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Dynamic instabilities in the interaction of transverse modes in a class-B laser

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Abstract. A theoretical investigation is made of the dynamics of a class-B laser emitting the transverse modes TEM_{00}, TEM_{10}, and TEM_{01}. Bifurcation mechanisms of transitions between several lasing regimes are investigated for various intermode intervals and values of the excess above the threshold .

In spite of the fact that nontrivial lasing dynamics, resulting from the interaction of transverse modes, was discovered back in the sixties [1], right up to the late eighties the majority of the investigations of dynamic instabilities in lasers were being made in the plane-wave approximation [2], i.e. ignoring inhomogeneities of the spatial distribution of the electromagnetic field perpendicular to the optic axis. The recent upsurge of interest in the mechanisms of the formation and annihilation of transverse structures in lasers [3, 4] is associated with the fundamental problems of transition to spatiotemporal chaos (turbulence) and of spontaneous formation of transverse structures [5]. Moreover, there are also potential applications such as the use of multistable transverse structures in the fabrication of optical memories [6].

A nonlinear system of the Maxwell – Bloch equations in terms of partial derivatives, describing the dynamics of a laser and taking account of the transverse distribution of the electromagnetic field in the cavity, can either be solved directly or it can be reduced to a system of ordinary differential equations by expanding in terms of the empty-cavity modes. The former approach is best suited to wide-aperture systems [7, 8] in which excitation of a large number of transverse modes is possible. In the case of narrow-aperture systems with a small number of excited transverse modes, it is convenient to use these modes as the basis functions [4, 9, 10].

We shall use this approach in a study of dynamic instabilities of generation of the fundamental TEM_{00} mode in a class-B laser (such as an Nd : YAG laser, CO_2 laser, etc.) when this mode interacts with the modes TEM_{10} and TEM_{01} which are closest in frequency. The complex spatiotemporal dynamics of a laser observed at large Fresnel numbers for a considerable excess above the threshold [7, 8] will not be considered. Our attention will be concentrated on the range of the small excess of the pumping rate above the threshold. Transverse inhomogeneities in the population inversion distribution are the main source of nontrivial dynamics in the case we shall consider.

We shall assume that one longitudinal mode and a corresponding set of transverse modes are excited in a laser. We shall consider unidirectional emission from a ring laser in order to avoid the influence of the inversion dips along the optic axis on laser dynamics and to concentrate on the transverse effects. In the derivation of the main equations we shall assume that the diffraction losses are the same for all the modes. The relevant parameters can then be introduced readily into the investigated equations.

We shall represent the field E and the polarisation P as:

$$E = \sum_{m,n} E_{mn}(x, y, z)\psi_{mn}(t)\exp(-\mathrm{i}t\omega_{mn}) \,,$$
$$P = \sum_{m,n} E_{mn}(x, y, z)\phi_{mn}(t)\exp(-\mathrm{i}t\omega_{mn}) \,. \tag{1}$$

Here, x and y are the transverse coordinates; z is the coordinate along the optic axis; E_{mn} are the Hermite – Gauss modes; ω_{mn} are the frequencies of the Hermite – Gauss TEM_{nm} modes. We shall use the approximation of class-B lasers ($\gamma_\perp \gg \varkappa, \gamma_\|$) in order to simplify the material equations, i.e. we shall assume that $\partial_t \phi_{mn} = 0$. Here, $\varkappa$, $\gamma_\|$, and $\gamma_\perp$ are the decay constants of the electromagnetic field, of the population inversion, and of the polarisation, respectively. In a nonastigmatic cavity, the modes with the same sum of indices $m + n = q$ are frequency-degenerate. Astigmatism lifts this degeneracy. We shall assume that the frequency interval occupied by a family of transverse modes with the specific index q is much less than the interval $\delta\omega$ between the adjacent mode families. We shall make two more assumptions, which are usually satisfied in practice: $|\varkappa - n_{\mathrm{th}}| \ll \delta\omega$ and $\gamma_\| \ll \delta\omega$ (where n_{th} is the unsaturated population inversion). Then, in the lasing equations we can ignore the term that oscillates at a frequency $\omega_{m_1 n_1} - \omega_{mn}$ such that $m_1 + n_1 \neq m + n$. This means that we are neglecting the phase coupling between different transverse-mode groups and ignoring locking of their frequencies (transverse mode locking) [11].

The Maxwell – Bloch system of equations can now be reduced to [12]

$$(\partial_t + \varkappa)\psi_{mn} = \mathcal{L}_{mn}(1 - \mathrm{i}\Delta_{mn})\left(n_{\mathrm{th}}\psi_{mn} + \sum_{\substack{m_1, n_1 \\ (m_1+n_1=m+n)}} \psi_{m_1 n_1} N_{mn\, m_1 n_1}\right),$$
$$(\partial_t + \gamma_{||})N_{mn\; m_1 n_1} = -n_{\mathrm{th}} \sum_{\substack{m_2 n_2,\, m_3 n_3 \\ (m_2+n_2=m_3+n_3)}} \mathcal{L}_{m_3 n_3}\psi_{m_2 n_2}\psi^*_{m_3 n_3}\Theta^{m_2 n_2\; m_3 n_3}_{mn\; m_1 n_1} \,. \tag{2}$$

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Here,

$$\mathcal{L}_{mn} = \frac{1}{\gamma_\perp(1+\Delta_{mn}^2)}\ ; \qquad \Delta_{mn} = \frac{\omega_{ab}-\omega_{mn}}{\gamma_\perp}\ ;$$

$$N_{mnm_1n_1} = \int_V \mathrm{d}VE^*_{mn}NE_{m_1n_1}\ ;$$

$$\Theta^{m_2n_2m_3n_3}_{mnm_1n_1} = \int_V \mathrm{d}VE^*_{mn}E_{m_1n_1}E^*_{m_2n_2}E_{m_3n_3}\ ;$$

ω_{ab} is the frequency of a lasing transition.

When the TEM_{00}, TEM_{10}, and TEM_{01} modes are generated, the total field in a transverse cross section is

$$E = \frac{1}{\sqrt{\pi}}\{\psi_{00}(t)\exp(-\mathrm{i}\omega_{00}t) + \sqrt{2}[x\psi_{10}(t)\exp(-\mathrm{i}\omega_{10}t) + y\psi_{01}(t)\exp(-\mathrm{i}\omega_{01}t)]\}\exp[-\tfrac{1}{2}(x^2+y^2)] + \text{c.c.}\ . \tag{3}$$

The coordinates x and y are assumed to be normalised to the corresponding beam radii w_x and w_y. The exponential factors which govern the wavefront curvature are unimportant for our analysis and, therefore, they are omitted. Going over from the Cartesian to the polar coordinates in accordance with the formulas $x=\rho\cos\vartheta$, $y=\rho\sin\vartheta$, we obtain the field distribution not in the Hermite–Gauss mode basis, as in expression (3), but the field in the Laguerre–Gauss mode basis:

$$E = \frac{1}{\sqrt{\pi}}\left\{\psi_{00}(t)\exp(-\mathrm{i}\omega_{00}t) + \frac{\rho}{\sqrt{2}}[\psi_+(t)\exp(\mathrm{i}\vartheta) + \psi_-(t)\exp(-\mathrm{i}\vartheta)]\exp\left[-\mathrm{i}\frac{(\omega_{10}+\omega_{01})t}{2}\right]\right\} \times\exp\left(-\frac{\rho^2}{2}\right) + \text{c.c.}\ , \tag{4}$$

where

$$\psi_\pm = \psi_{10}\exp\left[\mathrm{i}\frac{(\omega_{10}-\omega_{01})t}{2}\right] \mp \mathrm{i}\psi_{01}\exp\left[\mathrm{i}\frac{(\omega_{01}-\omega_{10})t}{2}\right]\ .$$

The Laguerre–Gauss modes with the amplitudes $\psi_\pm$ are usually denoted by TEM^*_{10} and TEM^*_{01}. We can easily demonstrate that the splitting of the frequencies of the TEM_{10} and TEM_{01} modes in an astigmatic cavity means that there is a linear coupling between the TEM^*_{10} and TEM^*_{01} modes. This case is analogous to a bidirectional ring laser in which the total field can be represented on the basis of standing or travelling waves [13].

If we ignore the difference between Δ_{10} and Δ_{01} (because of the large width of the gain line), and if we introduce new variables and parameters

$$\varepsilon_0 = \mathcal{L}_{00}n_{th} - \varkappa\ , \quad \varepsilon_1 = \mathcal{L}_{10}n_{th} - \varkappa\ , \quad \tau = t(\varepsilon_1\gamma_{||})^{1/2}\ ,$$

$$F_0 = \psi_{00}\mathcal{L}_{10}\left(\frac{n_{\rm th}\Theta^{0110}_{1001}}{\varepsilon_1\gamma_{||}}\right)^{1/2}\exp(-\mathrm{i}\varkappa\Delta_{00}t)\ ,$$

$$F_\pm = \psi_\pm\mathcal{L}_{10}\left(\frac{n_{\rm th}\Theta^{0110}_{1001}}{\varepsilon_1\gamma_{||}}\right)^{1/2}\exp(-\mathrm{i}\varkappa\Delta_{10}t)\ , \tag{5}$$

$$M_0 = \frac{\mathcal{L}_{10}N_{0000}+\varepsilon_0}{(\varepsilon_1\gamma_{||})^{1/2}}\ , \quad N_0 = \frac{\mathcal{L}_{10}N_{1010}+\varepsilon_1}{(\varepsilon_1\gamma_{||})^{1/2}}\ ,$$

$$N_2 = \frac{\mathcal{L}_{10}N_{1001}}{(\varepsilon_1\gamma_{||})^{1/2}}\ , \quad \mathcal{L} = \frac{\mathcal{L}_{00}}{\mathcal{L}_{10}} = \frac{1+\Delta_{10}^2}{1+\Delta_{00}^2}\ , \quad \varepsilon = \frac{\varepsilon_0}{\varepsilon_1}\ , \quad \gamma = \left(\frac{\gamma_{||}}{\varepsilon_1}\right)^{1/2},$$

and calculate the overlap integrals $\Theta^{m_2n_2m_3n_3}_{mnm_1n_1}$, we obtain the following system of equations:

$$\begin{aligned}
&\partial_\tau F_0 = 2\mathcal{L}(1-\mathrm{i}\Delta_0)F_0M_0\ ,\\
&\partial_\tau F_+ = (1-\mathrm{i}\Delta_1)(F_+N_0 + F_-N_2) + \mathrm{i}R\exp(\mathrm{i}\psi)F_-\ ,\\
&\partial_\tau F_- = (1-\mathrm{i}\Delta_1)(F_-N_0 + F_+N_2^*) + \mathrm{i}R\exp(\mathrm{i}\psi)F_+\ ,\\
&\partial_\tau M_0 = \varepsilon - \gamma M_0 - 2\mathcal{L}|F_0|^2 - |F_+|^2 - |F_-|^2\ ,\\
&\partial_\tau N_0 = 1 - \gamma N_0 - \mathcal{L}|F_0|^2 - |F_+|^2 - |F_-|^2\ ,\\
&\partial_\tau N_2 = -\gamma N_2 - F_+F_-^*\ .
\end{aligned} \tag{6}$$

Here and later, we have $\Delta_{00}=\Delta_0$, $\Delta_{10}=\Delta_1$. The system of equations (6) implies that the losses can be different for each of the three modes. The parameter ε represents the ratio of the excess of the gain above the fundamental-mode losses to the average excess for the TEM_{10} and TEM_{01} modes. The quantities $R\cos\psi$ and $R\sin\psi$ characterise respectively the difference between the frequencies and between the losses of the TEM_{10} and TEM_{01} modes. If $\psi=0$, then $R=|\omega_{10}-\omega_{01}|/2(\varepsilon_1\gamma_{||})^{1/2}$. It should be mentioned that if $F_0=0$, the system of equations (6) becomes identical with the system describing generation of counterpropagating waves in a ring laser with backscattering [14].

The solutions of the system of equations (6) can be separated into two classes: with zero and nonzero intensities of the TEM_{00} mode. We shall consider only the situation when $|F_0|\neq 0$. A detailed study of the generation of the two modes TEM^*_{10} and TEM^*_{01} will be published elsewhere.

We shall first consider the steady-state solutions. The solution corresponding to the generation of the TEM_{00} fundamental mode is

$$|F_0|^2 = \frac{\varepsilon}{2\mathcal{L}}\ , \qquad F_\pm = 0\ .$$

A pair of solutions of the standing-wave type, corresponding to simultaneous generation of the TEM_{00}, TEM^*_{01}, and TEM^*_{10} modes, is

$$|F_0|^2 = \frac{1}{2\mathcal{L}}\left(\frac{3\varepsilon}{2} - 1 \mp \gamma R\sin\psi\right)\ ,$$

$$|F_+|^2 = |F_-|^2 = \frac{1}{2}\left(1 - \frac{\varepsilon}{2} \pm \gamma R\sin\psi\right)\ .$$

The upper (lower) sign corresponds to $\arg F_+ - \arg F_- = 0(\pi)$. The total intensity of the radiation generated by a laser under these conditions is modulated at a frequency $|\frac{1}{2}(\omega_{10}+\omega_{01}) - \omega_{00}|$. We shall refer to these solutions as the modulated standing waves MSW1 and MSW2.

If $R=0$, analytic formulas for solutions of the $|F_0|\neq 0$, $|F_+|\neq 0$, $F_-=0$ and $|F_0|\neq 0$, $F_+=0$, $|F_-|\neq 0$ type can be obtained quite readily. Astigmatism transforms these solutions to those of the $|F_0|\neq 0$, $|F_+|>|F_-|$ and $|F_0|\neq 0$, $|F_+|<|F_-|$ type, where $|F_\pm|\neq 0$. They will be called the modulated travelling waves (MTW1 and MTW2). It is clear from numerical calculations that if astigmatism is weak, these solutions are stable in a wide range near the centre of the gain line.

We shall now assume that the frequency of the TEM_{00} mode is in resonance with the frequency of a laser transition ($\Delta_0=0$), so that the parameter Δ_1 represents the frequency interval between the adjacent mode families and that Δ_1 is always positive. The parameter ε can be controlled experimentally with the aid of an aperture and ψ will be assumed to be always $\pi/10$, which takes account of the slight difference between the losses of the TEM^*_{10} and TEM^*_{01} modes. We shall now estimate the parameters occurring in the investigated equations and corresponding to, for example, CO_2 and Nd:YAG lasers for which the above approximations are valid: $\gamma_{\parallel} \sim 10^2 - 10^4$ Hz, $\gamma_{\perp} \sim 10^9 - 10^{11}$ Hz, $\varkappa \sim 10^6 - 10^8$ Hz, $|\omega_{10} - \omega_{00}| \sim 10^8$ Hz. Splitting of the frequencies of the TEM^*_{10} and TEM^*_{01} modes is governed by the cavity astigmatism and it does not usually exceed $10^6 - 10^7$ Hz.

We investigated numerically the stability of the steady-state solutions. A bifurcation diagram in the (ε, R) plane is plotted in Fig. 1 for $\Delta_1 = 0.05$, $\gamma = 0.1$. Small changes in Δ_1 near zero do not alter qualitatively this diagram. The first instability of the TEM_{00} mode, which appears on reduction in ε, is a transcritical bifurcation which occurs on the AB line. This bifurcation ensures stability of the MSW1 solution, for which the total intensity of the TEM^*_{01} and TEM^*_{10} modes is higher than for MSW2. In turn, the solution MSW1 becomes unstable as a result of a supercritical Hopf bifurcation on the AC line. In the region FEGDAF the pair of solutions MTW1 and MTW2 is stable. On the EF and AD lines these solutions experience subcritical Hopf bifurcations and in the segment GD they undergo supercritical bifurcations. The point D represents a bifurcation of codimensionality of 2.

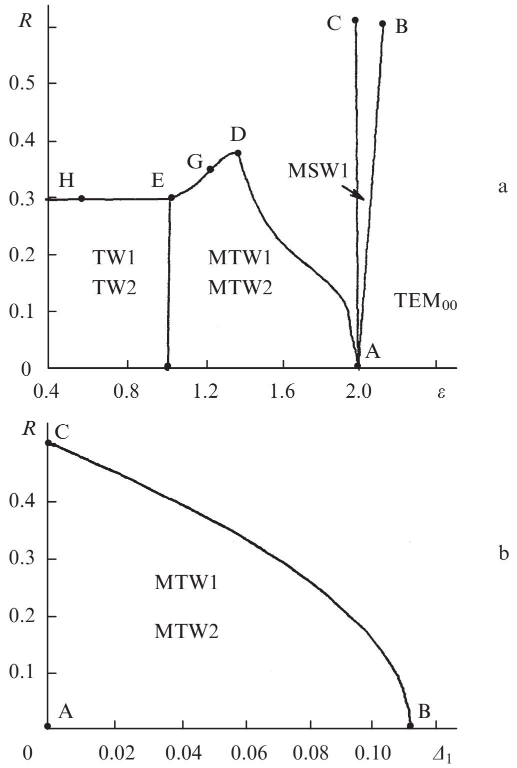

Figure 1. Bifurcation diagrams of stationary solutions in the (ε, R) (a) and (Δ_1, R) (b) planes.

Bifurcation phenomena in the vicinity of the point of intersection of two Hopf bifurcation lines are extremely varied and their detailed investigation requires an analysis of what are known as the normal forms [15] of the set of equations (6). In general, there may be quasiperiodic and chaotic attractors in the vicinity of the point of intersection of two Hopf bifurcation lines [15]. On the EF line the solutions MTW1 and MTW2 undergo transcritical bifurcations with the result that the stable pair of solutions is $F_0 = 0$, $|F_+| > |F_-|$ and, correspondingly, $F_0 = 0$, $|F_+| > |F_-|$. We shall call them the travelling waves TW1 and TW2. These solutions in turn experience a subcritical Hopf bifurcation on the HE line.

A characteristic feature of a laser generating the TEM_{00}, TEM^*_{10}, and TEM^*_{01} modes is the possibility of existence of an optical vortex in a transverse cross section of the laser beam [4]. The coordinates of this vortex x_v and y_v are found by equating to zero the real and imaginary parts of the field intensity in the laser:

$$x_v + iy_v = \frac{F_- F_0^* \exp\left\{it[-\frac{1}{2}(\omega_{10} + \omega_{01}) + \omega_{00}]\right\}}{\sqrt{2}\left(|F_+|^2 - |F_-|^2\right)} - \frac{F_+^* F_0 \exp\left\{it[-\frac{1}{2}(\omega_{10} + \omega_{01}) + \omega_{00}]\right\}}{\sqrt{2}\left(|F_+|^2 - |F_-|^2\right)} . \quad (7)$$

It is evident from the above expression that when the TEM_{00} mode is not excited, the vortex is on the optic axis. If the regimes with $|F_+| = |F_-|$ are excited, there is no vortex in a transverse section, or, more exactly, the vortex is at infinity. In the range of stability of MTW1 and MTW2 the vortex rotates about the optic axis at a frequency $\frac{1}{2}(\omega_{10} + \omega_{01}) - \omega_{00}$ (Fig. 2). In the vicinity of the point D there is an interesting bistability of the vortex motion. For example, for $\varepsilon = 1.3$ and $R = 0.25$ the initial condition is assumed to be the solution MSW1, a quasiperiodic regime shown in Fig. 3a is found to

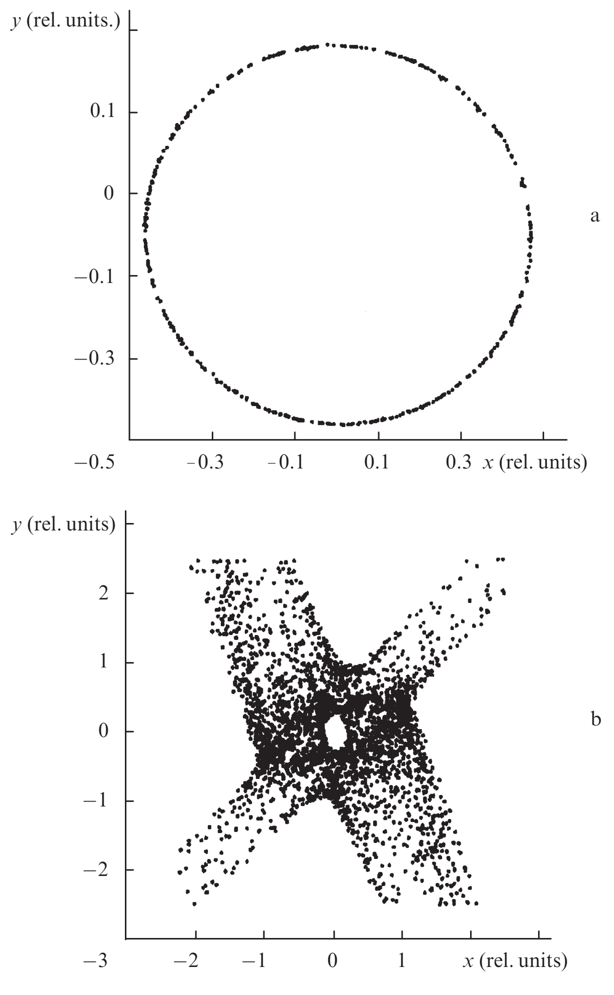

Figure 2. Bistability of the motion of an optical vortex, calculated for $\varepsilon = 1.3$, $R = 0.25$, $\gamma = 0.1$, $\psi = \pi/10$, $\Delta_1 = 0.05$. The points represent the position of the vortex at consecutive moments in time.

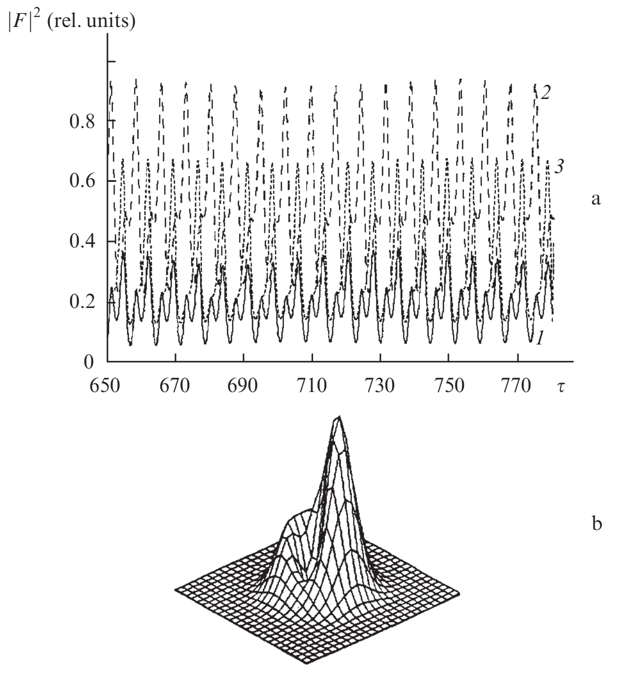

Figure 3. Quasiperiodic time dependences of the intensities $|F|^2$ of the TEM^*_{10} (*1*), TEM^*_{01} (*2*), and TEM_{00} (*3*) modes, corresponding to the motion of a vortex in Fig. 2b (a) and the transverse intensity profile for structures with a rotating vortex at a fixed moment in time (b).

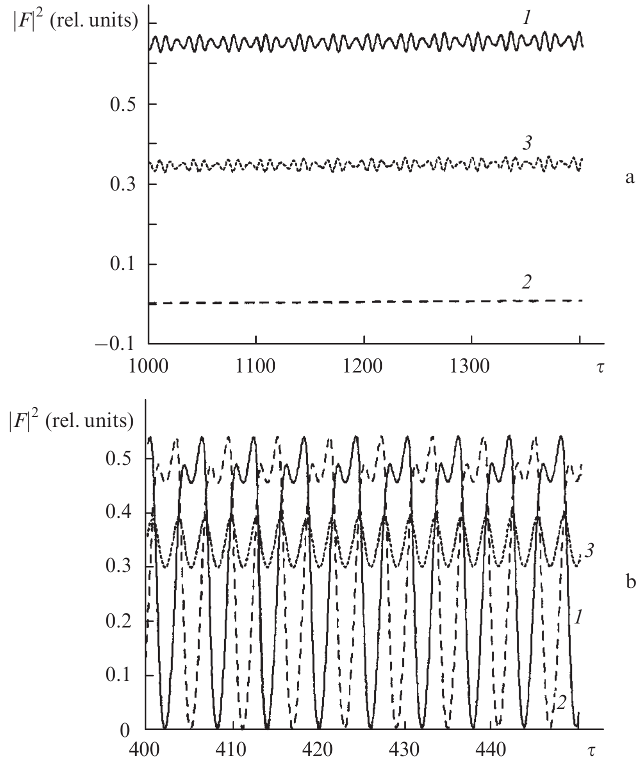

Figure 4. Bistable regimes predicted for the TEM^*_{10} (*1*), TEM^*_{01} (*2*), and TEM_{00} (*3*) modes, calculated for $\varepsilon = 1.34$, $R = 0.385$, $\gamma = 0.1$, $\psi = \pi/10$, $\Delta_1 = 0.05$.

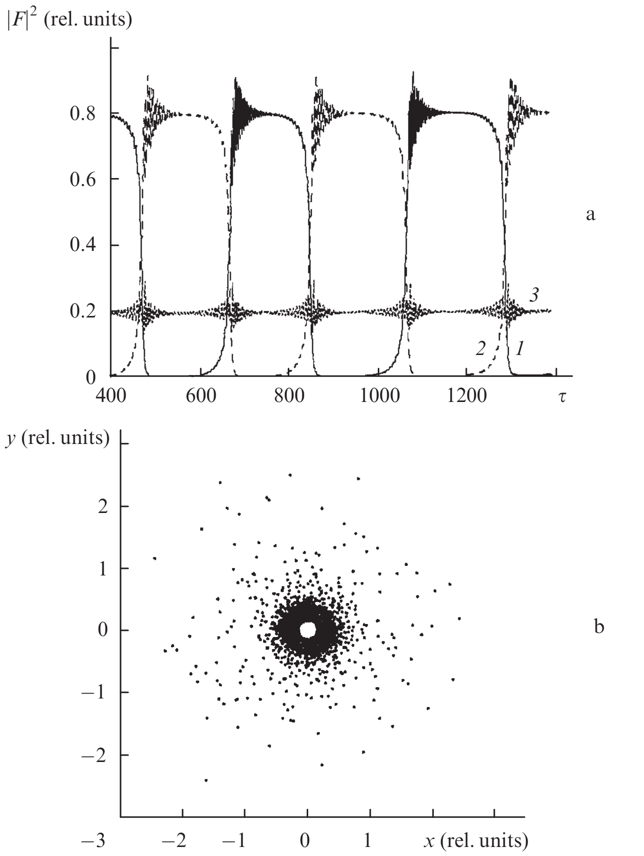

Figure 5. Intensities of the TEM^*_{10} (*1*), TEM^*_{01} (*2*), and TEM_{00} (*3*) modes (a) and the corresponding motion of a vortex (b), calculated for $\varepsilon = 1.2$, $R = 0.05$, $\gamma = 0.1$, $\psi = \pi/10$, $\Delta_1 = 016$.

be stable. The corresponding positions of the vortex during a time interval $\delta\tau = 1000$ s are represented by the points in Fig. 2b, spaced in steps of 0.5.

At the moments when $|F_+| = |F_-|$, the vortex escapes to infinity. For the same values of the parameters the stable solutions are MTW1 and MTW2. They correspond to the motion of a vortex on a circle (Fig. 2a). If $\varepsilon = 1.34$ and $R = 0.385$, the regimes shown in Fig. 4 are bistable. The corresponding vortex motion is similar to that shown in Fig. 2. Numerical calculations show that when astigmatism is strong, i.e. above the HEGD line, and for $l < t < 2$ the stable dynamic transverse structure is similar to that shown in Fig. 2b, but it has very definite mutually orthogonal directions along which the vortex is moving.

We shall now consider a bifurcation diagram in the (Δ_1, R) plane when $\varepsilon = 1.2$. The MTW1 and MTW2 solutions are stable in the region ABCA and they experience a subcritical Hopf bifurcation on the BC line. In the region where they are unstable, corresponding to small R, typical behaviour is represented by the regime shown in Fig. 5. In respect of the nature of antiphase oscillations of the TEM^*_{10} and TEM^*_{01} modes, this regime is similar to a self-oscillatory regime of the second kind in a ring class-B laser [14]. At high values of R there is a regime similar to that shown in Fig. 2b. A comparison of the paths of motion of a vortex in the case of strong and weak astigmatism shows that in the former case the vortex moves along two orthogonal directions, whereas in the latter case the vortex motion is almost uniformly smeared out over all the angles ϑ.

An approach to a study of the interaction of several transverse modes more rigorous than that adopted above involves reducing the Maxwell–Bloch system to a system of integrodifferential equations [4, 9]. However, the system (2) of ordinary differential equations makes it possible to analyse in detail the bifurcation mechanisms of the excitation of various dynamic regimes without significant distortion of the qualitative nature of the solutions. For example, the vortex motion paths similar to those shown in Fig. 2 had been obtained earlier for a three-mode laser by a more rigorous approach [4].

We thus considered the problem of generation of three transverse modes belonging to the $m = n = 0$ and $m + n = 1$ families in a class-B laser. An important feature of our analysis is that, on the one hand, the frequency interval between mode families is assumed to be sufficiently large so that they cannot become mutually locked and, on the other, that the frequency splitting within the $m + n = 1$ family is small and the TEM_{10} and TEM_{01} modes can be mutually locked or not locked. The main source of the instability of the regimes with constant mode intensities is an increase in both the above-mentioned frequency intervals. Stable generation of the fundamental mode requires that the excess above the threshold should be at least twice as high as that for the TEM^*_{01} and TEM^*_{10} modes ($\varepsilon > 2$). On the other hand, complete suppression of the fundamental mode simply requires that the excess above the threshold should be less than for the TEM^*_{01} and TEM^*_{10} modes ($\varepsilon < 1$). This is related to the circumstance that the modes of the $m + n = 1$ family occupy a larger volume of the active medium than the fundamental mode.

When all three modes TEM_{00}, TEM^*_{01}, and TEM^*_{10} are generated, an optical vortex moving across the beam is observed for a wide range of laser parameters. The motion of this vortex can be finite or infinite. The vortex escapes to infinity at the moments in time when the intensities of the TEM^*_{01} and TEM^*_{10} modes become equal. For some values of the parameters a bistability between finite and infinite motion of the vortex is established.

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