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Exact controllability on a curve for a semilinear parabolic equation

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ABSTRACT. Motivated by the growing number of industrially important laser material treatments we investigate the controllability on a curve for a semilinear parabolic equation. We prove the local exact controllability and a global stability result in the two-dimensional setting. As an application we consider the control of laser surface hardening. We show that our theory applies to this situation and present numerical simulations for a PID control of laser hardening. Moreover, the result of an industrial case study is presented confirming that the numerically derived temperature in the hot-spot of the laser can indeed be used as set-point for the machine-based process control.

1. INTRODUCTION AND MAIN RESULT

In this paper we investigate the controllability on a curve for a semilinear parabolic equation. Our research is motivated by an application related to thermal laser material treatments like hardening or coating. The new generation of laser heat treatment equipments usually includes a device for process control. Its mode of operation is shown in Figure 1. While the laser moves along the workpiece surface, the temperature $u_{hs}(t)$ in the hot-spot of the laser beam is measured by a pyrometer. A PID-controller is employed to adjust the laser power $p(t)$ to approximate a desired set-point temperature $h(t)$.

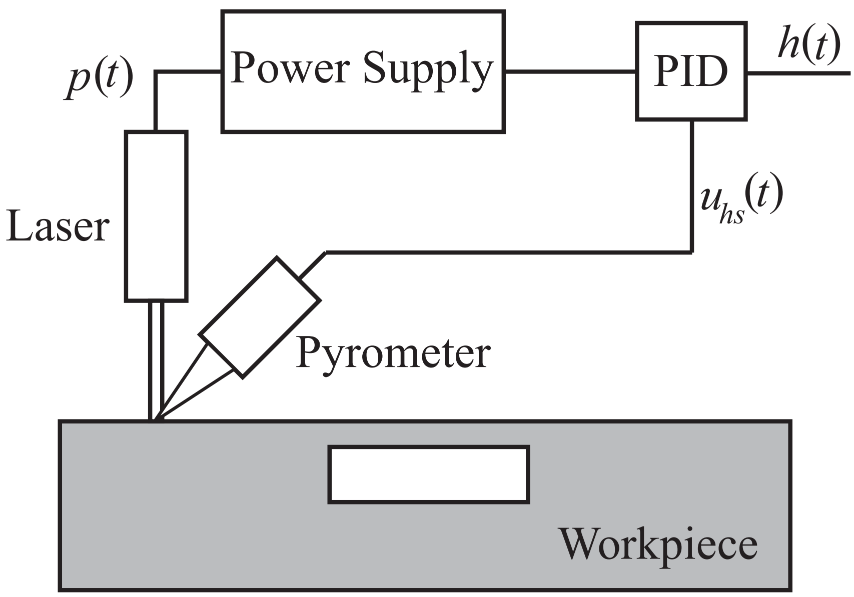

FIGURE 1. Pyrometer process control of laser surface hardening.

In many applications it suffices to choose the set-point temperature as a constant below the melting temperature of the respective workpiece material. However, recently we have shown that in case of workpieces with varying thickness or with cavities below the surface a constant set-point temperature is not sufficient to achieve a uniform penetration depth of the laser beam. In [4] we have shown that it is much more favourable to keep the temperature constant on a curve inside the workpiece in a fixed distance from the workpiece surface penetrated by the laser beam.

To fix the mathematical setting we assume the workpiece $\Omega \subset \mathbb{R}^2$ to be a bounded domain with smooth boundary $\partial\Omega$. The evolution of temperature u is described by the semilinear heat equation

$$\begin{aligned}
&(1.1a) \quad & \partial_t u(x,t) &= \Delta u(x,t) + F(u(x,t)) + p(t)G(x,t), \quad \text{in } \Omega \times (0,T),\\
&(1.1b) \quad & u(x,0) &= 0, \quad \text{in } \Omega,\\
&(1.1c) \quad & \frac{\partial u}{\partial \nu}(x,t) &= 0, \quad \text{on } \partial\Omega \times (0,T).
\end{aligned}$$

Here, all physical constants have been normalized. As in [2] we assume the laser energy to be absorbed volumetrically, modelled by the term $p(t)G(x,t)$. Here, $p(t)$ is the laser power, serving as our control parameter and G is the normalized intensity profile. In the case of a gas-laser the profile would be of Gaussian shape decaying exponentially with decreasing distance from the surface.

Typically, the heating induces further effects inside the workpiece, e.g. melting or solid-solid phase transitions. These in turn usually lead to a release or consumption of heat. This is modelled by the function $F(u)$, for which we assume

(H1) For any $M > 0$, there exists a constant $C = C_M > 0$ such that

$$\left| \frac{d^k F}{dy^k}(y) - \frac{d^k F}{dy^k}(y') \right| \le C_M |y - y'|, \quad k = 0, 1, \quad |y|, |y'| \le M.$$

By $u_p = u_p(x,t)$ we denote the strong solution to (1.1a) – (1.1c) with prescribed real-valued laser power $p = p(t)$, provided that it exists for $0 < t < t_0$. Then, the mathematical formulation of laser process control reads as follows:

(P) Exact control on a curve.
Let $\gamma(t) \in \Omega$, $0 < t < T$, be a smooth curve and let $h = h(t) \in H^1(0,T)$ be a given set-point function. Then determine an input $p(t)$ such that

$$u_p(\gamma(t), t) = h(t), \qquad 0 < t < t_0.$$

Inverse problems of determining t-functions in evolution equations by observation data in t are discussed in [6, Chapter 6], but our problem takes a different type of observations and requires more analysis.

The paper is organized as follows: In Section 2 we formulate and prove the main result. In Section 3 we show a global stability result. Section 4 is devoted to an application in laser surface hardening followed by a short concluding section on future research.

2. THE MAIN RESULT

We fix $\theta > 0$ and define an operator A in $L^2(\Omega)$ by $-Au(x) = \Delta u(x)$, $x \in \Omega$ with $\mathcal{D}(A) = \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \right\}$. Henceforth $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L^2(\Omega)$, respectively. Then $-A$ generates an analytic semigroup in $L^2(\Omega)$ and the fractional power $(A+\theta)^\alpha$, $\alpha \in \mathbb{R}$, can be defined (e.g., Pazy [5]). Moreover

$$(2.1) \qquad \begin{cases} \|(A+\theta)^\alpha e^{-tA}\| \le C_0 t^{-\alpha}, \quad \alpha > 0, \\ \|u\|_{L^\infty(\Omega)} \le C_0 \|(A+\theta)^\alpha u\|, \quad \alpha > \dfrac{1}{2}, \\ \|e^{-tA} a\|_{L^\infty(\Omega)} \le C_T \|a\|_{L^\infty(\Omega)}, \quad 0 \le t \le T. \end{cases}$$

The last estimate can be seen by the fundamental solution. For the function G we assume

(H2)

$$\begin{cases} G \in C([0,T];\mathcal{D}((A+\theta)^{\gamma})), & \alpha > \frac{1}{2},\ \gamma > \alpha + \frac{1}{2}, \\ G(\gamma(t),t) \neq 0, & 0 \le t \le T. \end{cases}$$

Then we can state the first main result on the unique local existence of the control $p(t)$.

Theorem 2.1. *Assume* (H1) *and* (H2). *Then we can choose small $t_0 > 0$ satisfying: for an arbitrary $h \in H^1(0,t_0)$ with $h(0) = 0$, there exists a unique $p = p(t) \in L^2(0,t_0)$ such that $u_p \in C([0,t_0];L^\infty(\Omega)) \cap H^1(0,t_0;L^\infty(\Omega))$ and*

$$u_p(\gamma(t),t) = h(t), \qquad 0 \le t \le t_0,$$
$$\|p\|_{L^2(0,t_0)} \le C(\|h'\|_{L^2(0,t_0)} + \|F(0)\|_{L^\infty(\Omega)}).$$

Proof.

First Step. Let $T > 0$ be arbitrarily fixed. For $M > 0$, we can choose $t_0 \in (0,T)$ such that there exists a unique solution $u_p = u_p(t)$ to (1.1a) – (1.1c) in $t \in (0,t_0)$ as long as $\|p\|_{L^2(0,T)} \le M$. Moreover

$$\|u_p'(t)\|_{L^\infty(\Omega)} \le C\left(|p(t)| + 1 + \int_0^t (|p(s)|+1)ds\right), \quad 0 \le t \le t_0,$$
$$\|u_p\|_{C([0,t_0];L^\infty(\Omega))} \le C(\|p\|_{L^2(0,t_0)} + 1).$$

Here and henceforth $C > 0$ denotes generic constants which are dependent on G, M, F, α, γ, but independent of choices of p.

The proof is done by a usual argument by the semigroup theory (Henry [1], Pazy [5]), and for completeness we will give the details. We set

$$B(R) = \{u \in C([0,t_0];L^\infty(\Omega));\ \|u\|_{C([0,t_0];L^\infty(\Omega))} \le R\}.$$

Since u_p formally satisfies

$$u_p(t) = \int_0^t e^{-(t-s)A} p(s)G(s)ds + \int_0^t e^{-(t-s)A}F(u_p(s))ds, \qquad t > 0,$$

we introduce an operator $K : B(R) \longrightarrow C([0,t_0];L^\infty(\Omega))$ defined by

$$(Ku)(t) = \int_0^t p(s)e^{-(t-s)A}G(s)ds + \int_0^t e^{-(t-s)A}F(u(s))ds, \qquad t > 0.$$

Here and henceforth we set $G(t) = G(\cdot,t)$, which is considered as a map from $\mathbb{R}$ to $L^2(\Omega)$.

Let $u \in B(R)$ for given $R > 0$. Then, by (H2), we have

$$\begin{aligned} \|Ku(t)\|_{L^\infty(\Omega)} &= \left\| \int_0^t p(s)e^{-(t-s)A}G(s)ds + \int_0^t e^{-(t-s)A}F(u(s))ds \right\|_{L^\infty(\Omega)} \\ &\le C\int_0^t |p(s)|ds \|G\|_{C([0,T];\mathcal{D}((A+\theta)^\alpha))} + \int_0^t \|F(u(s))\|_{L^\infty(\Omega)}ds \\ &\le C\sqrt{t_0}M + Ct_0C_R. \end{aligned}$$

For given $R > 0$, we choose $t_0 > 0$ sufficiently small such that

(2.2) $$C\sqrt{t_0}M + Ct_0C_R < R.$$

Hence

$$KB(R) \subset B(R). \tag{2.3}$$

Next we will prove that K is a contraction for small $t_0 > 0$. Let $u, v \in B(R)$. We have

$$\|Ku(t) - Kv(t)\|_{L^\infty(\Omega)} \le \left\| \int_0^t e^{-(t-s)A}\big(F(u(s)) - F(v(s))\big)ds \right\|_{L^\infty(\Omega)}$$
$$\le C_R \int_0^t \|u(s) - v(s)\|_{L^\infty(\Omega)} ds \le C_R t_0 \|u - v\|_{C([0,t_0];L^\infty(\Omega))},$$

that is,

$$\|Ku - Kv\|_{C([0,t_0];L^\infty(\Omega))} \le C_R t_0 \|u - v\|_{C([0,t_0];L^\infty(\Omega))}.$$

Therefore, if $t_0 > 0$ is sufficiently small, then $K : B(R) \longrightarrow B(R)$ is a contraction. By the contraction mapping theorem, K has a unique fixed point:

$$u_p(t) = \int_0^t e^{-(t-s)A} p(s)G(s)ds + \int_0^t e^{-(t-s)A} F(u_p(s))ds, \qquad 0 < t < t_0 \tag{2.4}$$

and

$$\|u_p\|_{C([0,t_0];L^\infty(\Omega))} \le R. \tag{2.5}$$

Next we will estimate $\|u_p'(t)\|_{L^\infty(\Omega)}$. By (2.4) and (2.5), we have

$$u_p'(t) = p(t)G(t) - \int_0^t p(s)Ae^{-(t-s)A}G(s)ds + e^{-tA}F(0)$$
$$+ \int_0^t e^{-sA}F'(u_p(t-s))u_p'(t-s)ds, \qquad t > 0.$$

Therefore, by (H2), (2.5) and the Schwarz inequality, we have

$$\|u_p'(t)\|_{L^\infty(\Omega)} \le C|p(t)| + C\int_0^t |p(s)| \|(A+\theta)^{1+\alpha-\gamma} e^{-(t-s)A} (A+\theta)^\gamma G(s)\| ds$$
$$+ C\|F(0)\|_{L^\infty(\Omega)} + \int_0^t \|F'(u_p(t-s))\|_{L^\infty(\Omega)} \|u_p'(t-s)\|_{L^\infty(\Omega)} ds$$
$$\le C|p(t)| + C\int_0^t |p(s)|(t-s)^{\gamma-\alpha-1} ds + C + C_R \int_0^t \|u_p'(t-s)\|_{L^\infty(\Omega)} ds$$
$$\le C|p(t)| + CM\left(\frac{t_0^{2\gamma-2\alpha-1}}{2\gamma-2\alpha-1}\right)^{\frac{1}{2}} + C + C_R\int_0^t \|u_p'(s)\|_{L^\infty(\Omega)} ds, \quad 0 \le t \le t_0.$$

Therefore the Gronwall inequality yields

$$\|u_p'(t)\|_{L^\infty(\Omega)} \le C(|p(t)| + 1) + C\int_0^t (|p(s)| + 1) ds, \quad 0 \le t \le t_0. \tag{2.6}$$

Second Step. Let $t_0 > 0$ be chosen in as in First Step. In this step we will estimate $\|u_p(t) - u_q(t)\|_{L^\infty(\Omega)}$ and $\|u_p'(t) - u_q(t)'\|_{L^\infty(\Omega)}$ for $\|p\|_{L^2(0,t_0)}, \|q\|_{L^2(0,t_0)} \le M$. Since

$$\begin{aligned} u_p(t) - u_q(t) = &\int_0^t e^{-(t-s)A}(p(s) - q(s))G(s)ds \\ &+ \int_0^t e^{-(t-s)A}\big(F(u_p(s)) - F(u_q(s))\big)ds, \qquad 0 < t < t_0, \end{aligned}$$

by (H1), (H2) and (2.1) we have

$$\begin{aligned} \|u_p(t) - u_q(t)\|_{L^\infty(\Omega)} \le C &\int_0^t |p(s) - q(s)|ds \\ &+ C\int_0^t \|u_p(s) - u_q(s)\|_{L^\infty(\Omega)}ds. \end{aligned}$$

Here we note that for $\|p\|_{L^2(0,t_0)}, \|q\|_{L^2(0,t_0)} \le M$, we have bounds

$$\|u_p\|_{C([0,t_0];L^\infty(\Omega))}, \quad \|u_q\|_{C([0,t_0];L^\infty(\Omega))} \le C,$$

so that

$$\|F(u_p(s)) - F(u_q(s))\|_{L^\infty(\Omega)} \le C\|u_p(s) - u_q(s)\|_{L^\infty(\Omega)}, \quad 0 \le s \le t_0,$$

by (H1). Hence the Gronwall inequality yields

$$\|u_p - u_q\|_{C([0,t_0];L^\infty(\Omega))} \le C\|p - q\|_{L^2(0,t_0)}. \tag{2.7}$$

Next we have

$$\begin{aligned} u_p'(t) - u_q'(t) &= (p-q)(t)G(t) - \int_0^t (p(s) - q(s))Ae^{-(t-s)A}G(s)ds \\ &+ \int_0^t e^{-sA}F'(u_p(t-s))(u_p'(t-s) - u_q'(t-s))ds \\ &+ \int_0^t e^{-sA}\big(F'(u_p(t-s)) - F'(u_q(t-s))\big)u_q'(t-s))ds \\ &\equiv (p-q)(t)G(t) + I_1 + I_2 + I_3. \end{aligned}$$

First, by (H2) we have

$$\begin{aligned} \|I_1\|_{L^\infty(\Omega)} &\le C\int_0^t |p(s) - q(s)|(t-s)^{\gamma-\alpha-1}\|(A+\theta)^\gamma G(s)\|ds \\ &\le C\left(\int_0^t |p(s) - q(s)|^2 ds\right)^{\frac{1}{2}} \left(\int_0^t (t-s)^{2\gamma-2\alpha-2}ds\right)^{\frac{1}{2}} \|(A+\theta)^\gamma G(s)\|_{C([0,t_0];L^2(\Omega))} \\ &\le C\|p - q\|_{L^2(0,t_0)}. \end{aligned}$$

Second, by (H1) and (2.5), we have

$$\|I_2\|_{L^\infty(\Omega)} \le C\int_0^t \|u_p'(s) - u_q'(s)\|_{L^\infty(\Omega)}ds.$$

Third, by (H1) and (2.5), we have

$$|F'(u_p(x,s)) - F'(u_q(x,s))| \le C|u_p(x,s) - u_q(x,s)|$$

for almost all $x \in \Omega$ and $t \in [0, t_0]$. Hence

$$\begin{aligned} &\|F'(u_p(s)) - F'(u_q(s))\|_{L^\infty(\Omega)} \le C\|u_p(s) - u_q(s)\|_{L^\infty(\Omega)} \\ &\le C\|(A+\theta)^\alpha u_p(s) - (A+\theta)^\alpha u_q(s)\|. \end{aligned}$$

Consequently it follows from (2.6) and (2.7) that

$$\begin{aligned} &\|I_3\|_{L^\infty(\Omega)} = \left\| \int_0^t e^{-(t-s)A}\big(F'(u_p(s)) - F'(u_q(s))\big)u_q'(s)ds \right\|_{L^\infty(\Omega)} \\ &\le C\int_0^t \|F'(u_p(s)) - F'(u_q(s))\|_{L^\infty(\Omega)} \|u_q'(s)\|_{L^\infty(\Omega)} ds \\ &\le C\|p-q\|_{L^2(0,t_0)} \int_0^t \|u_q'(s)\|_{L^\infty(\Omega)} ds \le C\|p-q\|_{L^2(0,t_0)}. \end{aligned}$$

Therefore

$$\begin{aligned} &\|u_p'(t) - u_q'(t)\|_{L^\infty(\Omega)} \\ &\le C|p(t)-q(t)| + C\|p-q\|_{L^2(0,t_0)} + C\int_0^t \|u_p'(s) - u_q'(s)\|_{L^\infty(\Omega)} ds, \quad 0 \le t \le t_0. \end{aligned}$$

The Gronwall inequality yields

$$\begin{aligned} &\|u_p'(t) - u_q'(t)\|_{L^\infty(\Omega)} \le C(|p(t)-q(t)| + \|p-q\|_{L^2(0,t_0)}) \\ &+ C\int_0^t (|p(s)-q(s)| + \|p-q\|_{L^2(0,t_0)}) ds, \end{aligned}$$

that is,

$$\begin{aligned} &\|u_p'(t) - u_q'(t)\|_{L^\infty(\Omega)} \le C(|p(t)-q(t)| + \|p-q\|_{L^2(0,t_0)}) \\ &+ \int_0^t |p(s)-q(s)| ds, \quad 0 \le t \le t_0. \end{aligned} \tag{2.8}$$

Third Step. By $(K_0 u)(t) = u(\gamma(t), t)$, $0 < t < t_0$, we define a linear operator K_0 from $C([0, t_0]; L^\infty(\Omega))$ to $L^2(0, t_0)$. Then

$$|(K_0 u)(t)| \le C\|u(t)\|_{L^\infty(\Omega)} \le C_1\|(A+\theta)^\alpha u(t)\| \tag{2.9}$$

by $\alpha > \frac{1}{2}$. For a given $h \in H^1(0, t_0)$, let $p \in L^2(0, t_0)$ satisfy

$$\begin{aligned} &h(t) = \int_0^t p(s)K_0(e^{-(t-s)A}G(s))ds + \int_0^t K_0(e^{-sA}F(u_p(t-s)))ds, \\ &0 < t < t_0. \end{aligned} \tag{2.10}$$

Then for this p, we can directly verify that $u_p(t)$ satisfies

$$u_p(\gamma(t), t) = h(t), \qquad 0 < t < t_0. \tag{2.11}$$

Differentiating (2.10) with respect to t, we obtain

$$\begin{aligned} &h'(t) = p(t)(K_0 G)(t) + \int_0^t p(s)\partial_t Q(t,s)ds + K_0(e^{-tA}F(0)) \\ &+ \int_0^t K_0(e^{-sA}F'(u_p(t-s))u_p'(t-s))ds, \quad 0 < t < t_0. \end{aligned} \tag{2.12}$$

Here and henceforth we set

$$Q(t,s) = K_0(e^{-(t-s)A}G(s)), \qquad 0 < s < t < t_0.$$

Then we have $\partial_t Q(t,s) = -K_0(Ae^{-(t-s)A}G(s))$, $0 < s < t < t_0$. Therefore

$$\begin{aligned}|\partial_t Q(t,s)| &\le C\|(A+\theta)^\alpha A e^{-(t-s)A}G(s)\| \le C\|(A+\theta)^{1+\alpha-\gamma}e^{-(t-s)A}(A+\theta)^\gamma G(s)\|\\ &\le C(t-s)^{\gamma-\alpha-1}\|(A+\theta)^\gamma G(s)\|_{C([0,t_0];L^2(\Omega))}, \quad 0 < s < t < t_0,\end{aligned}$$

so that

$$|\partial_t Q(t,s)| \le C(t-s)^{\gamma-\alpha-1}, \qquad 0 < s < t < t_0 \tag{2.13}$$

by (H2). Again by (H2), we can define an operator L defined in $L^2(0,t_0)$ by

$$\begin{aligned}(Lp)(t) = \frac{-1}{(K_0G)(t)}&\left\{\int_0^t p(s)(\partial_t Q)(t,s)ds\right.\\ &+\left.\int_0^t K_0\{e^{-sA}(F'(u_p(t-s))u_p'(t-s))\}ds\right\}, \quad 0 < t < t_0.\end{aligned} \tag{2.14}$$

We set

$$\begin{aligned}\mathcal{U}_{M_1} = &\left\{p \in L^2(0,t_0);\ \|p\|_{L^2(0,t_0)}\right.\\ &\left.\le \left\|\frac{h' - K_0(e^{-tA}F(0))}{K_0G}\right\|_{L^2(0,t_0)} + 1 \equiv 2M_1 + 1\right\}.\end{aligned} \tag{2.15}$$

Then, for sufficiently small $t_0 > 0$, we will prove that

$$L\mathcal{U}_{M_1} \subset \mathcal{U}_{M_1}. \tag{2.16}$$

Proof of (2.16)**.** Let $p \in \mathcal{U}_{M_1}$. Then, by (2.5) and (2.6), we can choose constants $t_0 > 0$ and $R_1 > 0$ such that

$$\|u_p\|_{C([0,t_0];L^\infty(\Omega))} \le R_1 \tag{2.17}$$

and

$$\|u_p'(t)\|_{L^\infty(\Omega)} \le C(|p(t)|+1) + C\int_0^t (|p(s)|+1)ds, \quad 0 \le t \le t_0. \tag{2.18}$$

By (2.1) and (2.13) we have

$$\begin{aligned}|Lp(t)| &\le C\int_0^t |p(s)|\cdot|\partial_t Q(t,s)|ds + C\int_0^t \|F'(u_p(t-s))\|_{L^\infty(\Omega)}\|u_p'(t-s)\|_{L^\infty(\Omega)}ds\\ &\le C\int_0^t |p(s)|(t-s)^{\gamma-\alpha-1}ds + C\int_0^t \left(\max_{|\eta|\le R_1}|F'(\eta)|\right)\|u_p'(s)\|_{L^\infty(\Omega)}ds.\end{aligned}$$

The Schwarz inequality, (H2) and (2.18) yield

$$
\begin{aligned}
|Lp(t)| &\le C\left(\int_0^t |p(s)|^2 ds\right)^{\frac{1}{2}} \left(\frac{t_0^{2\gamma-2\alpha-1}}{2\gamma-2\alpha-1}\right)^{\frac{1}{2}} \\
&+ C\int_0^t \left\{|p(s)| + 1 + \left(\int_0^s (|p(\xi)|+1)d\xi\right)\right\} ds \\
&\le Ct_0^{\gamma-\alpha-\frac{1}{2}} + Ct_0^{\frac{1}{2}}, \quad 0 \le t \le t_0.
\end{aligned}
$$

Hence

$$\|Lp\|_{L^2(0,t_0)} \le C(t_0^{\gamma-\alpha-\frac{1}{2}} + t_0^{\frac{1}{2}}),$$

so that if $t_0 > 0$ is sufficiently small such that

$$C(t_0^{\gamma-\alpha-\frac{1}{2}} + t_0^{\frac{1}{2}}) \le 2M_1 + 1,$$

then $Lp \in \mathcal{U}_{M_1}$. Thus the proof of (2.16) is complete.

Next we will prove that $L : \mathcal{U}_{M_1} \longrightarrow \mathcal{U}_{M_1}$ is a contraction if $t_0 > 0$ is sufficiently small. Let $p, q \in \mathcal{U}_{M_1}$. First by (2.13) and the Schwarz inequality, we obtain

$$
\begin{aligned}
&\left|\int_0^t p(s)\partial_t Q(t,s)ds - \int_0^t q(s)\partial_t Q(t,s)ds\right| \le C\int_0^t |p(s)-q(s)|(t-s)^{\gamma-\alpha-1}ds \\
&\le C\|p-q\|_{L^2(0,t_0)} \left(\frac{t_0^{2\gamma-2\alpha-1}}{2\gamma-2\alpha-1}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Second

$$
\begin{aligned}
&\left|\int_0^t \left\{K_0(e^{-sA}F'(u_p(t-s))u_p'(t-s)) - K_0(e^{-sA}F'(u_q(t-s))u_q'(t-s))\right\} ds\right| \\
&= \left|\int_0^t K_0(e^{-sA}F'(u_p(t-s))(u_p'(t-s) - u_q'(t-s)))ds \right. \\
&\left. + \int_0^t K_0 e^{-sA}\Big(F'(u_q(t-s)) - F'(u_q(t-s))u_q'(t-s)\Big)ds\right| \\
&\le C\int_0^t \|F'(u_p(s))\|_{L^\infty(\Omega)}\|u_p'(s) - u_q'(s)\|_{L^\infty(\Omega)}ds \\
&+ C\int_0^t \|F'(u_q(s)) - F'(u_q(s))\|_{L^\infty(\Omega)}\|u_q'(s)\|_{L^\infty(\Omega)}ds.
\end{aligned}
$$

By (2.17) and (H1) we have $\|F'(u_p(s))\|_{L^\infty(\Omega)} \le C$ and

$$\|F'(u_p(s)) - F'(u_q(s))\|_{L^\infty(\Omega)} \le C\|u_p(s) - u_q(s)\|_{L^\infty(\Omega)} \le C\|(A+\theta)^\alpha u_p(s) - (A+\theta)^\alpha u_q(s)\|.$$

Hence by (2.7) and (2.8)

$$
\begin{aligned}
&\left|\int_0^t K_0 e^{-sA}(F'(u_p(t-s))u_p'(t-s) - F'(u_q(t-s))u_q'(t-s))ds\right| \\
&\le C\int_0^t \|u_p'(s) - u_q'(s)\|_{L^\infty(\Omega)}ds + C\int_0^t \|(A+\theta)^\alpha u_p(s) - (A+\theta)^\alpha u_q(s)\|ds \\
&\le C\int_0^t \left(|p(s)-q(s)| + \|p-q\|_{L^2(0,t_0)} + \int_0^s |p(\xi)-q(\xi)|d\xi\right)ds \\
&+ Ct_0\|p-q\|_{L^2(0,t_0)} \le C\sqrt{t_0}\|p-q\|_{L^2(0,t_0)}.
\end{aligned} \tag{2.19}
$$

Consequently

$$
|Lp(t) - Lq(t)| \le C(t_0^{\gamma-\alpha-\frac{1}{2}} + t_0^{\frac{1}{2}})\|p-q\|_{L^2(0,t_0)}, \quad 0 \le t \le t_0,
$$

that is, L is a contraction from $\mathcal{U}_{M_1}$ to $\mathcal{U}_{M_1}$ for small $t_0 > 0$. Thus for $h \in H^1(0,t_0)$, there exists a unique $p \in L^2(0,t_0)$ such that (2.12) holds. Since $h(0) = 0$, by integrating (2.12) over $(0,t)$, we see that this p satisfies (2.10). Thus the proof of Theorem 2.1 is complete.

3. A global stability result

We discuss the global stability provided that u_p, u_q exist in $t \in (0,T)$ and satisfty a priori bounds:

$$
\begin{cases}
\|u_p\|_{C([0,T];L^\infty(\Omega))}, \quad \|u_q\|_{C([0,T];L^\infty(\Omega))} \le M, \\
\|p\|_{L^2(0,t_0)}, \|q\|_{L^2(0,t_0)} \le M.
\end{cases} \tag{3.1}
$$

Theorem 3.1. *We assume that* (3.1) *holds and* (p,u_p), (q,u_q) *satisfy*

$$
u_p(\gamma(t),t) = h(t), \quad u_q(\gamma(t),t) = j(t), \qquad 0 < t < T.
$$

Then there exists a constant $C > 0$ *such that*

$$
\|p-q\|_{L^2(0,T)} \le C\|h'-j'\|_{L^2(0,T)}.
$$

Proof. By (H1), (2.4) and (3.1), similarly to (2.6), we can prove

$$
\|u_p'(t)\|_{L^2(0,T;L^\infty(\Omega))}, \quad \|u_p'(t)\|_{L^2(0,T;L^\infty(\Omega))} \le C_M. \tag{3.2}
$$

Since (p,u_p) and (q,u_q) satisfy (2.12), we have

$$
\begin{aligned}
&(p(t)-q(t))(KG_0)(t) = h'(t) - j'(t) - \int_0^t (p(s)-q(s))\partial_t Q(t,s)ds \\
&- \int_0^t K_0 e^{-sA}(F'(u_p(t-s))u_p'(t-s) - F'(u_q(t-s))u_q'(t-s))ds.
\end{aligned}
$$

Similarly to (2.7), (2.8) and (2.19), we can obtain

$$
\begin{aligned}
&\left|\int_0^t K_0 e^{-sA}(F'(u_p(t-s))u_p'(t-s) - F'(u_q(t-s))u_q'(t-s))ds\right| \\
&\le C\left(\int_0^t |p(s)-q(s)|^2 ds\right)^{\frac{1}{2}}, \quad 0 \le t \le T.
\end{aligned} \tag{3.3}
$$

By (H2), (2.3) and (3.3) we can estimate to obtain

$$\begin{aligned}|p(t)-q(t)| &\le C|h'(t)-j'(t)| + C\int_0^t |p(s)-q(s)|(t-s)^{\gamma-\alpha-1}ds \\ &+ C\left(\int_0^t |p(s)-q(s)|^2 ds\right)^{\frac{1}{2}} \\ &\le C|h'(t)-j'(t)| + C\left(\int_0^t |p(s)-q(s)|^2 ds\right)^{\frac{1}{2}}, \quad 0 \le t \le T.\end{aligned}$$

Therefore squaring the both hand sides and applying the Gronwall inequality, we obtain

$$|p(t)-q(t)|^2 \le C|h'(t)-j'(t)|^2 + C\int_0^t |h'(s)-j'(s)|^2 ds, \quad 0 \le t \le T.$$

Thus the proof of Theorem 3.1 is complete.

4. An application: laser surface hardening

In laser surface hardening the heating induces phase transitions inside the workpiece that subsequently lead to a hardening of the penetrated surface layers. A simple way of describing this phenomenon is to keep track of the growth of the high-temperature phase *austenite* assuming that it will be transformed completely to the hard phase martensite upon cooling.

The corresponding mathematical model is

$$\partial_t u = \Delta u - \partial_t a + p(t)G(x,t) \quad \text{in } \Omega \times (0,T) \tag{4.1a}$$

$$\partial_t a = f(u,a) := \frac{1}{\tau(u)}\big[a_{eq}(u) - a\big]_+ \quad \text{in } \Omega \times (0,T) \tag{4.1b}$$

$$a(x,0) = 0, \quad \text{in } \Omega \tag{4.1c}$$

$$u(x,0) = 0, \quad \text{in } \Omega \tag{4.1d}$$

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \quad \text{in } \partial\Omega \times (0,T). \tag{4.1e}$$

Here, $F(u) = -\partial_t a$ describes the consumption of latent heat during the growth of austenite, τ is a time constant and $[x]_+ = \max\{x, 0\}$ the positive part function. The equilibrium volume fraction a_{eq} is monotone and satisfies $a_{eq}(x) = 0$ for $x < A_s$ and $a_{eq}(x) = 1$ for $x > A_f$, where A_s and A_f are the threshold temperatures for the beginning and the end of the transformation. Hence the austenite volume fraction increases during heating until it reaches some value $\tilde{a} \le 1$. During cooling we have $a_t = 0$, and the value $\tilde{a}$ is kept.

In [3, Lemma 2.2] we have shown that for any temperature evolution $u \in L^2(Q)$ (4.1b) – (4.1c) admits a unique solution a. After regularizing the positive part function we may assume that f as defined in (4.1b) is differentiable with bounded partial derivatives $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial a}$. Then it is a standard application of Gronwall's lemma to infer

$$\begin{aligned}|\partial_t a_1(x,t)| - \partial_t a_2(x,t) &\le c_1\Big(|u_1(x,t)-u_2(x,t)| + \int_0^t |u_1(x,\xi)-u_2(x,\xi)|\,d\xi\Big) \\ |\partial_{tt} a_1(x,t)| - \partial_{tt} a_2(x,t) &\le c_2\Big(|\partial_t u_1(x,t)-\partial_t u_2(x,t)| + |u_1(x,t)-u_2(x,t)| \\ &\quad + \int_0^t |u_1(x,\xi)-u_2(x,\xi)|\,d\xi\Big).\end{aligned}$$

Here a_1, a_2 are the solutions to (4.1b) – (4.1c) corresponding to u_1, u_2, respectively. Using these estimates instead of (H1) it is possible to recover Theorems 2.1 and 3.1.

As a numerical example we consider the surface hardening of a workpiece with rectangular cavity (cf. [4]). Figure 2 depicts the result of a numerical simulation of (4.1a) – (4.1e) with constant laser power $p(t) \equiv p_0$ and constant velocity. On top, the temperature in the

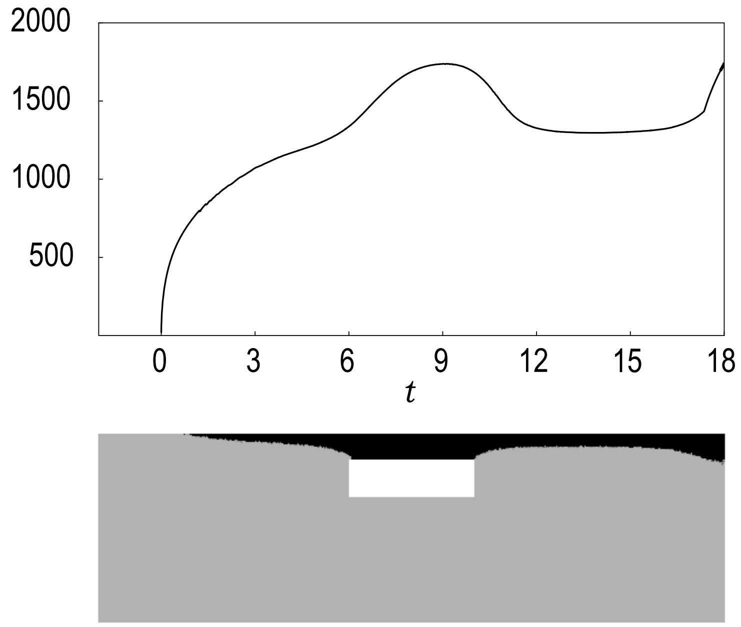

FIGURE 2. Uncontrolled case: Temperature in hot-spot of the laser beam (top) and resulting fraction of austenite (bottom).

hot-spot of the laser on the surface is described. Below, one can see the resulting austenite fraction $a(T)$ which we assume to coincide with the hardened zone. The situation shown in Figure 2 is unfavourable for two reasons. The high temperature above the cavity and at the end of the workpiece will lead to an undesired surface melting. Moreover, the through-hardening above the cavity will foster fatigue effects and eventually lead to crack formation.

In [4], we have shown that a way to obtain a uniform hardening depth is to control the temperature to be constant about one millimeter from the quenched end close to the lower boundary of the desired hardening strip. Theorems 2.1 and 3.1 show that it is indeed possible to control the temperature in such a way and thus provide a theoretical basis for this approach.

We define a corresponding curve $\gamma(t)$ and compute a solution to (4.1a) – (4.1e) where the laser power at timestep t_{i+1} is derived using a standard PID algorithm, i.e.,

$$e(t_i) = h(t_i) - u(\gamma(t_i), t_i) \tag{4.2a}$$

$$p(t_{i+1}) = k_1 e(t_i) + k_2 \int_{t_0}^{t_i} e(t)\, dt + k_3 \dot{e}(t_i). \tag{4.2b}$$

Here, h is the given constant set-point temperature, and the resulting laser power at time t_{i+1} is the sum of a term **p**roportional to the error between actual and set-point temperature, its **i**ntegral and its **d**erivative, which explains the name PID algorithm.

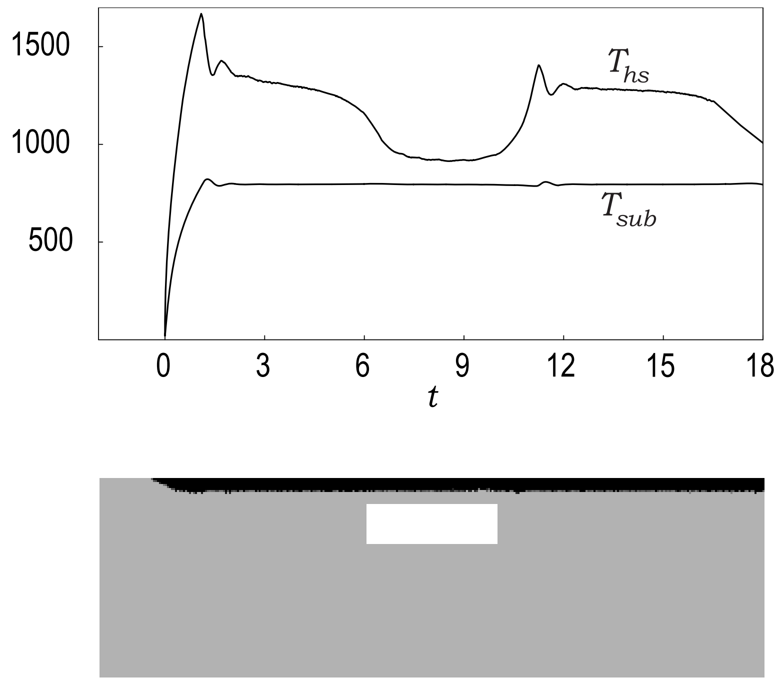

FIGURE 3. Controlled case: Temperature T_{hs} in hot-spot of laser beam and $T_s ub$ close to lower end of desired hardening depth (top). Resulting austenite fraction (bottom).

On top of Figure 3 the temperature T_{hs} in the hot-spot of the laser beam and the controlled temperature T_{sub} are depicted. Obviously, the PID algorithm works very well. The figure below shows that the desired constant hardening depth has been achieved. From practical point of view the non-constant temperature T_{hs} should be suitable as the set-point temperature for the machine-based control illustrated in Figure 1.

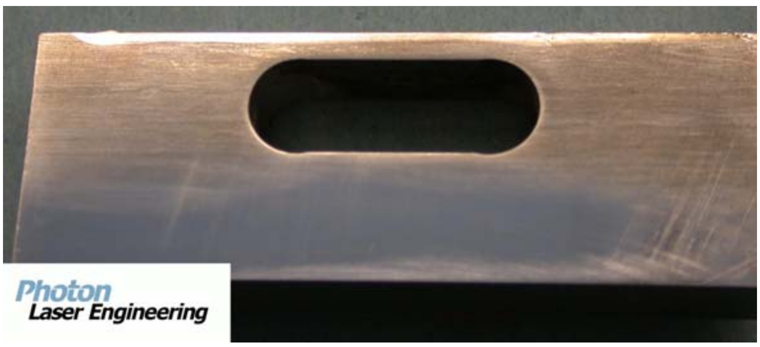

FIGURE 4. Result of a hardening experiment utilizing the computed temperature T_{hs} as set-point for the machine-based control (Courtesy of *Photon Laser Engineering*, Berlin, Germany).

Figure 4 shows the result of a corresponding experiment. The resulting hardening strip is nearly constant. Especially the dangerous situation of a through-hardening above the cavity does not occur.

5. Concluding remarks

In the present paper we have investigated the controllability on a curve for a semilinear parabolic equation. This problem is of particular importance for laser material treatments. Unfortunately, our analysis so far is limited to the twodimensional case. An extension to the 3D case requires further regularity analysis of the state equation.

From numerical point of view the PID algorithm in combination with a finite element solver for the state equations works very well. However, another challenging direction of further research is convergence analysis for this numerical strategy.

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