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Iterating Snowballs and related path dependent callables in a multi-factor Libor model

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Abstract

We propose a valuation method for callable structures in a multi-factor Libor model which are path-dependent in the sense that, after calling, one receives a sequence of cash-flows in the future, instead of a well specified cash-flow at the calling date. The method is based on a Monte Carlo procedure for standard Bermudans recently developed in Kolodko and Schoenmakers [8], and is applied to the cancelable snowball interest rate swap. The proposed procedure is quite generic, straightforward to implement, and can be easily adapted to other related path-dependent products.

1 Introduction

The pricing of callable instruments with respect to a high dimensional underlying system is considered a thorny issue. In the literature different approaches are proposed to treat this problem. Many of these methods, such as the stochastic mesh method of Broadie and Glasserman [4] and the regression method of Longstaff and Schwartz [9], are Monte Carlo procedures based on the so called backward dynamic program. Other popular methods, such as Andersen [1], optimize a suitable parametric family of exercise boundaries. Generally, all of these have their own shortcomings and merits. However, their application becomes increasingly problematic when the cash-flow at a calling date is not directly known but merely virtually known as the (conditional) expectation of a system of cash-flows in the future. In this article we show that a new iterative Monte Carlo procedure for pricing callable structures, recently developed in Kolodko and Schoenmakers [8] and further extended in Bender and Schoenmakers [3], can be easily adapted to a large class of callable products where the cash-flows by calling have to be considered as present values of future cash-flows. As a main example we consider the (cancelable) snowball, an exotic interest rate product with growing popularity, in a full-blown Libor market model. From the treatment of this example it will be clear how to design Monte Carlo valuation algorithms for related callable path-dependent products. The proposed approach is quite generic, as in principle it only requires a Monte Carlo simulation mechanism for an underlying Markovian system, for instance a Markovian system of SDEs. Moreover, by incorporating information obtained from another suboptimal method, for example Andersen's approach (see [1]), we may improve upon this method to obtain our target results more efficiently.

The paper is organized as follows. In Section 2 we introduce the Libor market model, the iterative procedure for Bermudan callables from [8], and formulate our target class of callable and cancelable products (as we will see a cancelable product can be translated to a callable one and vice versa). Then in Section 3 we tailor the iterative procedure presented in Section 2 to the cancelable snowball and give surprising numerical results.

2 Iterative valuation of callable and cancelable Libor products

2.1 Recap of the Libor market model

The Libor market model is a popular and advanced tool for modelling interest rates and pricing of interest rate products. Let us first recall the Libor Market Model with respect to a tenor structure $0 = T_0 < T_1 < \ldots < T_n$ in the spot Libor measure P^*. The dynamics of the forward Libors $L_i(t)$, defined in the interval $[0, T_i]$ for $1 \le i < n$, are governed by the following system of SDE's (e.g., see Jamshidian [7]),

$$dL_i = \sum_{j=\kappa(t)}^{i} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^*, \tag{1}$$

where $\delta_i = T_{i+1} - T_i$ are day count fractions, $t \to \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,d}(t))$ are deterministic volatility vector functions defined in $[0, T_i]$ (called factor loadings), and $\kappa(t) := \min\{m : T_m \ge t\}$ denotes the next reset date at time t. In (1), $(W^*(t) \mid 0 \le t \le T_{n-1})$ is a standard d-dimensional Wiener process under the spot Libor measure P^* with d, $1 \le d < n$, being the number of driving factors. This measure is induced by the numeraire

$$B_*(t) := \frac{B_{\kappa(t)}(t)}{B_1(0)} \prod_{i=1}^{\kappa(t)-1} (1 + \delta_i L_i(T_i)) \,, \; t > 0, \quad B_*(0) := 1,$$

with $\prod_{i=1}^{0} := 1$, and where $B_i(t)$ is the value of a zero coupon bond with face value \$1 at time $t \le T_i$. For further use we here also introduce the filtration $(\mathcal{F}_t)_{t \ge 0}$ (history information process) connected with the Libor process.

2.2 Path dependent callable and cancelable products

Consider a subset of tenor dates $\mathcal{T}_1 < \cdots < \mathcal{T}_k$, hence $\mathbb{T} := \{\mathcal{T}_1, \ldots, \mathcal{T}_k\} \subset \{T_0, \ldots, T_n\}$, and adapted cash-flows C_i defined at $\mathcal{T}_i$ for $i = 1, \ldots, k$. Let us consider a path dependent contract which involves the right to call a sequence of (possibly negative) cash-flows $C_{\tau+1}, \ldots, C_k$, at a date τ to be decided by the option holder. Obviously, the discounted time τ-value of the contracted cash-flows is equal to $\mathcal{Z}_\tau := E^\tau \sum_{j=\tau+1}^{k} Z_j$ (with $\mathcal{Z}_k := 0$), where discounted cash-flows are denoted by $Z_i := C_i / B_*(\mathcal{T}_i)$. Thus $\mathcal{Z}_\tau$ can be considered as a virtual (discounted) cash-flow and so by general arguments (see Duffie [5]) it follows that the value of this product at time zero is given by

$$V_0^{call} := \sup_{\tau \in \{1, \ldots, k\}} E^0 \mathcal{Z}_\tau = \sup_{\tau \in \{1, \ldots, k\}} E^0 \sum_{j=\tau+1}^{k} Z_j \,, \tag{2}$$

where the supremum is taken over all stopping indices with values in the set $\{1, \ldots, k\}$. Similarly, we may consider a path dependent cancelable contract which generates cash-flows (possibly negative) $C_1, \ldots, C_\sigma$, up to cancellation date σ. The cash-flows of this contract are equivalent to an aggregated cash-flow $B_*(\mathcal{T}_\sigma)\mathcal{U}_\sigma := B_*(\mathcal{T}_\sigma) \sum_{j=1}^{\sigma} Z_j$ at the cancellation date. Indeed, it is equivalent to replace each cash-flow C_i by an amount

$C_i/B_*(\mathcal{T}_i) = Z_i$ of the numeraire B_*, which in turn is worth $B_*(\mathcal{T}_\sigma)Z_i$ at the cancellation date. So, the value of this cancelable product at time zero is given by

$$V_0^{cancel} := \sup_{\sigma\in\{1,\ldots,k\}} E^0\mathcal{U}_\sigma = \sup_{\sigma\in\{1,\ldots,k\}} E^0\sum_{j=1}^{\sigma} Z_j \ . \tag{3}$$

Note that $\sigma = k$ may be interpreted as "not cancelled" in fact. Obviously, we have

$$V_0^{call} \;=\; E^0\sum_{j=1}^{k} Z_j + \sup_{\tau\in\{1,\ldots,k\}} E^0\sum_{j=1}^{\tau}(-Z_j),$$

hence the path dependent callable product can be seen as the sum of a non callable and a cancelable one, and vice versa.

2.3 Iterative pricing procedure

Both the path dependent cancallable and path dependent callable product introduced in Section 2.2 can be seen as a standard Bermudan product with respect to virtual cash-flows $\mathcal{U}_i$ and $\mathcal{Z}_i$, respectively. Therefore they can be evaluated by the iterative method developed in Kolodko and Schoenmakers [8], which is studied further concerning numerical stability and extended to multiple stopping in Bender and Schoenmakers [3]. Although in [8] cash-flows are assumed to be non-negative, it is not hard to see that the iterative construction in this article goes through for negative cash-flows as well.

Let us recall briefly the procedure in [8] and show how to evaluate callable and cancelable Libor products by plain Monte Carlo simulation using this method. Suppose we are given some (generally suboptimal) exercise policy τ_i, $i = 1,\ldots,k$ for a standard Bermudan product with cash-flow process $\mathcal{Z}$; τ_i is the stopping rule according to which the cash-flow should be called, provided the option has not been called before $\mathcal{T}_i$. We assume that the exercise policy τ has the following properties,

$$\begin{aligned} i \le \tau_i \le k,\ \tau_k = k,& \\ \tau_i > i \Rightarrow \tau_i = \tau_{i+1}, \qquad & 0 \le i < k, \end{aligned} \tag{4}$$

This policy provides a sequence of lower bounds Y_i for the discounted Bermudan prices Y_i^*, also called Snell envelope,

$$Y_i^* \ge Y_i := E^i\mathcal{Z}_{\tau_i}, \qquad i = 1,\ldots,k,$$

where E^i denotes conditional expectation with respect to $\mathcal{F}_{\mathcal{T}_i}$. For a fixed "window parameter" $\varkappa$, $1 \le \varkappa \le k$ (in applications we usually take $\varkappa = k$), we next construct a new exercise policy

$$\widehat{\tau}_i \;:=\; \inf\{j \ge i : \mathcal{Z}_j \ge \max_{p:\, j\le p\le \min(j+\varkappa,k)} E^j\mathcal{Z}_{\tau_p}\} \tag{5}$$

$$\;=\; \inf\{j \ge i : \mathcal{Z}_j \ge \max_{p:\, j< p\le \min(j+\varkappa,k)} E^j\mathcal{Z}_{\tau_p}\}, \qquad i = 1,\ldots,k, \tag{6}$$

which clearly satisfies (4) also, and consider the new lower bound process

$$\widehat{Y}_i := E^i\mathcal{Z}_{\widehat{\tau}_i}, \qquad i = 1,\ldots,k, \tag{7}$$

which is generally an improvement of Y,

$$Y_i \le \widehat{Y}_i \le Y_i^*, \qquad i = 1, \dots, k.$$

We note that expression (6) is identical with (5) (which is taken as definition of $\widehat{\tau}$ in [8]) due to consistency property (4), but (6) is somewhat more convenient for stability studies (see [3]).

Naturally, we may iterate the above procedure, i.e. improve $\widehat{\tau}$ in the same way and so forth. It is shown that after iterating this procedure $k-1$ times, the Snell envelope is attained. The iteration may be started with the canonical starting policy $\tau_i^{(0)} \equiv i$, but one also can start with any other policy which satisfies (4), for example a policy obtained by the Andersen [1] method. More explicitly, we construct via (5)-(7) iteratively a sequence of pairs

$$\Big((\tau_i^{(m)})_{0 \le i \le k}, (Y_i^{(m)})_{0 \le i \le k} \Big)_{m=0,1,2,\dots,}$$

starting with an initial stopping family $(\tau_i^{(0)})_{0 \le i \le k}$ satisfying (4), with

$$Y_i^{(m)} := E^i \mathcal{Z}_{\tau_i^{(m)}}, \quad i = 1, \dots, k,$$

and for $m \ge 0$,

$$\tau_i^{(m+1)} := \inf\{ j : i \le j \le k, \max_{p:\, j < p \le \min(j+\varkappa, k)} E^j \mathcal{Z}_{\tau_p^{(m)}} \le \mathcal{Z}_j \} \quad i = 1, \dots, k.$$

We thus have the monotonicity result

$$Y_i^{(0)} \le Y_i^{(m)} \le Y_i^{(m+1)} \le Y_i^*, \quad m \ge 1.$$

It is intuitively clear that for the choice $\varkappa = k$ the most information of the input stopping family is carried over to the next improvement. Taking $\varkappa = k$ is also optimal from a theoretical point of view but sometimes, for particular problems, it may be nevertheless computationally more efficient to take $\varkappa$ smaller. For notational convenience we henceforth take $\varkappa = k$.

Finally, based on the above constructed sequence of lower bound processes we may construct by the dual approach of Rogers [11], Haugh and Kogan [6] a sequence of upper bound processes converging to the Snell envelope as well,

$$Y_i^{(m),up} \;:=\; E^i \max_{i \le j \le k} \Big(\mathcal{Z}_j - \sum_{l=i+1}^{j} Y_l^{(m)} + \sum_{l=i+1}^{j} E^{l-1} Y_l^{(m)} \Big) \qquad 0 \le i \le k.$$

Monte Carlo implementation

In most cases the cash-flows $\mathcal{Z}_i$ are functions of an underlying Markovian process, for instance a Libor process. In such a situation the conditional expectations involved in the iterative procedure can be estimated by Monte Carlo simulation, which thus leads to a Monte Carlo algorithm in a natural way. We refer to [8] for a detailed description of the general algorithm and to [12], Section 5.4.3, for a linear implementation of a one step improvement. The numerical stability of these procedures is proved in [3]. By Remark 1 below the efficiency of the iterative Monte Carlo procedure can be improved. Moreover, in [2] we investigate how computation time can be reduced further when additional information (for example closed form lower approximations of Europeans) is available.

Remark 1 *(variance reduced Monte Carlo simulation of $Y^{(m)}$)* We can reduce the number of Monte Carlo simulations for $Y^{(m)}$ by using the following variance reduced representation,

$$Y_i^{(m)} = E^i \mathcal{Z}_{\tau_i^{(m)}} = E^i \mathcal{Z}_{\tau_i^{(m-1)}} + E^i(\mathcal{Z}_{\tau_i^{(m)}} - \mathcal{Z}_{\tau_i^{(m-1)}}).$$

One can expect that $\mathcal{Z}_{\tau_i^{(m-1)}}$ and $\mathcal{Z}_{\tau_i^{(m)}}$ are strongly correlated and thus the variance of $(\mathcal{Z}_{\tau_i^{(m)}} - \mathcal{Z}_{\tau_i^{(m-1)}})$ will be less than the variance of $\mathcal{Z}_{\tau_i^{(m)}}$. So, the computation of $E^i(\mathcal{Z}_{\tau_i^{(m)}} - \mathcal{Z}_{\tau_i^{(m-1)}})$ for a given accuracy, can usually be done with less Monte Carlo simulations than needed for direct simulation of $E^i \mathcal{Z}_{\tau_i^{(m)}}$.

2.4 Iterating path dependent callable and cancelables

For the callable product (2) the policy iteration (5) (with now $\varkappa = k$) can be written as

$$\begin{aligned}
\widehat{\tau}_i &= \inf\{j \geq i : \mathcal{Z}_j \geq \max_{p: j<p\leq k} E^j \mathcal{Z}_{\sigma_p}\} \\
&= \inf\{j \geq i : E^j \sum_{q=j+1}^{k} Z_q \geq \max_{p: j<p\leq k} E^j E^{\tau_p} \sum_{q=\tau_p+1}^{k} Z_q\} && (8) \\
&= \inf\{j \geq i : 0 \geq \max_{p: j+1\leq p\leq k} E^j \sum_{q=j+1}^{\tau_p} -Z_q\}, \qquad i = 1, \ldots, k. && (9)
\end{aligned}$$

Interestingly, although the cash-flows $\mathcal{Z}_i$ in (2) are virtual in the sense that they generally have to be evaluated by computing or estimating conditional expectations, the complexity of the iterative method is not affected by this complication since the inner conditional expectations in (8) drop by the well-known tower property, and thus (9) follows.

For the cancelable product (3) we obtain from (5) for an input cancellation policy σ, the improved policy

$$\begin{aligned}
\widehat{\sigma}_i &:= \inf\{j \geq i : \mathcal{U}_j \geq \max_{p: j<p\leq k} E^j \mathcal{U}_{\sigma_p}\} \\
&= \inf\{j \geq i : \sum_{q=1}^{j} Z_q \geq \max_{p: j<p\leq k} E^j \sum_{q=1}^{\sigma_p} Z_q\} \\
&= \inf\{j \geq i : 0 \geq \max_{p: j+1\leq p\leq k} E^j \sum_{q=j+1}^{\sigma_p} Z_q\}, \qquad i = 1, \ldots, k. && (10)
\end{aligned}$$

An important issue of course is the choice of the input stopping family τ and σ in (9) and (10). Due to the similarity of (9) and (10) let us only consider (10). By choosing the trivial family $\sigma_i \equiv i$ we are faced with the evaluation of the conditional expectations $E^j Z_q$ for $q > j$ in (10). When these are available in closed form we may compute (estimate) $\widehat{Y}_i := E^i \mathcal{U}_{\widehat{\sigma}_i}$ via (standard) Monte Carlo simulation. After improving $\widehat{\sigma}$ in turn via (10) again, we obtain a next improved stopping family $\widehat{\widehat{\sigma}}$ via Monte Carlo simulation along each simulated Libor trajectory. We so arrive at a next improved estimation $\widehat{\widehat{Y}}_i$ via a nested Monte Carlo simulation.

Not in all situations closed form solutions (or close approximations) for the Europeans $E^j Z_q$, $q > j$, are known. For such cases it is usually better not to start (10) with the trivial stopping family above. As an alternative one could take

$$\sigma_i := \inf\{j \geq i : 0 \geq Z_j\}, \tag{11}$$

hoping that the family $\widehat{\sigma}$ obtained via (10) gives a good exercise policy in the sense that $\widehat{Y}_i$, which now requires nested Monte Carlo simulation, is close enough to Y^*. The starting policy (11) may be refined to

$$\sigma_i := \inf\{j \geq i : H_j \geq Z_j\}, \tag{12}$$

where the deterministic sequence H is pre-computed via a standard optimization procedure as studied in Andersen [1] for Bermudan swaptions (see also [12]).

Remark 2 In case where the cash-flow Z_{j+1} is already known at T_j from the contract (like in the example in Section 3) we replace Z_j by Z_{j+1} in (11) and (12).

3 Example: Iterating cancelable Snowballs

3.1 Product specification and valuation

Let us consider a *snowball swap* contract on a 1\$ nominal loan. According to this contract one has to pay, instead of floating Libor, so called Snowball coupons which follow the following term sheet. One pays on a semi-annual base a constant rate I over the first year and in the forthcoming years (Previous Coupon+A-Libor)$^+$, where A increases as specified in the contract. A *cancelable snowball swap* is a snowball swap which may be cancelled after first year. We here consider this cancelable snowball product in a (semi-annual) Libor model (1). The snowball coupons K_i, settled at T_{i+1} ($i = 0, \ldots, n-1$), are thus specified by

$$\begin{aligned} K_i &:= I, \quad i = 0, 1, \\ K_i &:= (K_{i-1} + A_i - L_i(T_i))^+ \quad i = 2, \ldots, n-1. \end{aligned}$$

We consider a contract where A increases on an annual base according to $A_2 := S$, , A_{i+1} $= A_i$ if i is even, and $A_{i+1} = A_i + s$ if i is odd, where S and s are given in the contract. The value V_0 of the cancelable snowball swap at $t = T_0 = 0$ is given by (3) with

$$Z_q := \frac{L_{q-1}(T_{q-1})\delta_{q-1} - K_{q-1}\delta_{q-1}}{B_*(T_q)}, \quad q = 1, \ldots, n.$$

Note that $Z_1, .., Z_n$ is an adapted (even predictable) sequence of cash-flows.

We now present different methods for computing V_0 based on Section 2.4. To this end we consider the Markov process $(B_*(T_j), L_j(T_j), K_j)$ and evaluate V_0 using a Monte Carlo algorithm, see Section 2.3. Unfortunately, we don't have closed form expressions for

$$E^j \sum_{q=j+1}^{p} Z_q = E^j \sum_{q=j+1}^{p} \frac{L_{q-1}(T_{q-1})\delta_{q-1} - K_{q-1}\delta_{q-1}}{B_*(T_q)} \tag{13}$$

so at first glance it is not efficient to apply (10) to the trivial initial stopping family $\sigma_i :\equiv i$. However, due to the standard relationship

$$B_*(T_j)E^j \sum_{q=j+1}^{p} \frac{L_{q-1}(T_{q-1})\delta_{q-1}}{B_*(T_q)} = 1 - B_p(T_j)$$

we may modify (13) into

$$E^j \sum_{q=j+1}^{p} Z_q = \frac{1-B_p(T_j)}{B_*(T_j)} - E^j \sum_{q=j+1}^{p} \frac{K_{q-1}\delta_{q-1}}{B_*(T_q)}$$

and then approximate $E^j \sum_{q=j+1}^{p} Z_q$ by

$$E^j \sum_{q=j+1}^{p} \widetilde{Z}_q := \frac{1-B_p(T_j)}{B_*(T_j)} - \frac{K_j\delta_j}{B_*(T_{j+1})} - E^j \sum_{q=j+2}^{p} \frac{\widetilde{K}_{q-1}\delta_{q-1}}{B_*(T_q)} \tag{14}$$

with

$$\widetilde{K}_{q-1} := (K_j + A_{q-1} - L_{q-1}(T_{q-1}))^+, \qquad j+2 \le q \le p.$$

The last term in (14) may be rewritten as

$$\begin{aligned} E^j \sum_{q=j+2}^{p} \frac{\widetilde{K}_{q-1}\delta_{q-1}}{B_*(T_q)} &= \frac{1}{B_*(T_j)} \sum_{q=j+2}^{p} B_*(T_j)E^j \frac{(K_j + A_{q-1} - L_{q-1}(T_{q-1}))^+\delta_{q-1}}{B_*(T_q)} \\ &= \frac{1}{B_*(T_j)} \sum_{q=j+2}^{p} B_q(T_j)E^j_{B_q}(K_j + A_{q-1} - L_{q-1}(T_{q-1}))^+\delta_{q-1}, \end{aligned}$$

where E_{B_q} denotes the T_q-forward measure, and can thus be evaluated in closed form by Black's 76 formula in a Libor market model. Instead of the via (10) improved trivial stopping family we may so consider the policy

$$\widetilde{\sigma}_i := \inf\{j \ge i : 0 \ge \max_{p:j+1\le p\le n} E^j \sum_{q=j+1}^{p} \widetilde{Z}_q\}, \tag{15}$$

yielding $\widetilde{Y}_i := E^i \mathcal{U}_{\widetilde{\sigma}_i}$ and then take this policy as input family for (10) again, thus yielding $\widehat{\widetilde{Y}}_i := E^i \mathcal{U}_{\widehat{\widetilde{\sigma}}_i}$. As a refinement of (15) we can also consider

$$\widetilde{\sigma}_{H,i} := \inf\{j \ge i : H_j \ge \max_{q\ge j+1} E^j \sum_{p=j+1}^{q} \widetilde{Z}_p\} \tag{16}$$

where the sequence H is pre-computed via an optimization procedure as for (12). The corresponding approximation for the Snell envelope is denoted by $\widetilde{Y}_{H,i} := E^i \mathcal{U}_{\widetilde{\sigma}_{H,i}}$ and its improvement by $\widehat{\widetilde{Y}}_{H,i} := E^i \mathcal{U}_{\widehat{\widetilde{\sigma}}_{H,i}}$.

For the example in Section 3.2 below we further investigate the families (11) and (12), taking into account Remark 2. The corresponding approximations of the Snell envelope and their improvements are denoted by $\overline{Y}_i$, $\overline{Y}_{H,i}$, $\widehat{\overline{Y}}_i$, and $\widehat{\overline{Y}}_{H,i}$, respectively.

3.2 Numerical results

In this section we carry out simulation experiments, where we consider a 6yr Snowball with $\delta_i = 0.5$yr and take

$$I = 0.079, \quad S = 0.01, \quad s = 0.005,$$

with the exercise possibilities at $T_2, \ldots, T_{11}$.

In the Libor model (1) we take the following volatility structure,

$$\gamma_i(t) = c_i g(T_i - t)e_i, \quad \text{where} \quad g(s) = g_\infty + (1 - g_\infty + as)e^{-bs} \tag{17}$$

is a parametric volatility function proposed by Rebonato [10], and e_i are d-dimensional unit vectors, decomposing some input correlation matrix of rank d. For generating Libor models with different numbers of factors d, we take as basis an endogenously full-rank correlation structure of the form

$$\rho_{ij} = \exp\left[\frac{|j-i|}{n-2}\ln\rho_\infty\right], \quad 1 \leq i, j \leq n-1. \tag{18}$$

with $n > 2$ (at least two Libors). For more general correlation structures we refer to Schoenmakers and Coffey [13]. For a particular choice of d we then deduce from ρ in (18) a rank-d correlation matrix ρ^d with decomposition $\rho^d_{ij} = e_i \cdot e_j$, $1 \leq i, j < n$, by principal component analysis.

We consider a Libor market model with typical model parameters which are obtained from a calibration to at-the-money (ATM) caps and swaptions volatilities. See for example Schoenmakers [12], Chapters 1-3, for details on calibration of a parametric Libor market model. For the correlation structure (18) we have $\rho_\infty = 0.663$, the parameters of the volatility function (17) are given in Table 1 and Table 2, and the initial Libor curve is given in Table 3.

Table 1. Vector c_i.

i	$d=1$	$d=3$	$d=5$	$d=11$
1	0.127	0.143	0.147	0.153
2	0.118	0.134	0.137	0.143
3	0.119	0.132	0.135	0.140
4	0.121	0.133	0.136	0.140
5	0.123	0.134	0.136	0.139
6	0.124	0.134	0.135	0.138
7	0.125	0.134	0.135	0.137
8	0.125	0.133	0.134	0.136
9	0.125	0.132	0.133	0.135
10	0.125	0.132	0.132	0.134
11	0.124	0.130	0.130	0.132

Table 2. Parameters a, b and g_∞.

d	a	b	g_∞
1	2.958	2.000	1.500
3	1.698	2.000	1.487
5	1.382	2.000	1.500
11	0.976	2.000	1.500

Table 3. Initial Libor curve.

Tenors	L_0
0.0	0.023
0.5	0.025
1.0	0.027
1.5	0.027
2.0	0.031
2.5	0.031
3.0	0.033
3.5	0.034
4.0	0.036
4.5	0.036
5.0	0.038
5.5	0.039

For comparison we state the prices of the Europeans on the snowball with the different maturities in Table 4. They are calculated via standard Monte-Carlo using 10^7 simulated paths. Noteworthy all Europeans have negative value while we will see from Tables 5–8 below that the Bermudan has positive value.

Table 4. Values of European options on the Snowball contract for different maturities and number of factors (in base points).

Maturity	$d=1$	$d=3$	$d=5$	$d=11$
0.5yr	-540.646(0.007)	-540.656(0.007)	-540.644(0.007)	-540.639(0.007)
1.0yr	-713.665(0.023)	-713.692(0.023)	-713.657(0.023)	-714.641(0.023)
1.5yr	-802.502(0.051)	-802.562(0.051)	-802.501(0.051)	-802.466(0.051)
2.0yr	-802.090(0.088)	-802.318(0.088)	-802.261(0.088)	-802.137(0.051)
2.5yr	-750.240(0.125)	-750.704(0.126)	-750.694(0.126)	-750.396(0.125)
3.0yr	-671.058(0.163)	-671.645(0.164)	-671.591(0.164)	-671.109(0.163)
3.5yr	-579.752(0.202)	-580.263(0.201)	-580.044(0.201)	-579.326(0.200)
4.0yr	-483.586(0.243)	-483.707(0.241)	-483.200(0.241)	-482.224(0.240)
4.5yr	-388.742(0.286)	-388.006(0.283)	-387.237(0.283)	-385.962(0.282)
5.0yr	-299.562(0.335)	-297.540(0.330)	-296.532(0.329)	-294.928(0.328)
5.5yr	-219.654(0.389)	-215.979(0.380)	-214.592(0.379)	-212.613(0.378)

We now investigate the stopping family (15) and its refinement (16). In Tables 5 and 6 we present the lower bounds $\widetilde{Y}_0$ and $\widetilde{Y}_{H,0}$, their improvements $\widehat{\widetilde{Y}}_0$ and $\widehat{\widetilde{Y}}_{H,0}$ and their dual upper bounds $\widetilde{Y}_0^{up}$ and $\widetilde{Y}_{H,0}^{up}$ (all the values are in base points). We construct 10^7 trajectories for $\widetilde{Y}_0$ and $\widetilde{Y}_{H,0}$. Next, we construct $\widehat{\widetilde{Y}}_0$ and $\widehat{\widetilde{Y}}_{H,0}$ using variance reduction (see Remark 1). Here we estimate $\widehat{\widetilde{Y}}_0 - \widetilde{Y}_0$ and $\widehat{\widetilde{Y}}_{H,0} - \widetilde{Y}_{H,0}$ using $3 \cdot 10^5$ and 10^5 trajectories, respectively, in order to keep the standard deviation within 0.5% relative to the value. For each trajectory we used 500 inner simulations. Further, we simulate $\widetilde{Y}_0^{up} - \widetilde{Y}_0$ and $\widetilde{Y}_{H,0}^{up} - \widetilde{Y}_{H,0}$ using $3.5 \cdot 10^4$ and 10^4 outer trajectories, respectively, in order to keep the standard deviation within 0.5% relative again, and 500 inner trajectories. The vector H in (12) is computed using $5 \cdot 10^5$ pre-simulations. We see, that the refined exercise policy (16) provides a higher lower bound (7.5%–9% relatively), than (15). However, the improved lower bounds are almost the same and coincide with $\widetilde{Y}_{H,0}^{up}$ within one standard deviation.

Table 5.

d	$\widetilde{Y}_0$ (SD)	$\widehat{\widetilde{Y}}_0$ (SD)	$\widetilde{Y}_0^{up}$ (SD)
1	62.575(0.233)	68.293(0.357)	68.932(0.323)
3	58.226(0.230)	64.312(0.345)	65.231(0.338)
5	58.799(0.229)	65.187(0.345)	65.795(0.338)
11	59.742(0.229)	64.815(0.341)	66.710(0.335)

Table 6.

d	$\widetilde{Y}_{H,0}$ (SD)	$\widehat{\widetilde{Y}}_{H,0}$ (SD)	$\widetilde{Y}_{H,0}^{up}$ (SD)
1	68.089(0.236)	67.957(0.303)	68.491(0.240)
3	62.691(0.232)	64.222(0.350)	64.343(0.259)
5	63.134(0.232)	64.432(0.358)	64.964(0.265)
11	64.164(0.231)	65.621(0.348)	65.600(0.263)

We also consider (a naive) stopping family (11) and its refinement (12), see Tables 7 and 8 (all the values are in base points). Here we use 10^7 trajectories for $\overline{Y}_0$ and $\overline{Y}_{H,0}$ and $3.5 \cdot 10^5$ trajectories (with 500 nested simulations) for $\overline{Y}_0^{up} - \overline{Y}_0$ and $\overline{Y}_{H,0}^{up} - \overline{Y}_{H,0}$. Then, we construct the improved lower bounds $\widehat{\overline{Y}}_0$ and $\widehat{\overline{Y}}_{H,0}$ using variance reduction (see Remark 1), where we estimate $\widehat{\widetilde{Y}}_0 - \widetilde{Y}_0$ and $\widehat{\widetilde{Y}}_{H,0} - \widetilde{Y}_{H,0}$ by $5 \cdot 10^5$ trajectories (with 500 nested simulations). The vector H in (12) is constructed using $5 \cdot 10^5$ pre-simulations. We see, that the exercise policy (11) provides a rather crude lower bound $\overline{Y}_0$ and a rather crude dual upper bound. This is due to the fact that for most trajectories the exercise policy (11) exercises too early. Interestingly, the iteration procedure provides an substantial improvement of (11), which still differs from the Bermudan price about 5%–15% relatively, however. Finally we note, that the refinement (12) provides much better lower bounds (and corresponding dual upper bounds). The difference of the refined lower bound $\overline{Y}_{H,0}$ with the Bermudan price is approximately 20% relatively, and its improvement coincides with the Bermudan price within one standard deviation.

Table 7.

d	$\overline{Y}_0$ (SD)	$\widehat{\overline{Y}}_0$ (SD)	$\overline{Y}_0^{up}$ (SD)
1	-526.155(0.059)	63.572(1.011)	119.460(3.344)
3	-526.835(0.056)	56.347(0.995)	120.259(3.339)
5	-526.735(0.056)	58.484(0.991)	119.756(3.347)
11	-526.778(0.056)	62.441(0.986)	114.240(3.333)

Table 8.

d	$\overline{Y}_{H,0}$ (SD)	$\widehat{\overline{Y}}_{H,0}$ (SD)	$\overline{Y}_{H,0}^{up}$ (SD)
1	66.389(0.237)	67.673(0.261)	68.281(0.242)
3	53.254(0.235)	64.142(0.321)	65.488(0.316)
5	53.301(0.235)	64.883(0.322)	66.570(0.327)
11	53.373(0.235)	65.295(0.325)	68.026(0.339)

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