) shows that w is invariant under rotations and translations. For this reason, we can assume without loss of generality that $z=0$ and $\vec{n}(z)=(-1,0,0)$. For obvious geometrical reasons, all points $y\in\Sigma$ such that $w(z,y)\neq 0$ belong to the half space $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \,\Big|\, x_1<0\}$. Observe also that the ray through the origin and an arbitrary point on the unit half-sphere $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \,\Big|\, x_1<0\,,\,|x|=1\}$ intersects Σ_z in at most one point, and in a unique point if we suppose that the surface is closed. Thus, passing to polar coordinates, we can parameterize the surface by the mapping

$$\begin{aligned} \Psi :\ \left]\frac{\pi}{2},\pi\right[\times]0,2\pi[\quad &\longrightarrow \quad \Sigma_z\,,\\ z=\Psi(\phi_1,\phi_2) \quad &:= \quad \begin{pmatrix} r(\phi_1,\phi_2)\,\cos(\phi_1)\\ r(\phi_1,\phi_2)\,\sin(\phi_1)\,\cos(\phi_2)\\ r(\phi_1,\phi_2)\,\sin(\phi_1)\,\sin(\phi_2)\end{pmatrix}\,, \end{aligned}$$

where the function $r:]\frac{\pi}{2},\pi[\times]0,2\pi[\longmapsto\mathbb{R}$ is Lipschitz continuous according to Lemma 1.1. Straightforward computations lead to

$$\vec{n}(\Psi) = \frac{1}{\left(r^2\frac{\partial r}{\partial\phi_1}^2\sin^2(\phi_1)+r^2\frac{\partial r}{\partial\phi_2}^2+r^4\sin^2(\phi_1)\right)^{\frac{1}{2}}}$$

$$\times\begin{pmatrix} r^2\,\sin(\phi_1)\,\cos(\phi_1)+r\,\frac{\partial r}{\partial\phi_1}\,\sin^2(\phi_1)\\ r\,\frac{\partial r}{\partial\phi_2}\,\sin(\phi_2)-r\,\frac{\partial r}{\partial\phi_1}\sin(\phi_1)\,\cos(\phi_1)\,\cos(\phi_2)+r^2\,\sin^2(\phi_1)\,\cos(\phi_2)\\ -r\,\frac{\partial r}{\partial\phi_2}\,\cos(\phi_2)-r\,\frac{\partial r}{\partial\phi_1}\,\sin(\phi_1)\,\cos(\phi_1)\,\sin(\phi_2)+r^2\,\sin^2(\phi_1)\,\sin(\phi_2)\end{pmatrix}\,,$$

$$\sqrt{G_\Psi} = \left(r^2\frac{\partial r}{\partial\phi_1}^2\,\sin^2(\phi_1)+r^2\frac{\partial r}{\partial\phi_2}^2\,+r^4\sin^2(\phi_1)\right)^{\frac{1}{2}}\,,$$

$$\vec{n}(\Psi)\cdot\Psi = \frac{r^3\,\sin(\phi_1)}{\left(r^2\frac{\partial r}{\partial\phi_1}^2\sin^2(\phi_1)+r^2\frac{\partial r}{\partial\phi_2}^2+r^4\sin^2(\phi_1)\right)^{\frac{1}{2}}}\,,$$

for λ_2-almost every $(\phi_1,\,\phi_2)\in]\frac{\pi}{2},\pi[\times]0,2\pi[$. By the symbol G_Ψ, we denote the Gram determinant of the matrix Ψ'. We thus have that

$$w(y\,,\,\Psi) \;=\; \frac{(-n(\Psi)\cdot\Psi)\ (n(y)\cdot\Psi)}{\pi\,|\Psi|^4} = \frac{-r^4\,\sin(\phi_1)\,\cos(\phi_1)}{\pi\,r^4\,\left(r^2\frac{\partial r}{\partial\phi_1}^2\sin^2(\phi_1)+r^2\frac{\partial r}{\partial\phi_2}^2+r^4\sin^2(\phi_1)\right)^{\frac{1}{2}}}\,.$$

Taking into consideration that $\phi_1 \in]\frac{\pi}{2}, \pi[$, this proves the nonnegativity of w. We still have to compute the integral. If Σ_z is a closed surface, we have

$$\begin{aligned}\int_{\Sigma} w(z,y)\, dS_y &= \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} w(z,\, \Psi)\, \sqrt{G_\Psi}\, d\phi_1\, d\phi_2 \\ &= \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{-\sin(\phi_1)\, \cos(\phi_1)}{\pi}\, d\phi_1\, d\phi_2 = -\sin^2(\phi_1)\Big|_{\frac{\pi}{2}}^{\pi} = 1\,,\end{aligned}$$

proving the lemma in the case that the range of vision Σ_z of the point z is a closed surface. If the surface Σ_z is not closed, we can argue exactly as above with a smaller domain of parameterization $\Psi :\, O \subset \left]\frac{\pi}{2}, \pi\right[\times]0, 2\pi[\longrightarrow \Sigma$. □

Definition 1.3. (1) We say that two points $z, y \in \partial\Omega_{\text{Rad}}$ *see each other* if and only if $w(z, y) \neq 0$.

(2) We call Ω an *enclosure* if and only if for $S-$almost all $z \in \Sigma$ we have $\int_\Sigma w(z, y)\, dS_y = 1$.

Remark 1.4. If Ω is an enclosure, we can assume without loss of generality that the surface Σ *consists of one part*, i.e. that Σ is the boundary of a unique connected transparent cavity. Technically, we say if $A \subset \Sigma$ is such that for almost all $z \in A$, $\int_A w(z, y)\, dS_y = 1$, then we can assume that either $A = \Sigma$ or $A = \emptyset$.

In view of the integrability of w stated in Lemma 1.2, we see that the definition (8) of the operator K is well-posed at least for $f \in L^\infty(\Sigma)$.

In the next Lemma, we recall the basic properties of the operator K that were proved in [Tii97b].

Lemma 1.5. (1) For every $1 \leq p \leq \infty$, the operator K extends to a linear bounded operator from $L^p(\Sigma)$ into itself.

(2) The norm estimate $\| K \|_{\mathcal{L}(L^p(\Sigma), L^p(\Sigma))} \leq 1$ is valid.

(3) The operator K is positive, in the sense that $K(f) \geq 0$ almost everywhere on Σ if $f \geq 0$ almost everywhere on Σ; K is selfadjoint and positive semi-definite from $L^2(\Sigma)$ into itself.

(4) If the emissivity ϵ is a function such that (15) is satisfied, then for $1 \leq p \leq \infty$, the operator $(I - (1 - \epsilon)K)$ has an inverse in $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$, with the representation

$$(I - (1 - \epsilon)K)^{-1} = \sum_{i=0}^{\infty} (1 - \epsilon)^i K^i\,. \tag{18}$$

Proof. See [Tii97b]. □

Thanks to Lemma 1.5, we see that the operator G introduced in (10) is well-defined as an element of $\mathcal{L}(L^p(\Sigma), L^p(\Sigma))$. Note the following equivalent representations of the operator G:

$$G := (I-K)\,(I-(1-\epsilon)K)^{-1}\epsilon = \epsilon - \epsilon K(I-(1-\epsilon)K)^{-1}\epsilon \tag{19}$$

Lemma 1.6. (1) The operator G can be represented as $I-H$, where the operator H is positive and selfadjoint in $L^2(\Sigma)$.

(2) For $1 \leq p \leq \infty$, the norm estimate $\| H \|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} \leq 1$ is true.

Proof. See [LT98]. □

In the next Lemma, we present some further elementary properties of G, K, and H. They turn out to be essential for the discussion of the coercivity. In the original version of the preprint, and in the paper [Dru08] the point (5) of the next Lemma is incorrect.

Lemma 1.7. (1) The equivalence $H(\psi) = \psi \iff K(\psi) = \psi$ is valid.

(2) If $\psi \in L^p(\Sigma)$ $(1 < p \leq \infty)$ satisfies $K(\psi) = \psi$, then ψ is a constant.

(3) If Ω is not an enclosure, then for $1 \leq p \leq \infty$, the strict estimate $\| H \|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} < 1$ is true.

(4) Let Ω be an enclosure. For some $r+s \geq 1$ $(r,s > 0)$, let $\psi \in L^{r+s}(\Sigma)$ satisfy $\int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi = 0$. Then ψ is a constant.

(5) Let Ω be an enclosure. Define $\text{sign}(0) := 0$. If $\psi \in L^1(\Sigma)$ satisfies $\int_\Sigma G(\psi)\,\text{sign}(\psi) = 0$, then $\text{sign}(\psi)$ is almost everywhere a constant on Σ.

Proof. (1): Assume first that $H(\psi) = \psi$. By definition, this means that $(1-\epsilon)\,\psi + \epsilon\, K\,(I-(1-\epsilon)K)^{-1}\,\epsilon\,(\psi) = \psi$, which implies that $K\,(I-(1-\epsilon)K)^{-1}\,\epsilon\,(\psi) = \psi$. Define

$$v \;:=\; (I-(1-\epsilon)K)^{-1}\epsilon\,(\psi)\,.$$

We then have $v - (1-\epsilon)K(v) = \epsilon\,\psi$ and $K(v) = \psi$. Hence $v = \psi$ and $K(\psi) = \psi$.

If we now start from $K(\psi) = \psi$, then we immediately see that $\epsilon K(\psi) = (I-(1-\epsilon)K)(\psi)$, so that $(I-(1-\epsilon)K)^{-1}\epsilon K(\psi) = \psi$. It follows that $H(\psi) = (1-\epsilon)\,\psi + \epsilon K(\psi) = \psi$. This proves the first point.

(2): By assumption, we have for almost all $z \in \Sigma$ that $\psi(z) = \int_\Sigma w(z,y)\,\psi(y)\,dS_y$. First, let $p = 2$. Then,

$$\begin{aligned} |\psi(z)|^2 &= \left|\int_\Sigma w(z,y)\,\psi(y)\,dSy\right|^2 \leq \left(\int_\Sigma w(z,y)\,dSy\right)\left(\int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy\right) \\ &\leq \int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy\,, \end{aligned} \tag{20}$$

by the triangle inequality, the Cauchy-Schwarz inequality, and the elementary properties (16) of the kernel w. Suppose now that there exists a set $M \subset \Sigma$ with positive surface measure such that strict inequality holds. This would imply that

$$|\psi(z)|^2 < \int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy \ \text{ on } M, \qquad |\psi(z)|^2 \le \int_\Sigma w(z,y)\,|\psi(y)|^2\,dSy \ \text{ on } \Sigma\setminus M\,.$$

Integrating over Σ, it follows that

$$\begin{aligned}\int_\Sigma |\psi(z)|^2\,dSz &< \int_\Sigma \left(\int_M w(z,y)\,dSz + \int_{\Sigma\setminus M} w(z,y)\,dSz\right)|\psi(y)|^2\,dSy \\ &\le \int_\Sigma |\psi(y)|^2\,dSy\,,\end{aligned}$$

which is a contradiction. Thus, for almost all $z \in \Sigma$ we must have the equality sign in (20).

This at first means that

$$\left|\int_\Sigma w(z,y)\,\psi(y)\,dSy\right| = \int_\Sigma w(z,y)\,|\psi(y)|\,dSy\,,$$

and for almost all $z \in \Sigma$ we must have

$$w(z,y)\,\psi(y)^- = 0, \qquad [\text{resp. } w(z,y)\,\psi(y)^+ = 0] \ \text{ for almost all } y \in \Sigma\,.$$

Without loss of generality, let $\psi^- = 0$.

Second, we have for almost all z the equality

$$\int_\Sigma w(z,y)^{1/2}\,w(z,y)^{1/2}\,\psi(y)dS_y = \left(\int_\Sigma w(z,y)dS_y\right)^{\frac12}\left(\int_\Sigma w(z,y)\,\psi^2(y)dS_y\right)^{\frac12}.$$

By a well-known property of the Cauchy-Schwarz inequality, this implies that

$$w(z,y)^{\frac12} = \lambda(z)\,w(z,y)^{\frac12}\,\psi(y)\,,$$

with a real number $\lambda(z)$, for almost all z. Thus, for almost all y and z that can see each other, we get $\psi(y) = \lambda(z)^{-1}$, which obviously leads to the claim.

In the case $1 < p < 2$, we can argue just the same. For almost all $z \in \Sigma$, we must have the equation

$$\int_\Sigma w(z,y)^{\frac{1}{p'}}\,w(z,y)^{\frac1p}\,\psi(y)\,dSy = \left(\int_\Sigma w(z,y)\,dSy\right)^{\frac{p}{p'}}\left(\int_\Sigma w(z,y)\,|\psi(y)|^p\,dSy\right),$$

which implies, with some $\lambda(z)$, the equality $w(z,y)^{\frac{1}{p'}} = \lambda(z)\left[w(z,y)^{\frac1p}\,\psi(y)\right]^{\frac{p}{p'}}$. The claim follows analogously.

(3): The third claim was proved in [Tii97a], [LT98]. We give an analogous simpler proof. Since Ω is no enclosure, we have $K(1) \not\equiv 1$. Thus, by (2), there exists no $\psi \in L^2(\Sigma)$

such that $K(\psi) = \psi$. By (1), we obtain that also $H(\psi) \neq \psi$ for all $\psi \in L^2(\Sigma)$, i. e. 1 is not an eigenvalue of H. But as H is selfadjoint in $L^2(\Sigma)$, $\| H \|_{\mathcal{L}(L^2(\Sigma),L^2(\Sigma))}$ must be an eigenvalue of H. It follows that

$$\| H \|_{\mathcal{L}(L^2(\Sigma),L^2(\Sigma))} < 1 \,,$$

and by classical interpolation arguments for linear positive operators, the claim even follows for all $1 \leq p \leq \infty$.

(4): By the triangle inequality and Hölder's inequality, we at first have

$$\begin{aligned} 0 = \int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi &\geq \int_\Sigma |\psi|^{r+s} - \left|\int_\Sigma H(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi\right| \\ &\geq \int_\Sigma |\psi|^{r+s} - \int_\Sigma \left|H(|\psi|^{r-1}\,\psi)\right|\,|\psi|^s \\ &\geq \int_\Sigma |\psi|^{r+s} - \| H(|\psi|^{r-1}\,\psi) \|_{L^{\frac{r+s}{r}}(\Sigma)} \| |\psi|^s \|_{L^{\frac{r+s}{s}}(\Sigma)} \\ &\geq (1 - \| H \|_{\mathcal{L}(L^{\frac{r+s}{r}}(\Sigma),L^{\frac{r+s}{r}}(\Sigma))}) \int_\Sigma |\psi|^{r+s}\,. \end{aligned} \tag{21}$$

Thus, we must have everywhere the equality sign. This at first means that

$$H(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \geq 0, \qquad [\text{resp. } H(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \leq 0] \text{ a. e. on } \Sigma\,, \tag{22}$$

and, at second, that we have in particular

$$\int_\Sigma H(|\psi|^{r-1}\,\psi)\,|\psi|^s = \| H(|\psi|^{r-1}\,\psi) \|_{L^{\frac{r+s}{r}}(\Sigma)} \| |\psi|^s \|_{L^{\frac{r+s}{s}}(\Sigma)}\,.$$

The latter point immediately implies that

$$|H(|\psi|^{r-1}\,\psi)| = c\,[|\psi|^s]^{\frac{(r+s)/s}{(r+s)/r}} = c\,|\psi|^r\,. \tag{23}$$

Because of (21), we clearly have $|c| \leq 1$. Since $-1 \leq c < 1$ implies $\psi \equiv 0$ again by (21), we just have to discuss the case $c = 1$.

Now, (23) gives that $|\psi|^r = |H(|\psi|^{r-1}\,\psi)| \leq H(|\psi|^r)$, so by definition $G(|\psi|^r) \leq 0$. Since Ω is an enclosure, $G(1) = 0$. By the fact that G is selfadjoint, we can write $0 \geq \int_\Sigma G(|\psi|^r) = \int_\Sigma G(1)\,|\psi|^r = 0$. The first and second points of this lemma now imply that $|\psi|^r \equiv C^r$, for some positive constant C.

Returning to (23), where we can assume $c = 1$, with this information, we get that $|\psi| = C = |H(\psi)|$. Using (22), we have in addition $\text{sign}(H(\psi)) = \pm\text{sign}(\psi)$. Thus, $H(\psi) = \pm\psi$.

Again, because of the first line in relation (21), we see that $H(\psi) = -\psi$ implies that $\psi = 0$. On the other hand, because of (1) and (2), $H(\psi) = \psi$ implies that ψ is constant. This proves point (4).

(5): Observe that

$$\psi\,G(\text{sign}(\psi)) = |\psi| - \psi\,H(\text{sign}(\psi)) \geq (1 - \|H\|_{\mathcal{L}(L^\infty(\Sigma),L^\infty(\Sigma))})\,|\psi| \geq 0\,,$$

almost everywhere on Σ. On the other hand, since G is selfadjoint, we have

$$0 = \int_\Sigma G(\psi)\,\mathrm{sign}(\psi) = \int_\Sigma \psi\, G(\mathrm{sign}(\psi)) \geq 0\,,$$

and we see that $\psi\, G(\mathrm{sign}(\psi))$ vanishes almost everywhere on Σ. This means that $|\psi| = \psi\, H(\mathrm{sign}(\psi))$, and we deduce that

$$H(\mathrm{sign}(\psi)) = \mathrm{sign}(\psi) \qquad \text{for almost all } z \in \Sigma \text{ such that } |\psi(z)| > 0\,.$$

In particular, we have for $z \in \Sigma$ such that $\psi(z) > 0$

$$1 = H(\mathrm{sign}(\psi))(z) = H(\chi_{\{z\in\Sigma\,:\,\psi>0\}})(z) - H(\chi_{\{z\in\Sigma\,:\,\psi<0\}})(z)\,.$$

Since H is a positive operator, the last idendity is only possible assuming that for almost all $z \in \Sigma$ such that $\psi(z) > 0$

$$1 = H(\chi_{\{z\in\Sigma\,:\,\psi>0\}})(z), \quad 0 = H(\chi_{\{z\in\Sigma\,:\,\psi<0\}})(z)\,.$$

Thus, we can write that

$$H(\chi_{\{z\in\Sigma\,:\,\psi>0\}}) \geq \chi_{\{z\in\Sigma\,:\,\psi>0\}} \qquad \text{almost everywhere on } \Sigma\,,$$

and it follows that $G(\chi_{\{z\in\Sigma\,:\,\psi>0\}}) \leq 0$ on Σ. But $G(\chi_{\{z\in\Sigma\,:\,\psi>0\}})$ has mean-value zero on Σ, and thus, $G(\chi_{\{z\in\Sigma\,:\,\psi>0\}}) = 0$ almost everywhere on Σ. Owing to (1) and (2), it follows that $\chi_{\{z\in\Sigma\,:\,\psi>0\}}$ is almost everywhere a constant. Analogously, we can deduce that $\chi_{\{z\in\Sigma\,:\,\psi<0\}}$ is almost everywhere a constant. The claim follows. $\square$

We recall that for Banach spaces X, Y, we denote by $\mathcal{K}(X, Y)$ the set of all linear bounded compact mappings from X into Y.

Lemma 1.8. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. For $1 < p < \infty$, the operator K belongs to the class $\mathcal{K}(L^p(\Sigma), L^p(\Sigma))$.

Proof. This assertion was stated in [Tii97b], [LT98] and follows from classical arguments about weakly singular integral operators. $\square$

For the discussion of L^1 right-hand sides, another compactness property of K turns out to be important.

Lemma 1.9. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Then for $\frac{1}{\alpha} < p$, we have $K \in \mathcal{K}(L^p(\Sigma), C(\Sigma))$.

Proof. The continuity and the compactness of K into $C(\Sigma)$ follow from standard arguments about weakly singular integral operators (see for example the part about Schur integral operators of the book [Alt85]) in the case of a convex surface Σ. The proof relies on the one hand on the estimate

$$|(K(f))(z_1) - (K(f))(z_2)| \leq \|f\|_{L^p(\Sigma)} \left(\int_\Sigma |w(z_1,\, y) - w(z_2,\, y)|^{p'}\, dS_y\right)^{1/p'}\,,$$

for $z_1,\, z_2 \in \Sigma$, and on the other hand on the uniform continuity

$$\max_{|z_1-z_2|\le\delta} \left(\int_\Sigma |w(z_1,\, y) - w(z_2,\, y)|^{p'}\, dS_y\right)^{1/p'} \longrightarrow 0\,,$$

as $\delta \longrightarrow 0$.

Due to the discontinuous factor Θ in the definition (2) of the kernel w, the proof is slightly more involved in the case of nonconvex Σ. Note that for each $z \in \Sigma$, the set of the jumps of the function $\Theta(z, \cdot)$ is a one-dimensional submanifold of Σ, that is, a set of zero surface measure. At the expense of some technical complications, we thus can adapt the standard arguments to this situation. □

Lemma 1.10. (1) The operator G has the representation $G = \epsilon\,(I - \tilde{H})$. If $\Sigma \in \mathcal{C}^{1,\alpha}$, then for $1 < p < \infty$, the operator $\tilde{H}$ belongs to $\mathcal{K}(L^p(\Sigma), L^p(\Sigma))$.

(2) For $1 \le p \le \infty$ let $1/p + 1/p' = 1$. Define

$$\tilde{H}_p \;:=\; \epsilon^{\frac{1}{p}}\, K\,(I - (1-\epsilon)K)^{-1}\, \epsilon^{\frac{1}{p'}}\,.$$

Then, the norm estimate $\|\tilde{H}_p\|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} \le 1$ is valid.

(3) Let $\epsilon < 1$ almost everywhere on Σ. Let $\psi \in L^\infty(\Sigma)$ satisfy $|\psi| \le 1$ almost everywhere on Σ. Then, if neither $\psi = 1$ nor $\psi = -1$ almost everywhere on Σ, we must have $|\tilde{H}(\psi)| < 1$ almost everywhere on Σ.

Proof. (1): The first claim follows from representation (19) and Lemma 1.8, setting $\tilde{H} := K\,(I - (1-\epsilon)K)^{-1}\,\epsilon$.

(2): We readily verify that

$$(I - K(1-\epsilon))^{-1}\, K \;=\; K\,(I - (1-\epsilon)K)^{-1}\,. \tag{24}$$

For an arbitrary $f \in L^2(\Sigma)$, define $(I - (1-\epsilon)K)^{-1}(f) \;=:\; v$. Then, obviously, $(1-\epsilon)K(v) = v - f$. This enables us to write that

$$\Big[\big(I - K(1-\epsilon)\big)\, K\Big](v) = K(v) - K\big((1-\epsilon)K(v)\big) = K(v) - (K(v) - K(f)) = K(f)\,.$$

It follows that $(I - K(1-\epsilon))\,K\,(I - (1-\epsilon)K)^{-1}\,(f) = K(f)$, which proves (24).

We at first consider the case $1 < p < \infty$.

By definition, we have $\tilde{H}_p = \epsilon^{\frac{1}{p}} K\,(I - (1-\epsilon)K)^{-1}\,\epsilon^{\frac{1}{p'}}$, and because of the relation (24), we can also write this in the form

$$\tilde{H}_p = \epsilon^{\frac{1}{p}} (I - K(1-\epsilon))^{-1}\, K\, \epsilon^{\frac{1}{p'}}\,.$$

For an arbitrary $f \in L^p(\Sigma)$, we define

$$R := \big[\epsilon^{\frac{1}{p}} (I - K(1-\epsilon))^{-1}\, K\, \epsilon^{\frac{1}{p'}}\big](f)\,.$$

This definition allows to write that $\frac{R}{\epsilon^{1/p}} - K\left(\frac{1-\epsilon}{\epsilon^{1/p}}\, R\right) = K\Big(\epsilon^{1/p'} f\Big)$, which is equivalent to the equality $\frac{R}{\epsilon^{1/p}} = K\Big(\epsilon^{1/p'}\, f + \frac{1-\epsilon}{\epsilon^{1/p}}\, R\Big)$. Thus, using the fact that $\|K\|_{\mathcal{L}(L^p(\Sigma),\, L^p(\Sigma))} \leq 1$, we deduce the inequality

$$\int_\Sigma \frac{|R|^p}{\epsilon} = \int_\Sigma \left|K\Big(\epsilon^{1/p'}\, f + \frac{1-\epsilon}{\epsilon^{1/p}}\, R\Big)\right|^p \leq \int_\Sigma \left|\epsilon^{\frac{1}{p'}}\, f + \frac{(1-\epsilon)}{\epsilon^{1/p}} R\right|^p = \int_\Sigma \frac{1}{\epsilon}\, |\epsilon\, f + (1-\epsilon)\, R|^p \, .$$

Using the convexity of the function $g(s) = s^p$ and the triangle inequality, we obtain that

$$\int_\Sigma \frac{|R|^p}{\epsilon} \leq \int_\Sigma \frac{1}{\epsilon}\, (\epsilon\, |f|^p + (1-\epsilon)\, |R|^p) \, .$$

It follows that

$$\| \, \tilde{H}_p(f) \, \|^p_{L^p(\Sigma)} = \| \, R \, \|^p_{L^p(\Sigma)} \leq \| \, f \, \|^p_{L^p(\Sigma)} \, ,$$

proving the result. The cases $p = 1$ and $p = \infty$ are straightforward exercises.

(3): Consider an arbitrary function $\psi \in L^\infty(\Sigma)$ such that $|\psi| \leq 1$ almost everywhere on Σ. We introduce two functions R, J by

$$R = \epsilon\, \psi + (1-\epsilon) J, \qquad J = K(R) \, . \tag{25}$$

Note that $\tilde{H}(\psi) = J$. In view of (2), we thus have $|J| \leq 1$ almost everywhere on Σ. Since by assumption $0 < \epsilon < 1$ on Σ, our definition (25) obviously implies the set identity

$$A := \Big\{ z \in \Sigma \, : \, R(z) = 1 \Big\} = \Big\{ z \in \Sigma \, : \, R(z) = 1 = \psi(z) = J(z) \Big\} \, . \tag{26}$$

Taking $z \in A$ arbitrary, we can write, on the other hand,

$$1 = J(z) = \int_\Sigma w(z,y)\, R(y)\, dS_y = \int_A w(z,y)\, dS_y + \int_{\{R<1\}} w(z,y)\, R(y)\, dS_y \, . \tag{27}$$

The latter equality is only possible if $\int_A w(z,y)\, dS_y = 1$. Since this is valid for any $z \in A$, we have by definition that the set A sees only itself. Therefore, by Remark 1.4, it follows either that $\mathrm{meas}(A) = 0$, or that $\mathrm{meas}(\Sigma \setminus A) = 0$.

Assume finally that $J(z) = 1$ for a $z \in \Sigma$. Writing (27) in this point gives a contradiction if $\mathrm{meas}(A) = 0$. This means that either $\tilde{H}(\psi)(z) = J(z) < 1$ a. e. on Σ or $\mathrm{meas}(\Sigma \setminus A) = 0$.

We can argue analogously with the set $B := \{z \in \Sigma \, : \, R(z) = -1\}$. We conclude that if neither A nor B are the whole of Σ, then they must both have zero measure, and that $-1 < \tilde{H}(\psi) < 1$ a. e. on Σ, proving the claim. □

2 Coercivity Inequalities

For the remainder of the paper, we assume that the boundary Γ is not empty. In this section we want to study the coercivity of the operator

$$\langle A\, T,\, \psi \rangle + \int_{\Gamma_3} \sigma\, \epsilon\, |T|^3\, T\, \psi + \int_\Sigma G(\sigma\, |T|^3\, T)\, \psi \, .$$

Here, the symbol $\langle \cdot, \cdot \rangle$ denotes the duality product between a suitable Banach space and its dual. The operator A was defined in (12) in the following way:

$$\langle A\,T,\,\psi\rangle = \int_{\Omega} \kappa(T)\,\nabla T \cdot \nabla\psi + \int_{\Gamma_2} \alpha\,T\,\psi\,.$$

Note that, the domain Ω being disconnected, the expression $\langle A\,T,\,T\rangle$ does not define an equivalent norm on $W^{1,2}(\Omega)$ as soon as there exists some domain $\Omega_i \subset \Omega$ that does not touch Γ_2. However, as was shown in [Tii97a], one easily obtains coercivity if the domain Ω is not an enclosure. In the latter case, $\parallel H \parallel_{\mathcal{L}(L^{5/4}(\Sigma),L^{5/4}(\Sigma))} < 1$, and one has

$$\int_{\Sigma} G(\sigma\,|T|^3\,T)\,T \geq (1- \parallel H \parallel_{\mathcal{L}(L^{5/4}(\Sigma),L^{5/4}(\Sigma))}) \int_{\Sigma} |T|^5\,,$$

(see also Lemma 1.7, (3) above). For the case that Ω might be an enclosure, we now prove a first general coercivity result.

Lemma 2.1. Assume that $\Sigma \in \mathcal{C}^{0,1}$. Let $r, s > 0$ be two numbers such that $r + s < 4$. Then there exists a constant $c = c_{r,s} > 0$ such that

$$\langle A\,\psi,\,\psi\rangle + \int_{\Gamma_3} \sigma\,\epsilon\,|\psi|^{r+s} + \int_{\Sigma} G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \geq c\,\min\left\{ \parallel \psi \parallel^2_{W^{1,2}_{\Gamma_1}(\Omega)},\, \parallel \psi \parallel^{r+s}_{W^{1,2}_{\Gamma_1}(\Omega)} \right\},$$

for all $\psi \in W^{1,2}_{\Gamma_1}(\Omega)$.

Proof. We at first show that there exists a constant $\bar{c} > 0$ such that

$$\langle A\,\psi,\,\psi\rangle + \int_{\Gamma_3} \sigma\,\epsilon\,|\psi|^{r+s} + \int_{\Sigma} G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi \geq \bar{c}\,\parallel \psi \parallel^2_{W^{1,2}_{\Gamma_1}(\Omega)}\,,$$

for all $\psi \in W^{1,2}_{\Gamma_1}(\Omega)$ such that

$$\parallel \psi \parallel_{W^{1,2}_{\Gamma_1}(\Omega)} \;\geq\; 1\,. \tag{28}$$

Suppose that the latter claim is not true. Then we can find a sequence $\{\psi_n\} \subset W^{1,2}_{\Gamma_1}(\Omega)$ such that

$$\langle A\,\psi_n,\,\psi_n\rangle + \int_{\Gamma_3} \sigma\,\epsilon\,|\psi_n|^{r+s} + \int_{\Sigma} G(|\psi_n|^{r-1}\,\psi_n)\,|\psi_n|^{s-1}\,\psi_n \;\leq\; \frac{1}{n}\,\parallel \psi_n \parallel^2_{W^{1,2}_{\Gamma_1}(\Omega)}\,.$$

Setting $\tilde{\psi}_n := \psi_n/\|\psi_n\|_{W^{1,2}_{\Gamma_1}(\Omega)}$, we observe that $\parallel \tilde{\psi}_n \parallel_{W^{1,2}_{\Gamma_1}(\Omega)} = 1$. Thus, $\tilde{\psi}_n \rightharpoonup \tilde{\psi}$ in $W^{1,2}_{\Gamma_1}(\Omega)$, and for a subsequence $\tilde{\psi}_n \to \tilde{\psi}$ almost everywhere on $\Sigma \cup \Gamma_3$. Considering the property (28), we find that

$$\langle A\,\tilde{\psi}_n,\,\tilde{\psi}_n\rangle + \int_{\Gamma_3} \sigma\,\epsilon\,|\tilde{\psi}_n|^{r+s} + \int_{\Sigma} G(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,|\tilde{\psi}_n|^{s-1}\,\tilde{\psi}_n \leq \frac{1}{n}\,. \tag{29}$$

Since the choice $r + s < 4$ implies that $\frac{4s}{4-r} < 4$, we can again pass to a subsequence to obtain that

$$|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n \rightharpoonup |\tilde{\psi}|^{r-1}\,\tilde{\psi} \text{ in } L^{\frac{4}{r}}(\Sigma \cup \Gamma_3)\,,$$
$$|\tilde{\psi}_n|^{s-1}\,\tilde{\psi}_n \to |\tilde{\psi}|^{s-1}\,\tilde{\psi} \text{ in } L^{\frac{4}{4-r}}(\Sigma \cup \Gamma_3)\,,$$

which allows us to pass to the limit in (29). Taking into account Lemma 1.7, we now have

$$\tilde{\psi} = c_i \text{ in each } \Omega_i\,, \qquad \tilde{\psi} = c \text{ on } \Sigma, \qquad \tilde{\psi} = 0 \text{ on } \Gamma\,.$$

This leads to $\tilde{\psi} = 0$. As a matter of fact, we can always find a part $\Omega_{i_0} \subset \Omega$ such that both $\partial\Omega_{i_0} \cap \Sigma$ and $\partial\Omega_{i_0} \cap \Gamma$ are not empty. Considering (29), we find that $\tilde{\psi}_n \to 0 \in W^{1,2}_{\Gamma_1}(\Omega)$, which is a contradiction.

In the case that $\| \psi \|_{W^{1,2}_{\Gamma_1}(\Omega)} < 1$, we use an analogous argument replacing $\| \psi_n \|^2_{W^{1,2}_{\Gamma_1}(\Omega)}$ by $\| \psi_n \|^{r+s}_{W^{1,2}_{\Gamma_1}(\Omega)}$. The claim follows. ∎

Remark 2.2. For $1 \leq p \leq \infty$, define the Sobolev embedding exponent for traces p^*_b by

$$p^*_b := \begin{cases} \frac{2\,p}{3-p} & \text{if } p < 3\,, \\ 1 \leq s < \infty & \text{arbitrary if } p = 3\,, \\ +\infty & \text{if } p > 3\,. \end{cases}$$

Then, we can show by analogous arguments that for any r, $s > 0$ such that $r + s < p^*_b$, there exists a constant $c_{r,s,p} > 0$ such that for all ψ in $W^{1,p}_{\Gamma_1}(\Omega)$,

$$\int_\Omega |\nabla\psi|^p + \int_{\Gamma_2} \alpha\,\psi^2 + \int_{\Gamma_3} \epsilon\,\sigma\,|\psi|^{r+s} + \int_\Sigma G(\sigma\,|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi$$
$$\geq c\,\min\left\{ \|\psi\|^2_{W^{1,p}_{\Gamma_1}(\Omega)}\,,\ \|\psi\|^p_{W^{1,p}_{\Gamma_1}(\Omega)}\,,\ \|\psi\|^{r+s}_{W^{1,p}_{\Gamma_1}(\Omega)} \right\}\,.$$

In the case that the operator K, is compact a better coercivity result was proven in [LT01].

Lemma 2.3. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Let $r, s > 0$. Then there exists a constant $c = c_{r,s} > 0$ such that for all $\psi \in V^{2,r+s}_{\Gamma_1}(\Omega)$,

$$\langle A\,\psi,\,\psi\rangle + \int_{\Gamma_3} \sigma\,\epsilon\,|\psi|^{r+s} + \int_\Sigma G(|\psi|^{r-1}\,\psi)\,|\psi|^{s-1}\,\psi$$
$$\geq c\,\min\{\ \|\ \psi\ \|^2_{V^{2,r+s}_{\Gamma_1}(\Omega)},\ \|\ \psi\ \|^{r+s}_{V^{2,r+s}_{\Gamma_1}(\Omega)}\ \}\,.$$

Proof. See [LT01]. ∎

The inequalities in Lemma 2.1 and Lemma 2.3 establish coercivity properties of the operator of heat radiation taken in connection with the heat conduction. The next statements show that the radiation operator by itself already exerts some coercivity.

Lemma 2.4. Let $\Sigma \in \mathcal{C}^{1,\alpha}$. Let $r, s > 0$ be to numbers with $s \leq r+1$. Then the following statements are valid:

(1) There exists a positive constant $c_{r,s}$ such that for all $\psi \in L^{r+1}(\Sigma)$,

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\psi + \left(\int_\Sigma |\psi|^s\right)^{\frac{r+1}{s}} \geq c \parallel \psi \parallel^{r+1}_{L^{r+1}(\Sigma)} .$$

(2) If the domain Ω is an enclosure, there exists a positive constant $\bar{c}_{r,s}$ such that

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,\psi \geq \bar{c} \parallel \psi \parallel^{r+1}_{L^{r+1}(\Sigma)} ,$$

for all $\psi \in L^{r+1}(\Sigma)$ such that $\int_\Sigma \psi \, dS = 0$.

Proof. (1): We assume that the assertion is false, and we seek a contradiction. We can construct a sequence $\{\tilde{\psi}_n\} \subset L^{r+1}(\Sigma)$ such that $\parallel \tilde{\psi}_n \parallel_{L^{r+1}(\Sigma)} = 1$ and

$$\int_\Sigma G(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,\tilde{\psi}_n + \left(\int_\Sigma |\tilde{\psi}_n|^s\right)^{\frac{r+1}{s}} < \frac{1}{n} . \tag{30}$$

Extracting subsequences, we find that

$$\tilde{\psi}_n \rightharpoonup \tilde{\psi} \text{ in } L^{r+1}(\Sigma) , \qquad |\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n \rightharpoonup w \text{ in } L^{\frac{r+1}{r}}(\Sigma) .$$

Passing to the limit in (30), we can write

$$\limsup_{n\to\infty} \int_\Sigma \epsilon\,|\tilde{\psi}_n|^{r+1} - \lim_{n\to\infty} \int_\Sigma \epsilon\,\tilde{H}(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,\tilde{\psi}_n \leq 0 ,$$

and, using the compactness of $\tilde{H}$ from $L^{1+1/r}(\Sigma)$ into itself, we get

$$\limsup_{n\to\infty} \int_\Sigma \epsilon\,|\tilde{\psi}_n|^{r+1} - \int_\Sigma \epsilon\,\tilde{H}(w)\,\tilde{\psi} \leq 0 .$$

On the other hand, we have by the same tools that

$$\begin{aligned} \int_\Sigma \epsilon\,\tilde{H}(w)\,\tilde{\psi} &= \liminf_{n\to\infty} \int_\Sigma \epsilon\,\tilde{H}(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n)\,\tilde{\psi} \\ &\leq \liminf_{n\to\infty} \left\| \epsilon^{\frac{r}{r+1}}\,\tilde{H}(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n) \right\|_{L^{1+1/r}(\Sigma)} \parallel \epsilon^{\frac{1}{r+1}}\,\tilde{\psi} \parallel_{L^{r+1}(\Sigma)} . \end{aligned} \tag{31}$$

In view of Lemma 1.10 we can write

$$\begin{aligned} \left\| \epsilon^{\frac{r}{r+1}}\,\tilde{H}(|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n) \right\|_{L^{1+1/r}(\Sigma)} &= \left\| \tilde{H}_{\frac{r+1}{r}}\,(\epsilon^{\frac{r}{r+1}}\,|\tilde{\psi}_n|^{r-1}\,\tilde{\psi}_n) \right\|_{L^{1+1/r}(\Sigma)} \leq \parallel \epsilon^{\frac{r}{r+1}}\,|\tilde{\psi}_n|^r \parallel_{L^{1+1/r}(\Sigma)} \\ &= \left(\int_\Sigma \epsilon\,|\tilde{\psi}_n|^{r+1}\right)^{\frac{r}{r+1}} . \end{aligned}$$

Thus, we can continue the estimate (31) by

$$\int_\Sigma \epsilon\, \tilde{H}(w)\, \tilde{\psi} \quad \leq \quad \| \, \epsilon^{\frac{1}{r+1}}\, \tilde{\psi} \, \|_{L^{r+1}(\Sigma)} \, \liminf_{n\to\infty} \left(\int_\Sigma \epsilon\, |\tilde{\psi}_n|^{r+1} \right)^{\frac{r}{r+1}}$$

It follows that

$$\limsup_{n\to\infty} \int_\Sigma \epsilon\, |\tilde{\psi}_n|^{r+1} \leq \| \, \epsilon^{\frac{1}{r+1}}\, \tilde{\psi} \, \|_{L^{r+1}(\Sigma)} \, \left(\limsup_{n\to\infty} \int_\Sigma \epsilon\, |\tilde{\psi}_n|^{r+1} \right)^{\frac{r}{r+1}} \quad ,$$

which implies that $\limsup_{n\to\infty} \| \, \epsilon^{\frac{1}{r+1}}\, \tilde{\psi}_n \, \|^{r+1}_{L^{r+1}(\Sigma)} \leq \| \, \epsilon^{\frac{1}{r+1}}\, \tilde{\psi} \, \|^{r+1}_{L^{r+1}(\Sigma)}$. Combining this with the usual lower semicontinuity of the norm, we obtain for a subsequence that $\lim_{n\to\infty} \| \, \epsilon^{\frac{1}{r+1}}\, \tilde{\psi}_n \, \|^{r+1}_{L^{r+1}(\Sigma)} = \| \, \epsilon^{\frac{1}{r+1}}\, \tilde{\psi} \, \|^{r+1}_{L^{r+1}(\Sigma)}$, which, in its turn, yields

$$\tilde{\psi}_n \to \tilde{\psi} \text{ in } L^{r+1}(\Sigma)\,. \tag{32}$$

Reconsidering (30) for this subsequence, we now obtain that

$$\int_\Sigma G(|\tilde{\psi}|^{r-1}\, \tilde{\psi})\, \tilde{\psi} = 0\,. \tag{33}$$

By Lemma 1.7, it follows that $\tilde{\psi}$ is constant. But since $s \leq r+1$, (30) also gives that $\left(\int_\Sigma |\tilde{\psi}|^s \right)^{\frac{r+1}{s}} = 0$. Thus, $\psi \equiv 0$ on Σ, a contradiction by the strong convergence (32).

(2): We prove the second estimate by the same arguments, obtaining the consequence (33). Then, by the strong convergence (32), we find that $\tilde{\psi}$ has mean value zero on Σ. We can finish the proof analogously. □

We now prove a last coercivity result, which will in particular help us to produce estimates in the case that f belongs only to L^1.

Lemma 2.5. Let $\Sigma \in \mathcal{C}^{1,\alpha}$, and let the emissivity satisfy $\epsilon < 1$ on Σ. Then there exists a positive constant c such that

$$\int_\Sigma G(\psi)\, \mathrm{sign}(\psi) \geq c \, \| \, \psi \, \|_{L^1(\Sigma)}\,,$$

for all $\psi \in L^1(\Sigma)$ such that $\int_\Sigma \psi\, dS = 0$.

Proof. Again, suppose that the claim is not true. Then, it is possible to construct a sequence $\{\psi_n\} \subset L^1(\Sigma)$ with the properties

$$\| \, \psi_n \, \|_{L^1(\Sigma)} = 1\,, \qquad \int_\Sigma \psi_n = 0\,, \qquad \int_\Sigma G(\psi_n)\, \mathrm{sign}(\psi_n) \leq \frac{1}{n}\,.$$

Now, since $\psi_n\, G(\mathrm{sign}(\psi_n)) = |\psi_n| - \psi_n\, H(\mathrm{sign}(\psi_n)) \geq 0$, and using also the fact that G is selfadjoint, we can write that

$$\frac{1}{n} \geq \int_\Sigma G(\psi_n)\,\mathrm{sign}(\psi_n) = \int_\Sigma \psi_n\, G(\mathrm{sign}(\psi_n)) = \int_\Sigma |\psi_n|\,|G(\mathrm{sign}(\psi_n))|$$
$$= \int_\Sigma \epsilon\,|\psi_n|\,|\mathrm{sign}(\psi_n) - \tilde{H}(\mathrm{sign}(\psi_n))|\,. \tag{34}$$

Choosing a $q > \frac{1}{\alpha}$, we can find a subsequence $\mathrm{sign}(\psi_n) \rightharpoonup u \in L^q(\Sigma)$. We have, in particular, that $|u| \leq 1$ almost everywhere on Σ. By Lemma 1.9, we can again pass to a subsequence if necessary to find that

$$\tilde{H}(\mathrm{sign}(\psi_n)) \longrightarrow \tilde{H}(u) \qquad \text{in } C(\Sigma)\,. \tag{35}$$

We distinguish two cases.

For the first case, we assume that $u = 1$ almost everywhere on Σ. By the uniform convergence of $\{\tilde{H}(\mathrm{sign}(\psi_n))\}$, and by (34), we obtain that

$$\lim_{n\to\infty}\int_\Sigma |\psi_n| = \lim_{n\to\infty}\int_\Sigma |\psi_n| - \psi_n = \lim_{n\to\infty}\int_\Sigma |\psi_n|\,|\mathrm{sign}(\psi_n) - 1|$$
$$= \lim_{n\to\infty}\int_\Sigma |\psi_n|\,|\mathrm{sign}(\psi_n) - \tilde{H}(u)| = \lim_{n\to\infty}\int_\Sigma |\psi_n|\,|\mathrm{sign}(\psi_n) - \tilde{H}(\mathrm{sign}(\psi_n))| = 0\,.$$

This is a contradiction. We argue analogously if $u = -1$ almost everywhere on Σ.

Thus, we must have the second case $u \not\equiv 1, -1$. In this case we know, thanks to Lemma 1.10, that $|\tilde{H}(u)| < 1$ on Σ. This implies, by the continuity of $\tilde{H}(u)$, that $1 > \max_\Sigma |\tilde{H}(u)| =: \gamma_0$. We have

$$0 = \lim_{n\to\infty}\int_\Sigma \epsilon\,|\psi_n|\,|\mathrm{sign}(\psi_n) - \tilde{H}(\mathrm{sign}(\psi_n))| = \lim_{n\to\infty}\int_\Sigma \epsilon\,|\psi_n|\,|\mathrm{sign}(\psi_n) - \tilde{H}(u)|$$
$$\geq \epsilon_l\,(1-\gamma_0)\,\lim_{n\to\infty}\int_\Sigma |\psi_n|\,.$$

□

The following Lemma is usefull when we want to use test functions that depend non linearly on temperature. It generalizes properties proved in [LT01], [Mey06].

Lemma 2.6. Let Ω be an enclosure. Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, continuous function with $F(0) = 0$ and $|F(t)| \leq C_0\,(1+|t|^s)$ as $|t| \to \infty$ $(0 \leq s < \infty)$. Let $0 \leq r < \infty$ be an arbitrary number. Then for all $\psi \in L^{r+s}(\Sigma)$,

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\,F(\psi) \geq 0\,.$$

Proof. We fix $n \in \mathbb{N}$. For $i = 1, 2, \ldots$, we define

$$a_i^{(n)} := F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right), \qquad a_{-i}^{(n)} := F\left(\frac{-i-1}{n}\right) - F\left(\frac{-i}{n}\right).$$

Since F is nondecreasing, we obviously have $a_i^{(n)} \geq 0$ and $a_{-i}^{(n)} \leq 0$. Denoting by $\chi_{[a,b]}$ the characteristic function of the interval $[a, b]$, we introduce

$$F_n(t) := \sum_{i=1}^{\infty} a_i^{(n)} \, \chi_{[i/n,+\infty[}(t) + a_{-i}^{(n)} \, \chi_{]-\infty,-i/n]}(t).$$

We can write

$$\begin{aligned}
&\int_\Sigma G(|\psi|^{r-1}\,\psi)\, F_n(\psi) \\
&\quad = \sum_{i=1}^{\infty} \left\{ a_i^{(n)} \int_\Sigma G(|\psi|^{r-1}\,\psi)\, \chi_{[i/n,+\infty[}(\psi) + a_{-i}^{(n)} \int_\Sigma G(|\psi|^{r-1}\,\psi)\, \chi_{]-\infty,-i/n]}(\psi) \right\}.
\end{aligned}$$

Now, since Ω is an enclosure, $G(1) = 0$, and we have

$$\begin{aligned}
\int_\Sigma G(|\psi|^{r-1}\,\psi)\, \chi_{[i/n,+\infty[}(\psi) &= \int_\Sigma G\left(|\psi|^{r-1}\,\psi - \frac{i^r}{n^r}\right) \chi_{[i/n,+\infty[}(\psi) \\
&= \int_\Sigma \left(|\psi|^{r-1}\,\psi - \frac{i^r}{n^r}\right) G(\chi_{[i/n,+\infty[}(\psi)).
\end{aligned}$$

As usual, we observe that

$$G(\chi_{[i/n,+\infty[}(\psi)) = \begin{cases} 1 - H(\chi_{[i/n,+\infty[}(\psi)) & \geq 0 \text{ if } \psi \geq i/n, \\ -H(\chi_{[i/n,+\infty[}(\psi)) & \leq 0 \text{ if } \psi < i/n. \end{cases}$$

This means that $sign\Big((|\psi|^{r-1}\,\psi - \frac{i^r}{n^r})\, G(\chi_{[i/n,+\infty[}(\psi))\Big) = 1$, whence

$$a_i^{(n)} \int_\Sigma G(|\psi|^{r-1}\,\psi)\, \chi_{[i/n,\infty[}(\psi) \geq 0,$$

for all $i = 1, 2, \ldots$. In the same way we show that $a_{-i}^{(n)} \int_\Sigma G(|\psi|^{r-1}\,\psi)\, \chi_{]-\infty,-i/n]}(\psi) \geq 0$. We thus proved that

$$\int_\Sigma G(|\psi|^{r-1}\,\psi)\, F_n(\psi) \geq 0. \tag{36}$$

Observe that for any $t \in \mathbb{R}^+$, we can find $i_0^{(n)} \in \mathbb{N}$ such that $t \in \left[\frac{i_0^{(n)}}{n}, \frac{i_0^{(n)}+1}{n}\right[$. We have

$$\begin{aligned}
F(t) - F_n(t) = F(t) - \sum_{i=1}^{i_0} a_i^{(n)} \, \chi_{[i/n,+\infty[}(t) &= F(t) - \sum_{i=1}^{i_0^{(n)}} F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right) \\
= F(t) - F\left(\frac{i_0^{(n)}}{n}\right) \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

which is true for all $t \in \mathbb{R}$. By an analogous consideration for $t \in \mathbb{R}^-$, we easily obtain that $F_n(t) \to F(t)$ for all $t \in \mathbb{R}$. We also immediately see that $|F_n(t)| \leq |F(t)|$ for all $t \in \mathbb{R}$. It follows that

$$F_n(\psi) \to F(\psi) \text{ in } L^s(\Sigma) \text{ for all } \psi \in L^s(\Sigma)\,.$$

Passage to the limit as $n \to \infty$ in (36) proves the assertion. □

3 A Uniqueness Result

In the context of $(P1)$, $(P2)$ the heat conductivity is material dependent, and given as a function of the temperature κ_i in each subdomain Ω_i for $i = 0, \ldots, m$. Thus, our function κ is defined piecewise, and has the form

$$\kappa = \kappa_i \text{ in } \Omega_i \quad i = 0, \ldots, m\,.$$

Due to the correction made to Lemma 1.7, (5), the simplified proof of uniqueness in the paper [Dru08] is incomplete. We therefore come back to the more complicated method of proof employed in the original version of the preprint.

Lemma 3.1. Let $\kappa_i : \mathbb{R} \longrightarrow \mathbb{R}$ be globally Lipschitz continuous for $i = 1, \ldots, m$, and satisfy (14). Then there exists at most one weak solution of $(P1)$, $(P2)$ in the class $V^{2,4}(\Omega)$.

Proof. The uniqueness of the solution is proved using the same method as in [LT01]. However, note that since κ depends on temperature, we must estimate some new terms. Suppose that T_1 and T_2 are two weak solutions of $(P1)$, $(P2)$. Subtracting the respective integral identities, we obtain that

$$\int_\Omega \kappa(T_2)\, \nabla(T_2 - T_1) \cdot \nabla\psi + \int_{\Gamma_2} \alpha\, (T_2 - T_1)\, \psi + \int_{\Gamma_3} \sigma\, \epsilon \left[\, |T_2^3|\, T_2 - |T_1^3|\, T_1 \right] \psi$$
$$+ \int_\Sigma G\left(\sigma[\,|T_2^3|\, T_2 - |T_1^3|\, T_1]\right) \psi = - \int_\Omega [\kappa(T_2) - \kappa(T_1)]\, \nabla T_1 \cdot \nabla\psi\,, \tag{37}$$

for all $\psi \in W^{1,2}_{\Gamma_1}(\Omega) \cap L^\infty(\Sigma \cup \Gamma_3)$. Define

$$\hat{\Omega}_0 := \left\{ x \in \Omega \,\middle|\, T_2(x) - T_1(x) > 0 \right\}\,, \qquad \hat{\Sigma}_0 := \left\{ z \in \Sigma \,\middle|\, \gamma(T_2 - T_1)(z) > 0 \right\}\,,$$
$$\hat{\Omega}_\delta := \left\{ x \in \Omega \,\middle|\, T_2(x) - T_1(x) > \delta \right\}\,, \qquad \hat{\Sigma}_\delta := \left\{ z \in \Sigma \,\middle|\, \gamma(T_2 - T_1)(z) > \delta \right\}\,,$$

and observe that $\hat{\Omega}_\delta \nearrow \hat{\Omega}_0$, and $\hat{\Sigma}_\delta \nearrow \hat{\Sigma}_0$ as $\delta \searrow 0$. Here, γ denotes the trace operator. We introduce a function

$$v_\delta := \min\{(T_2 - T_1)^+, \delta\}\,.$$

We easily can show that $\gamma(v_\delta) = \min\{[\gamma(T_2 - T_1)]^+, \delta\}$. Thus, writing also on the boundary v_δ instead of $\gamma(v_\delta)$, we have

$$v_\delta \geq 0 \ \text{ in } \ \Omega\,, \qquad v_\delta = 0 \ \text{ on } \ \Gamma_1\,, \qquad G(v_\delta) = \begin{cases} \delta - H(v_\delta) \geq 0 & \text{on } \hat{\Sigma}_\delta \\ 0 - H(v_\delta) \leq 0 & \text{on } \Sigma \setminus \hat{\Sigma}_0\,. \end{cases} \tag{38}$$

Testing with $\psi = v_\delta$ in (37) is possible, since this function is bounded. Observing that $(T_2 - T_1)\, v_\delta \geq v_\delta^2$, and that the term $-\int_{\Gamma_3} \sigma\, \epsilon \left[|T_2^3|\, T_2 - |T_1^3|\, T_1 \right] v_\delta$ is negative we get the inequality

$$\int_\Omega \kappa(T_2)\, |\nabla v_\delta|^2 + \int_{\Gamma_2} \alpha\, v_\delta^2 \leq \tag{39}$$
$$- \int_\Sigma \sigma \left[|T_2^3|\, T_2 - |T_1^3|\, T_1 \right] G(v_\delta) - \int_\Omega [\kappa(T_2) - \kappa(T_1)]\, \nabla T_1 \cdot \nabla v_\delta\,.$$

By adding the term $\int_\Sigma G(\sigma\, v_\delta)\, v_\delta$ on both sides of this equation, we obtain that

$$\int_\Omega \kappa(T_2)\, |\nabla v_\delta|^2 + \int_\Sigma G(\sigma\, v_\delta)\, v_\delta \leq$$
$$- \int_\Sigma \sigma \left[|T_2^3|\, T_2 - |T_1^3|\, T_1 \right] G(v_\delta) + \int_\Sigma G(\sigma\, v_\delta)\, v_\delta - \int_\Omega [\kappa(T_2) - \kappa(T_1)]\, \nabla T_1 \cdot \nabla v_\delta\,.$$

Now, we use the disjoint decomposition of Σ,

$$\Sigma = \Sigma \setminus \hat{\Sigma}_0 \ \cup \ \hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta \ \cup \ \hat{\Sigma}_\delta\,,$$

and we observe that

$$- \int_{\Sigma \setminus \hat{\Sigma}_0} \sigma \left[|T_2^3|\, T_2 - |T_1^3|\, T_1 \right] G(v_\delta) \leq 0\,, \qquad \int_{\Sigma \setminus \hat{\Sigma}_0} \sigma\, G(v_\delta) v_\delta \leq 0\,. \tag{40}$$

On the other hand, using the inequality

$$\big| |t_1|^3\, t_1 - |t_2|^3\, t_2 \big| \leq 4 \left(|t_1|^3 + |t_2|^3 \right) |t_1 - t_2|\,, \ \text{for all } t_1,\, t_2 \in \mathbb{R}\,, \tag{41}$$

we can write

$$\int_{\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta} \sigma \left[|T_2^3|\, T_2 - |T_1^3|\, T_1 \right] G(v_\delta) \leq 4\, \sigma\, \delta \int_{\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta} \left(|T_2|^3 + |T_1|^3 \right) \left| G(v_\delta) \right|$$
$$\leq 8\, \sigma\, \delta \left(\int_{\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta} |T_2 + T_1|^4 \right)^{\frac{3}{4}} \| \, v_\delta \, \|_{L^4(\Sigma)} \leq c\, \delta \left(\int_{\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta} |T_2 + T_1|^4 \right)^{\frac{3}{4}} \| \, v_\delta \, \|_{W^{1,2}_{\Gamma_1}(\Omega)}\,. \tag{42}$$

We find easily that

$$\int_{\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta} \sigma\, G(v_\delta)\, v_\delta \leq c\, \delta \ \| \, v_\delta \, \|_{W^{1,2}_{\Gamma_1}(\Omega)} \ \text{meas}(\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta)^{\frac{1}{2}}\,. \tag{43}$$

In order to estimate the last terms on Σ, we introduce the set

$$\tilde{\Sigma}_\delta := \left\{ z \in \Sigma \,\middle|\, |T_2|^3\, T_2 - |T_1|^3\, T_1 < \delta \right\} .$$

Recalling that $G(v_\delta) \geq 0$ in $\hat{\Sigma}_\delta$ we have

$$\int_{\hat{\Sigma}_\delta} \sigma \left[v_\delta - \left(|T_2^3|\, T_2 - |T_1^3|\, T_1 \right) \right] G(v_\delta) \leq \int_{\tilde{\Sigma}_\delta \cap \hat{\Sigma}_\delta} \sigma \left[v_\delta - \left(|T_2^3|\, T_2 - |T_1^3|\, T_1 \right) \right] G(v_\delta)$$

$$\leq \int_{\tilde{\Sigma}_\delta \cap \hat{\Sigma}_\delta} \sigma\, v_\delta\, G(v_\delta) \leq c\, \delta\, \parallel v_\delta \parallel_{L^2(\Sigma)} \operatorname{meas}(\tilde{\Sigma}_\delta \cap \hat{\Sigma}_\delta)^{\frac{1}{2}} . \tag{44}$$

Summing up the results (40), (42), (43) and (44), we can write down the estimate

$$- \int_\Sigma \sigma \left[|T_2^3|\, T_2 - |T_1^3|\, T_1 \right] G(v_\delta) + \int_\Sigma G(\sigma\, v_\delta)\, v_\delta \leq c\, \delta\, \parallel v_\delta \parallel_{W^{1,2}_{\Gamma_1}(\Omega)}\; f_\delta\, , \tag{45}$$

with the sequence

$$f_\delta := \left(\int_{\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta} |T_2 + T_1|^4 \right)^{\frac{3}{4}} + \operatorname{meas}(\hat{\Sigma}_0 \setminus \hat{\Sigma}_\delta)^{\frac{1}{2}} + \operatorname{meas}(\tilde{\Sigma}_\delta \cap \hat{\Sigma}_\delta)^{\frac{1}{2}} ,$$

that converges to zero as $\delta \to 0$. Finally,

$$\int_\Omega [\kappa(T_2) - \kappa(T_1)]\, \nabla T_1 \cdot \nabla v_\delta = \int_{\hat{\Omega}_0 \setminus \hat{\Omega}_\delta} [\kappa(T_2) - \kappa(T_1)]\, \nabla T_1 \cdot \nabla v_\delta$$

$$\leq L_\kappa\, \delta \left(\int_{\hat{\Omega}_0 \setminus \hat{\Omega}_\delta} |\nabla T_1|^2 \right)^{\frac{1}{2}} \parallel v_\delta \parallel_{W^{1,2}_{\Gamma_1}} , \tag{46}$$

L_κ being a Lipschitz constant of κ.

By the inequalities (45) and (46), and taking into account Lemma 2.1, we can conclude that

$$\parallel v_\delta \parallel_{W^{1,2}_{\Gamma_1}} \leq c\, \delta \left(f_\delta + \tilde{f}_\delta + \left(\int_{\hat{\Omega}_0 \setminus \hat{\Omega}_\delta} |\nabla T_1|^2 \right)^{\frac{1}{2}} \right) .$$

It follows that

$$|\Omega \cap \hat{\Omega}_\delta| = \frac{1}{\delta} \left(\int_{\Omega \cap \hat{\Omega}_\delta} \delta^2 \right)^{\frac{1}{2}} \leq \frac{1}{\delta} \parallel v_\delta \parallel_{L^2(\Omega)} \longrightarrow 0 .$$

This gives $|\Omega \cap \hat{\Omega}_0| = 0$, that is, $T_2 \leq T_1$ *a.e.* in Ω. By exchanging the roles of T_2 and T_1, one gets their equality in Ω. □

4 Existence Results

We recall our purpose to obtain estimates that involve the L^p−norm of the *heat sources*. Throughout this section, we assume that there exist an extension of the imposed temperature T_0 on Γ_1, that we still denote by T_0, such that

$$T_0(x) \geq \operatorname*{ess\,inf}_{z\in\Gamma_1} T_0(z) \quad \text{almost everywhere in } \Omega\,. \tag{47}$$

We also assume that Ω is such that

$$\operatorname{dist}(\Gamma\,,\,\Sigma) > 0\,. \tag{48}$$

This assumption implies no loss of generality in the type of applications considered (see the Introduction). If the assumption (48) is satisfied, we choose a fixed $\phi_0 \in C^\infty(\overline{\Omega})$ such that $\phi_0 \equiv 1$ on Γ and $\phi_0 \equiv 0$ on Σ, and we set

$$\hat{T}_0 := T_0\,\phi_0\,. \tag{49}$$

The function $\hat{T}_0$ is an extension of T_0 that does not perturb the non local terms on Σ. We can state our first main result.

Theorem 4.1. Let the heat conductivity κ satisfy (14) and assume that ϵ satisfies (15). Assume that $f \in L^p(\Omega)$ with $1 < p \leq \infty$, and let the assumptions (47) on T_0 and (48) on the domain Ω be satisfied.

(1) If $\Sigma \in \mathcal{C}^{0,1}$ and $p > \frac{9}{7}$, then Problem (P) has a weak solution T. In addition we have the following *a priori* estimates. If $p \geq \frac{3}{2}$, then for all $1 \leq r < \infty$, we can find a continuous function P_r such that

$$\left\| |T|^r \right\|_{W^{1,2}(\Omega)} \leq P_r\Big(\,\|\, f\,\|_{L^p(\Omega)}\,,\,\|\,\tilde{f}\,\|_{L^2(\Gamma)}\,,\,\|T_0\|_{W^{1,2}(\Omega)}\,,\,\|T_0\|_{L^5(\Gamma_3)}\Big)\,.$$

If $p \in \left]9/7, 3/2\right[$, then

$$\|\,T\,\|_{V^{2,\frac{2p}{3-2p}}(\Omega)} \leq P\Big(\,\|\,f\,\|_{L^p(\Omega)}\,,\,\|\,\tilde{f}\,\|_{L^{\frac{2p}{3-p}}(\Gamma)}\,,\,\|T_0\|_{W^{1,2}(\Omega)}\,,\,\|T_0\|_{L^5(\Gamma_3)}\Big)\,.$$

(2) If $\Sigma \in \mathcal{C}^{1,\alpha}$, Problem (P) has a weak solution T. If $p > 9/7$, then (1) is valid. In addition, we find that for $p \in \left[6/5, 9/7\right]$,

$$\|\,T\,\|_{V^{2,\frac{9-5p}{3-2p}}(\Omega)} \leq P\Big(\,\|\,f\,\|_{L^p(\Omega)}\,,\,\|\,\tilde{f}\,\|_{L^{\frac{2p}{3-p}}(\Gamma)}\,,\,\|T_0\|_{W^{1,2}(\Omega)}\,,\,\|T_0\|_{L^5(\Gamma_3)}\Big)\,,$$

and that for $p \in \left]1, 6/5\right[$

$$\|\,T\,\|_{V^{\frac{3p}{3-p},\frac{9-5p}{3-2p}}(\Omega)} \leq P\Big(\,\|\,f\,\|_{L^p(\Omega)}\,,\,\|\,\tilde{f}\,\|_{L^{\frac{2p}{3-p}}(\Gamma)}\,,\,\|T_0\|_{W^{1,\frac{3p}{4p-3}}(\Omega)}\,,\,\|T_0\|_{L^{\frac{9-5p}{3-2p}}(\Gamma_3)}\Big)\,.$$

Here P, P_r are continuous functions, which depend on p, $\text{dist}(\Gamma, \Sigma)$ and on $\operatorname*{ess\,inf}_{z\in\Gamma_1} T_0(z)$. If the right-hand side f is positive in Ω, then $T \geq \inf\{\operatorname*{ess\,inf}_{z\in\Gamma_1} T_0(z), \operatorname*{ess\,inf}_{z\in\Gamma} T_{\text{Ext}}(z)\}$ almost everywhere in Ω.

We essentially carry out the proof in the next two propositions. We will make use of the following notations. The space $W^{1,p}_{\Gamma_1}(\Omega)$ contains the elements of $W^{1,p}(\Omega)$ whose trace vanishes on the boundary part Γ_1. Recalling the notation (49), we define for $T \in W^{1,p}_{\Gamma_1}(\Omega)$ $(1 < p < \infty)$

$$\hat{T} := T + \hat{T}_0 \, .$$

For $\delta > 0$, we introduce the operators

$$\begin{aligned} \langle A\,T\,,\,\psi\rangle &:= \int_\Omega \kappa(\hat{T})\,\nabla\hat{T}\cdot\nabla\psi + \int_{\Gamma_2}\alpha\,\hat{T}\,\psi \\ \langle A_\delta\,T,\,\psi\rangle &:= \delta\int_\Omega \left|\nabla\hat{T}\right|^{p-2}\,\nabla\hat{T}\cdot\nabla\psi\,. \end{aligned}$$

Proposition 4.2. We fix $3 < p < \infty$. For an arbitrary number $\delta > 0$ there exists $T \in W^{1,p}_{\Gamma_1}(\Omega)$ such that

$$\langle A_\delta\,T,\,\psi\rangle + \langle A\,T\,,\,\psi\rangle + \int_{\Gamma_3}\epsilon\,\sigma\,|\hat{T}|^3\,\hat{T}\,\psi + \int_\Sigma G(\sigma\,|\hat{T}|^3\,\hat{T})\,\psi = \int_\Omega f\,\psi + \int_\Gamma \tilde{f}\,\psi\,, \qquad (50)$$

for all $\psi \in W^{1,p}_{\Gamma_1}(\Omega)$.

Proof. For $T, \psi \in W^{1,p}_{\Gamma_1}(\Omega)$, define

$$\langle\tilde{A}\,T\,,\,\psi\rangle := \langle A_\delta\,T,\,\psi\rangle + \int_{\Gamma_3}\epsilon\,\sigma\,|\hat{T}|^3\,\hat{T}\,\psi\,, \qquad \langle B\,T\,,\,\psi\rangle := \int_\Sigma G(\sigma|\hat{T}|^3\,\hat{T})\,\psi\,.$$

We show that the sum $\tilde{A} + A + B$ defines a coercive, pseudomonotone operator from $W^{1,p}_{\Gamma_1}(\Omega)$ into $\left[W^{1,p}_{\Gamma_1}(\Omega)\right]^*$. We at first discuss coercivity. In view of (49) we have

$$\langle B\,T\,,\,T\rangle = \int_\Sigma G(\sigma|\hat{T}|^3\,\hat{T})\,\hat{T} = \int_\Sigma G(\sigma|T|^3\,T)\,T\,.$$

Using Remark 2.2, we easily find that

$$\langle(\tilde{A} + A + B)\,T\,,\,T\rangle \;\geq\; \frac{\delta}{2}\,\min\left\{\|\,T\,\|^2_{W^{1,p}_{\Gamma_1}(\Omega)}\,,\,\|\,T\,\|^p_{W^{1,p}_{\Gamma_1}(\Omega)}\,,\,\|\,T\,\|^5_{W^{1,p}_{\Gamma_1}(\Omega)}\right\} - \tilde{C}_{0,\delta}\,,$$

with a positive constant $C_{0,\delta}$ that depends on δ whose exact value is not needed. This proves the coercivity.

In order to show the pseudomonotonicity of $\tilde{A}+A+B$, we at first prove that B is compact. Let

$$T_k \rightharpoonup T \;\text{ in }\; W^{1,p}_{\Gamma_1}(\Omega)\,. \qquad (51)$$

For $\psi \in W^{1,p}_{\Gamma_1}(\Omega)$, we have the estimate

$$\begin{aligned}\left|\langle B\,T_k - B\,T, \psi\rangle\right| &= \left|\int_\Sigma \left[G(\sigma|\hat{T}_k|^3\,\hat{T}_k) - G(\sigma|\hat{T}|^3\,\hat{T})\right]\psi\right| = \sigma\,\left|\int_\Sigma \left[|\hat{T}_k|^3\,\hat{T}_k - |\hat{T}|^3\,\hat{T}\right] G(\psi)\right| \\ &\le 4\,\sigma\,\max_{\overline{\Omega}}\{|\hat{T}_k|^3 + |\hat{T}|^3\}\int_\Sigma |T_k - T|\,|G(\psi)| \\ &\le c^4\,C^3\,\|\,G\,\|_{\mathcal{L}(L^\infty(\Sigma),L^\infty(\Sigma))}\,\|\,\psi\,\|_{W^{1,p}_{\Gamma_1}(\Omega)}\|\,T_k - T\,\|_{L^1(\Sigma)}\,,\end{aligned}$$

where c is the embedding constant $W^{1,p}_{\Gamma_1}(\Omega) \hookrightarrow C(\overline{\Omega})$, and C is a bound for the norm of the sequence $\{T_k\}$ in $W^{1,p}_{\Gamma_1}(\Omega)$. We thus can write that

$$\|\,B\,T_k \,-\, B\,T\,\|_{[W^{1,p}_{\Gamma_1}(\Omega)]^*} \le \bar{C}\,\|\,T_k - T\,\|_{L^1(\Sigma)} \longrightarrow 0\,,$$

since the embedding $W^{1,p}_{\Gamma_1}(\Omega) \hookrightarrow L^1(\Sigma)$ is compact.

We now show that A is pseudomonotone. For the sequence (51) we assume that

$$\limsup_{k\to\infty}\,\langle A(T_k)\,,\,T_k - T\rangle \;\le 0\,. \tag{52}$$

From straightforward manipulations we get, using the monotonicity of the p-Laplace part, that

$$\begin{aligned}&\int_\Omega \kappa(\hat{T}_k)\,|\nabla(T_k - T)|^2 + \int_{\Gamma_2}\alpha\,(T_k - T)^2 \\ &\quad \le \langle A(T_k)\,,\,T_k - T\rangle - \int_\Omega \kappa(\hat{T}_k)\,\nabla\hat{T}\cdot\nabla(T_k - T) - \int_{\Gamma_2}\alpha\,\hat{T}\,(T_k - T)\,.\end{aligned}$$

Thanks also to (51) and (52), this yields

$$\limsup_{k\to\infty}\int_\Omega \kappa(\hat{T}_k)\,|\nabla(T_k - T)|^2 + \int_{\Gamma_2}\alpha\,(T_k - T)^2 \le 0\,.$$

This provides us with a (not relabelled) subsequence such that $\nabla T_k \longrightarrow \nabla T$ in $[L^2(\Omega)]^3$. For this subsequence and $\psi \in W^{1,p}_{\Gamma_1}(\Omega)$, one gets

$$\lim_{k\to\infty}\,\langle A\,T_k\,,\,T_k - \psi\rangle = \langle A\,T\,,\,T - \psi\rangle\,.$$

Thus A is pseudomonotone. Since it is well known that $\tilde{A}$ is monotone, we also get that $\tilde{A}+A$ is pseudomonotone (see [Lio69], remark 2.12). Since B is compact, we finally obtain that $\tilde{A} + A + B$ is pseudomonotone.

The assertion now follows from standard arguments. □

Remark 4.3. Proposition 4.2 states the existence of a solution of (P) with the following nonlinear Fourier law with respect to ∇T for the heat flux q:

$$q = -\big(\delta\,|\nabla T|^{p-2} + \kappa(T)\big)\,\nabla T\,.$$

In the next proposition, we obtain uniform estimates on the sequence of approximate solutions $\{T_\delta\}$ constructed in Proposition 4.2.

Proposition 4.4. (1) If $\Sigma \in \mathcal{C}^{0,1}$ we get, for all $2 < q < \infty$, the estimate

$$\|T_\delta\|_{W^{1,2}(\Omega)} + \left\| |T_\delta|^{\frac{q+1}{2}} \right\|_{W^{1,2}(\Omega)} + \|T_\delta\|_{L^{2\,(q+1)}(\Gamma_3\cup\Sigma)}$$
$$\leq P_q\Big(\parallel f \parallel_{L^{\frac{3\,(q+1)}{2q+3}}(\Omega)} , \parallel \tilde{f} \parallel_{L^{\frac{2\,(q+1)}{q+2}}(\Gamma)} , \|T_0\|_{W^{1,2}(\Omega)} , \|T_0\|_{L^5(\Gamma_3)}\Big) + C_\delta \,,$$

with a continuous function P_q of the data and a sequence C_δ of positive numbers that converge to zero for $\delta \to 0$.

(2) If $\Sigma \in \mathcal{C}^{1,\alpha}$, we have, for all $1 \leq q < \infty$,

$$\|T_\delta\|_{W^{1,2}(\Omega)} + \left\| |T_\delta|^{\frac{q+1}{2}} \right\|_{W^{1,2}(\Omega)} + \|T_\delta\|_{L^{q+4}(\Gamma_3\cup\Sigma)}$$
$$\leq \bar{P}_q\Big(\parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)} , \parallel \tilde{f} \parallel_{L^{\frac{2(q+1)}{q+2}}(\Gamma)} , \|T_0\|_{W^{1,2}(\Omega)} , \|T_0\|_{L^5(\Gamma_3)}\Big) + C_\delta \,,$$

and for all $0 < q < 1$,

$$\|T_\delta\|_{W^{1,s}(\Omega)} + \left\| |T_\delta|^{\frac{q+1}{2}} \right\|_{W^{1,2}(\Omega)} + \|T_\delta\|_{L^{q+4}(\Gamma_3\cup\Sigma)}$$
$$\leq \bar{P}_q\Big(\parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)} , \parallel \tilde{f} \parallel_{L^{\frac{2(q+1)}{q+2}}(\Gamma)} , \|T_0\|_{W^{1,s'}(\Omega)} , \|T_0\|_{L^{q+4}(\Gamma_3)}\Big) + C_\delta \,,$$

with $s = \frac{3\,(q+1)}{q+2}$, and $C_\delta \to 0$ as $\delta \to 0$.

Proof. For the sake of clarity, we present the proof in the homogeneous case $T_0 = 0$ and $T_{\text{Ext}} = 0$. The general estimates follow by similar techniques.

We would like to use $\psi = |T_\delta|^{q-1}\, T_\delta$ as test function in (50). We first consider a $q \geq 1$. As one easily computes,

$$\nabla\Big(|T_\delta|^{q-1}\, T_\delta\Big) = q\, |T_\delta|^{q-1}\, \nabla T_\delta \, \in L^p(\Omega) \,,$$

since $T_\delta \in L^\infty(\Omega)$. Thus, we can test with this function. Consider also the relation

$$\nabla\Big(|T_\delta|^{q-1}\, T_\delta\Big) \cdot \nabla T_\delta = \frac{4q}{(q+1)^2} \left|\nabla |T_\delta|^{\frac{q+1}{2}}\right|^2 .$$

We can write

$$\int_\Omega \frac{4q}{(q+1)^2} \left\{\delta\, |\nabla T_\delta|^{p-2} + \kappa(T_\delta)\right\} \left|\nabla |T_\delta|^{\frac{q+1}{2}}\right|^2 + \int_\Sigma G(\sigma |T_\delta^3|\, T_\delta)\, |T_\delta|^{q-1}\, T_\delta = \int_\Omega f\, |T_\delta|^{q-1}\, T_\delta \,.$$

It follows that

$$\int_\Omega \frac{4q}{(q+1)^2}\, \kappa(T_\delta) \left|\nabla |T_\delta|^{\frac{q+1}{2}}\right|^2 + \int_\Sigma G(\sigma |T_\delta^3|\, T_\delta)\, |T_\delta|^{q-1}\, T_\delta \leq \int_\Omega f\, |T_\delta|^{q-1}\, T_\delta \,. \tag{53}$$

Now, if we want to consider a $0 < q < 1$, we choose an arbitrary small $\alpha > 0$, and we test with the function $T_\delta\,(\alpha + |T_\delta|)^{q-1}$. We obtain the inequality

$$\int_\Omega \kappa(T_\delta)\,(|T_\delta| + \alpha)^{q-2}\,(q|T_\delta| + \alpha)\ |\nabla T_\delta|^2 + \int_\Sigma G(\sigma|T_\delta^3|\,T_\delta)\,(|T_\delta| + \alpha)^{q-1}\,T_\delta$$
$$\leq \int_\Omega f\,(|T_\delta| + \alpha)^{q-1}\,|T_\delta|\,.$$

Letting $\alpha \to 0$, we get, by Fatou's lemma,

$$\int_\Omega \kappa(T_\delta)\,q\,|T_\delta|^{q-1}\ |\nabla T_\delta|^2 + \int_\Sigma G(\sigma|T_\delta^3|\,T_\delta)\,|T_\delta|^{q-1}\,T_\delta \leq \int_\Omega f\,|T_\delta|^q\,. \tag{54}$$

Denoting by $\chi_{A(0)}$ the characteristic function of the set $A(0) := \{x \in \Omega\ :\ |T_\delta(x)| > 0\}$, and considering the relation $\nabla\,|T_\delta|^{\frac{q+1}{2}} = \frac{q+1}{2}\,|T_\delta|^{\frac{q-1}{2}}\,\mathrm{sign}(T_\delta)\,\chi_{A(0)}\,\nabla T_\delta$, we see that we can write (53) also if $0 < q < 1$.

Define $w_\delta := |T_\delta|^{\frac{q+1}{2}}$. Applying Young's inequality, we can write down the estimate

$$\int_\Omega f\,|T_\delta|^q = \int_\Omega |f|\,w_\delta^{\frac{2q}{q+1}} \leq \parallel w_\delta \parallel_{L^6(\Omega)}^{\frac{2q}{q+1}} \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)} \leq c \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)} \parallel w_\delta \parallel_{W^{1,2}_{\Gamma_1}(\Omega)}^{\frac{2q}{q+1}}$$
$$\leq c_\gamma\ \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{q+1} + \gamma\ \parallel w_\delta \parallel_{W^{1,2}_{\Gamma_1}(\Omega)}^2\,, \tag{55}$$

where γ is an arbitrary small positive number. Analogously, we prove the estimate

$$\int_\Omega f\,|T_\delta|^q \ \ \leq\ \ c_\gamma\ \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{\frac{q+4}{4}} + \gamma\ \parallel w_\delta \parallel_{W^{1,2}_{\Gamma_1}(\Omega)}^{\frac{2(q+4)}{q+1}}\ . \tag{56}$$

We also note that $G(\sigma|T_\delta|^3\,T_\delta)|T_\delta|^{q-1}\,T_\delta \geq G(\sigma|T_\delta|^4)|T_\delta|^q$, since the operator H is positive.

First Case: $\boldsymbol{\Sigma \in \mathcal{C}^{0,1}}$ only.

In view of the definition of w_δ, and of the estimates (53) and (55), we have

$$\int_\Omega \frac{4q}{(q+1)^2}\,\kappa(T_\delta)\ |\nabla w_\delta|^2 + \int_\Sigma G\left(\sigma\,w_\delta^{\frac{8}{q+1}}\right)\,w_\delta^{\frac{2q}{q+1}} \leq c_\gamma\ \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{q+1} + \gamma\ \parallel w_\delta \parallel_{W^{1,2}_{\Gamma_1}(\Omega)}^2\ . \tag{57}$$

If we choose $2 < q < \infty$, we have $\frac{2q}{q+1} + \frac{8}{q+1} < 4$. Then, Lemma 2.1 is applicable with $r = 2q/(q+1)$, $s = 8/(q+1)$. First assuming that $\||T_\delta|^{\frac{q+1}{2}}\|_{W^{1,2}_{\Gamma_1}(\Omega)} \geq 1$, we obtain from (57) that $\||T_\delta|^{\frac{q+1}{2}}\|^2_{W^{1,2}_{\Gamma_1}(\Omega)} \leq c_q\ \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{q+1}$.

If $\||T_\delta|^{\frac{q+1}{2}}\|_{W^{1,2}_{\Gamma_1}(\Omega)} < 1$, we obtain, replacing (55) by (56) and using the same argument, that $\||T_\delta|^{\frac{q+1}{2}}\|_{W^{1,2}_{\Gamma_1}(\Omega)}^{\frac{2(q+4)}{q+1}} \leq c_q\ \parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}^{\frac{q+4}{4}}$. We conclude asserting that

$$\left\||T_\delta|^{\frac{q+1}{2}}\right\|_{W^{1,2}_{\Gamma_1}(\Omega)} \leq P_q(\parallel f \parallel_{L^{\frac{3(q+1)}{2q+3}}(\Omega)})\,.$$

for all $2 < q < \infty$.

Second Case: $\boldsymbol{\Sigma} \in \mathcal{C}^{\mathbf{1},\alpha}$.

We can apply Lemma 2.3 instead of Lemma 2.1. By the same techniques, we achieve the better estimate

$$\left\| |T_\delta|^{\frac{q+1}{2}} \right\|_{W^{1,2}(\Omega)} + \| T_\delta \|_{L^{q+4}(\Sigma)} \leq P_q(\| f \|_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}) , \tag{58}$$

for all $0 < q < \infty$.

On the other hand, one has for $1 \leq s < 2$, $1 < r$, and $r' = r/(r-1)$, that

$$\int_\Omega |\nabla T_\delta|^s = \int_\Omega |\nabla T_\delta|^s \chi_{A(0)} = \int_\Omega \frac{|\nabla T_\delta|^s}{|T_\delta|^{(1-q)\frac{s}{2}}} |T_\delta|^{(1-q)\frac{s}{2}} \chi_{A(0)}$$
$$\leq \left(\int_\Omega f\, |T_\delta|^q \right)^{s/2} \left(\int_\Omega |T_\delta|^{(1-q)\frac{s}{2-s}} \right)^{(2-s)/2} \leq \| f \|_{L^{r'}(\Omega)}^{s/2} \| T_\delta \|_{L^{q\,r}(\Omega)}^{q\,s/2} \| T_\delta \|_{L^{(1-q)\frac{s}{2-s}}(\Omega)}^{\frac{(1-q)s}{2}} .$$

For $0 < q < 1$ and for the choice

$$r = \frac{3(q+1)}{q} , \qquad s = \frac{3(q+1)}{q+2} ,$$

we see that $r' = \frac{3(q+1)}{2q+3}$, and, using the embedding theorems, we obtain from (58) that $\| \nabla T_\delta \|_{L^s(\Omega)} \leq P_q(\| f \|_{L^{\frac{3(q+1)}{2q+3}}(\Omega)})$, which finally gives that

$$\| T_\delta \|_{W^{1,s}_{\Gamma_1}(\Omega)} + \| \, |T_\delta|^{\frac{q+1}{2}} \|_{W^{1,2}(\Omega)} + \| T_\delta \|_{L^{q+4}(\Sigma)} \leq P_q(\| f \|_{L^{\frac{3(q+1)}{2q+3}}(\Omega)}) .$$

In the general, nonhomogeneous case, we have to consider test functions of the type $|T_\delta|^{q-1} T_\delta - T_{0,q}$, where $T_{0,q} = T_0^q \, \phi_0$, with the function ϕ_0 according to (49). Making use of the assumption (47), we can then prove the general estimate stated by the proposition. □

Proof of Theorem 4.1. Suppose that $f \in L^p(\Omega)$. It is straightforward to calculate for which range of $q > 0$ we can obtain the estimates of Proposition 4.4. This are precisely the estimates stated by the theorem. In each case we get

$$T_\delta \rightharpoonup T \text{ in } V^{s,r}(\Omega) \tag{59}$$

with $s > \frac{3}{2}$ and $r > 4$. The passage to the limit with the sequence of approximate solutions constructed in Proposition 4.2 is then a straightforward exercise.

If $f \geq 0$ in Ω, we set $k_0 := \inf\{\operatorname*{ess\,inf}_{z \in \Gamma_1} T_0(z), \operatorname*{ess\,inf}_{z \in \Gamma} T_{\text{Ext}}(z)\}$ and use in (50) the test function $(T_\delta - k_0)^-$. It follows that

$$\limsup_{\delta \to 0} \Big(\int_\Omega |\nabla (T_\delta - k_0)^-|^2 + \int_{\Gamma_2} \alpha\, (T_\delta - T_{\text{Ext}})\, (T_\delta - k_0)^-$$
$$+ \int_{\Gamma_3} \epsilon\, \sigma\, (|T_\delta|^3\, T_\delta - |T_{\text{Ext}}|^3\, T_{\text{Ext}})\, (T_\delta - k_0)^- + \int_\Sigma G(\sigma\, |T_\delta|^3\, T_\delta)\, (T_\delta - k_0)^- \Big) \leq 0 .$$

Thus, by (59), and since by Lemma 2.6,

$$\int_\Sigma G(\sigma\,|T_\delta|^3\,T_\delta)\,(T_\delta - k_0)^- = \int_\Sigma G(\sigma\,|T_\delta|^3\,T_\delta)\,\big((T_\delta - k_0)^- + k_0\big) \geq 0\,,$$

we get that $(T - k_0)^- =$ constant in Ω, and that $(T - k_0)^- = 0$ on Γ and the claim is proved. □

5 L^1-Estimates

Since for the right-hand side f, we only want to assume the regularity $f \in L^1(\Omega)$, the theory of the precedent section do no longer apply. We have to look for other techniques in order to pass to the limit with approximate solutions. Throughout this section we will assume that $\Sigma \in \mathcal{C}^{1,\alpha}$, and that $\epsilon < 1$ almost everywhere on Σ.

Theorem 5.1. Let $f \in L^1(\Omega)$ and $\tilde{f} \in L^1(\Gamma)$. If (48) is satisfied for the domain Ω, then there exists $T \in V^{s,4}(\Omega)$, $1 \leq s < \frac{3}{2}$ arbitrary, such that $T = T_0$ almost everywhere on Γ_1 and such that

$$\langle A\,T\,,\,\psi\rangle + \int_{\Gamma_3} \epsilon\,\sigma\,|T|^3\,T\,\psi + \int_\Sigma G(\sigma\,|T|^3\,T)\,\psi = \int_\Omega f\,\psi + \int_\Gamma \tilde{f}\,\psi\,,$$

for all $\psi \in W^{1,r}_{\Gamma_1}(\Omega)\,(r > 3)$. In addition, one has the estimate

$$\|\,T\,\|_{W^{1,s}_{\Gamma_1}(\Omega)} + \|\,T\,\|_{L^4(\Gamma_3\cup\Sigma)} \leq P_s\Big(\,\|\,f\,\|_{L^1(\Omega)}\,,\,\|\,\tilde{f}\,\|_{L^1(\Gamma)}\,,\,\|T_0\|_{W^{1,2}(\Omega)}\,,\,\|\,T_0\,\|_{L^4(\Gamma_3)}\,\Big)\,,$$

with a continuous function P_s, for all $1 \leq s < \frac{3}{2}$.

It is easy to construct approximate solutions. Setting $f^{[\delta]} := \text{sign}(f)\,\min\{|f|\,,\,\delta\}$, we find by Theorem 4.1 a $T \in V^{2,5}(\Omega)$ such that $T = T_0$ on Γ_1 and

$$\langle A\,T\,,\,\psi\rangle + \int_{\Gamma_3} \epsilon\,\sigma\,|T|^3\,T\,\psi + \int_\Sigma G(\sigma\,|T|^3\,T)\,\psi = \int_\Omega f^{[\delta]}\,\psi + \int_\Gamma \tilde{f}^{[\delta]}\,\psi\,, \tag{60}$$

for all $\psi \in V^{2,5}_{\Gamma_1}(\Omega)$. We define a sequence of numbers $\{M_\delta\}$ by

$$M_\delta := \frac{1}{\text{meas}(\Sigma)}\,\int_\Sigma |T_\delta|^3\,T_\delta\,. \tag{61}$$

.

Proposition 5.2. Under the assumptions of Theorem 5.1, we have or any sequence of approximate solutions $\{T_\delta\}$ according to (60) the following uniform estimates:

(1) For the temperature on the boundaries Γ, Σ, we have:

$$\|\,T_\delta\,\|_{L^1(\Gamma_2)} + \|\,T_\delta\,\|_{L^4(\Gamma_3)} + \|\,T_\delta^3\,T_\delta\,-\,M_\delta\,\|_{L^1(\Sigma)}$$
$$\leq P\Big(\,\|\,f\,\|_{L^1(\Omega)}\,,\,\|\,\tilde{f}\,\|_{L^1(\Gamma)}\,,\,\|\,T_0\,\|_{L^4(\Gamma_3)}\,\Big) + C_\delta\,,$$

where $C_\delta \to 0$ as $\delta \to 0$.

(2) For all $1 \leq s < \frac{3}{2}$, the temperature gradient is estimated by

$$\| T_\delta \|_{W^{1,s}_{\Gamma_1}(\Omega)} \leq P_s\Big(\| f \|_{L^1} \, , \, \| \tilde{f} \|_{L^1(\Gamma)} \, , \, \|T_0\|_{W^{1,2}(\Omega)} \, , \, \| T_0 \|_{L^4(\Gamma_3)} \Big) + \tilde{C}_\delta \, ,$$

where the sequence $\{\tilde{C}_\delta\}$ converges to zero.

In these estimates P, P_s are continuous functions of the data.

Proof. Again, we prove the propostion for the homogeneous case, and we only indicate how to obtain the general result. In relation (60) we use the test function

$$\psi = \psi_{\gamma,\delta} = \mathrm{sign}(T_\delta) \, \frac{\min\{|T_\delta|, \gamma\}}{\gamma} \, ,$$

where γ is a positive number. Note that $\nabla \psi_{\gamma,\delta} = \frac{1}{\gamma} \nabla T_\delta \, \chi_{\{x \in \Omega \, : \, |T_\delta| < \gamma\}}$. Therefore, we have that $\nabla T_\delta \cdot \nabla \psi_{\gamma,\delta} \geq 0$ almost everywhere in Ω. Since $|\psi_{\gamma,\delta}| \leq 1$ almost everywhere in Ω, we obtain the inequality

$$\int_{\Gamma_2} \alpha \, T_\delta \, \psi_{\gamma,\delta} + \int_{\Gamma_4} \epsilon \, \sigma \, |T_\delta|^3 \, T_\delta \, \psi_{\gamma,\delta} + \int_\Sigma G(\sigma \, |T_\delta|^3 \, T_\delta) \, \psi_{\gamma,\delta} \leq \int_\Omega |f| + \int_\Gamma |\tilde{f}| \, . \tag{62}$$

We see that $\psi_{\gamma,\delta} \to \mathrm{sign}(T_\delta)$ almost everywhere in Ω.
Observe also that $\int_\Sigma G(\sigma \, |T_\delta|^3 \, T_\delta) \, \mathrm{sign}(T_\delta) \geq 0$. Letting γ tend to zero in (62), we obtain from the dominated convergence theorem that

$$\min\{\epsilon_l \, \sigma, \, \alpha\} \, (\|T_\delta\|_{L^1(\Gamma_2)} + \|T_\delta\|^4_{L^4(\Gamma_3)}) \leq \|f\|_{L^1(\Omega)} + \|f\|_{L^1(\Gamma)} \, . \tag{63}$$

Now, we consider the test function

$$\psi_{\gamma,\delta} = \mathrm{sign}(|T_\delta|^3 \, T_\delta - M_\delta) \, \frac{\min\{||T_\delta|^3 \, T_\delta - M_\delta|, \gamma\}}{\gamma} + 1 \, ,$$

where γ is a positive number.
Note that $\nabla \psi_{\gamma,\delta} = \frac{4}{\gamma} \, |T_\delta|^3 \, \nabla T_\delta \, \chi_{\{x \in \Omega \, : \, ||T_\delta|^3 \, T_\delta - M_\delta| < \gamma\}}$ almost everywhere in Ω. Therefore,

$$\int_\Omega \kappa(T_\delta) \, \nabla T_\delta \cdot \nabla \psi_{\gamma,\delta} = \int_\Omega \frac{4 \, |T_\delta|^3}{\gamma} \, \kappa(T_\delta) \, \chi_{\{x \in \Omega \, : \, ||T_\delta|^3 \, T_\delta - M_\delta| < \gamma\}} \, |\nabla T_\delta|^2 \geq 0 \, ,$$

and since $|\psi_{\gamma,\delta}| \leq 2$, we obtain that

$$\int_\Sigma G(\sigma \, |T_\delta|^3 \, T_\delta) \, \left\{ \mathrm{sign}(|T_\delta|^3 \, T_\delta - M_\delta) \, \frac{\min\{||T_\delta|^3 \, T_\delta - M_\delta|, \gamma\}}{\gamma} + 1 \right\}$$
$$\leq c \, (\| \, f \, \|_{L^1(\Omega)} + \|\tilde{f}\|_{L^1(\Gamma)}) \, .$$

Here, we also made use of (63).

Now, since Ω is an enclosure and $G(1) = 0$ almost everywhere on Σ, we can also write

$$\int_\Sigma G\big(\sigma\,(|T_\delta|^3\,T_\delta - M_\delta)\big)\,\mathrm{sign}(|T_\delta|^3\,T_\delta - M_\delta)\,\frac{\min\{||T_\delta|^3\,T_\delta - M_\delta|, \gamma\}}{\gamma}$$

$$\leq c\,(\|\,f\,\|_{L^1} + \|\tilde{f}\|_{L^1(\Gamma)})\,.$$

Letting $\gamma \to 0$ we obtain that

$$\int_\Sigma G(\sigma\,(|T_\delta|^3\,T_\delta - M_\delta))\,\mathrm{sign}(|T_\delta|^3\,T_\delta - M_\delta) \leq c\,(\|\,f\,\|_{L^1(\Omega)} + \|\tilde{f}\|_{L^1(\Gamma)})\,.$$

Lemma 2.5 applies and gives the first estimate.

For $\gamma \in]0, 1[$, we finally use the test function $\psi_{\gamma,\delta} = \mathrm{sign}(T_\delta)\,\left(1 - \frac{1}{(1+T_\delta)^\gamma}\right)$. This leads to

$$\kappa_l\,\gamma \int_\Omega \frac{|\nabla T_\delta|^2}{(1+T_\delta)^{\gamma+1}} \leq (\|\,f\,\|_{L^1(\Omega)} + \|\tilde{f}\|_{L^1(\Gamma)})\,,$$

where we made use of Lemma 2.6 in order to verify that

$$\int_\Sigma G(\sigma\,|T_\delta|^3\,T_\delta)\,\mathrm{sign}(T_\delta)\,\left(1 - \frac{1}{(1+T_\delta)^\gamma}\right) \geq 0\,.$$

Now, using the arguments of [Rak91], we get for $1 \leq s < \frac{3}{2}$ and $\gamma = \frac{3-2\,s}{3-s}$ the estimate

$$\|\,\nabla T_\delta\,\|_{L^s(\Omega)} \leq c_s\,\left(\|\,f\,\|_{L^1(\Omega)} + \|\tilde{f}\|_{L^1(\Gamma)} + (\|\,f\,\|_{L^1(\Omega)} + \|\tilde{f}\|_{L^1(\Gamma)})^{\frac{2\,(3-s)}{s}}\right)\,.$$

□

Proof of Theorem 5.1. From Proposition 5.2, we get for any sequence of approximate solutions $\{T_\delta\}$ according to (60) the existence of a subsequence such that

$$T_\delta \rightharpoonup T \text{ in } W^{1,s}(\Omega)\,,\quad T_\delta \to T \text{ in } L^{\tilde{s}}(\partial\Omega)\,,\quad T_\delta \to T \text{ in } L^{s^*}(\Omega)\,,$$

$$T_\delta \to T \text{ almost everywhere in } \Omega \text{ and on } \Sigma\,,$$

with $1 \leq s < \frac{3}{2}$, $1 \leq \tilde{s} < 2$, and $1 \leq s^* < 3$ arbitrary.

The difficult point is the passage to the limit in the nonlocal boundary terms. For the sake of clarity, we prove the theorem in the case that $\Gamma_2 = \emptyset = \Gamma_3$, i. e. $\Gamma = \Gamma_1$. The general result is proved by the same method. Starting from Proposition 5.2, we can write, by Fatou's lemma,

$$C \geq \liminf_{\delta\to 0}\,\|\,|T_\delta|^3\,T_\delta - M_\delta\,\|_{L^1(\Sigma)} \geq \int_\Sigma \liminf_{\delta\to 0}\left|T_\delta|^3\,T_\delta - M_\delta\right|\,. \tag{64}$$

Now, suppose that there exists a subsequence $|M_\delta| \to \infty$. Then, for this subsequence, we have almost everywhere on Σ that

$$\liminf_{\delta\to 0}\left||T_\delta|^3\,T_\delta - M_\delta\right| = \lim_{\delta\to 0}\left||T_\delta|^3\,T_\delta - M_\delta\right| = \lim_{\delta\to 0}\left||T|^3\,T - M_\delta\right| = +\infty\,,$$

since the pointwise limes T must be finite almost everywhere on the boundary. This contradicts (64). Thus, the whole sequence $\{M_\delta\}$ must be bounded by some constant, and we have, by the definition (61), that $\| T_\delta \|_{L^4(\Sigma)} \leq C$.

Now, in view of Lemma 1.10, we write $G(\sigma\,|T_\delta|^3\,T_\delta) = \epsilon\,\sigma\,\big(|T_\delta|^3\,T_\delta - \tilde{H}(|T_\delta|^3\,T_\delta)\big)$. Considering χ_A, the characteristic function of an arbitrary measurable subset $A \subseteq \Sigma$, we can write

$$\int_\Sigma \Big|\tilde{H}(|T_\delta|^3\,T_\delta)\Big|\,\chi_A \leq \int_\Sigma \tilde{H}(|T_\delta|^4)\,\chi_A = \int_\Sigma \big[(I-(1-\epsilon)K)^{-1}\,\epsilon\big](|T_\delta|^4)\;K(\chi_A)$$
$$\leq c\,\|\,T_\delta^4\,\|_{L^1}\,\max_\Sigma K(\chi_A) \leq C\,\max_\Sigma K(\chi_A)\,.$$

If we now assume that $\text{meas}(A) \to 0$, that is $\chi_A \longrightarrow 0$ in $L^q(\Sigma)$ for $q < \infty$ arbitrary, then by Lemma 1.9, we obtain that $K(\chi_A) \to 0$ in $L^\infty(\Sigma)$. This yields

$$\sup_{\delta\in\mathbb{R}} \int_\Sigma \Big|\tilde{H}(|T_\delta|^3\,T_\delta)\Big|\,\chi_A \to 0 \qquad \text{as meas}(A) \to 0\,.$$

Thus, the sequence $\{\tilde{H}(|T_\delta|^3\,T_\delta)\}$ is equi-integrable, and therefore weakly compact in $L^1(\Sigma)$. We now find $u \in L^1(\Sigma)$ and a subsequence such that

$$\tilde{H}(|T_\delta|^3\,T_\delta) \rightharpoonup u \text{ in } L^1(\Sigma)\,. \tag{65}$$

Passing to the limit in (60), we obtain, for all $\psi \in W^{1,r}_{\Gamma_1}(\Omega)$, $r > 3$ arbitrary, that

$$\int_\Omega \kappa(T)\,\nabla T\cdot\nabla\psi + \int_{\Gamma_2} \alpha\,T\,\psi + \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,|T_\delta|^3\,T_\delta\,\psi - \int_\Sigma \epsilon\,\sigma\,u\,\psi = \int_\Omega f\,\psi + \int_\Gamma \tilde{f}\,\psi\,. \tag{66}$$

We now want to compute $\lim_{\delta\to 0}\int_\Sigma G(\sigma\,|T_\delta|^3\,T_\delta)\,\psi$. For $t \in \mathbb{R}$ and $\gamma > 0$, we introduce the function

$$g_\gamma(t) := \begin{cases} 1 & \text{if } t < 0\,,\\ \frac{1}{1+\gamma\,t^4} & \text{if } t \geq 0\,. \end{cases}$$

For an arbitrary $\psi \in C^\infty(\overline{\Omega})$, such that $\psi \geq 0$ in Ω, and $\psi = 0$ on Γ, we use in (60) the test function $g_\gamma(T_\delta)\,\psi$. We obtain

$$\int_\Omega \kappa(T_\delta)\,\nabla T_\delta\cdot\nabla\psi\,g_\gamma(T_\delta) + \int_\Sigma G(\sigma\,|T_\delta|^3\,T_\delta)\,g_\gamma(T_\delta)\,\psi + R_{\gamma,\delta}$$
$$= \int_\Omega f^{[\delta]}\,\psi\,g_\gamma(T_\delta) + \int_\Gamma \tilde{f}^{[\delta]}\,\psi\,g_\gamma(T_\delta)\,,$$

with the notation $R_{\gamma,\delta} := \int_\Omega \kappa(T_\delta)\,|\nabla T_\delta|^2\,g'_\gamma(T_\delta)\,\psi$. Since for each $\gamma > 0$, the function g_γ is monotonely decreasing, we have that $R_{\gamma,\delta} \leq 0$. This gives that

$$\int_\Omega \kappa(T_\delta)\,\nabla T_\delta\cdot\nabla\psi\,g_\gamma(T_\delta) + \int_\Sigma G(\sigma\,|T_\delta|^3\,T_\delta)\,\psi\,g_\gamma(T_\delta) \geq \int_\Omega f^{[\delta]}\,\psi\,g_\gamma(T_\delta) + \int_\Gamma \tilde{f}^{[\delta]}\,\psi\,g_\gamma(T_\delta)\,. \tag{67}$$

We can write

$$G(\sigma\,|T_\delta|^3\,T_\delta)\,g_\gamma(T_\delta) = \Big(\epsilon\,\sigma\,|T_\delta|^3\,T_\delta - \epsilon\,\sigma\,\tilde H\big(|T_\delta|^3\,T_\delta\big)\Big)\,g_\gamma(T_\delta)$$
$$= \frac{\epsilon\,\sigma\,T_\delta^{+^4}}{1+\gamma\,T_\delta^{+^4}} + \epsilon\,\sigma\,|T_\delta|^3\,T_\delta^- - \epsilon\,\sigma\,\tilde H\big(|T_\delta|^3\,T_\delta\big)\,g_\gamma(T_\delta)$$

For fixed γ, g_γ is contiunous and bounded. Using the dominated convergence theorem and Lemma 5.3 at the end of this proof, we can take the limit $\delta \to 0$ in (67) and obtain that

$$\int_\Omega \kappa(T)\,\nabla T\cdot\nabla\psi\,g_\gamma(T) + \int_\Sigma \sigma\,\epsilon\,\frac{T^{+^4}}{1+\gamma\,T^{+^4}}\,\psi + \lim_{\delta\to 0}\int_\Sigma \epsilon\,\sigma\,|T_\delta|^3\,T_\delta^-\,\psi - \int_\Sigma \sigma\,\epsilon\,u\,\psi\,g_\gamma(T)$$
$$\geq \int_\Omega f\,\psi\,g_\gamma(T) + \int_\Gamma \tilde f\,\psi\,g_\gamma(T)\,.$$

Letting now $\gamma \to 0$ and observing that $g_\gamma \nearrow 1$, we find that

$$\int_\Omega \kappa(T)\,\nabla T\cdot\nabla\psi + \int_\Sigma \sigma\,\epsilon\,T^{+^4}\,\psi + \lim_{\delta\to 0}\int_\Sigma \epsilon\,\sigma\,|T_\delta|^3\,T_\delta^-\,\psi - \int_\Sigma \sigma\,\epsilon\,u\,\psi \geq \int_\Omega f\,\psi + \int_\Gamma \tilde f\,\psi\,. \tag{68}$$

Recalling our choice of ψ, we compare (66) and (68) to find that

$$\int_\Sigma \sigma\,\epsilon\,T^{+^4}\,\psi \geq \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,T_\delta^{+^4}\,\psi\,.$$

for all $\psi \in C^\infty(\overline{\Omega})$ such that $\psi \geq 0$ in Ω and $\psi = 0$ on Γ. Fatou's lemma gives for such ψ that even

$$\int_\Sigma \sigma\,\epsilon\,T^{+^4}\,\psi = \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,T_\delta^{+^4}\,\psi\,. \tag{69}$$

In order to study the convergence of the negative part, we can for $\gamma > 0$ consider the functions

$$\hat g_\gamma(t) := \begin{cases} \frac{-1}{1+\gamma\,t^4} & \text{for } t \leq 0\,,\\ -1 & \text{for } t > 0\,. \end{cases}$$

Using the test function $\hat g_\gamma(T_\delta)\,\psi$ for $\psi \in C^\infty(\overline{\Omega})$ such that $\psi \geq 0$ in Ω and $\psi = 0$ on Γ, we obtain in a similar manner that $\int_\Sigma \sigma\,\epsilon\,T^{-^4}\,\psi \geq \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,T_\delta^{-^4}\,\psi$, which implies that

$$\int_\Sigma \sigma\,\epsilon\,T^{-^4}\,\psi = \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,T_\delta^{-^4}\,\psi\,. \tag{70}$$

In view of (69) and (70), we obtain that $\int_\Sigma \sigma\,\epsilon\,T^4\,\psi = \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,T_\delta^4\,\psi$. Because of (48), we can, in particular, choose $\psi \equiv 1$ on Σ, which yields $\int_\Sigma \sigma\,\epsilon\,T^4 = \lim_{\delta\to 0}\int_\Sigma \sigma\,\epsilon\,T_\delta^4$. In view of Lemma 5.4 at the end of this proof, this suffices to establish the strong convergence

$$T_\delta \to T \text{ in } L^4(\Sigma)\,. \tag{71}$$

As a matter of consequence, we now have $u = \tilde{H}\big(|T|^3\,T\big)$. Coming back to (66) with this knowledge, we find that

$$\int_\Omega \kappa(T)\,\nabla T\cdot\nabla\psi + \int_\Sigma G(\sigma\,|T|^3\,T)\,\psi = \int_\Omega f\,\psi + \int_\Gamma \tilde{f}\,\psi\,,$$

proving the integral relation. □

The two following Lemmas are proved in [GMS98].

Lemma 5.3. Let $a_k,\ a \in L^\infty(\Omega)$ such that $\|a_k\|_{L^\infty(\Omega)} \leq A$ for all $k\in\mathbb{N}$. Let $b_k,\ b\in L^1(\Omega)$. Suppose that $a_k \to a$ almost everywhere and that $b_k \rightharpoonup b$ in $L^1(\Omega)$. Then, $a_k\,b_k \rightharpoonup a\,b$ in $L^1(\Omega)$.

Lemma 5.4. Let $u_k,\ u\in L^1(\Omega)$ be such that $u_k \to u$ almost everywhere and $\|u_k\|_{L^1(\Omega)} \to \|u\|_{L^1(\Omega)}$. Then $u_k \to u$ strongly in $L^1(\Omega)$.

6 Concluding Remarks

In the two main theorems 4.1 and 5.1 of the paper, we have presented new results on the weak solvability of the stationary heat conduction-radiation problem. The practically relevant case of L^p- heat source densities, with p close to one is covered by the theory. Continuous estimates in terms of the data are obtained in each case for the temperature gradient, and for the total emitted heat radiation on the surface Σ. The estimates of the fifth section only involve the term $\|f\|_{L^1(\Omega)}$. They are especially attractive, since the total heating power is the quantity that is actually controlled in industrial applications.

The proof of these theorems relies on coercivity properties of the nonlocal radiation operators that had not been stated before (in particular Lemma 2.4, 2.5 and 2.6) and have been derived in the first two sections.

Throughout the paper, the regularity of the surface Σ has also been an issue. Theorem 4.1 shows that the existence of weak solutions can be proved in the case of general Lipschitz boundaries, which is a small improvement on previous results. However, if the heat sources are in L^1, we cannot prove existence if the surface Σ is less than $\mathcal{C}^{1,\alpha}$. In the case of interfaces that are only piecewise smooth, the smoothing properties of the operator K are much more difficult to establish, so that a further publication would be necessary to discuss that case.

Finally, note that the regularity of the solution has not been at discussion in the paper. In the standard case of say a L^2-right-hand side, further regularity results, such as boundedness and continuity of weak solutions are known (see [LT01], [Mey06]), which we have not recalled here. Thus, it should be emphasized that Theorem 4.1 does not state optimal results concerning regularity. On the contrary, the integrability $s < 3/2$ stated for the temperature gradient in Theorem 5.1 is known to constitute an upper bound for the regularity of elliptic problems with L^1- right-hand sides ([BG92], [Rak91]), and is therefore optimal.

The question of the uniqueness of the weak solution in the case that $f \notin [W^{1,2}(\Omega)]^*$ is closely related to the regularity issue, and is still open to discussion.

References

[Alt85] H.-W. Alt. *Lineare Funktionalanalysis.* Springer, Berlin, Heidelberg, 1985.

[BG92] L. Boccardo and T. Gallouët. Nonlinear elliptic equations with right-hand side measures. *Commun. Partial Diff. Equs.*, 17:641–655, 1992.

[Dru08] P.-E. Druet. Weak solutions to a stationary heat equation with nonlocal radiation boundary condition and right-hand side in L^p ($p \geq 1$). To appear in Math. Meth. Appl. Sci., `http://dx.doi.org/10.1002/mma.1029`, 2008.

[GMS98] M. Giaquinta, L. Modica, and J. Souček. *Cartesian Currents in the Calculus of Variations I.* Springer, Berlin, Heidelberg, 1998.

[KP05] O. Klein and P. Philip. Transient conductive-radiative heat transfer: Discrete existence and uniqueness for a finite volume scheme. *Math. Mod. Meth. Appl. Sci.*, 15(2):227–258, 2005.

[KPS04] O. Klein, P. Philip, and J. Sprekels. Modelling and simulation of sublimation growth in sic bulk single crystals. *Interfaces and Free Boundaries*, 6(1):295–314, 2004.

[Lio69] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris, 1969.

[LT98] M. Laitinen and T. Tiihonen. Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials. *Math. Meth. Appl. Sci.*, 21(4):375–392, 1998.

[LT01] M. Laitinen and T. Tiihonen. Conductive-radiative heat transfer in grey materials. *Quart. Appl. Math.*, 59(4):737–768, 2001.

[Mey06] C. Meyer. *Optimal Control of Semilinear Elliptic Equations with Applications to Sublimation Crystal Growth.* PhD thesis, Technische-Universität Berlin, Germany, 2006.

[MPT06] C. Meyer, P. Philip, and F. Tröltzsch. Optimal control of a semilinear pde with nonlocal radiation interface conditions. *SIAM Journal On Control and Optimization (SICON)*, 45:699–721, 2006.

[Phi03] P. Philip. *Transient Numerical Simulation of Sublimation Growth of SiC Bulk Single Crystals. Modeling, Finite Volume Method, Results.* PhD thesis, Department of Mathematics, Humboldt University of Berlin, Germany, 2003. Report No. 22, Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Berlin.

[Rak91] J.-M. Rakotoson. Quasilinear elliptic problems with measures as data. *Diff. Integral Eqs.*, 4(1):449–457, 1991.

[Tii97a] T. Tiihonen. A nonlocal problem arising from heat radiation on non-convex surfaces. *Eur. J. App. Math.*, 8(4):403–416, 1997.

[Tii97b] T. Tiihonen. Stefan-boltzmann radiation on non-convex surfaces. *Math. Meth. in Appl. Sci.*, 20(1):47–57, 1997.

[Voi01] A. Voigt. *Numerical Simulation of Industrial Crystal Growth.* PhD thesis, Technische-Universität München, Germany, 2001.