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ON MODERATE DEVIATIONS FOR MARTINGALES

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ABSTRACT. Let $X^n = (X_t^n, \mathcal{F}_t^n)_{0\le t\le 1}$ be the square integrable martingales with the quadratic characteristics $\langle X^n\rangle$, $n = 1, 2, \dots$. We have proved that the large deviation relation $P(X_1^n \ge r)/(1-\Phi(r)) \to 1$ is valid with r growing to infinity at some rate depending on $L_{2\delta}^n = E\sum_{0\le t\le 1} |\Delta X_t^n|^{2+2\delta}$ and $N_{2\delta}^n = E|\langle X^n\rangle_1 - 1|^{1+\delta}$, where $\delta > 0$ and $L_{2\delta}^n \to 0$, $N_{2\delta}^n \to 0$ as $n \to \infty$. The exact bound for the remainder is obtained too.

1. Introduction

Suppose we are given the triangular array of square integrable martingales

$$X^n = (X_k^n, \mathcal{F}_k^n)_{0\le k\le n}, \quad X_0^n = 0 \quad \text{a.s.}, \qquad n = 1, 2, \dots.$$

Denote $\xi_k^n = X_k^n - X_{k-1}^n$ and

$$\langle X^n\rangle_k = \sum_{0<i\le k} E((\xi_i^n)^2 | \mathcal{F}_{i-1}^n),$$

where $k = 1, \dots, n$ and $n = 1, 2, \dots$.

The celebrated Central Limit Theorem (CLT) for martingales gives us conditions for the weak convergence of the distributions $P(X_n^n \le x)$ to the standard normal distribution $\Phi(x)$ in terms of the asymptotic negligibility of r.v. ξ_k^n, $k = 1, \dots, n$ and $\langle X^n\rangle_n - 1$. The exact bounds for the departure from normality of $P(X_n^n \le x)$ under such type of conditions were obtained by many authors among them Brown and Heyde (1970), Liptser and Shiryaev (1982), (1989), Bolthausen (1982), Haeusler (1988), Haeusler and Joos (1988), Kubilius (1990), Grama (1988a), (1988b), (1990), (1993). We particularly point out the results of

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Haeusler (1988) and Haeusler and Joos (1988), where exact bounds of the rate of convergence are obtained under the assumption that for some $\delta > 0$

$$
\begin{aligned}
(1.1) \qquad & L^n_{2\delta} = E \sum_{0<i\leq n} |\xi^n_i|^{2+2\delta} \to 0, \\
& N^n_{2\delta} = E|\langle X^n\rangle_n - 1|^{1+\delta} \to 0,
\end{aligned}
$$

as $n \to \infty$, which clearly imply the conditions of the CLT for martingales. This yields that the relation

$$
(1.2) \qquad P(X^n_n \geq r) = (1 - \Phi(r))\{1 + o(1)\}
$$

holds true uniformly in r only in the range $0 \leq r \leq C$, C being some constant not depending on n, although the stronger assumptions (1.1) allow us to obtain (1.2) in a growing range as n goes to ∞ i.e. to prove moderate deviation results for martingales. The case of sums of independent r.v. is studied in Rubin and Sethuraman (1965), Amosova (1972) (see also Petrov (1972) p. 309), but until recently, however, this problem for martingales has not been properly settled. It should be pointed out that some moderate deviation results for martingales were also obtained in Bose (1986a), (1986b). These results are under rather stringent assumptions on the prelimiting martingales this making comparison with ours a difficult task. In any case they do not provide us with the optimal rate and do not allow us to manage the general case considered here.

The aim of the paper is to prove a moderate deviation relation for martingales in the normal zone that is to prove the relation (1.2) uniformly on r in a possibly wider range the only assumptions on prelimiting martingales being (1.1). Exact bounds for the remainder will be given as well.

Main results of the paper obtained in this direction are presented (for continuous time martingales) in the next section (see Theorems 2.1, 2.2, 2.3, 2.4). Let us write down some of these results in the discrete case under consideration.

Assume that x is such that $1 \leq x \leq \alpha(L^n_{2\delta} + N^n_{2\delta})^{-1}$, where $\alpha > 0$. Then by virtue of Theorem 2.1 and Remark 2.1 we have uniformly on x

$$
(1.3) \qquad P(X^n_n \geq r) = (1 - \Phi(r))\{1 + \theta C(\alpha,\delta) x^{1/(3+2\delta)} (L^n_{2\delta} + N^n_{2\delta})^{1/(3+2\delta)}\},
$$

where $|\theta| \leq 1$, $C(\alpha,\delta)$ is a constant depending only on α and δ and

$$
(1.4) \qquad r^2 = 2\ln x - \theta_1 2c(\delta)\ln(1 + \sqrt{2\ln x}\,)
$$

with $0 \leq \theta_1 \leq 1$, $c(\delta) = 3 + 6\delta$.

The first term in the above expansion for r^2 is exact. Unfortunately the constant $c(\delta) = 3 + 6\delta$ in (1.4) is not the best one. We conjecture that the best possible value for $c(\delta)$ is $3 + 2\delta$ but our method of the proof does not allow us to reach it.

The remainder in (1.3) is the best one since with $x = 1$ we get exactly the rate of convergence in the CLT for martingales (see Lemma 3.4 below).

In particular the above relations imply that for any $0 < q < 1$ and x subject to $1 < x \leq \alpha(L^n_{2\delta} + N^n_{2\delta})^{-1}$ the following relation

$$P(X^n_n \geq \sqrt{2q \ln x}\,) = \frac{1}{\sqrt{2\pi}\, x^q \sqrt{2 \ln x}} \{1 + \theta C(\alpha, \delta, q) \frac{1}{\ln x}\}, \tag{1.5}$$

holds, where $|\theta| \leq 1$, $C(\alpha, \delta, q)$ is a constant depending only on α, δ and q. For the case of independent r.v. (1.5) improve the result of Amosova (1972) from which the relation (1.5) turns out to be exact too.

Relations (1.3), (1.5) allow us to derive limit theorems on moderate deviation for martingales. For instance it follows from (1.5) that if $L^n_{2\delta} + N^n_{2\delta} < 1$ then for any $0 < q < 1$ uniformly in r subject to $0 \leq r \leq \sqrt{2q|\ln(L^n_{2\delta} + N^n_{2\delta})|}$ we have

$$\frac{P(X^n_n \geq r)}{(1 - \Phi(r))} \to 1$$

as $n \to \infty$.

Let us examine the case when $\xi^n_k = \frac{1}{\sqrt{n}} \eta_k$, $k = 1, \ldots, n$, where r.v. $\eta_1, \ldots, \eta_n$ form an i.i.d. sequence of r.v. with $E\eta_1 = 0$, $E\eta_1^2 = 1$ and $m_{2\delta} = E|\eta_1|^{2+2\delta} < \infty$. In this case $N^n_{2\delta} = 0$ and $L^n_{2\delta} = n^{-\delta} m_{2\delta}$. What we can get from (1.3) and (1.5) is the following results. Uniformly on x in the range $1 \leq x \leq \alpha n^\delta$ we have

$$P(\frac{1}{\sqrt{n}} \sum_{0<i\leq n} \eta_i \geq r) = (1 - \Phi(r))\{1 + \theta C(\alpha, \delta, m_{2\delta}) x^{1/(3+2\delta)} (\frac{1}{\sqrt{n}})^{2\delta/(3+2\delta)}\},$$

where r is defined by (1.4), and with $0 < q < 1$

$$P(\frac{1}{\sqrt{n}} \sum_{0<i\leq n} \eta_i \geq \sqrt{2q\delta \ln n}\,) = \frac{1}{\sqrt{2\pi}\, n^{q\delta} \sqrt{2\delta \ln n}} \{1 + \theta C(\delta, q, m_{2\delta}) \frac{1}{\ln n}\},$$

$C(\alpha, \delta, m_{2\delta})$, $C(\delta, q, m_{2\delta})$ being constants depending on α, δ, q, $m_{2\delta}$ respectively.

We are going to pay some attention to the methods of the proof and to the related works now.

For the proofs we make use of the composition method which originally goes back to Bergstrom (1944). It was developed for the discrete time martingales by Bolthausen (1982) and Haeusler (1988) to get rates of convergence in the CLT. For the case of continuous time semimartingales the composition method was extended in Grama (1988a), (1988b). This method turns out to be useful for obtaining large and moderate deviation results for martingales too. Roughly speaking the main idea behind the technique we propose is as follows. Consider the two-dimensional semimartingale $(X^n_k, 1 - V^n_k)$. Here V^n is an increasing process on k, $V^n_0 = 0$, $V^n_n =$

1., We apply Ito's formula in order to give an expansion for $\Phi(f, X_k^n, 1 - V_k^n) - \Phi(f, X_0^n, 1)$, where $\Phi(f, x, y) = \int_{-\infty}^{\infty} f(x + z\sqrt{y})\varphi(z)\, dz$ with a smooth function f. This approach leads us to some Gronwall-Bellman type inequalities. We should also point out that the present proof employs stopping time technique rather than smoothing inequalities. The reason to proceed in such a way is that it gives us a better result for this method.

A similar approach was used in Grama (1994) to get large deviation results with the bounds for the remainder. Closely related papers belong to Bentkus (1986), Bentkus and Rackauskas (1990) (both deal with Banach space valued independent r.v.) Rackauskas (1990) (for 1-dimensional martingales), where large deviation results were established in the discrete case. Under the general conditions the exponential type inequalities for large deviation probabilities for semimartingales were proved in the book of Liptser and Shiryaev (1989). For large deviation results for independent r.v. we refer the reader to the books of Ibragimov and Linnik (1965), Petrov (1972), Saulis and Statulevicius (1989).

2. The results

We begin this section by settling some notations which we make use all over the paper. Throughout the paper $\Phi(x)$ denotes the distribution function of the standard normal r.v. N. Let C, C_i, $i = 1, 2, \dots$ be the absolute constants and $C_i(\alpha, \beta, \dots)$, $i = 1, 2, \dots$ be the constants depending only on the arguments $\alpha, \beta, \dots$, whose values may differ from place to place.

Suppose that on the probability space $(\Omega, \mathcal{F}, P)$ we are given the square integrable martingale

$$X = (X_t, \mathcal{F}_t)_{0 \le t \le 1}, \qquad X_0 = 0 \quad a.s.,$$

under the usual conditions. Corresponding to the martingale X is the quadratic characteristic

$$\langle X \rangle = (\langle X \rangle_t, \mathcal{F}_t)_{0 \le t \le 1}.$$

Let us introduce the following notations

$$L_{2\delta} = E \sum_{0 < s \le 1} |\Delta X_s|^{2+2\delta},$$

$$N_{2\delta} = E|\langle X \rangle_1 - 1|^{1+\delta},$$

where $\delta > 0$. Of course if we want to obtain non-trivial results we have to assume that both $L_{2\delta}$ and $N_{2\delta}$ are finite for some $\delta > 0$.

Our main result concerning the moderate deviation for martingales is formulated as

Theorem 2.1. *Assume that x, r are such that $r \geq 0$, $x = (1+r)^{c(\delta)} e^{r^2/2}$, $x \leq \alpha(L_{2\delta} + N_{2\delta})^{-1}$, where $c(\delta) = 3 + 6\delta$, $\alpha > 0$. Then the following moderate deviation relations*

$$
\begin{aligned}
P(X_1 \geq r) &= (1 - \Phi(r))\{1 + \theta C(\alpha, \delta) x^{1/(3+2\delta)} (L_{2\delta} + N_{2\delta})^{1/(3+2\delta)}\}, \\
P(X_1 \leq -r) &= \Phi(-r)\{1 + \theta C(\alpha, \delta) x^{1/(3+2\delta)} (L_{2\delta} + N_{2\delta})^{1/(3+2\delta)}\}
\end{aligned} \tag{2.1}
$$

hold, where $|\theta| \leq 1$.

Remark 2.1. *Let us observe that if r and x are such that $r \geq 0$ and $x = (1+r)e^{c(\delta)} e^{r^2/2}$, then*

$$
r = \sqrt{2 \ln x - \theta_1 2c(\delta) \ln(1 + \sqrt{2 \ln x}\,)}, \tag{2.2}
$$

where $0 \leq \theta_1 \leq 1$. With this relation we reformulate Theorem 2.1 in the following form. Relations (2.1) hold true for any x in the range $1 \leq x \leq \alpha(L_{2\delta} + N_{2\delta})^{-1}$, where r satisfies (2.2).

We easily derive from Theorem 2.1 the following

Theorem 2.2. *Assume that x is such that $1 < x \leq \alpha(L_{2\delta} + N_{2\delta})^{-1}$, where $\alpha > 0$. Then for any $0 < q < 1$ the following moderate deviation relations*

$$
\begin{aligned}
P(X_1 \geq \sqrt{2q \ln x}\,) &= \frac{1}{\sqrt{2\pi}\, x^q \sqrt{2 \ln x}} \{1 + \theta C(\alpha, \delta, q) \frac{1}{\ln x}\}, \\
P(X_1 \leq -\sqrt{2q \ln x}\,) &= \frac{1}{\sqrt{2\pi}\, x^q \sqrt{2 \ln x}} \{1 + \theta C(\alpha, \delta, q) \frac{1}{\ln x}\}
\end{aligned}
$$

hold, where $|\theta| \leq 1$.

Remark 2.2. *In particular if $\varepsilon = L_{2\delta} + N_{2\delta} < 1$ then for any $0 < q < 1$ the following relations*

$$
\begin{aligned}
P(X_1 \geq \sqrt{2q |\ln \varepsilon|}\,) &= \frac{\varepsilon^q}{\sqrt{2\pi}\, \sqrt{2 |\ln \varepsilon|}} \{1 + \theta C(\delta, q) \frac{1}{|\ln \varepsilon|}\}, \\
P(X_1 \leq -\sqrt{2q |\ln \varepsilon|}\,) &= \frac{\varepsilon^q}{\sqrt{2\pi}\, \sqrt{2 |\ln \varepsilon|}} \{1 + \theta C(\delta, q) \frac{1}{|\ln \varepsilon|}\}
\end{aligned}
$$

hold, where $|\theta| \leq 1$.

The above statements allow us to formulate some new limit theorems on moderate deviation for martingales.

Theorem 2.3. *Let $X^n = (X_t^n, \mathcal{F}_t^n)_{0\le t\le 1}$, $X_0^n = 0$ a.s., be the square integrable martingales under the usual conditions with quadratic characteristics $\langle X^n\rangle = (\langle X^n\rangle_t, \mathcal{F}_t^n)_{0\le t\le 1}$ respectively. Denote*

$$L_{2\delta}^n = E \sum_{0<s\le 1} |\Delta X_s^n|^{2+2\delta},$$
$$N_{2\delta}^n = E|\langle X^n\rangle_1 - 1|^{1+\delta},$$

where $\delta > 0$. Assume that x, r are such that $r \ge 0$, $x = (1+r)^{c(\delta)}e^{r^2/2}$, with $c(\delta) = 3 + 6\delta$. Then uniformly in r such that $x = o((L_{2\delta}^n + N_{2\delta}^n)^{-1})$ the following moderate deviation relations

$$\frac{P(X_1^n \ge r)}{(1-\Phi(r))} \to 1,$$
$$\frac{P(X_1^n \le -r)}{\Phi(-r)} \to 1$$

hold as $n \to \infty$.

Theorem 2.4. *With the notations of Theorem 2.3 if $L_{2\delta}^n + N_{2\delta}^n < 1$ for any $0 < q < 1$, then uniformly in r subject to $0 \le r \le \sqrt{2q|\ln(L_{2\delta}^n + N_{2\delta}^n)|}$ the following moderate deviation relations*

$$\frac{P(X_1^n \ge r)}{(1-\Phi(r))} \to 1,$$
$$\frac{P(X_1^n \le -r)}{\Phi(-r)} \to 1$$

hold as $n \to \infty$.

3. Preliminary statements

Before to proceed with the proofs let us state some background assertions to be used latter.

The following lemma is almost obvious modification of the time change formula in Dellacherie (1972) and is related to Lemma 3.1 in Grama (1994).

Lemma 3.1. *Let $A = (A_s)_{0\le s\le 1}$, $A_0 = 0$, $A_1 = T$ be the right continuous increasing function, where $T > 0$. For any $s \in [0,T]$ denote*

$$\tau_s = \inf\{0 \le t \le 1 : A_t > s\}, \qquad \text{where} \qquad \inf\emptyset = 1.$$

Then for any $0 \le t \le T$ and any non-negative real measurable function $f = (f(u))_{0\le u\le 1}$

$$\int_0^{\tau_t} f(s)\,dA_s \le \int_0^t f(\tau_s)\,ds + f(\tau_t)\Delta A_{\tau_t}.$$

Proof. It is obvious that

$$\int_0^{\tau_t} f(s)\,dA_s = \int_0^1 1(s < \tau_t) f(s)\,dA_s + f(\tau_t)\Delta A_{\tau_t}. \tag{3.1}$$

By applying the change time formula (see Dellacherie (1972))

$$\int_0^1 1(s < \tau_t) f(s)\,dA_s = \int_0^{A_1} 1(\tau_s < \tau_t) f(\tau_s)\,ds$$

(since $\tau_s < \tau_t$ implies $s < t$)

$$\leq \int_0^T 1(s < t) f(\tau_s)\,ds.$$

This concludes the proof.

The following two elementary formulas are well known.

Lemma 3.2. *For any $r \geq 0$ and $\varepsilon \geq 0$*

(a) $$\frac{\sqrt{2/\pi}}{1+r} e^{-r^2/2} \leq P(|N| \geq r) \leq \frac{4}{3} \frac{\sqrt{2/\pi}}{1+r} e^{-r^2/2},$$

(b) $$P(r - \varepsilon \leq |N| \leq r + \varepsilon) \leq C\varepsilon(1+r)(1-\Phi(r))e^{\varepsilon r}.$$

We shall need in what follows well-known Gronwall-Bellman inequality.

Lemma 3.3. *Assume that function $g = (g_t)_{0 \leq t \leq T}$, $T \geq 0$ is bounded by a constant not depending on t and satisfies for any $t \in [0, T]$ the inequality*

$$g_t \leq C_1 \int_0^t g_s a_s\,ds + C_2,$$

where $a = (a_t)_{0 \leq t \leq T}$ is nonnegative integrable function. Then for any $t \in [0, T]$

$$g_t \leq C_2 \exp\{C_1 \int_0^t a_s\,ds\}.$$

We shall make use of the following exact estimate in CLT for continuous time martingales due to Haeusler (1988) (see also Haeusler and Joos (1988)).

Lemma 3.4. *Let $X = (X_t, \mathcal{F}_t)_{0 \leq t \leq 1}$, $X_0 = 0$ a.s. be the square integrable martingale under the usual conditions with quadratic characteristic $\langle X \rangle = (\langle X \rangle_t, \mathcal{F}_t)_{0 \leq t \leq 1}$. Then for any $\delta > 0$ the bound holds*

$$\sup_{x \in R^1} |P(X_1 \leq x) - \Phi(x)| \leq C(\delta)(L_{2\delta} + N_{2\delta})^{\frac{1}{3+2\delta}}.$$

Lemma 3.5. *Let X, ξ be random variables and X be $\mathcal{G}$-measurable, where $\mathcal{G} \subseteq \mathcal{F}$. Then for any $\varepsilon \geq 0$*

$$\sup_{x\in R^1} |P(X \leq x) - \Phi(x)| \leq 2 \sup_{x\in R^1} |P(X + \xi \leq x) - \Phi(x)| + \frac{5}{\sqrt{2\pi}}\varepsilon + 2P(E(\xi^2|\mathcal{G}) > \varepsilon^2).$$

Proof. This assertion is a small improvement of Lemma 1 of Bolthausen (1982) or Lemma 2 of Haeusler and Joos (1988) and therefore the proof is left it to the reader.

Throughout the rest of the paper we shall be using the notations that we proceed to introduce. Let $\varphi(x)$ be the standard normal density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Given any bounded function $f : R^1 \to R^1$ denote

$$\Phi(f, x, y) = \int_{-\infty}^{\infty} f(x + z\sqrt{y})\varphi(z)\, dz.$$

For any Borel set G in R^1 put $\Phi(G, x, y) = \Phi(1_G, x, y)$, where 1_G is the indicator of the set G. By straightforward calculations we state for any bounded function f, having four bounded derivatives, and for any $x, y \in R^1$, $y \geq 0$ the equalities

$$\frac{\partial^2}{\partial x^2}\Phi(f, x, y) = 2\frac{\partial}{\partial y}\Phi(f, x, y) = \int_{-\infty}^{\infty} f''(x + z\sqrt{y})\varphi\, dz, \tag{3.1}$$

$$\frac{\partial^3}{\partial x^3}\Phi(f, x, y) = \int_{-\infty}^{\infty} f'''(x + z\sqrt{y})\varphi(z)\, dz, \tag{3.2}$$

$$\frac{\partial^2}{\partial y^2}\Phi(f, x, y) = \frac{1}{4}\int_{-\infty}^{\infty} f''''(x + z\sqrt{y})\varphi(z)\, dz \tag{3.3}$$

and if $y > 0$

$$\frac{\partial^2}{\partial x^2}\Phi(f, x, y) = \frac{1}{y}\int_{-\infty}^{\infty} f(x + z\sqrt{y})\varphi(z)(z^2 - 1)\, dz, \tag{3.4}$$

$$\frac{\partial^3}{\partial x^3}\Phi(f, x, y) = \frac{1}{y}\int_{-\infty}^{\infty} f'(x + z\sqrt{y})\varphi(z)(z^2 - 1)\, dz, \tag{3.5}$$

$$\frac{\partial^2}{\partial y^2}\Phi(f, x, y) = \frac{2}{y}\int_{-\infty}^{\infty} f''(x + z\sqrt{y})\varphi(z)(z^2 - 1)\, dz. \tag{3.6}$$

4. Auxiliary results

In this section we shall prove a technical result formulated as Theorem 4.1 which play the key role in the proof of main result of the paper Theorem 2.1. Before stating this result it is the appropriate place to develop some more notations to be involved in the formulation of this result and in the proofs as well.

Suppose we are given the square integrable martingale $X = (X_t, \mathcal{F}_t)_{0\le t\le 1}$, $X_0 = 0$ a.s. under the usual conditions. Let $\langle X\rangle = (\langle X\rangle_t, \mathcal{F}_t)_{0\le t\le 1}$ be the quadratic characteristic of martingale X and let $\varepsilon = L_{2\delta} + N_{2\delta}$ be finite for some $\delta > 0$. Assume that $r, x \in R^1$ are such that

$$1 \le x = (1+|r|)^{c(\delta)} e^{r^2/2} \le \alpha\varepsilon^{-1}, \tag{4.1}$$

where $c(\delta) = 3 + 6\delta$, $\alpha > 0$. Denote for $i = 1, 2$

$$g_1(r) = (1+|r|)^{-6}, \quad g_2(r) = (1+|r|)^{4\delta},$$
$$\varepsilon_i = \varepsilon_i(r) = \frac{1}{2}(\alpha^{-1} g_i(r) e^{r^2/2}\varepsilon)^{1/(3+2\delta)}.$$

Let $T = 1 + \varepsilon_1^2$ (of course T is depending on r). Introduce the process $V = (V_t, \mathcal{F}_t)_{0\le t\le 1}$ as follows

$$V = \langle X\rangle 1_{[\![0,\tau [\![} + T 1_{[\![\tau, 1]\!]},$$

where

$$\tau = \inf\{0 \le s \le 1 : \langle X\rangle_s > T\}, \qquad \text{with} \qquad \inf\emptyset = 1.$$

Define the random time change $\tau = (\tau_t, \mathcal{F}_t)_{0\le t\le T}$ as

$$\tau_s = \inf\{0 \le u \le 1 : V_u > s\}, \qquad \text{with} \qquad \inf\emptyset = 1,$$

and non-negative process $\lambda = (\lambda_t, \mathcal{F}_t)_{0\le t\le 1}$ as

$$\lambda_t = T - V_t, \qquad 0 \le t \le 1.$$

And finally let $B_x(a)$ be the 1-dimensional ball of radius a with the center in x, i.e. $B_x(a) = [x-a, x+a]$.

Main result of this section is formulated below.

Theorem 4.1. *Assume that $r, x \in R^1$ are such that condition (4.1) is satisfied with some $\alpha > 0$. Then for a fixed $\beta \ge 1$ and any $0 \le t \le T$*

$$E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) \le C(\alpha,\beta,\delta)\frac{1}{\sqrt{t^*}}(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(|r|)),$$

where $t^* = t \wedge 1$.

The following auxiliary assertion will be used in the proof of Theorem 4.1. Unfortunately we can't derive it directly from Lemma 3.4 so we have to give a proof involving Lemma 3.5 this, by the way, being the only reason to include it in the paper. If we would take $T = 1$ in the above definitions of V, τ and τ_s, then the assertion of Lemma 4.1 (with $\varepsilon^{1/(3+2\delta)}$ instead of ε_2) would be an immediate consequence of Lemma 3.4 without making use of Lemma 3.5 at all. But in this case we are not able to give a simple estimate for the term I_3 (see (4.15) below) thus making the proof of Theorem 4.1 much more complicated.

Lemma 4.1. *For any $v \geq 0$ and $0 \leq t \leq T$*

$$\sup_{y \in R^1} P(X_{\tau_t} \in B_y(v)) \leq C(\alpha, \delta) \frac{v \vee \varepsilon_2}{\sqrt{t^*}},$$

where $t^* = t \wedge 1$.

Proof. Since $\tau_{t^*} \leq \tau_t$ for $t \in [0, T]$, then by Lemma 3.5

$$\begin{aligned} \sup_{y \in R^1} &|P(X_{\tau_t} \leq y) - \Phi(y/\sqrt{t^*})| \\ &\leq 2 \sup_{y \in R^1} |P(X_{\tau_{t^*}} \leq y) - \Phi(y/\sqrt{t^*})| \\ &+ \frac{5}{\sqrt{2\pi}} \frac{\sqrt{5}\,\varepsilon_2}{\sqrt{t^*}} + 2P(\langle X \rangle_{\tau_t} - \langle X \rangle_{\tau_{t^*}} > 5\varepsilon_2^2)\}. \end{aligned} \tag{4.2}$$

First we note that for any $s \in [0, 1]$

$$\Delta V_s \leq \Delta \langle X \rangle_s + |\langle X \rangle_1 - T| 1(s = 1) \tag{4.3}$$

and

$$|\langle X \rangle_1 - T| \leq \varepsilon_1^2 + |\langle X \rangle_1 - 1|. \tag{4.4}$$

Together with $V_{\tau_t} \leq \Delta V_{\tau_t} + t$ and $V_{\tau_t} \geq t$ which hold for any $t \in [0, T]$ these inequalities imply (note that $\varepsilon_1 \leq \varepsilon_2$)

$$\begin{aligned} \langle X \rangle_{\tau_t} - \langle X \rangle_{\tau_{t^*}} & \\ &= (\langle X \rangle_{\tau_t} - V_{\tau_t}) + (V_{\tau_{t^*}} - \langle X \rangle_{\tau_{t^*}}) + (V_{\tau_t} - V_{\tau_{t^*}}) \\ &\leq 4\varepsilon_2^2 + 3|\langle X \rangle_1 - 1| + \Delta \langle X \rangle_{\tau_t}. \end{aligned}$$

It is not hard to see that

$$E \sum_{0 < s \leq 1} \Delta \langle X \rangle_s^{1+\delta} \leq L_{2\delta} \tag{4.5}$$

From the above inequalities it follows that the last probability in (4.2) do not exceed

$$\varepsilon_2^{-2-2\delta} \{ E \Delta \langle X \rangle_{\tau_t}^{1+\delta} + E|\langle X \rangle_1 - 1|^{1+\delta} \} \leq \varepsilon_2^{-2-2\delta} \varepsilon \leq C(\alpha, \delta) \varepsilon_2, \quad C(\alpha, \delta) = \alpha 2^{3+2\delta}.$$

On the other hand for any $t \in [0,T]$ we have

$$
\begin{aligned}
&|\langle X\rangle_{\tau_{t^*}} - t^*| \\
&\qquad \le 1(\langle X\rangle_1 > t^*)|\langle X\rangle_{\tau_{t^*}} - t^*| \\
&\qquad + 1(\langle X\rangle_1 \le t^*)|\langle X\rangle_{\tau_{t^*}} - t^*|
\end{aligned}
$$

(since $\tau_s = \inf\{0 \le u \le 1 : \langle X\rangle_u > s\}$, with $\inf\emptyset = 1$ for $s < T$ and $\tau_{t^*} = 1$ provided $\langle X\rangle_1 \le t^*$))

$$
\begin{aligned}
&\le \Delta\langle X\rangle_{\tau_{t^*}} + 1(\langle X\rangle_1 \le t^*)(t^* - \langle X\rangle_1) \\
&\le \Delta\langle X\rangle_{\tau_{t^*}} + |\langle X\rangle_1 - 1|,
\end{aligned}
$$

and therefore

$$E|\langle X\rangle_{\tau_{t^*}} - t^*|^{1+\delta} \le 2^{1+\delta}(L_{2\delta} + N_{2\delta}).$$

From this inequality by using the exact estimate of the rate of convergence in CLT for martingales (Lemma 3.4) it follows that

$$\sup_{y\in R^1} |P(X_{\tau_{t^*}} \le y) - \Phi(y/\sqrt{t^*})| \le C(\delta)\left(\frac{1}{\sqrt{t^*}}\right)^{(2+2\delta)/(3+2\delta)} \varepsilon^{1/(3+2\delta)}. \tag{4.6}$$

The requested assertion of Lemma 4.1 can be obtained now from (4.2), (4.6) and

$$\sup_{y\in R^1} P(N \in B_y(\frac{v}{\sqrt{t^*}})) \le \frac{v}{\sqrt{2\pi t^*}},$$

which holds for any $v \ge 0$.

Proof of Theorem 4.1.

We can assume that $\varepsilon = L_{2\delta} + N_{2\delta} > 0$ since otherwise the assertion of Theorem 4.1 becomes trivial. For the proof we consider a fixed pair $r, x \in R$ such that condition (4.1) is satisfied with some fixed $\alpha > 0$. Let $C_* > 1$ be a constant which value will be determined latter and let $h = C_*\varepsilon_2$ and $h_i = h_i(\beta) = (\beta + i)h$, $i = 0, 1, \ldots$, where $\beta \ge 1$. In the sequel we shall make use of the function $f : R^1 \to R^1$ defined as

$$f(y) = \hat{f}(\frac{y-r}{h}), \qquad y \in R^1,$$

where $\hat{f} : R^1 \to R^1$ is a fixed function with four bounded derivatives and such that $0 \le \hat{f}(y) \le 1$ and $\hat{f}(y) = 0$ if $|y| \ge \beta + 1$, $\hat{f}(y) = 1$ if $|y| \le \beta$. It is easy to see that function f satisfies for any $y \in R^1$ and $i = 1, \ldots, 4$ the relations

$$|f^{(i)}(y)| \le Ch^{-i} 1_{B_r(h_1)}(y), \quad f(y) \le 1, \tag{4.7a}$$

$$1_{B_r(h_0)}(y) \le f(y) \le 1_{B_r(h_1)}(y). \tag{4.7b}$$

First we note that because of the obvious inequality $h = C_*\varepsilon_2 > \varepsilon_2$ (valid since $C_* > 1$) we have that

$$E\Phi(B_r(\beta\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}) \leq E\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t}). \tag{4.8}$$

All we want to do at this stage is to prove that the function $g = (g_t)_{0\leq t\leq T}$ defined as

$$g_t = \sup_{\beta\geq 1} \frac{E\Phi(B_r(\beta h), X_{\tau_t}, \lambda_{\tau_t})}{\beta h \exp(-r^2/2 + \beta h|r|)}, \qquad t \in [0, T].$$

satisfies for any $t \in [0, T]$ the inequality

$$g_t \leq C(\alpha, \delta)\frac{1}{\sqrt{t^*}}, \tag{4.9}$$

where $t^* = t \wedge 1$. Having proved this and by taking into account the bounds (a) in Lemma 3.2 and the relation

$$\varepsilon_2|r| \leq \varepsilon_2(1 + |r|) = \frac{1}{2}(\alpha^{-1}x\varepsilon)^{1/(3+2\delta)} \leq \frac{1}{2} \tag{4.10}$$

one can easily derive the assertion of Theorem 4.1 from (4.8). Before to give a proof of (4.9) let us remark that function g is actually bounded above by a constant but which is depending on r and h. At this moment important for us is that this constant do not depend on t.

We start our estimation of function g by replacing an appropriate smooth function instead of the indicator of the interval $B_r(h_0) = [r-h_0, r+h_0]$ in the right-hand side of (4.8). Following this line we take into consideration (4.7b) to obtain

$$\begin{aligned} &E\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t}) \\ &\qquad\leq |E\{\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)\}| \\ &\qquad+P(\sqrt{\lambda_0}N \in B_r(h_1)). \end{aligned} \tag{4.11}$$

The most consuming part of the proof will be to give an estimate for the first term of the in the right side of (4.11) the second one being easily handled by applying Lemma 3.2. In order to give such an estimate we apply Ito's formula (see Liptser and Shiryaev (1989)) to the two-dimensional semimartingale $(X_{\tau_t}, \lambda_{\tau_t})$. According to this we can derive that for any $0 \le t \le T$

$$\begin{aligned}\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0) = \\ = \int_0^{\tau_t} \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-})\, dX_s - \int_0^{\tau_t} \frac{\partial}{\partial y}\Phi(f, X_{s-}, \lambda_{s-})\, dV_s \\ + \frac{1}{2}\int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_{s-})\, d\langle X^c\rangle_s \\ + \sum_{0<s\le\tau_t} [\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_{s-}) \\ - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_{s-})\Delta X_s \\ + \frac{\partial}{\partial y}\Phi(f, X_{s-}, \lambda_{s-})\Delta V_s].\end{aligned}$$

Taking expectation and making use of (3.1) and of the obvious relation

$$\begin{aligned}E \sum_{0<s\le\tau_t} [\Phi(f, X_{s-}, \lambda_s) - \Phi(f, X_{s-}, \lambda_{s-})]\Delta X_s \\ = E\int_0^{\tau_t} [\Phi(f, X_{s-}, \lambda_s) - \Phi(f, X_{s-}, \lambda_{s-})] 1(\Delta V_s > 0)\, dX_s = 0,\end{aligned}$$

we get after some straightforward calculations that

$$E\{\Phi(f, X_{\tau_t}, \lambda_{\tau_t}) - \Phi(f, X_0, \lambda_0)\} = I_1 + I_2 + I_3, \tag{4.12}$$

where

$$\begin{aligned}I_1 = E \sum_{0<s\le\tau_t} [\Phi(f, X_s, \lambda_s) - \Phi(f, X_{s-}, \lambda_s) \\ - \frac{\partial}{\partial x}\Phi(f, X_{s-}, \lambda_s)\Delta X_s \\ - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\Delta X_s^2],\end{aligned} \tag{4.13}$$

$$\begin{aligned}I_2 = -E \sum_{0<s\le\tau_t} [\Phi(f, X_{s-}, \lambda_{s-}) - \Phi(f, X_{s-}, \lambda_s) \\ - \frac{\partial}{\partial y}\Phi(f, X_{s-}, \lambda_s)\Delta V_s],\end{aligned} \tag{4.14}$$

$$I_3 = \frac{1}{2}E\int_0^{\tau_t} \frac{\partial^2}{\partial x^2}\Phi(f, X_{s-}, \lambda_s)\, d(\langle X\rangle_s - V_s). \tag{4.15}$$

Now we proceed to produce bounds for I_1, I_2, I_3.

Estimate I_1. By applying Taylor's formula we arrive to

$$|I_1| \le J_1 + J_2,$$

where

$$J_1 = E \sum_{0<s\le\tau_t} \sup_{0\le\theta\le1} |\frac{\partial^2}{\partial x^2}\Phi(f, X_{s-} + \theta_s \Delta X_s, \lambda_s)| \Delta X_s^2 1(|\Delta X_s| > \varepsilon_s),$$

$$J_2 = \frac{1}{6} E \sum_{0<s\le\tau_t} |\frac{\partial^3}{\partial x^3}\Phi(f, X_{s-} + \theta_s \Delta X_s, \lambda_s)| \, |\Delta X_s|^3 1(|\Delta X_s| \le \varepsilon_s),$$

with $0 \le \theta_s \le 1$ and

$$\varepsilon_s = \varepsilon_1 \left(\frac{\varepsilon_2}{\sqrt{\lambda_s^*}}\right)^{1/\delta}, \qquad \lambda_s^* = \varepsilon_2^2 \vee \lambda_s.$$

Estimate J_1. Relations (3.1), (3.4) and (4.7a) imply

$$|\frac{\partial^2}{\partial x^2}\Phi(f, x, y)| \le C(y \vee \varepsilon_2^2)^{-1}. \tag{4.16}$$

Taking into account that

$$\begin{gathered} (\lambda_s^*)^{-1}\varepsilon_s^{-2\delta} \le \varepsilon_1^{-2\delta}\varepsilon_2^{-2}, \\ \varepsilon_2^{-2-2\delta}\varepsilon \le \varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon = C(\alpha,\delta)\varepsilon_2 e^{-r^2/2}, \quad C(\alpha,\delta) = \alpha 2^{3+2\delta} \end{gathered} \tag{4.17}$$

one can easily obtain the following estimate

$$J_1 \le CE \sum_{0<s\le1} (\lambda_s^*)^{-1}\varepsilon_s^{-2\delta}|\Delta X_s|^{2+2\delta} \le C\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon = C(\alpha,\delta)\varepsilon_2 e^{-r^2/2}.$$

Estimate J_2. It follows from (3.2), (3.5) and (4.7a) that

$$|\frac{\partial^3}{\partial x^3}\Phi(f, x, y)| \le Ch^{-1}(y \vee \varepsilon_2^2)^{-1} \int_{-\infty}^{\infty} 1(|z\sqrt{y} + x - r| \le h_1)\psi(z)\, dz, \tag{4.18}$$

where $\psi(z) = \varphi(z)(z^2+1)$. Therefore implementing this estimate in J_2 and using the inequality $|\Delta X_s| \le \varepsilon_s \le \varepsilon_2$ we obtain

$$\begin{aligned} J_2 &\le Ch^{-1}E \sum_{0<s\le\tau_t} \frac{\varepsilon_s}{\lambda_s^*} \int_{-\infty}^{\infty} 1(|z\sqrt{\lambda_s} + X_{s-} - r| \le h_2)\psi(z)\, dz\, \Delta X_s^2 \\ &\le Ch^{-1}E \int_0^{\tau_t} \frac{\varepsilon_s}{\lambda_s^*} \int_{-\infty}^{\infty} 1(|z\sqrt{\lambda_s} + X_{s-} - r| \le h_2)\psi(z)\, dz\, d\langle X\rangle_s. \end{aligned}$$

Since for any bounded non-negative predictable process $U = (U_t, \mathcal{F}_t)_{0\le t\le 1}$

$$\int_0^{\tau_t} U_s d\langle X\rangle_s \le \int_0^{\tau_t} U_s\, dV_s + 1(\langle X\rangle_1 > T)\int_0^{\tau_t} U_s\, d(\langle X\rangle_s - V_s)$$

we have

$$J_2 \le H_1 + H_2 + H_3,$$

where

$$H_1 = Ch^{-1}E\int_0^{\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\int_{-\infty}^{\infty} 1(|z\sqrt{\lambda_s} + X_{s-} - r| \le h_2)\psi(z)\,dz 1(\Delta V_s < 2\varepsilon_1^2)\,dV_s,$$

$$H_2 = C\varepsilon_2^{-2}E\sum_{0<s\le 1}\Delta V_s 1(\Delta V_s \ge 2\varepsilon_1^2),$$

$$H_3 = C\varepsilon_2^{-2}E1(\langle X\rangle_1 > T)\int_0^1 d(\langle X\rangle_s - V_s),$$

Estimate H_1. Let us introduce the following sets $S_1 = \{z : |z| \le 6|r|\}$ and $S_2 = \{z : |z| > 6|r|\}$. With these notations we have

$$H_1 \le L_1 + L_2,$$

where for $i = 1, 2$

$$L_i = Ch^{-1}E\int_0^{\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\int_{S_i} 1(|z\sqrt{\lambda_s} + X_{s-} - r| \le h_2)\psi(z)\,dz 1(\Delta V_s < 2\varepsilon_1^2)\,dV_s.$$

Estimate L_1. Since $\psi(z) \le 6\varphi(z)(1 + r^2)$ on the set S_1 it is obvious that

$$L_1 = Ch^{-1}(1 + r^2)E\int_0^{\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\Phi(B_r(h_2), X_{s-}, \lambda_s)1(\Delta V_s < 2\varepsilon_1^2)\,dV_s$$

(apply random change time formula from Lemma 3.1 and use the inequality $(1 + r^2)h^{-1}\varepsilon_1 \le C_*^{-1}$)

$$\le CC_*^{-1}\varepsilon_2^{1/\delta}E\{\int_0^t \Phi(B_r(h_2), X_{\tau_s-}, \lambda_{\tau_s})(\lambda_{\tau_s}^*)^{-1-\frac{1}{2\delta}}1(\Delta V_{\tau_s} < 2\varepsilon_1^2)\,ds$$
$$+\Phi(B_r(h_2), X_{\tau_t-}, \lambda_{\tau_t})(\lambda_{\tau_t}^*)^{-1-\frac{1}{2\delta}}1(\Delta V_{\tau_t} < 2\varepsilon_1^2)\Delta V_{\tau_t}\}.$$

It is not hard to see that on the set $\{\Delta V_{\tau_s} < 2\varepsilon_1^2\}$ for any $s \in [0, T]$

$$\lambda_{\tau_s} \ge \underline{\lambda}_s, \qquad \lambda_{\tau_s}^* \ge \underline{\lambda}_s^*, \tag{4.19}$$

where

$$\underline{\lambda}_s = -\varepsilon_1^2 + 1 - s, \qquad \underline{\lambda}_s^* = \underline{\lambda}_s \vee \varepsilon_2^2.$$

On the other hand let us observe that

$$(4.20) \qquad \begin{aligned} E\Phi(B_r(h_k), X_{\tau_t-}, \lambda_{\tau_t}) \leq & E\Phi(B_r(h_{k+1}), X_{\tau_t}, \lambda_{\tau_t}) + \varepsilon_2^{-2-2\delta} L_{2\delta}, \\ E\Phi(B_r(h_k), X_{\tau_t}, \lambda_{\tau_t}) \leq & (k+1) g_t h_0 e^{-r^2/2 + h_0|r| + k\varepsilon_2|r|}. \end{aligned}$$

for $k = 0, 1, ...,$ and

$$(4.21) \qquad \varepsilon_2^{1/\delta} \int_0^T (\underline{\lambda}_s^*)^{-1-\frac{1}{2\delta}} \, ds \leq C(\delta) = 2\delta + 3.$$

Then from the bounds (4.19), (4.20), (4.21), (4.3), (4.4), (4.17), (4.10) it follows that

$$L_1 \leq C(\alpha, \delta) C_*^{-1} h_0 e^{-r^2/2 + h_0|r|} \{ \int_0^t g_s \varepsilon_2^{1/\delta} (\underline{\lambda}_s^*)^{-1-\frac{1}{2\delta}} \, ds + g_t + 1 \}.$$

Estimate L_2. Applying the random change time formula (Lemma 3.1) we arrive to the inequality

$$L_2 \leq$$

$$C C_*^{-1} \varepsilon_2^{1/\delta} E\{ \int_0^t \int_{S_2} 1(|z\sqrt{\lambda_{\tau_s}} + X_{\tau_s-} - r| \leq h_2) \psi(z) \, dz 1(\Delta V_{\tau_s} < 2\varepsilon_1^2) \lambda_{\tau_s}^{-1-\frac{1}{2\delta}} \, ds$$

$$+ \int_{S_2} 1(|z\sqrt{\lambda_{\tau_t}} + X_{\tau_t-} - r| \leq h_2) \psi(z) \, dz 1(\Delta V_{\tau_t} < 2\varepsilon_1^2) \lambda_{\tau_t}^{-1-\frac{1}{2\delta}} \Delta V_{\tau_t} \}.$$

Now we observe that on the set $\{\Delta V_{\tau_s} < 2\varepsilon_1^2\}$ for any $s \in [0, T]$

$$\lambda_{\tau_s} \leq 2\varepsilon_1^2 + \underline{\lambda}_s,$$

which together with (4.19) yield

$$|\sqrt{\lambda}_{\tau_s} - \sqrt{\underline{\lambda}_s}| \leq \sqrt{2} \varepsilon_1.$$

Therefore by taking into consideration that from Lemma 4.1 we have

$$\sup_{z \in R^1} P(|X_{\tau_s} - r + z\sqrt{\underline{\lambda}_s}| \leq h_2 + |z| \sqrt{2} \varepsilon_2) \leq \frac{C(\delta)}{\sqrt{s^*}} h_0 (1 + |z|),$$

where $s^* = s \wedge 1$ and that for any $x \in R$ and $v \geq 0$

$$P(|X_{\tau_t-} - x| \leq v) \leq P(|X_{\tau_t} - x| \leq v + \varepsilon_2) + \varepsilon_2^{-2-2\delta} L_{2\delta}.$$

we can obtain

$$L_2 \le CC_*^{-1}\{h_0 \int_0^t \Psi(r)\varepsilon_2^{1/\delta}(\underline{\lambda}_s^*)^{-1-\frac{1}{2\delta}} \frac{ds}{\sqrt{s^*}} + \frac{h_0}{\sqrt{t^*}}\Psi(r) + \varepsilon_2^{-2-2\delta}\varepsilon\},$$

where

$$\Psi(r) = \int_{S_2} \psi(z)(|z|+1)\,dz \le Ce^{-r^2/2}.$$

Since

$$\varepsilon_2^{1/\delta} \int_0^T \underline{\lambda}_s^{-1-\frac{1}{2\delta}} \frac{ds}{\sqrt{s^*}} \le C(\delta) = 4\delta + 2 + 3\sqrt{2}, \tag{4.22}$$

then using (4.17) we arrive to the following bound for L_2

$$L_2 \le C(\alpha,\delta)C_*^{-1}\frac{1}{\sqrt{t^*}}h_0 e^{-r^2/2}.$$

Estimate H_2. The estimate for H_2 easily follows from (4.3), (4.4), (4.5) and (4.17)

$$\begin{aligned} H_2 \le& C\varepsilon_1^{-2\delta}\varepsilon_2^{-2}E(\sum_{0<s\le 1} \Delta\langle X\rangle_s^{1+\delta} + |\langle X\rangle_1 - 1|^{1+\delta}) \\ \le& C\varepsilon_1^{-2\delta}\varepsilon_2^{-2}\varepsilon \le C(\alpha,\delta)\varepsilon_2 e^{-r^2/2}. \end{aligned}$$

Estimate H_3. Since $V = \langle X\rangle$ on $[\![0,\tau[\![$ and $V = T$ on $[\![\tau,1]\!]$

$$\begin{aligned} H_3 &\le C\varepsilon_2^{-2}E(\langle X\rangle_1 - T)1(\langle X\rangle_1 \ge T) \\ &\le C\varepsilon_2^{-2}E(\langle X\rangle_1 - 1)1(\langle X\rangle_1 - 1 \ge \varepsilon_2^2) \\ &\le C\varepsilon_2^{-2-2\delta}\varepsilon \le C(\alpha,\delta)\varepsilon_2 e^{-r^2/2}, \end{aligned}$$

by using (4.17). Now we put together the bounds for J_1, J_2, H_1, H_2, H_3, L_1, L_2 to obtain

$$|I_2| \le C(\alpha,\delta)C_*^{-1}h_0 e^{-r^2/2+h_0|r|}\{\int_0^t g_s\varepsilon_2^{1/\delta}(\underline{\lambda}_s^*)^{-1-\frac{1}{2\delta}}\,ds + g_t + \frac{1}{\sqrt{t^*}}\}.$$

Estimate I_2. From (3.1), (3.3), (3.4), (3.6) and (4.7a) we have

$$|\frac{\partial}{\partial y}\Phi(f,x,y)| \le C(y \vee \varepsilon_2^2)^{-1} \tag{4.23}$$

and

$$(4.24)\qquad |\frac{\partial^2}{\partial y^2}\Phi(f,x,y)| \le Ch^{-2}(y \vee \varepsilon_2^2)^{-1}\int_{-\infty}^{\infty} 1(|z\sqrt{y}+x-r| \le h_1)\psi(z)\,dz.$$

Therefore applying Taylor's formula we get

$$|I_2| \le K_1 + K_2,$$

where

$$K_1 = C\varepsilon_2^{-2}E\sum_{0<s\le 1}\Delta V_s 1(\Delta V_s \ge 2\varepsilon_1^2),$$

$$K_2 = Ch^{-2}E\sum_{0<s\le \tau_t}\frac{1}{\lambda_s^*}\int_{-\infty}^{\infty} 1(|z\sqrt{\lambda_s}+X_{s-}-r| \le h_1+|z|\varepsilon_1)\psi(z)\,dz\times$$
$$1(\Delta V_s < 2\varepsilon_1^2)\,\Delta V_s^2.$$

In order to estimate K_1 we proceed exactly in the same way as for H_2.
Estimate K_2. Obviously

$$K_2 \le L_1' + L_2',$$

where for $i = 1, 2$

$$L_i' \le Ch^{-1}E\int_0^{\tau_t}\frac{\varepsilon_s}{\lambda_s^*}\int_{S_i} 1(|z\sqrt{\lambda_s}+X_{s-}-r| \le h_1+|z|\varepsilon_1)\psi(z)\,dz\,1(\Delta V_s < 2\varepsilon_1^2)\,dV_s.$$

Since $|z|\varepsilon_1 \le \varepsilon_2$ if $z \in S_1$, then we can produce for L_1' the same bound as for L_1. As to L_2' it can be bounded in the manner similar to L_2 too. Therefore for I_3 the same bound as for I_2 can be established.

Estimate I_3. It follows from (3.1) and (4.7a) that for any $x \in R$, $y > 0$

$$(4.25)\qquad |\frac{\partial^2}{\partial x^2}\Phi(f,x,y)| \le Ch^{-2}\Phi(B_r(h_1),x,y)$$

Then obviously

$$|I_3| \le G_1 + G_2,$$

where

$$G_1 = Ch^{-2}1(\langle X\rangle_1 \ge T)E\int_0^1 d(\langle X\rangle_s - V_s),$$

$$G_2 = Ch^{-2}E\Phi(B_r(h_1), X_{\tau_t -}, \lambda_{\tau_t})|V_s - \langle X\rangle_s|.$$

We have for G_1 exactly the same estimate as for H_3.

Estimate G_2. It is clear that with (4.4) and $\varepsilon_1 \le \varepsilon_2$ we have

$$G_2 \le Ch^{-2} E\Phi(B_r(h_1), X_{\tau_t -}, \lambda_{\tau_t})\{2\varepsilon_2^2 + |\langle X\rangle_1 - 1| 1(|\langle X\rangle_1 - 1| > \varepsilon_2^2)\}$$

(use (4.20))

$$\le C\{C_*^{-1} h_0 e^{-r^2/2 + h_0|r| * \varepsilon_2 |r|} + \varepsilon_2^{-2-2\delta}\varepsilon\}$$

(use (4.10), (4.17) and $\varepsilon_2 \le C_*^{-1} h_0$)

$$\le C(\alpha,\delta) C_*^{-1} h_0 e^{-r^2/2 + h_0|r|}\{g_t + 1\}.$$

Putting together these two bounds for G_1 and G_2 we obtain the following estimate

$$|I_3| \le C(\alpha,\delta) C_*^{-1} h_0 e^{-r^2/2 + h_0|r|}\{g_t + 1\}.$$

Only that remains to estimate is the probability in the right-hand side of (4.11). By virtue of Lemma 3.2 and (4.10) we have

$$P(\sqrt{\lambda_0} N \in B_r(h_1)) \le C h_0 e^{-r^2/2 + h_0|r|}. \tag{4.26}$$

From this and from (4.11), (4.12) with the above bounds for I_i, $i = 1, 2, 3$ we derive

$$E\Phi(B_r(h_0), X_{\tau_t}, \lambda_{\tau_t})$$
$$\le C(\alpha,\delta) C_*^{-1} h_0 e^{-r^2/2} e^{h_0|r|}\{\int_0^t g_s \varepsilon_2^{1/\delta} (\underline{\lambda}_s^*)^{-1-\frac{1}{2\delta}}\, ds + g_t + \frac{1}{\sqrt{t^*}} + C_*\}.$$

Dividing both sides by $h_0 e^{-r^2/2 + h_0|r|}$ and taking sup on $\delta \ge 1$ and then choosing C_* large enough (so that $C_*^{-1} C(\alpha,\delta) \le 1/2$) it follows that for any $t \in (0, T]$

$$g_t \le C(\alpha,\delta)\{\int_0^t g_s \varepsilon_2^{1/\delta} (\lambda_s^*)^{-1-\frac{1}{2\delta}}\, ds + \frac{1}{\sqrt{t^*}}\}.$$

Multiplying both sides by $\sqrt{t^*}$ and denoting

$$\bar{g}_t = g_t - \frac{1}{\sqrt{t^*}}$$

we obtain using the obvious inequality $t^* \le 1$ that for every $t \in [0, T]$

$$\bar{g}_t \le C(\alpha,\delta)\{\int_0^t \bar{g}_s \varepsilon_2^{1/\delta} (\lambda_s^*)^{-1-\frac{1}{2\delta}} \frac{ds}{\sqrt{s^*}} + 1\}.$$

Since $\bar{g}_t \le g_t$ and the function g is bounded by a constant not depending on t, then by virtue of Gronwall-Bellman inequality (Lemma 3.3) and of (4.22)

$$\bar{g}_t \le C_1(\alpha,\delta) \exp\{C_2(\alpha,\delta) \int_0^t \varepsilon_2^{1/\delta} (\lambda_s^*)^{-1-\frac{1}{2\delta}} \frac{ds}{\sqrt{s^*}}\} \le C_3(\alpha,\delta),$$

and finally

$$g_t \le \frac{1}{\sqrt{t^*}} C_3(\alpha,\delta),$$

consequently the inequality (4.9) is proved and therefore we complete the proof of Theorem 4.1.

5. Proof of the main result

We proceed to prove Theorem 2.1 now. We give a proof only for the first inequality in Theorem 2.1 the second being proved by the same way.

Assume that $\varepsilon > 0$ since otherwise the assertion of Theorem 2.1 becomes trivial.

Let x, r be such that conditions of Theorem 2.1 are satisfied. Introduce the functions $f_i : R^1 \to R^1$, $i = 1, 2$ defined as

$$f_1(y) = \hat{f}(\frac{y-r}{\varepsilon_2}), \qquad f_2(y) = \hat{f}(\frac{y-r-\varepsilon_2}{\varepsilon_2}), \qquad y \in R^1,$$

where $\hat{f} : R^1 \to R^1$ is the function with four bounded derivatives and such that $0 \le \hat{f}(y) \le 1$ and $\hat{f}(y) = 0$ if $y \le 0$, $\hat{f}(y) = 1$ if $y \ge 1$. The functions f_i satisfy for any $y \in R^1$ and $i = 1, ...4$

$$(5.1) \qquad \begin{gathered} |f_i^{(i)}(y)| \le C\varepsilon_2^{-i} 1_{B_r(2\varepsilon_2)}(y), \\ 1(r - \varepsilon_2 \le y) \le f(y) \le 1(r + \varepsilon_2 \le y) \le 1. \end{gathered}$$

These inequalities give rise

$$(5.2) \qquad \begin{aligned} &E\Phi([r,\infty), X_1, \lambda_1) - E\Phi([r,\infty), X_0, \lambda_0) \\ &\quad \le \max_{i=1,2} |E\{\Phi(f_i, X_1, \lambda_1) - \Phi(f_i, X_0, \lambda_0)\}| \\ &\quad + P(|\sqrt{\lambda_0} N - r| \le \varepsilon_2). \end{aligned}$$

Since $\tau_T = 1$ a.s. then by Ito's formula we have for $i = 1, 2$

$$(5.3) \qquad E\{\Phi(f_i, X_1, \lambda_1) - \Phi(f_i, X_0, \lambda_0)\} = I_1 + I_2 + I_3,$$

where the quantities I_1, I_2, I_3 are defined as in (4.15), (4.13), (4.14) with $t = T$ and f_i instead of f. Now we can estimate I_1, I_2, I_3 in the same way as in Theorem 4.1 using (5.1) instead of (4.7a) since inequalities (5.1) allows us to get exactly the same bounds for derivatives as in (4.16), (4.18), (4.23), (4.24), (4.25). Therefore we arrive to the following bounds

$$(5.4) \qquad \begin{aligned} &|I_j| \le C(\alpha, \delta)\{\int_0^T \rho_s \varepsilon_2^{1/\delta} (\underline{\lambda}_s^*)^{-1-\frac{1}{2\delta}} \, ds + \rho_T + \varepsilon_2 e^{-r^2/2}\}, \\ &|I_3| \le C(\alpha, \delta)\{\rho_T + \varepsilon_2 e^{-r^2/2}\}, \end{aligned}$$

where $j = 1, 2$ and

$$\rho_t = E\Phi(B_r(3\varepsilon_2), X_{\tau_t}, \lambda_{\tau_t}),$$

for $t \in [0, T]$. Since by Theorem 4.1 we have for any $t \in [0, T]$

$$\rho_t \le C(\alpha, \delta) \frac{1}{\sqrt{t^*}} (x\varepsilon)^{1/(3+2\delta)} (1 - \Phi(|r|)).$$

then implementing this bound in (5.4) and taking into account (4.22) and inequality (a) in Lemma 3.2 we obtain for $j = 1, 2, 3$

$$|I_j| \le C(\alpha, \delta)(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(|r|)). \tag{5.5}$$

As in (4.26) the probability in the right-hand side of (5.2) do not exceed

$$C\varepsilon_2 e^{-r^2/2} \le C(\alpha, \delta)(x\varepsilon)^{1/(3+2\delta)}(1 - \Phi(|r|)).$$

Then the assertion of Theorem 2.1 follows from (5.2), (5.3), (5.5). Theorem 2.1 is proved.

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