Universität Stuttgart

Sonderforschungsbereich 404

Mehrfeldprobleme in der Kontinuumsmechanik

Marita Thomas
Anna-Margarete Sändig

Energy release rate for interface-cracks in compounds of p-Laplacian type – Griffith formula and J-integral

Bericht 2006/13

Universität Stuttgart

Sonderforschungsbereich 404

Mehrfeldprobleme in der Kontinuumsmechanik

Marita Thomas
Anna-Margarete Sändig

Energy release rate for interface-cracks in compounds of p-Laplacian type – Griffith formula and J-integral

Bericht 2006/13

SFB 404
- Geschäftsstelle -
Pfaffenwaldring 57
70550 Stuttgart

Telefon: 0711/685-5554
Telefax: 0711/685-5599
E-Mail: sfb404@mathematik.uni-stuttgart.de
http://www.mathematik.uni-stuttgart.de/SFB404

ISSN 0949-2046

Abstract

Main goal of this paper is to prove a Griffith formula and a J-integral to determine the energy release rate for a quasistatic interface crack in a compound of materials of p-Laplacian type. The first mentioned formula is given as a volume-integral, where only quantities, which can be determined via FEM, occur in the integrands. The second formula corresponds to a path integral derived from Griffith formula by application of Gauss' theorem, but due to the low regularity of the solution it is expressed by dual pairings. In order to prove these formulae, the existence and uniqueness of minimizers as well as their regularity are discussed.

Key words: energy release rate, Griffith formula, Griffith fracture criterion, interface-crack, J-integral, p-Laplacian operator, p-structures, strain-hardening.

1 Introduction

In this paper, Griffith fracture criterion is applied on a quasistatic mode-III-interface crack in a compound of geometrically linear but physically nonlinear elastic materials of p-Laplacian type. This energetic fracture criterion was introduced by A. Griffith in 1921 and it reads as follows: *The crack is stationary under external loadings, if the total energy of the body in the current configuration is minimal in comparison to the total energy referring to any other possible configuration.*

Due to a quasistatic crack growth one introduces the energy release rate, which states the amount of energy that is released if the crack would extend in an infinitesimally small length:

$$ERR(\Omega_{\delta_0}) := \lim_{\delta\to 0} \frac{E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})}{\delta} = -\frac{\mathrm{d}E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})}{\mathrm{d}\delta}\bigg|_{\delta=0} . \tag{1}$$

Thereby $E(\cdot,\cdot)$ denotes the potential energy referring to the reference configuration Ω_{δ_0} or to a configuration $\Omega_{\delta_0+\delta}$ with a crack extended by the length δ. The functions $u_{[\delta_0]}$, $u_{[\delta_0+\delta]}$ are the corresponding minimizers and it will be shown in section 3 that they are unique.

Griffith fracture criterion can be expressed as follows: *The crack is stationary under external loadings, if*

$$ERR(\Omega_{\delta_0}) < \frac{\mathrm{d}D(\Omega_{\delta_0+\delta})}{\mathrm{d}\delta}\bigg|_{\delta=0} , \tag{2}$$

where $D(\cdot)$ is the dissipative energy of a configuration, taking into account all the irreversible processes that occur during cracking. In a simple model $D(\cdot)$ is assumed to be proportional to the crack length. In engineering applications inequality (2) involving the energy release rate is used to realize Griffith fracture criterion. The problem is now to derive formulae to compute the energy release rate efficiently, since definition equation (1) is not applicable for numerical computations. Thus, the main focus of this paper lies in the proof of a so called Griffith formula for the compound of p-Laplacian type, which is carried out in section 4. Griffith formula allows to calculate the energy release rate as a volume integral in dependence of the given volume force density and of quantities that can be determined via FEM, like the minimizer of the transmission bondary value problem, stresses, elastic strain energy density.

So, Griffith formula for the considered compound reads as follows:

$$\begin{aligned} ERR(\Omega_{\delta_0}, u_{[\delta_0]}) = &\sum_{j=1}^{2} \int_{\Omega_j} \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right)\partial_{y_1} u_{[\delta_0]\,j} \cdot \nabla\theta \,\mathrm{d}\mathbf{y} \\ &- \sum_{j=1}^{2} \int_{\Omega_j} W_{el\,j}(\nabla u_{[\delta_0]\,j})\partial_{y_1}\theta \,\mathrm{d}\mathbf{y} + \sum_{j=1}^{2} \int_{\Omega_j} u_{[\delta_0]\,j}\partial_{y_1}(f_j\theta)\,\mathrm{d}\mathbf{y}\,, \end{aligned} \tag{3}$$

where ∂_{y_1} is the derivative in crack extension direction $(1,0)^\top$, that has been chosen arbitrarily but fixed in the following sections.
In section 4.2 some numerical examples are discussed, where Griffith formula is used in the postprocessing of a FEM code in order to compute the energy release rate in dependence of the crack length and of material parameters.
Finally, in section 5 the J-integral for compounds of p-Laplacian types is proved rigorously. If the minimizer is regular enough, this is a path integral formula to determine the energy release rate, that can be formally derived by applying Gauss' theorem on Griffith formula. Since our problem shows less regularity, the J-integral can only be expressed by dual pairings in corresponding Sobolev spaces defined on the path.

2 Preliminaries

In this paper we consider a domain Ω_δ consisting of two subdomains with a crack on the interface. This setting is specified more precisely in the following:

Definition 2.1 (The considered domain)
Let Ω_δ be an open, bounded and connected 2D domain consisting of two nonlinear, hyperelastic materials located in the subdomains Ω_1 and Ω_2, such that $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega_\delta}$. Both subdomains have Lipschitz boundaries with piecewise continuous outer unit normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$. The interface between Ω_1 and Ω_2 is assumed to contain a straight crack of the length δ, which may straightly expand along the interface, see fig. 1.
We furthermore introduce $\Omega = \operatorname{int}\overline{\Omega_\delta}$. Its boundary $\partial\Omega$ can be separated into a Dirichlet-boundary Γ_D and a Neumann-boundary Γ_N, so that $\partial\Omega_\delta = \partial\Omega \cup R_\delta$, where R_δ denotes the crack of length δ. Thus the crack lips are defined by $R_{\delta j} = R_\delta \cap \Omega_j$. Analogously, we set $\Gamma_{Dj} = \Gamma_D \cap \partial\Omega_j$ and $\Gamma_{Nj} = \Gamma_N \cap \partial\Omega_j$ for $j = 1, 2$.
We claim that:

1. *Γ_D is a set of non-zero Lebesgue-measure: $\mathcal{L}^1(\Gamma_D) \neq 0$.*
2. *$\overline{\Gamma_D} \cap \overline{R_\delta} = \emptyset$.*

The elastic strain energy densities referring to the subdomains Ω_1, Ω_2 are supposed to be of p-Laplacian type:

Definition 2.2 (Elastic strain energy density)
The elastic strain energy density $W_{el\,j}(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ referring to the subdomain Ω_j, $j = 1, 2$, has the form

$$W_{el\,j}(\boldsymbol{a}) = \frac{\mu_j}{p_j}(\kappa_j + |\boldsymbol{a}|^2)^{\frac{p_j}{2}}\ , \tag{4}$$

where $\mu_j \in (0, \infty)$, $p_j \in (1, \infty)$ are material constants and $\kappa_j \in [0, 1]$ are parameters that are introduced in order to avoid numerical instabilities and which can be chosen sufficiently small.

Due to hyperelasticity the stresses $\mathrm{D}W_{el\,j}(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ on Ω_j can be expressed by the derivative of $W_{el\,j}(\cdot)$ with respect to the argument:

$$\mathrm{D}W_{el\,j}(\boldsymbol{a}) = \mu_j(\kappa_j + |\boldsymbol{a}|^2)^{\frac{p_j-2}{2}}\boldsymbol{a} \tag{5}$$

In the case $1 < p_j < 2$ formula (5) can be interpreted as the approximation of the true stress-strain curve for a strain-hardening material. Then μ_j [MPa] and p_j describe material parameters for a given metal or alloy depending on its thermomechanical history, e.g. degree of mechanical working or heat treatment. $(p_j - 1)$ is called strain-hardening exponent [18]. In section 4, composites of the specimen shown in Table 1 (see [18] Table 6.3) are investigated

Table 1:

Alloy	μ_j [MPa]	p_j
Copper Cu (annealed)	315	1.54
Brass, 70 Cu-30 Zn (annealed)	895	1.49
Stainless steel 14301 (annealed)	1275	1.45

numerically. The approximation of the true stress-strain curves for these three alloys have the following appearance, fig. 2:

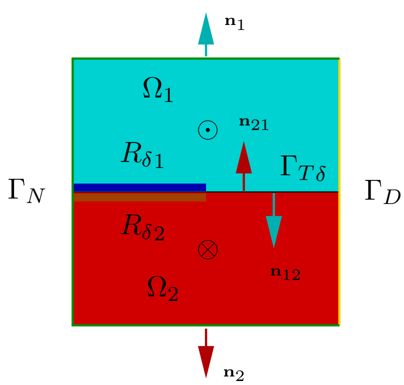

Figure 1: Domain Ω_δ and loadings

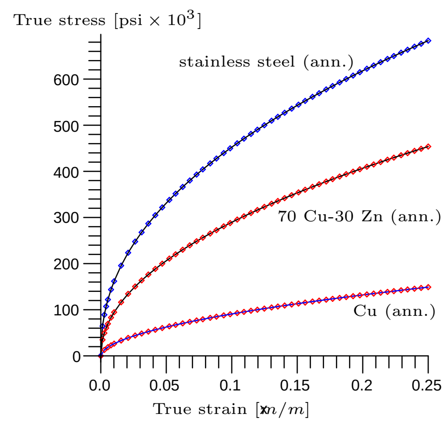

Figure 2: True stress-strain curves

The case $p_j = 2$ corresponds to a linear elastic material under anti-plane strain loadings whereas the exponent $p_j > 2$ is more important in fluid mechanics as in solid mechanics, but it is covered with the analytical studies as well. Thus the definition of materials of p-Laplacian type stands not only for strain-hardening and linear elastic materials but for a much larger class of energy functionals. As it will be shown in section 2.2, the elastic strain energy densities of p-Laplacian type are p-structures. For the analysis of transmission boundary value problems this fact favors the application of variational approaches more than the application of the theory on monotone operators.

Due to equation (5) the transmission boundary value problem for the configuration Ω_δ can be stated as follows (where $i, j \in \{1, 2\}$ in the sequel):

Definition 2.3 (Transmission boundary value problem)
Find $u : \Omega_\delta \to \mathbb{R}$, with $u|_{\Omega_j} = u_j$ for given functions $f : \Omega_\delta \to \mathbb{R}$ with $f|_{\Omega_j} = f_j$, $h : \Gamma_N \to \mathbb{R}$ with $h|_{\Gamma_N \cap \partial\Omega_j} = h_j$ and $g : \Gamma_D \to \mathbb{R}$ with $g|_{\Gamma_D \cap \partial\Omega_j} = g_j$, such that (compare fig. 1):

$$\begin{aligned} -\mathrm{div}\,\mathrm{D}W_{elj}\left(\nabla u_j\right) &= f_j \ \ \text{in } \Omega_j\,, & (6)\\ u_1 - u_2 &= 0 \ \ \text{on } \Gamma_{T\delta}\,, & (7)\\ \mathrm{D}W_{el1}\left(\nabla u_1\right)\cdot \mathbf{n}_{12} + \mathrm{D}W_{el2}\left(\nabla u_2\right)\cdot \mathbf{n}_{21} &= 0 \ \ \text{on } \Gamma_{T\delta}\,, & (8)\\ u_j &= g_j \ \ \text{on } \Gamma_{Dj} = \Gamma_D \cap \partial\Omega_j\,, & (9)\\ \mathrm{D}W_{elj}\left(\nabla u_j\right)\cdot \mathbf{n}_j &= h_j \ \ \text{on } \Gamma_{Nj} = \Gamma_N \cap \partial\Omega_j\,, & (10)\\ \mathrm{D}W_{elj}\left(\nabla u_j\right)\cdot \mathbf{n}_j &= 0 \ \ \text{on } R_{\delta j}\,. & (11) \end{aligned}$$

Thereby $\mathbf{n}_j$ and $\mathbf{n}_{ji}$ denote the piecewise continuous outer unit normal vectors of Ω_j.
Furthermore we introduce the following notation:
On the total Neumann boundary $\Gamma_{\tilde{N}}$ with $\Gamma_{\tilde{N}j} = \Gamma_{Nj} \cup R_{\delta j}$, $j = 1, 2$, the Neumann condition is given by:

$$\mathrm{D}W_{elj}\left(\nabla u_j\right)\cdot \mathbf{n}_j = \tilde{h}_j = \begin{cases} h_j & \text{on } \Gamma_{Nj}\,, \\ 0 & \text{on } R_{\delta j}\,. \end{cases} \qquad (12)$$

Remark 2.1
For $\kappa_j = 0$, and $\boldsymbol{a} = \nabla u_j$, $j = 1, 2$, formula (6) leads to the p_j-Laplacian equation on Ω_j.

Remark 2.2
The configuration Ω_δ together with the given transmission boundary value problem can be interpreted as a 2D cut-out of a 3D body subjected to mode-III loadings, see fig. 1: $\odot$ stands for a loading pointing orthogonally out of Ω_1 and $\otimes$ indicates a loading pointing orthogonally into Ω_2, which means that out-of-plane loadings are applied.

The energies referring to the configuration Ω_δ are explained in the following.

Definition 2.4 (Energies)
The elastic strain energy referring to Ω_δ is given by

$$J_{el}(\Omega_\delta, u) = \sum_{j=1}^{2} J_{elj}(\Omega_j, u_j) = \sum_{j=1}^{2} \int_{\Omega_j} W_{elj}(\nabla u_j)\,\mathrm{d}\mathbf{y}\,. \qquad (13)$$

The work of the applied forces on Ω_δ is defined by

$$W(\Omega_\delta, u) = \sum_{j=1}^{2} \left(\langle f_j, u_j \rangle_{W^{1,p_j}(\Omega_j)} + \left\langle \tilde{h}_j, u_j \right\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j})} \right)\,, \qquad (14)$$

where $\langle\cdot,\cdot\rangle_{W^{1,p_j}(\Omega_j)}$, $\langle\cdot,\cdot\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j})}$ are dual pairings in the Sobolev spaces introduced in the definitions 3.1, 3.2.
The potential energy of Ω_δ has the form

$$\begin{aligned} E(\Omega_\delta, u) &= J_{el}(\Omega_\delta, u) - W(\Omega_\delta, u) \\ &= \sum_{j=1}^{2} \int_{\Omega_j} W_{elj}(\nabla u_j)\,\mathrm{d}\mathbf{y} - \sum_{j=1}^{2} \left(\langle f_j, u_j \rangle_{W^{1,p_j}(\Omega_j)} + \left\langle \tilde{h}_j, u_j \right\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j})} \right)\,. \qquad (15) \end{aligned}$$

The dissipative energy is the amount of energy that refers to all irreversible processes that occur during cracking, like the creation of a new crack surface or the braking of atomic bonds. As a simplified model in the case of an interface crack it is considered in dependence of the fracture toughness of the interface $\mathcal{G}_c$ and of the crack length:

$$D(\Omega_\delta) = \mathcal{G}_c \delta \ . \tag{16}$$

The total energy of the compound Ω_δ is given by:

$$\begin{aligned}\Pi(\Omega_\delta, u) &= E(\Omega_\delta, u) + D(\Omega_\delta) \\ &= \sum_{j=1}^{2} \int_{\Omega_j} W_{elj}(\nabla u_j)\,\mathrm{d}\mathbf{y} - \sum_{j=1}^{2} \left(\langle f_j, u_j \rangle_{W^{1,p_j}(\Omega_j)} + \left\langle \tilde{h}_j, u_j \right\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j})} \right) + \mathcal{G}_c \delta \ .\end{aligned} \tag{17}$$

3 Minimization of the potential energy

In this section we want to prove the existence and uniqueness of a minimizer u in a set of functions M for the potential energy functional $E(\Omega_\delta, \cdot)$ referring to a cracked domain Ω_δ.

Remark 3.1
Since $D(\Omega_\delta) = \Pi(\Omega_\delta, v) - E(\Omega_\delta, v)$ is independent of v it holds:

$$\min_{v \in M} \Pi(\Omega_\delta, v) = \Pi(\Omega_\delta, u) = \min_{v \in M} E(\Omega_\delta, v) + D(\Omega_\delta) = E(\Omega_\delta, u) + D(\Omega_\delta) \ ,$$

which means that the minimizers of $\Pi(\Omega_\delta, \cdot)$ and $E(\Omega_\delta, \cdot)$ coincide.

The first step towards this aim is to specify the set of functions M in an appropriate function space corresponding to the above minimization problem.

3.1 Function spaces

Since the domain Ω_δ consists of a compound of two materials with different growth properties, we have to introduce special function spaces that reflect this constitution in a proper way:

Definition 3.1 (Function spaces)
For $\vec{p} = (p_1, p_2)$ and $p_{\min} = \min\{p_1, p_2\}$ we define

$$W^{1,\vec{p}}(\Omega_\delta) = \left\{ u \in W^{1,p_{\min}}(\Omega_\delta) \,:\, u_j = u|_{\Omega_j} \in W^{1,p_j}(\Omega_j),\ j = 1,2 \right\} \tag{18}$$

provided with the norm $\|u\|_{W^{1,\vec{p}}(\Omega_\delta)} = \|u_1\|_{W^{1,p_1}(\Omega_1)} + \|u_2\|_{W^{1,p_2}(\Omega_2)}$, (19)

$$V^{\vec{p}}_{(g)}(\Omega_\delta) = \left\{ u \in W^{1,\vec{p}}(\Omega_\delta) \,:\, \gamma_{\Gamma_D} u = g \right\} , \tag{20}$$

$$L^{\vec{p}}(\Omega_\delta) = \left\{ u \in L^{p_{\min}}(\Omega_\delta) \,:\, u_j = u|_{\Omega_j} \in L^{p_j}(\Omega_j),\ j = 1,2 \right\} \tag{21}$$

provided with the norm $\|u\|_{L^{\vec{p}}(\Omega_\delta)} = \|u_1\|_{L^{p_1}(\Omega_1)} + \|u_2\|_{L^{p_2}(\Omega_2)}$. (22)

In this context γ_{Γ_D} denotes the trace operator onto the Dirichlet boundary, which will be explained more detailed below.

These function spaces were introduced by W.B. Liu for the first time, see [15], and also [11] for the properties stated in the next lemma.

Lemma 3.1
Let $\vec{p} = (p_1, p_2)$ with $p_j \in (1, \infty)$ for $j \in \{1, 2\}$. Then the normed function spaces $\left(W^{1,\vec{p}}(\Omega_\delta), \|\cdot\|_{W^{1,\vec{p}}(\Omega_\delta)}\right)$, $\left(L^{\vec{p}}(\Omega_\delta), \|\cdot\|_{L^{\vec{p}}(\Omega_\delta)}\right)$, from definition 3.1 are separable, reflexive Banach spaces.

Recall that $\Omega_\delta = \Omega \backslash R_\delta = \mathrm{int}(\overline{\Omega_1 \cup \Omega_2}) \backslash R_\delta$. Because of the crack R_δ, the domain Ω_δ is no Lipschitz domain and therefore traces of functions defined on Ω_δ can't be taken from the domain onto the boundary $\partial\Omega_\delta$ at it's whole. But since the subdomains Ω_1, Ω_2 are claimed to be Lipschitz domains with piecewise continous outer unit normal vectors, the trace theorem holds for any $W^{1,p_j}(\Omega_j)$-function on any part Γ_j of the boundary $\partial\Omega_j$, $j = 1, 2$. Thus, for a part of the boundary $\Gamma \subset \partial\Omega_\delta$ with $\Gamma = \cup_{j=1}^2 \Gamma_j$ and $\mathcal{L}^1\left((\Gamma_1 \backslash \Gamma_2) \cup (\Gamma_2 \backslash \Gamma_1)\right) = 0$ there exists a linear, continuous trace operator $\gamma_\Gamma : W^{1,\vec{p}}(\Omega_\delta) \to W^{1-\frac{1}{p_{\min}}, p_{\min}}(\Gamma)$, where $p_{\min} = \min\{p_1, p_2\}$. It can obviously be defined by the trace operators γ_{Γ_j}, that exist on $W^{1,p_j}(\Omega_j)$:

$$\gamma_\Gamma := \begin{cases} \gamma_{\Gamma_1} & \text{on } W^{1,p_1}(\Omega_1) \\ \gamma_{\Gamma_2} & \text{on } W^{1,p_2}(\Omega_2) \end{cases} .$$

Definition 3.2 (Trace spaces)
Let $\vec{p} = (p_1, p_2)$ and $p_{min} = \min\{p_1, p_2\}$. For a part of the boundary $\Gamma \subset \partial\Omega$ we define

$$W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma) = \{u \in L^{p_{\min}}(\Gamma) \,:\, \exists \hat{u} \in W^{1,\vec{p}}(\Omega_\delta),\ \gamma_{\Gamma_j}\hat{u} = u|_{\Gamma_j},\ \gamma_\Gamma \hat{u} = u\}$$

provided with the norm $\|u\|_{W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma)} = \inf_{\substack{\hat{u} \in W^{1,\vec{p}}(\Omega_\delta) \\ \gamma_\Gamma \hat{u} = u}} \|\hat{u}\|_{W^{1,\vec{p}}(\Omega_\delta)}$,

With regard to traces of $W^{1,\vec{p}}(\Omega_\delta)$-functions onto the total Neumann-boundary $\Gamma_{\widetilde{N}}$, where $\Gamma_{\widetilde{N}j} = \Gamma_{\widetilde{N}} \cap \partial\Omega_j$, $j = 1, 2$, we refer to the well known spaces

$$W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j}) = \{u \in L^{p_j}(\Gamma_{\widetilde{N}j}) \,:\, \exists \hat{u} \in W^{1,p_j}(\Omega_j),\ \gamma_{\Gamma_{\widetilde{N}j}}\hat{u} = u\}$$

provided with the norm $\|u\|_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})} = \inf_{\substack{\hat{u} \in W^{1,p_j}(\Omega_j) \\ \gamma_{\Gamma_{\widetilde{N}j}} \hat{u} = u}} \|\hat{u}\|_{W^{1,p_j}(\Omega_j)}$ *and*

$$\widetilde{W}^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j}) = \{u \in W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j}) \,:\, \exists \tilde{u} \in W^{1-\frac{1}{p_j},p_j}(\partial\Omega_j) \text{ with } \tilde{u}|_{\Gamma_{\widetilde{N}j}} = u,\ \tilde{u}|_{\partial\Omega_j \backslash \overline{\Gamma_{\widetilde{N}j}}} = 0\}$$

provided with the norm $\|u\|_{\widetilde{W}^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})} = \|\tilde{u}\|_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})} = \inf_{\substack{\hat{u} \in W^{1,p_j}(\Omega_j) \\ \gamma_{\Gamma_{\widetilde{N}j}} \hat{u} = u}} \|\hat{u}\|_{W^{1,p_j}(\Omega_j)}$.

Remark 3.2
For the function $\tilde{h}_j \in \left(W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})\right)' = \widetilde{W}^{-\frac{1}{q_j},q_j}(\Gamma_{\widetilde{N}j})$ the Neumann condition (12) on the total Neumann boundary has to be understood in the following sense:

$$\left\langle \tilde{h}_j, v_j \right\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})} = 0 \quad \text{for any } v_j \in W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j}) \quad \text{with } v_j = 0 \text{ on } \Gamma_{Nj} \text{ and}$$

$$\left\langle \tilde{h}_j, w_j \right\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})} = \langle h_j, w_j \rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j})} \quad \text{for any } w_j \in W^{1-\frac{1}{p_j},p_j}(\Gamma_{\widetilde{N}j}) \text{ with } w_j = 0 \text{ on } R_{\delta j} .$$

Transmission condition (8) has to be treated analogously.

With the aid of traces the following property of the space $V^{\vec{p}}_{(0)}(\Omega_\delta)$ can be shown:

Lemma 3.2
The function space $\left(V^{\vec{p}}_{(0)}(\Omega_\delta), \|\cdot\|_{W^{1,\vec{p}}(\Omega_\delta)}\right)$ *from definition 3.1 is a separable, reflexive Banach space.*

Proof:
At first it shown that $\left(V^{\vec{p}}_{(0)}(\Omega_\delta), \|\cdot\|_{W^{1,\vec{p}}(\Omega_\delta)}\right)$ is a Banach space.
Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(V^{\vec{p}}_{(0)}(\Omega_\delta), \|\cdot\|_{W^{1,\vec{p}}(\Omega_\delta)}\right)$. Since $V^{\vec{p}}_{(0)}(\Omega_\delta) \subset W^{1,\vec{p}}(\Omega_\delta)$ there exists a limit $f \in W^{1,\vec{p}}(\Omega_\delta)$. Now $\gamma_{\Gamma_D} f = 0$ has to be shown.
It holds

$$
\begin{aligned}
0 \le \|\gamma_{\Gamma_D} f\|_{W^{1-\frac{1}{p},\vec{p}}(\Gamma_D)} &= \|\gamma_{\Gamma_D} f - \gamma_{\Gamma_D} f_n\|_{W^{1-\frac{1}{p},\vec{p}}(\Gamma_D)} = \|\gamma_{\Gamma_D}(f - f_n)\|_{W^{1-\frac{1}{p},\vec{p}}(\Gamma_D)} \\
&= \inf_{\substack{v\in W^{1,\vec{p}}(\Omega_\delta) \\ \gamma_{\Gamma_D} v = \gamma_{\Gamma_D}(f-f_n)}} \|v\|_{W^{1,\vec{p}}(\Omega_\delta)} \le \|f - f_n\|_{W^{1,\vec{p}}(\Omega_\delta)} \longrightarrow 0 \quad \text{for } n \to \infty\,.
\end{aligned}
$$

This yields $\gamma_{\Gamma_D} f = 0$ and $f \in V^{\vec{p}}_{(0)}(\Omega_\delta)$. Therefore $V^{\vec{p}}_{(0)}(\Omega_\delta)$ is a closed subspace of the separable, reflexive Banach space $W^{1,\vec{p}}(\Omega_\delta)$ with respect to the $W^{1,\vec{p}}(\Omega_\delta)$-norm.
Separability und reflexivity of $V^{\vec{p}}_{(0)}(\Omega_\delta)$ follow from the properties of $W^{1,\vec{p}}(\Omega_\delta)$ [1] p.7, Th.1.21. ∎

Now we are able to specify the set of functions M:

Definition 3.3 (The set of functions M)
For a minimization problem corresponding to the transmission boundary value problem (6)-(11) *the set of functions is given by:*

$$
M = V^{\vec{p}}_{(g)}(\Omega_\delta)\,, \tag{23}
$$

where g is the right hand side of the Dirichlet condition (9).

The next lemma gives an important property of M:

Lemma 3.3 (weak sequentially closedness of M)
For any sequence $\{u_k\}_{k\in\mathbb{N}} \subset M = V^{\vec{p}}_{(g)}(\Omega_\delta)$ with $u_k \rightharpoonup u$ in $W^{1,\vec{p}}(\Omega_\delta)$ as $k \to \infty$ it holds $u \in M$.

Proof:
Since $V^{\vec{p}}_{(0)}(\Omega_\delta)$ is a Banach space, it is a closed and convex subset of $W^{1,\vec{p}}(\Omega_\delta)$. Thus it is weak sequentially closed, see [20] theorem III.3.8. ∎

At last, we point out that Poincaré-Friedrichs inequality is valid for the space $V^{\vec{p}}_{(0)}(\Omega_\delta)$. This is due to the fact, that Poincaré-Friedrichs inequality [16, 17] holds on the subdomain, where the Dirichlet boundary is located. Thus, for a function v satisfying $\nabla v = 0\ \mathcal{L}^2 - a.e.$ it holds $v = 0\ \mathcal{L}^2 - a.e.$. This property is passed onto the other subdomain by transmission condition (7). Therefore the following norm equivalence holds on $V^{\vec{p}}_{(0)}(\Omega_\delta)$:

Theorem 3.1 (Norm equivalence)
For every $u \in V^{\vec{p}}_{(0)}(\Omega_\delta)$ holds:

$$
\|\nabla u\|_{L^{\vec{p}}(\Omega_\delta)} \le \|u\|_{W^{1,\vec{p}}(\Omega_\delta)} \le c_N \|\nabla u\|_{L^{\vec{p}}(\Omega_\delta)}\,.
$$

3.2 The p-structural properties of the elastic strain energy density

In this subsection we show that the elastic strain energy density of the considered problem is a p-structure on each subdomain. This means that the elastic strain energy densities $W_{el\,j}(\cdot)$ satisfy the properties mentioned in the theorem below:

Theorem 3.2
The elastic strain energy densities fulfill the properties of p-structures, i.e. for $j \in \{1,2\}$ the following items hold:

(H0) $W_{el\,j}(\cdot) \in C^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$.

(H1) *There exist constants $c_{1j},\ c_{2j} \in \mathbb{R}^+$ such that for every $\boldsymbol{a} \in \mathbb{R}^2$:*

$$c_{1j}|\boldsymbol{a}|^{p_j} \leq W_{el\,j}(\boldsymbol{a}) \leq c_{2j}(1+|\boldsymbol{a}|^{p_j})\ .$$

(H2) *There exists a constant $c_j \in \mathbb{R}^+$ such that for every $\boldsymbol{a} \in \mathbb{R}^2$:*

$$|\mathrm{D}W_{el\,j}\,(\boldsymbol{a})| \leq c_j(1+|\boldsymbol{a}|^{p_j-1})\ .$$

(H3) *There exists a constant $k_j \in \mathbb{R}^+$ such that for every $\boldsymbol{a} \in \mathbb{R}^2\backslash\{\boldsymbol{0}\}$:*

$$|\mathrm{D}(\mathrm{D}W_{el\,j}\,(\boldsymbol{a}))| \leq k_j(1+|\boldsymbol{a}|^{p_j-2})\ .$$

(H4) *$W_{el\,j}(\cdot)$ is convex with respect to $\boldsymbol{a}$, i.e. there exist constants $l_j \in \mathbb{R}^+$ and $\tilde{k}_j \in \{0,1\}$ such that for all $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^2$ the following inequality holds:*

$$\big\langle D_{\boldsymbol{a}}\big\langle \mathrm{D}W_{el\,j}\,(\boldsymbol{a}),\boldsymbol{b}\big\rangle,\boldsymbol{b}\big\rangle \geq l_j\left(\tilde{k}_j+|\boldsymbol{a}|\right)^{p_j-2}|\boldsymbol{b}|^2\ , \tag{24}$$

where $\tilde{k}_j = 0$ if $p_j \geq 2$ and $\tilde{k}_j = 1$ if $1 < p_j < 2$.

Proof:
Concerning **(H0)**: The first and second Fréchet derivatives of $W_{el\,j}(\cdot)$ can be easily calculated:

$$\big\langle \mathrm{D}W_{el\,j}\,(\boldsymbol{a}),\boldsymbol{b}\big\rangle = \mu_j(\kappa_j+|\boldsymbol{a}|^2)^{\frac{p_j-2}{2}}\boldsymbol{a}\cdot\boldsymbol{b} \tag{25}$$

$$\begin{aligned}&\big\langle\big\langle D_{\boldsymbol{a}}\mathrm{D}W_{el\,j}\,(\boldsymbol{a}),\boldsymbol{b}\big\rangle,\boldsymbol{c}\big\rangle = \\ &\qquad\mu_j(p_j-2)\,(\kappa_j+|\boldsymbol{a}|^2)^{\frac{p_j-4}{2}}(\boldsymbol{a}\cdot\boldsymbol{b})\,(\boldsymbol{a}\cdot\boldsymbol{c})+\mu_j\,(\kappa_j+|\boldsymbol{a}|^2)^{\frac{p_j-2}{2}}(\boldsymbol{a}\cdot\boldsymbol{b})\end{aligned} \tag{26}$$

Concerning **(H1)**: Since $\kappa_j \in [0,1]$ it is obvious that

$$\frac{\mu_j}{p_j}|\boldsymbol{a}|^{p_j} \leq \frac{\mu_j}{p_j}(\kappa_j+|\boldsymbol{a}|^2)^{\frac{p_j}{2}} = W_{el\,j}(\boldsymbol{a}) \leq \frac{\mu_j}{p_j}(1+|\boldsymbol{a}|^2)^{\frac{p_j}{2}} \overset{(105)}{\leq} 2^{\frac{p_j-2}{2}}\frac{\mu_j}{p_j}(1+|\boldsymbol{a}|^{p_j})\ .$$

Concerning **(H2)**: If $p_j \geq 2$ we get

$$\begin{aligned}|\mathrm{D}W_{el\,j}\,(\boldsymbol{a})| &= |\mu_j(\kappa_j+|\boldsymbol{a}|^2)^{\frac{p_j-2}{2}}\boldsymbol{a}| \leq \mu_j(1+|\boldsymbol{a}|^2)^{\frac{p_j-2}{2}}(1+|\boldsymbol{a}|) \leq \mu_j(1+|\boldsymbol{a}|)^{p_j-1} \\ &\overset{(105)}{\leq} 2^{p_j-2}\mu_j(1+|\boldsymbol{a}|^{p_j-1})\ .\end{aligned}$$

In the case $1 < p_j < 2$ it holds

$$|\mathrm{D}W_{el\,j}(\boldsymbol{a})| = \left|\frac{\mu_j \boldsymbol{a}}{(\kappa_j + |\boldsymbol{a}|^2)^{\frac{2-p_j}{2}}}\right| \leq \frac{\mu_j |\boldsymbol{a}|}{(0 + |\boldsymbol{a}|^2)^{\frac{2-p_j}{2}}} = \mu_j(1 + |\boldsymbol{a}|^{p-1}) \ .$$

Concerning **(H3)**: Under consideration of (26) we immediately obtain

$$\begin{aligned} |\mathrm{D}^2 W_{el\,j}(\boldsymbol{a})||\boldsymbol{b}||\boldsymbol{c}| &= \mu_j |p_j - 2|(\kappa_j + |\boldsymbol{a}|^2)^{\frac{p_j-4}{2}} |\boldsymbol{a}|^2 |\boldsymbol{b}||\boldsymbol{c}| + \mu_j(\kappa_j + |\boldsymbol{a}|^2)^{\frac{p_j-2}{2}} |\boldsymbol{b}||\boldsymbol{c}| \\ &\leq \mu_j(p_j + 3)(1 + |\boldsymbol{a}|^2)^{\frac{p_j-2}{2}} |\boldsymbol{b}||\boldsymbol{c}| \\ &\overset{(105)}{\leq} \mu_j(p_j + 3)2^{\frac{p_j-4}{2}}(1 + |\boldsymbol{a}|^{p_j-2})|\boldsymbol{b}||\boldsymbol{c}| \end{aligned}$$

Concerning **(H4)**: For $p_j \geq 2$ we have $\mu_j(p_j - 2)(\kappa_j + |\boldsymbol{a}|^2)^{\frac{p_j-4}{2}}(\boldsymbol{a} \cdot \boldsymbol{b})^2 \geq 0$ and since $\kappa_j \geq 0$ it turns out that

$$\mathrm{D}_{\boldsymbol{a}}(\mathrm{D}W_{el\,j}(\boldsymbol{a}) \cdot \boldsymbol{b}) \cdot \boldsymbol{b} \geq \mu_j(\kappa_j + |\boldsymbol{b}|^2)^{\frac{p_j-2}{2}} |\boldsymbol{b}|^2 \geq \mu_j |\boldsymbol{a}|^{p-2} |\boldsymbol{b}|^2 \ .$$

If $1 < p < 2$ it is $(p_j - 2) < 0$ and the following estimate holds

$$\begin{aligned} \mu_j(p_j - 2)\frac{(\boldsymbol{a} \cdot \boldsymbol{b})^2}{(\kappa_j + |\boldsymbol{a}|^2)^{\frac{4-p_j}{2}}} + \mu_j \frac{|\boldsymbol{b}|^2}{(\kappa_j + |\boldsymbol{a}|^2)^{\frac{2-p_j}{2}}}. &\geq \mu_j(p_j - 2)\frac{|\boldsymbol{a}|^2|\boldsymbol{b}|^2}{|\boldsymbol{a}|^{4-p_j}} + \mu_j \frac{|\boldsymbol{b}|^2}{(1 + |\boldsymbol{a}|^2)^{\frac{2-p_j}{2}}} \\ &\geq \mu_j(p_j - 2)|\boldsymbol{a}|^{p_j-2}|\boldsymbol{b}|^2 + \mu_j(1 + |\boldsymbol{a}|)^{p_j-2}|\boldsymbol{b}|^2 \\ &\geq \mu_j(p_j - 1)(1 + |\boldsymbol{a}|)^{p_j-2}|\boldsymbol{b}|^2 \ . \end{aligned}$$

■

Inequality (24) of property **(H4)** provides another convexity inequality involving the elastic strain energy densities:

Lemma 3.4
From the convexity inequality (24) *follows that there exist constants* $d_j \in \mathbb{R}^+$, $\tilde{k}_j \in \{0, 1\}$ *such that for all* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$ *holds:*

$$W_{el\,j}(\boldsymbol{a}) - W_{el\,j}(\boldsymbol{b}) \geq \mathrm{D}W_{el\,j}(\boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) + d_j(\tilde{k}_j + |\boldsymbol{a}| + |\boldsymbol{b}|)^{p_j-2}|\boldsymbol{a} - \boldsymbol{b}|^2 \,. \tag{27}$$

and furthermore it holds for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$*:*

$$\big\langle \mathrm{D}W_{el\,j}(\boldsymbol{a}) - \mathrm{D}W_{el\,j}(\boldsymbol{b}), \boldsymbol{a} - \boldsymbol{b} \big\rangle \geq \tilde{d}_j(\tilde{k}_j + |\boldsymbol{a}| + |\boldsymbol{b}|)^{p_j-2}|\boldsymbol{a} - \boldsymbol{b}|^2 \,. \tag{28}$$

Estimate (28) directly follows from estimate (27), which is proved in [11] p. 149.

Remark 3.3
From inequality (27) *directly follows that* $W_{el\,j}(\cdot)$ *is even strictly convex, i.e. for all* $\boldsymbol{a} \neq \boldsymbol{b} \in \mathbb{R}^2$ *holds:*

$$W_{el\,j}(\boldsymbol{a}) - W_{el\,j}(\boldsymbol{b}) > \mathrm{D}W_{el\,j}(\boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b}) \ , \tag{29}$$

which states the uniqueness of a minimizer, provided it exists.

3.2.1 Existence and uniqueness of the minimizer

In the following we want to state the existence and the uniqueness for a solution of the following minimization problem for the configuration Ω_δ:

Definition 3.4 (Minimization problem)
Find $u \in M = V_{(g)}^{\vec{p}}(\Omega_\delta)$ for given functions $f \in W^{-1,\vec{q}}(\Omega_\delta)$, $\tilde{h}_j \in \widetilde{W}^{-\frac{1}{q_j},q_j}(\Gamma_{\tilde{N}j})$, $j = 1, 2$, and $g \in W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma_D)$, such that

$$E(\Omega_\delta, u) = \min_{v \in M} E(\Omega_\delta, v) \;, \tag{30}$$

where the potential energy functional is defined by

$$\begin{aligned} E(\Omega_\delta, u) &= J_{el}(\Omega_\delta, u) - W(\Omega_\delta, u) \\ &= \sum_{j=1}^{2} \int_{\Omega_j} W_{elj}(\nabla u_j)\,\mathrm{d}\mathbf{y} - \sum_{j=1}^{2} \left(\langle f_j, u_j \rangle_{W^{1,p_j}(\Omega_j)} + \left\langle \tilde{h}_j, u_j \right\rangle_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j})} \right) \;. \end{aligned} \tag{31}$$

Existence and uniqueness of a minimizer can be analyzed following the ideas of [22] p.229ff. The necessary and sufficient conditions for the existence of a unique minimizer are summarized in the following theorem:

Theorem 3.3 (Existence and uniqueness of a minimizer, compare [22] p. 232)
The elastic strain energy $J_{el}(\Omega_\delta, \cdot) : M = V_{(g)}^{\vec{p}}(\Omega_\delta) \subset W^{1,\vec{p}}(\Omega_\delta) \to \mathbb{R}$ satisfies the following assumpions:

(i) $W^{1,\vec{p}}(\Omega_\delta)$ is a real reflexive Banach space due to lemma 3.1.

(ii) $J_{el}(\Omega_\delta, \cdot)$ is weak sequentially lower semicontinuous, since $W_{elj}(\cdot)$ is bounded from below by 0 and strictly convex due to lemma 3.4.

(iii) $M = V_{(g)}^{\vec{p}}(\Omega_\delta)$ is weak sequentially closed due to lemma 3.3.

(iv) For each sequence $\{u_n\}_{n\in\mathbb{N}} \subset M$ with $\|u_n\|_{W^{1,\vec{p}}(\Omega_{\Omega_\delta})} \to \infty$ as $n \to \infty$ we have

$$\overline{\lim}_{n\to\infty} \left(J_{el}(\Omega_\delta, u_n) - \sum_{j=1}^{2} \langle b_j, u_{nj} \rangle_{W^{1,p_j}(\Omega_j)} \right) = +\infty$$

due to the growth condition **(H1)** *for $W_{elj}(\cdot)$ in theorem 3.2.*
Thereby $\langle b_j, u_{nj} \rangle_{W^{1,p_j}(\Omega_j)} = \left\langle f_j + \gamma^{\star}_{\Gamma_{\tilde{N}j}} \tilde{h}_j, u_{nj} \right\rangle_{W^{1,p_j}(\Omega_j)}$, where

$$\gamma^{\star}_{\Gamma_{\tilde{N}j}} : (W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j}))' \to (W^{1,p_j}(\Omega_j))'$$

denotes the adjoint operator of the trace operator $\gamma_{\Gamma_{\tilde{N}j}} : W^{1,p_j}(\Omega_j) \to W^{1-\frac{1}{p_j},p_j}(\Gamma_{\tilde{N}j})$.

Then the minimization problem (30) possesses a unique solution $u \in M$ for any given data $f \in W^{-1,\vec{q}}(\Omega_\delta)$, $\tilde{h}_j \in \widetilde{W}^{-\frac{1}{q_j},q_j}(\Gamma_{\tilde{N}j})$, $j = 1, 2$, and $g \in W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma_D)$.

4 Griffith Formula for p-Laplacian compounds

Main goal of this section is to prove a Griffith formula for the compound of p-Laplacian type in the reference domain Ω_{δ_0}, see subsection 4.1.. This notation is introduced in order to indicate that this reference domain contains the straight crack of length δ_0, and to make it distinguishable from the current domains $\Omega_{\delta_0+\delta}$ with the crack enlarged by any sufficiently small $\delta > 0$.

Theorem 4.1 (Griffith formula for the compound)
Let Ω_{δ_0} be a 2D domain, as in definition 2.1, containing the crack $R_{\delta_0} \subset \{\mathbf{x} \in \Omega_{\delta_0} : x_2 = 0\}$ with the crack propagation direction along the vector field $(1,0)^\top$ and the crack tip S_{δ_0}. Let $\theta \in C_0^\infty(\Omega)$ such that $\theta \equiv 1$ in B_1^θ and $\theta \equiv 0$ in $\Omega \backslash B_2^\theta$. Thereby B_k^θ for $k \in \{1,2\}$ denotes a connected Lipschitz domain around S_{δ_0} such that $\overline{B}_1^\theta \subset B_2^\theta \subset \operatorname{int} \overline{\Omega}_{\delta_0} = \Omega$. Then the Griffith formula for the cracked compound with a volume force density $f \in L^{\vec{q}}_{\partial_{y_1}}(\Omega_{\delta_0})$, where

$$L^{\vec{q}}_{\partial_{y_1}}(\Omega_{\delta_0}) = \{f \in L^{\vec{q}}(\Omega_{\delta_0}), \partial_{y_1} f \in L^{\vec{q}}(\Omega_{\delta_0})\},$$

surface force density $h \in L^{\vec{q}}(\Gamma_N)$ and the minimizer $u_{[\delta_0]} \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0})$ reads as follows:

$$\begin{aligned} ERR(\Omega_{\delta_0}, u_{[\delta_0]}) = & \sum_{j=1}^{2} \int\limits_{\Omega_j} \partial_{y_1} u_{[\delta_0]j} \, \mathrm{D}W_{el\,j} \left(\nabla u_{[\delta_0]j} \right) \cdot \nabla \theta \, \mathrm{d}\mathbf{y} \\ & - \sum_{j=1}^{2} \int\limits_{\Omega_j} W_{el\,j}(\nabla u_{[\delta_0]j}) \partial_{y_1} \theta \, \mathrm{d}\mathbf{y} + \sum_{j=1}^{2} \int\limits_{\Omega_j} u_{[\delta_0]j} \partial_{y_1}(f_j \theta) \, \mathrm{d}\mathbf{y} \,. \end{aligned} \tag{32}$$

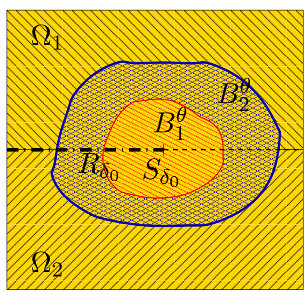

Figure 3: Support of the functions $\theta \in C_0^\infty(\Omega)$

This formula allows to calculate the energy release rate as a sum of volume integrals only involving known quantities and such quantities that can be determined by FEM, like the minimizer, stresses, elastic strain energy density.

In subsection 4.2. a numerical result for the behavior of the energy release rate obtained with Griffith formula and FEM is presented.

4.1 Proof of Griffith Formula

The proof of theorem 4.1 is quite long and technical. Firstly, we point out the main ideas. The particular steps are carried out in the succeeding subsections.

Sketch of the proof:
For $\delta \in [0, a]$, see definition 4.1 for the definition of a, a diffeomorphism $T_\delta : \Omega_{\delta_0+\delta} \to \Omega_{\delta_0}$ is introduced, which maps the current configuration $\Omega_{\delta_0+\delta}$ onto the reference configuration Ω_{δ_0}.

Thereby the potential energy of the current configuration $E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})$ can be mapped onto the reference configuration so that the limit

$$\lim_{\delta\to 0} \frac{E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})}{\delta} \tag{33}$$

can be taken there. The proof is divided into several steps:
Section 4.1.1: The mapping T_δ

1. Definition of T_δ and its properties.

2. Properties of the transformed functions.

Section 4.1.2: Boundedness and convergence of the transformed minimizers as $\delta \to 0$

- Uniform boundedness of the minimizers $u_{[\delta_0+\delta]}$ in $W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$ and of the transformed minimizers $u^{[\delta_0+\delta]}$ in $W^{1,\vec{p}}(\Omega_{\delta_0})$ respectively.

- Convergence of the minimizers in $W^{1,\vec{p}}(\Omega_{\delta_0})$ as $\delta \to 0$.

Section 4.1.3: Derivation of the Griffith formula by consideration of minimization problems

- Taking the limit in formula (33) on the reference configuration considering the uniqueness of the minimizers.

This approach is based on the work of A.M. Khludnev and J. Sokolowski, who verified a Griffith formula for the Laplacian [9] and for linear elasticity [10]. Analogously a Griffith formula for Ramberg/Osgood materials was shown by D. Knees [11, 12].

4.1.1 The mapping T_δ

Definition of T_δ and its properties

Definition 4.1
Let $\theta \in C_0^\infty(\Omega)$ as in theorem 4.1 with $\theta \equiv 1$ in B_1^θ and $\theta \equiv 0$ in $\Omega \backslash B_2^\theta$. Let $\delta \in [0, a]$, where a is given by the requirement $P_a := S_{\delta_0} + a(1, 0)^\top \in \overline{B_1^\theta}$. Then the mapping T_δ is defined by

$$T_\delta : \Omega_{\delta_0+\delta} \to \Omega_{\delta_0},\, \mathbf{x} \mapsto \mathbf{y} = T_\delta(\mathbf{x}) = \mathbf{x} - \delta \begin{pmatrix} \theta(\mathbf{x}) \\ 0 \end{pmatrix} . \tag{34}$$

The Jacobian of this mapping is given by

$$J_{T_\delta}(\mathbf{x}) = \begin{pmatrix} \partial_{x_1} y_1 & \partial_{x_2} y_1 \\ \partial_{x_1} y_2 & \partial_{x_2} y_2 \end{pmatrix} = \begin{pmatrix} 1 - \delta\partial_{x_1}\theta(\mathbf{x}) & -\delta\partial_{x_2}\theta(\mathbf{x}) \\ 0 & 1 \end{pmatrix} \tag{35}$$

and the Jacobian determinant reads as follows

$$\det J_{T_\delta}(\mathbf{x}) = 1 - \delta\partial_{x_1}\theta(\mathbf{x}) . \tag{36}$$

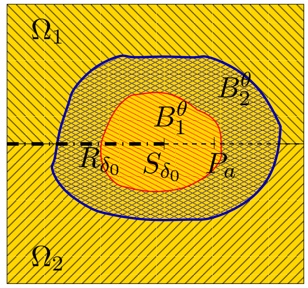

Figure 4: Definition of the mapping T_δ

In order to guarantee that T_δ maps properly, a has to be chosen sufficiently small so that firstly no point of the dark patterned area is mapped out of the domain Ω. Secondly, the Jacobian determinant $\det J_{T_\delta}(\mathbf{x})$ has to be greater than 0 so that T_δ is a diffeomorphism.

Lemma 4.1
T_δ is a diffeomorphism.

Proof:
It holds $T_\delta \in \left(C^\infty(\Omega_{\delta_0+\delta})\right)^2$. If a is chosen sufficiently small, then $\det J^\delta_{T_\delta} > 0$ for every $\delta \in [0,a]$. From this follows that T_δ is a diffeomorphism, see [21] p. 53. ■

Remark 4.1
Note that for every $\mathbf{y} \in \Omega_{\delta_0+\delta}$ and $\delta \in [0,a]$ holds:

$$|T_\delta^{-1}(\mathbf{y}) - \mathbf{y}| = |\mathbf{x} - T_\delta(\mathbf{x})| = |\mathbf{x} - \mathbf{x} + \delta(\theta(\mathbf{x}),0)^\top| \le \delta\|\theta\|_\infty \,. \tag{37}$$

Therefore we know that $T_\delta^{-1}(\mathbf{y}) \in B(\mathbf{y}, \delta\|\theta\|_\infty) = \{\mathbf{z} \in \mathbb{R}^2, |\mathbf{z}-\mathbf{y}| \le \delta\|\theta\|_\infty\}$ for every $\mathbf{y} \in \Omega_\theta := \Omega \cap \operatorname{supp}\theta$ and $\delta \in [0,a]$.

4.1.2 Properties of the transformed functions

Definition 4.2
Let $\mathbf{x} \in \Omega_{\delta_0+\delta}$ and $\mathbf{y} \in \Omega_{\delta_0}$. For $v_{[\delta_0+\delta]} : \Omega_{\delta_0+\delta} \to \mathbb{R}$ we define

$$v^{[\delta_0+\delta]}(\mathbf{y}) = v_{[\delta_0+\delta]}(T_\delta^{-1}(\mathbf{y})) = v_{[\delta_0+\delta]}(\mathbf{x}) \quad \text{for } \mathbf{y} \in \Omega_{\delta_0} \,. \tag{38}$$

The derivative of such a transformed function can be calculated from the following relation:

$$\begin{aligned}
\nabla_{\mathbf{x}} v^{[\delta_0+\delta]}(\mathbf{y}) &= \nabla_{\mathbf{x}} v_{[\delta_0+\delta]}(T_\delta^{-1}(\mathbf{y})) = \left(\nabla_{\mathbf{y}} v^{[\delta_0+\delta]}(\mathbf{y})\right)^\top \nabla_{\mathbf{x}}\mathbf{y} \\
&= \left(\nabla_{\mathbf{y}} v^{[\delta_0+\delta]}(\mathbf{y})\right)^\top J_{T_\delta} = \nabla_{\mathbf{y}} v^{[\delta_0+\delta]}(\mathbf{y}) - \delta\partial_{y_1} v^{[\delta_0+\delta]}(\mathbf{y}) \nabla_{\mathbf{x}}\theta(T_\delta^{-1}(\mathbf{y})) \\
&= \nabla_{\mathbf{y}} v^{[\delta_0+\delta]}(\mathbf{y}) - \delta\partial_{y_1} v^{[\delta_0+\delta]}(\mathbf{y}) \nabla_{\mathbf{x}}\theta^\delta((\mathbf{y})) \,.
\end{aligned} \tag{39}$$

Now we state some properties of the transformed functions.

Lemma 4.2
Let $u_{[\delta_0+\delta]} \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0+\delta})$ *be the minimizer of the potential energy* $E(\Omega_{\delta_0+\delta}, \cdot)$ *referring to the current configuration. Then for every* $\delta \in [0, a]$ *holds:*

$$u^{[\delta_0+\delta]} = u_{[\delta_0+\delta]} \circ T_\delta^{-1} \ \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0}) \ .$$

Let $u_{[\delta_0]} \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0})$ *be the minimizer of the potential energy* $E(\Omega_{\delta_0}, \cdot)$ *referring to the reference configuration. Then for every* $\delta \in [0, a]$ *holds:*

$$u_{[\delta_0]} \circ T_\delta \ \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0+\delta}) \ .$$

Proof:
We have $u_{[\delta_0+\delta]} \in W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$ and for every $\mathbf{y} \in \Omega_{\delta_0}$ it holds $T_\delta^{-1}(\mathbf{y}) \in \Omega_{\delta_0+\delta}$. Since T_δ is a diffeomorphism and $u^{[\delta_0+\delta]}(\mathbf{y}) = u_{[\delta_0+\delta]}(T_\delta^{-1}(\mathbf{y}))$ it follows $u^{[\delta_0+\delta]} \in W^{1,\vec{p}}(\Omega_{\delta_0})$. Furthermore we have $\operatorname{supp}\theta \cap \partial\Omega = \emptyset$ and especially $\operatorname{supp}\theta \cap \Gamma_D = \emptyset$. Therefore $T_\delta(\mathbf{y}) = \mathbf{y} = T_\delta^{-1}(\mathbf{y})$ for every $\mathbf{y} \in \Gamma_D$. Because $u_{[\delta_0+\delta]} \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0+\delta})$ and $u^{[\delta_0+\delta]}(\mathbf{y}) = u_{[\delta_0+\delta]}(\mathbf{y}) = g(\mathbf{y})$ for $\mathbf{y} \in \Gamma_D$ we have $u^{[\delta_0+\delta]} \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0})$.
Analogously $u_{[\delta_0]} \circ T_\delta \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0+\delta})$ can be shown. ∎

Properties of the applied force densities and boundary displacements

Theorem 4.2 (Convergence of the volume force densities)
For $f_j \in L^{q_j}(\Omega_j)$, $q_j \in (1, \infty)$, $j = 1, 2$, *it holds*

$$\left\| f_j^{[\delta_0+\delta]} - f_j \right\|_{L^{q_j}(\Omega_{\delta_j})} \to 0 \quad \text{as } \delta \to 0 \, . \tag{40}$$

To prove this theorem it is necessary to introduce the Lebesgue-Besicovitch differentiation theorem, compare [4] p.43:

Theorem 4.3 (Lebesgue-Besicovitch differentiation theorem)
Let $\mathcal{L}^2$ *be the Lebesgue measure on* $\mathbb{R}^2$ *and* $f \in L^1_{loc}(\mathbb{R}^2, \mathcal{L}^2)$. *Then*

$$\lim_{r \to 0} \fint_{B(\mathbf{x},r)} f(\mathbf{y}) \mathrm{d}\mathbf{y} = f(\mathbf{x}) \quad \text{for } \mathcal{L}^2\text{-a.e. } \mathbf{x} \in \mathbb{R}^2 \, . \tag{41}$$

Here $\fint_E f(\mathbf{y}) d\mathbf{y} := \frac{1}{\mu(E)} \int_E f(\mathbf{y}) \mathrm{d}\mathbf{y}$ *denotes the average of* f *over the set* E *with respect to the measure* $\mathcal{L}^2$, *provided* $\mathcal{L}^2(E) < \infty$.

Additionally, the following corollary holds, see [4] p.44:

Corollary 4.1
If $f \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{L}^2)$ *for some* $1 \le p < \infty$, *then*

$$\lim_{B \downarrow \{x\}} \fint_B |f(\mathbf{y}) - f(\mathbf{x})|^p \mathrm{d}\mathbf{y} = 0 \quad \text{for } \mathcal{L}^2\text{-a.e. } \mathbf{x} \, , \tag{42}$$

where the limit is taken over all closed balls B *containing* x *as* $\operatorname{diam} B \to 0$.

With these two auxilliary means theorem 4.2 can be proved:

Proof:
We define $\Omega_j^\theta := \Omega_j \cap \operatorname{supp}\theta$. The norm in line (40) is rewritten with the aid of theorem 4.3 and estimated from above by Hölder's inequality:

$$
\begin{aligned}
\lim_{\delta\to 0}\left\| f_j^{[\delta_0+\delta]} - f_j\right\|^{q_j}_{L^{q_j}(\Omega_{\delta_0})} &= \lim_{\delta\to 0}\int\limits_{\Omega_j^\theta} |f_j(T_\delta^{-1}(\mathbf{y})) - f_j(\mathbf{y})|^{q_j}\,\mathrm{d}\mathbf{y} \\
\overset{(41)}{=} & \lim_{\delta\to 0}\int\limits_{\Omega_j^\theta} |\lim_{r\to 0} \fint\limits_{B(T_\delta^{-1}(\mathbf{y}),r)} f_j(\mathbf{z})\mathrm{d}\mathbf{z} - f_j(\mathbf{y})|^{q_j}\,\mathrm{d}\mathbf{y} \\
= & \lim_{\delta\to 0}\int\limits_{\Omega_\theta} |\lim_{r\to 0} \fint\limits_{B(T_\delta^{-1}(\mathbf{y}),r)} f_j(\mathbf{z}) - f_j(\mathbf{y})\mathrm{d}\mathbf{z}|^{q_j}\,\mathrm{d}\mathbf{y} \\
\leq & \lim_{\delta\to 0}\int\limits_{\Omega_j^\theta} \lim_{r\to 0}\left(\fint\limits_{B(T_\delta^{-1}(\mathbf{y}),r)} |f_j(\mathbf{z}) - f_j(\mathbf{y})|\mathrm{d}\mathbf{z}\right)^{q_j}\mathrm{d}\mathbf{y} \\
\leq & \lim_{\delta\to 0}\int\limits_{\Omega_j^\theta} \lim_{r\to 0}\left(\frac{1}{(2r)^2}\right)^{q_j}\left(\int\limits_{B(T_\delta^{-1}(\mathbf{y}),r)} |f_j(\mathbf{z}) - f_j(\mathbf{y})|^{q_j}\mathrm{d}\mathbf{z}\right)\left(\int\limits_{B(T_\delta^{-1}(\mathbf{y}),r)} 1^{p_j}\mathrm{d}\mathbf{z}\right)^{\frac{q_j}{p_j}}\mathrm{d}\mathbf{y} \\
= & \lim_{\delta\to 0}\int\limits_{\Omega_j^\theta} \lim_{r\to 0}\frac{1}{(2r)^2} \int\limits_{B(T_\delta^{-1}(\mathbf{y}),r)} |f_j(\mathbf{z}) - f_j(\mathbf{y})|^{q_j}\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{y} \qquad (43)
\end{aligned}
$$

Since theorem 4.3 holds for $r \to 0$ in any arbitrary order, the variable r in line (43) can be chosen in the following way: Taking into account remark 4.1 we know that $T_\delta^{-1}(\mathbf{y}) \in B(\mathbf{y}, \delta\|\theta\|_\infty)$. Therefore we chose r in dependence of δ such that $B(T_\delta^{-1}(\mathbf{y}), r) \subset B(\mathbf{y}, \delta)$, see fig. 5:

$$
\begin{aligned}
r &= |\mathbf{y} + \delta(T_\delta^{-1}(\mathbf{y}) - \mathbf{y}) - T_\delta^{-1}(\mathbf{y})| = |(\mathbf{y} - T_\delta^{-1}(\mathbf{y})) - \delta(\mathbf{y} - T_\delta^{-1}(\mathbf{y}))| \\
&= (1-\delta)|\mathbf{y} - T_\delta^{-1}(\mathbf{y})| = (1-\delta)|T_\delta(\mathbf{x}) - \mathbf{x}| = (1-\delta)|\mathbf{x} - \delta(\theta(\mathbf{x}), 0)^\top - \mathbf{x}| \\
&\leq (1-\delta)\delta\|\theta\|_\infty
\end{aligned}
$$

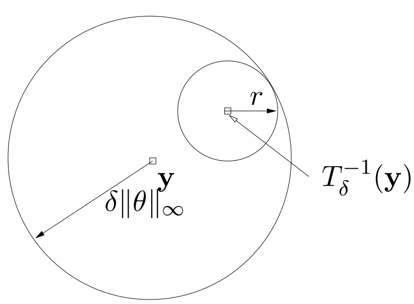

Figure 5: choice of r in dependence of δ

Choosing $r = (1-\delta)\delta\|\theta\|_\infty$ we have:

$$
\begin{aligned}
(43) &= \lim_{\delta\to 0}\int_{\Omega_\theta} \lim_{\delta\to 0} \frac{1}{(2(1-\delta)\delta\|\theta\|_\infty)^2} \int_{B(T_\delta^{-1}(\mathbf{y}),(1-\delta)\delta\|\theta\|_\infty)} |f_j(\mathbf{z}) - f_j(\mathbf{y})|^{q_j}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{y} \\
&\leq \lim_{\delta\to 0}\frac{1}{(1-\delta)^2}\int_{\Omega_\theta} \lim_{\delta\to 0} \fint_{B(\mathbf{y},\delta\|\theta\|_\infty)} |f_j(\mathbf{z}) - f_j(\mathbf{y})|^{q_j}\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{y} \overset{(42)}{=} 0
\end{aligned}
$$

■

Remark 4.2

The volume force density defined on Ω_{δ_0} is always denoted with f. In our context the volume force density related to $\Omega_{\delta_0+\delta}$ has the property $f_{[\delta_0+\delta]} = f|_{\Omega_{\delta_0+\delta}}$.

Lemma 4.3 (Norms of the applied force densities)

The norms of the applied force densities , f, h, are independent of the crack length, i.e.:

$$\left\|f_{[\delta_0+\delta]}\right\|_{L^{\vec{q}}(\Omega_{\delta_0+\delta})} = \|f\|_{L^{\vec{q}}(\Omega_{\delta_0})} .$$

Proof:

Recall that the reference configuration Ω_{δ_0} and a current configuration $\Omega_{\delta_0+\delta}$ only differ in the crack length:

$$\Omega_{\delta_0}\backslash\Omega_{\delta_0+\delta} = (\Gamma_{T\delta_0}\backslash\Gamma_{T\delta_0+\delta}) \subset \partial\Omega_j \quad \text{for;} \, j\in\{1,2\}\,.$$

Therefore it is easy to see that

$$\left\|f_{[\delta_0+\delta]}\right\|_{L^{\vec{q}}(\Omega_{\delta_0+\delta})} = \sum_{j=1}^{2}\|f_j\|_{L^{q_j}(\Omega_j)} = \|f\|_{L^{\vec{q}}(\Omega_{\delta_0})}$$

Since $\operatorname{supp} h \subset \partial\Omega$, the norm of the surface force density $\|h\|_{L^{\vec{q}}(\Gamma_N)}$ is independent of the crack length. Furthermore h is not affected by the transformation T_δ for any $\delta\in[0,a]$. ■

Lemma 4.4 (Extension of the applied boundary displacements)

For a given $g\in W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma_D)$ and every $\delta\in[0,a]$ there exists an extension $\hat{g}_{[\delta_0+\delta]}\in V^{\vec{p}}_{(g)}(\Omega_{\delta_0+\delta})$ with the property:

$$\left\|\hat{g}_{[\delta_0+\delta]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta})} = \left\|\hat{g}_{[\delta_0]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0})} . \tag{44}$$

Proof:

By definition of the trace spaces for every function $g\in W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma_D)$ there exists an extension $\hat{g}^\star_{[\delta_0+\delta]}\in W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$. Multiplication with a function $\eta\in C^\infty(\overline{\Omega})$ leads to
$(\hat{g}^\star_{[\delta_0+\delta]}\eta)\in W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$, where $\operatorname{supp}(\hat{g}^\star_{[\delta_0+\delta]}\eta) = \operatorname{supp}\hat{g}^\star_{[\delta_0+\delta]}\cap\operatorname{supp}\eta$.
Let $B_a = \{\mathbf{x}, |\mathbf{x}-S_{\delta_0}|\leq a\}$. Choose η in such a way that $B_a\cap\operatorname{supp}\eta = \emptyset, \eta\equiv 1$ on Γ_D, then $\hat{g} = (\hat{g}^\star_{[\delta_0+\delta]}\eta)$ is an admissible extension for every $\delta\in[0,a]$. See fig. 6. Therefore assumption (44) holds. ■

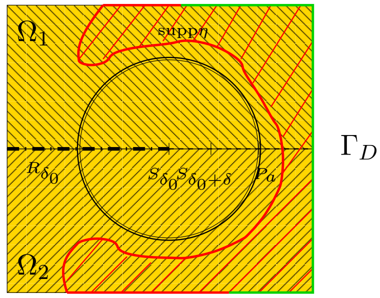

Figure 6: Choice of the function η

4.1.3 Boundedness and convergence of the transformed minimizers as $\delta \to 0$

We begin with the verification of the uniform boundedness of the minimizers $u_{[\delta_0+\delta]}$ in $W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$ and of the transformed minimizers $u^{[\delta_0+\delta]}$ in $W^{1,\vec{p}}(\Omega_{\delta_0})$ respectively.

Lemma 4.5 (Uniform boundedness of the minimizers)
Let $f \in L^{\vec{q}}(\Omega_{\delta_0+\delta}), h \in L^{\vec{q}}(\Gamma_N)$ and $g \in W^{1-\frac{1}{\vec{p}},\vec{p}}(\Gamma_D)$. Then the solution $u_{[\delta_0+\delta]}$ of the minimization problem referring to the current configuration $\Omega_{\delta_0+\delta}$ is bounded for all $\delta \in [0,a]$ by a constant $c > 0$:

$$\left\| u_{[\delta_0+\delta]} \right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta})} < c \, . \tag{45}$$

For the minimizers transformed to the reference configuration respectively holds:

$$\left\| u^{[\delta_0+\delta]} \right\|_{W^{1,\vec{p}}(\Omega_{\delta_0})} < c \tag{46}$$

Proof:
Let $\hat{g} \in W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$ be an extension of the boundary displacement g as in lemma 4.4. Then it holds $u^0_{[\delta_0+\delta]} = (u_{[\delta_0+\delta]} - \hat{g}) \in V^{\vec{p}}_{(0)}(\Omega_{\delta_0+\delta})$. In the following we define for arbitrary $\delta \in [0,a]$:

$$u^0_j = u^0_{[\delta_0+\delta]_j} = u^0_{[\delta_0+\delta]}|_{\Omega_j} \quad \text{and} \quad u_j = u_{[\delta_0+\delta]_j} = u_{[\delta_0+\delta]}|_{\Omega_j} \, .$$

Because $\|u_j\|_{W^{1,p_j}(\Omega_j)} \leq \left\|u^0_j\right\|_{W^{1,p_j}(\Omega_j)} + \|\hat{g}_j\|_{W^{1,p_j}(\Omega_j)}$ we now want to show the uniform boundedness of $\left\|u^0_j\right\|_{W^{1,p_j}(\Omega_j)}$. Since $u = u_{[\delta_0+\delta]}$ is the minimizer of the potential energy $E(\Omega_{\delta_0+\delta}, \cdot)$ it holds

$$\left\langle \mathrm{D}E(\Omega_{\delta_0+\delta}, u), u^0 \right\rangle_{V^{\vec{p}}_{(0)}(\Omega_{\delta_0+\delta})} = 0$$

and therefore

$$\left\langle \mathrm{D}J_{el}(\Omega_{\delta_0+\delta}, u), u^0 \right\rangle_{V^{\vec{p}}_{(0)}(\Omega_{\delta_0+\delta})} = \sum_{j=1}^{2} \int\limits_{\Omega_j} f_j u^0_j \, \mathrm{d}\mathbf{x} + \int\limits_{\Gamma_{Nj}} h_j u^0_j \, \mathrm{d}\mathbf{s} \, . \tag{47}$$

See theorem B.1 for the Fréchet derivatives of $J_{el}(\Omega_\delta, \cdot)$ and $E(\Omega_\delta, \cdot)$.

In the following we make the left hand side of equation (47) smaller using estimate (27) with $\boldsymbol{a} = \nabla \hat{g}_j, \boldsymbol{b} = \nabla u_j$ and **(H1)**:

$$
\begin{aligned}
\left\langle \mathrm{D}J_{el}(\Omega_{\delta_0+\delta}, u), u^0 \right\rangle_{V^{\vec{p}}_{(0)}(\Omega_{\delta_0+\delta})} &\overset{(27)}{\geq} \sum_{j=1}^{2} \int\limits_{\Omega_j} W_{elj}(\nabla u_j) - W_{elj}(\nabla \hat{g}_j)\,\mathrm{d}\mathbf{x} \\
&\quad + \int\limits_{\Omega_j} d_j(\tilde{k}_j + |\nabla u_j| + |\nabla \hat{g}_j|)^{p_j-2}\,|\nabla \hat{g}_j - \nabla u_j|^2\,\mathrm{d}\mathbf{x} \\
&\overset{(\mathbf{H1})}{\geq} \sum_{j=1}^{2} c_{1j}\|\nabla u_j\|^{p_j}_{L^{p_j}(\Omega_j)} - \int\limits_{\Omega_j} W_{elj}(\nabla \hat{g}_j)\,\mathrm{d}\mathbf{x} \\
&\overset{(\mathbf{H1})}{\geq} \sum_{j=1}^{2} c_{1j}\|\nabla u_j\|^{p_j}_{L^{p_j}(\Omega_j)} - c_{2j}\mathcal{L}^2(\Omega_j) - c_{2j}\|\nabla \hat{g}_j\|^{p_j}_{L^{p_j}(\Omega_j)} \quad (48)
\end{aligned}
$$

The right hand side of equation (47) can be enlarged using Hölder's inequality (103)

$$
\sum_{j=1}^{2} \int\limits_{\Omega_j} f_j u_j^0\,\mathrm{d}\mathbf{x} + \int\limits_{\Gamma_{Nj}} h_j u_j^0\,\mathrm{d}\mathbf{s} \leq \sum_{j=1}^{2} \|f_j\|_{L^{q_j}(\Omega_j)}\left\|u_j^0\right\|_{L^{p_j}(\Omega_j)} + \|h_j\|_{L^{q_j}(\Gamma_{Nj})}\|u_j^0\|_{L^{p_j}(\Gamma_{Nj})}\ , \quad (49)
$$

where $\left\|u_j^0\right\|_{L^{p_j}(\Omega_j)} \leq \left\|u_j^0\right\|_{W^{1,p_j}(\Omega_j)}$ and according to definition 3.2 holds

$$
\|u_j^0\|_{L^{p_j}(\Gamma_{Nj})} \leq \|u_j^0\|_{W^{1-\frac{1}{p_j},p_j}(\Gamma_{Nj})} = \inf_{\substack{v_j^0 \in W^{1,p_j}(\Omega_j) \\ v_j^0|_{\Gamma_{Nj}} = u_j^0|_{\Gamma_{Nj}}}} \|v_j^0\|_{W^{1,p_j}(\Omega_j)} \leq \|u_j^0\|_{W^{1,p_j}(\Omega_j)}\ .
$$

Under consideration of the norm equivalence in theorem 3.1 this leads to

$$
\begin{aligned}
(49) &\leq \sum_{j=1}^{2} \|f_j\|_{L^{q_j}(\Omega_j)}\|u_j^0\|_{W^{1,p_j}(\Omega_j)} + \|h_j\|_{L^{q_j}(\Gamma_{Nj})}\|u_j^0\|_{W^{1,p_j}(\Omega_j)} && (50) \\
&\leq \|u^0\|_{W^{1,\vec{p}}(\Omega_\delta)}\left(\|f\|_{L^{\vec{q}}(\Omega_\delta)} + \|h\|_{L^{\vec{q}}(\Gamma_N)}\right) && (51) \\
&\leq c_N(\|f\|_{L^{\vec{q}}(\Omega_{\Omega_{\delta_0+\delta}})} + \|h\|_{L^{\vec{q}}(\Gamma_N)})\|\nabla u^0\|_{L^{\vec{p}}(\Omega_{\delta_0+\delta})}\ . && (52)
\end{aligned}
$$

Notify, that the constant $c_N = \max_{j=1,2}\{c_{Nj}\}$ is independent of δ, since the crack is situated on $\partial\Omega_j,\ j = 1, 2$.
Putting together estimates (48) and (52) we get:

$$
\begin{aligned}
&c_N(\|f\|_{L^{\vec{q}}(\Omega_{\Omega_{\delta_0+\delta}})} + \|h\|_{L^{\vec{q}}(\Gamma_N)})\|\nabla u^0\|_{L^{\vec{p}}(\Omega_{\delta_0+\delta})} + \sum_{j=1}^{2} c_{2j}\mathcal{L}^2(\Omega_j) + c_{2j}\|\nabla \hat{g}_j\|^{p_j}_{L^{p_j}(\Omega_j)} \\
\geq\ &\sum_{j=1}^{2} c_{1j}\|\nabla u_j\|^{p_j}_{L^{p_j}(\Omega_j)} \geq c_{1m}\sum_{j=1}^{2} \left|\|\nabla u_j^0\|_{L^{p_j}(\Omega_j)} - \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}\right|^{p_j}\ , \qquad (53)
\end{aligned}
$$

where $c_{1m} = \min_j\{c_{1j}\}$. Because of the absolute value in line (53) we have to distinguish between two cases:

If $\left\|\nabla u_j^0\right\|_{L^{p_j}(\Omega_j)} - \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)} \leq 0$ we immediately see

$$\left\|\nabla u_j^0\right\|_{L^{p_j}(\Omega_j)} \leq \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)} \tag{54}$$

In the case $\left\|\nabla u_j^0\right\|_{L^{p_j}(\Omega_j)} > \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}$ we get

$$\begin{aligned}
\left|\|\nabla u_j^0\|_{L^{p_j}(\Omega_j)} - \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}\right|^{p_j} &= \left(\|\nabla u_j^0\|_{L^{p_j}(\Omega_j)} - \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}\right)^{p_j} \\
&\geq \left(\|\nabla u_j^0\|_{L^{p_j}(\Omega_j)} - (1 + \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)})\right)^{p_j} \\
&\geq \|\nabla u_j^0\|_{L^{p_j}(\Omega_j)}^{p_j} - (1 + \|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)})^{p_j} \\
&\overset{(105)}{\geq} \|\nabla u_j^0\|_{L^{p_j}(\Omega_j)}^{p_j} - 2^{p_j-1} - 2^{p_j-1}\|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}^{p_j}
\end{aligned} \tag{55}$$

Thus it holds

$$\begin{aligned}
&\sum_{j=1}^{2} \|\nabla u_j^0\|_{L^{p_j}(\Omega_j)}^{p_j} \\
&\leq \frac{1}{2c_{1m}}\left(\sum_{j=1}^{2} c_{2j}\|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}^{p_j} + c_{2j}\mathcal{L}^2(\Omega_j) + c_m\left(\|f\|_{L^{\vec{q}}(\Omega_{\delta_0+\delta})} + \|h\|_{L^{\vec{q}}(\Gamma_N)}\right)\left\|\nabla u^0\right\|_{L^{\vec{p}}(\Omega_{\delta_0+\delta})}\right. \\
&\qquad \left. + 2^{p_j-2} + \frac{1}{2}(1 + 2^{p_j-1})\|\nabla \hat{g}_j\|_{L^{p_j}(\Omega_j)}^{p_j}\right) \\
&= A(\hat{g}, \Omega_1, \Omega_2) + B(f,h)\left\|\nabla u^0\right\|_{L^{\vec{p}}(\Omega_{\delta_0+\delta})}
\end{aligned} \tag{56}$$

Estimate (56) can be used to show $\left\|u^0\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta})} < c$ for every $\delta \in [0,a]$ by contradiction. Thus we assume that $\left\|u^0\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta})}$ is unbounded. Then there exists a sequence $\{u_{\delta_n}^0\}_{n\in\mathbb{N}}$ with $\delta_n \in [0,a], n \in \mathbb{N}$, having the property

$$\left\|u_{\delta_n}^0\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta_n})} \to \infty \quad \text{as } n \to \infty$$

and due to the norm equivalence 3.1 $\left\|\nabla u_{\delta_n}^0\right\|_{L^{\vec{p}}(\Omega_{\delta+\delta_n})} \to \infty$ as $n \to \infty$, respectively. This implies that at least one of the norms $\left\|\nabla u_{\delta_n j}^0\right\|_{L^{p_j}(\Omega_j)}$, $j = 1, 2$, tends to ∞ as $n \to \infty$.
On the other hand we get from estimate (56)

$$\begin{aligned}
1 &\leq \frac{A(\hat{g}, \Omega_1, \Omega_2)}{\sum_{j=1}^{2}\left\|\nabla u_{\delta_n j}^0\right\|_{L^{p_j}(\Omega_j)}^{p_j}} + \frac{2B(f,h)\max_j\{\left\|\nabla u_{\delta_n j}^0\right\|_{L^{p_j}(\Omega_j)}\}}{\max_j\{\left\|\nabla u_{\delta_n j}^0\right\|_{L^{p_j}(\Omega_j)}^{p_j}\}} \\
&= \frac{A(\hat{g}, \Omega_1, \Omega_2)}{\sum_{j=1}^{2}\left\|\nabla u_{\delta_n j}^0\right\|_{L^{p_j}(\Omega_j)}^{p_j}} + \frac{2B(f,h)}{\max_j\{\left\|\nabla u_{\delta_n j}^0\right\|_{L^{p_j}(\Omega_j)}^{p_j-1}\}} \\
&\longrightarrow 0 \quad \text{as } n \to \infty\ ,
\end{aligned}$$

which states the contradiction.
So it is proved that the norms $\left\|u_{[\delta_0+\delta]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta})}$ are uniformly bounded for $\delta \in [0,a]$. Since T_δ is a diffeomorphism it holds $T_\delta^{-1}(\Omega_{\delta_0}) = \Omega_{\delta_0+\delta}$ and because of $u^{[\delta_0+\delta]} = u_{[\delta_0+\delta]} \circ T_\delta^{-1}$ the norms $\|u^{[\delta_0+\delta]}\|_{W^{1,\vec{p}}(\Omega_{\delta_0})}$ and $\left\|u_{[\delta_0+\delta]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0+\delta})}$ are equivalent. Therefore the transformed minimizers are uniformly bounded, too. ■

Now, with the aid of lemma 4.5 the convergence in $W^{1,\vec{p}}(\Omega_{\delta_0})$ of the transformed minimizers $u^{[\delta_0+\delta]}$ for the current configurations $\Omega_{\delta_0+\delta}$ to the minimizer $u_{[\delta_0]}$ for the reference configuration as $\delta \to 0$ can be shown.

Theorem 4.4 (Convergence of the transformed minimizers in $W^{1,\vec{p}}(\Omega_{\delta_0})$)
Let $u^{[\delta_0+\delta]} \in W^{1,\vec{p}}(\Omega_{\delta_0})$ be the transformed solutions of the minimization problem for the current configurations $\Omega_{\delta_0+\delta}$ and $u_{[\delta_0]} \in W^{1,\vec{p}}(\Omega_{\delta_0})$ the minimizer for the reference configuration. Then it holds:

$$\left\|u^{[\delta_0+\delta]} - u_{[\delta_0]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0})} \to 0 \qquad \text{as } \delta \to 0 \,. \tag{57}$$

Proof:
We proceed as follows: The Fréchet derivatives of the potential energies of both the reference and the current configuration are set up and their difference is estimated on the reference configuration using the difference of the minimizers $(u^{[\delta_0+\delta]} - u_{[\delta_0]}) \in V_{(0)}^{\vec{p}}(\Omega_{\delta_0})$ as special test functions.
For the current configurations the Fréchet derivatives of the potential energies applied to a test function $v \in V_{(0)}^{\vec{p}}(\Omega_{\delta_0+\delta})$ read as follows, see theorem B.1:

$$\left\langle \mathrm{D}E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]}), v\right\rangle_{V_{(0)}^{\vec{p}}(\Omega_{\delta_0+\delta})} = \sum_{j=1}^{2} \int\limits_{\Omega_j} \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0+\delta]}\right) \cdot \nabla v_j \,\mathrm{d}\mathbf{x} - \int\limits_{\Omega_j} f_j v_j \,\mathrm{d}\mathbf{x} - \int\limits_{\Gamma_{N\,j}} h_j v_j \,\mathrm{d}\mathbf{s} = 0 \,.$$

Transformation to the reference configuration and application to a test function $v \in V_{(0)}^{\vec{p}}(\Omega_{\delta_0})$ leads to:

$$\begin{aligned} &\left\langle \mathrm{D}E_{[\delta_0+\delta]}(\Omega_{\delta_0}, u^{[\delta_0+\delta]}), v\right\rangle_{V_{(0)}^{\vec{p}}(\Omega_{\delta_0})} \\ &= \sum_{j=1}^{2} \int\limits_{\Omega_j} \mathrm{D}W_{el\,j}\left(\mathfrak{T}_{u_j}^{\delta}\right) \cdot \frac{\nabla v_j}{\det J_{T_\delta}^{\delta}} \,\mathrm{d}\mathbf{y} - \int\limits_{\Omega_j} f_j^{[\delta_0+\delta]} \frac{v_j}{\det J_{T_\delta}^{\delta}} \,\mathrm{d}\mathbf{y} - \int\limits_{\Gamma_{N\,j}} h_j \frac{v_j}{1} \,\mathrm{d}\mathbf{s} = 0 \,, \end{aligned} \tag{58}$$

where $\mathfrak{T}_u^{\delta}(\mathbf{y})$ is according to (39) given by

$$\begin{aligned} \nabla_{\mathbf{x}} u_{[\delta_0+\delta]}(\mathbf{x}) &= \left(\nabla_{\mathbf{y}} u^{[\delta_0+\delta]}(\mathbf{y})\right)^{\top} \nabla_{\mathbf{x}} \mathbf{y} = \left(\nabla_{\mathbf{y}} u^{[\delta_0+\delta]}(\mathbf{y})\right)^{\top} J_{T_\delta} \\ &= \nabla_{\mathbf{y}} u^{[\delta_0+\delta]}(\mathbf{y}) - \delta \partial_{y_1} u^{[\delta_0+\delta]}(\mathbf{y}) \nabla_{\mathbf{x}} \theta^{\delta}(\mathbf{y}) = \mathfrak{T}_u^{\delta}(\mathbf{y}) \,. \end{aligned} \tag{59}$$

For the Fréchet derivative of the potential energy for the reference configuration applied to a test function $v \in V_{(0)}^{\vec{p}}(\Omega_{\delta_0})$ we get:

$$\left\langle \mathrm{D}E(\Omega_{\delta_0}, u_{[\delta_0]}), v\right\rangle_{V_{(0)}^{\vec{p}}(\Omega_{\delta_0})} = \sum_{j=1}^{2} \int\limits_{\Omega_j} \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]}\right) \cdot \nabla v_j \,\mathrm{d}\mathbf{y} - \int\limits_{\Omega_j} f_j v_j \,\mathrm{d}\mathbf{y} - \int\limits_{\Gamma_{N\,j}} h_j v_j \,\mathrm{d}\mathbf{s} = 0 \,. \tag{60}$$

Setting up the difference ((58) – (60)) and using $w^0 = (u^{[\delta_0+\delta]} - u_{[\delta_0]}) \in V^{\vec{p}}_{(0)}(\Omega_{\delta_0})$ as special test functions results in:

$$\sum_{j=1}^{2} \int_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \cdot \frac{\nabla w^0_j}{\det J^{\delta}_{T_\delta}} - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) \cdot \nabla w^0_j \right) \mathrm{d}\mathbf{y} \tag{61}$$

$$= \sum_{j=1}^{2} \int_{\Omega_j} \left(\frac{f^{[\delta_0+\delta]}_j}{\det J^{\delta}_{T_\delta}} - f_j \right) w^0_j \, \mathrm{d}\mathbf{y} \; . \tag{62}$$

At first expression (62) will be estimated from above. Thereto the following identity is used:

$$\frac{A}{1-\delta Z} - B = A - B + \frac{\delta Z A}{1-\delta Z} \; . \tag{63}$$

Applying this identity to expression (62) leads to:

$$(62) \;=\; \underbrace{\sum_{j=1}^{2} \int_{\Omega_j} \left(f^{[\delta_0+\delta]}_j - f_j \right) w^0_j \, \mathrm{d}\mathbf{y}}_{F_{1j}} + \underbrace{\sum_{j=1}^{2} \int_{\Omega_j} \frac{\delta \partial_{x_1} \theta^{\delta}}{\det J^{\delta}_{T_\delta}} w^0_j f^{[\delta_0+\delta]}_j \, \mathrm{d}\mathbf{x}}_{F_{2j}} \; .$$

On the integrals in expression F_{1j} Hölder's inequality (103) is applied. Since $\theta^\delta \in C^\infty_0(\Omega)$, we have $(\partial_{x_1}\theta^\delta) \in C^\infty_0(\Omega)$ and $(1 - \partial_{x_1}\theta^\delta) \in C^\infty(\Omega)$. Furthermore $\det J^{\delta}_{T_\delta} > 0$ for every $\delta \in [0, a]$ and the supremum norm exists and is independent of δ:

$$\left\| \frac{\partial_{x_1}\theta^\delta}{\det J^{\delta}_{T_\delta}} \right\|_{\infty} = \sup_{\mathbf{y} \in \Omega_{\delta_0}} \left| \frac{\partial_{x_1}\theta^\delta(\mathbf{y})}{\det J^{\delta}_{T_\delta}(\mathbf{y})} \right| = \mathfrak{s} < \infty \; . \tag{64}$$

Taking this supremum and applying Hölder's inequality (103) to expression F_{2j} leads to:

$$\begin{aligned}
(62) \;\le\;& |F_{1j} + F_{2j}| \le \sum_{j=1}^{2} \|w^0_j\|_{W^{1,p_j}(\Omega_j)} \left(\left\| f^{[\delta_0+\delta]}_j - f_j \right\|_{L^{q_j}(\Omega_j)} + \delta\mathfrak{s} \left\| f^{[\delta_0+\delta]}_j \right\|_{L^{q_j}(\Omega_j)} \right) \\
\le\;& \sum_{j=1}^{2} \left(\left\| u^{[\delta_0+\delta]}_j \right\|_{W^{1,p_j}(\Omega_j)} + \left\| u_{[\delta_0]\,j} \right\|_{W^{1,p_j}(\Omega_j)} \right) \left(\left\| f^{[\delta_0+\delta]}_j - f_j \right\|_{L^{q_j}(\Omega_j)} + \delta\mathfrak{s} \|f_j\|_{L^{q_j}(\Omega_j)} \right) \\
\le\;& 2c \left(\left\| f^{[\delta_0+\delta]} - f \right\|_{L^{\vec{q}}(\Omega_{\delta_0})} + \delta\mathfrak{s} \|f\|_{L^{\vec{q}}(\Omega_{\delta_0})} \right) \; ,
\end{aligned} \tag{65}$$

where the uniform boundedness of the minimizers, see lemma 4.5, has been considered.

In the following, expression (61) is estimated from below using identity (63) once more:

$$\begin{aligned}
(61) =& \sum_{j=1}^{2} \int_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \cdot \frac{\nabla w^0_j}{\det J^{\delta}_{T_\delta}} - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) \cdot \nabla w^0_j \right) \mathrm{d}\mathbf{y} \\
=& \sum_{j=1}^{2} \int_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) \right) \cdot \nabla w^0_j \, \mathrm{d}\mathbf{y} + \int_{\Omega_j} \frac{\delta \partial_{x_1}\theta^\delta}{\det J^{\delta}_{T_\delta}} \mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \cdot \nabla w^0_j \, \mathrm{d}\mathbf{y} \\
=& \sum_{j=1}^{2} I_{1j} + I_{2j} \; ,
\end{aligned} \tag{66}$$

where

$$I_{1j} = \int\limits_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) \cdot \nabla w^0_j \right) \mathrm{d}\mathbf{y}$$

and

$$I_{2j} = \int\limits_{\Omega_j} \frac{\delta \partial_{x_1}\theta^{\delta}}{\det J^{\delta}_{T_\delta}} \mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \cdot \nabla w^0_j \,\mathrm{d}\mathbf{y}\ .$$

At first, intergral I_{2j} is made smaller taking into account Hölder's inequality (103) and **(H2)**:

$$\begin{aligned}
I_{2j} = & \int\limits_{\Omega_j} \frac{\delta \partial_{x_1}\theta^{\delta}}{\det J^{\delta}_{T_\delta}} \mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \cdot \nabla w^0_j \,\mathrm{d}\mathbf{y} \geq -\delta \int\limits_{\Omega_j} \left| \frac{\partial_{x_1}\theta^{\delta}}{\det J^{\delta}_{T_\delta}} \mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \cdot \nabla w^0_j \right| \mathrm{d}\mathbf{y} \\
\geq & -\delta\mathfrak{s} \left\| \mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) \right\|_{L^{q_j}(\Omega_j)} \left\|\nabla w^0_j\right\|_{L^{p_j}(\Omega_j)} \overset{\textbf{(H2)}}{\geq} -\delta\mathfrak{s}c_j \left\| 1 + |\mathfrak{T}^{\delta}_{u_j}|^{p_j-1} \right\|_{L^{q_j}(\Omega_j)} \left\|\nabla w^0_j\right\|_{L^{p_j}(\Omega_j)} \\
\geq & -\delta\mathfrak{s}c_j \left\| 1 + |\mathfrak{T}^{\delta}_{u_j}|^{p_j-1} \right\|_{L^{q_j}(\Omega_j)} \left(\left\|\nabla u^{[\delta_0+\delta]}_j\right\|_{L^{p_j}(\Omega_j)} + \left\|\nabla u_{[\delta_0]\,j}\right\|_{L^{p_j}(\Omega_j)} \right)\ . \qquad (67)
\end{aligned}$$

Now the following estimate can be applied since $q_j = \frac{p_j}{p_j - 1} > 1$:

$$\left\| 1 + |A|^{p_j-1} \right\|^{q_j}_{L^{q_j}(\Omega_j)} \overset{(105)}{\leq} \int\limits_{\Omega_j} 2^{q_j-1} \left(1^{q_j} + |A|^{p_j}\right) \mathrm{d}\mathbf{y} = 2^{q_j-1}\left(\mathcal{L}^2(\Omega_j) + \|A\|^{p_j}_{L^{p_j}(\Omega_j)}\right)\ . \qquad (68)$$

Under consideration of

$$\begin{aligned}
\left\|\mathfrak{T}^{\delta}_{u_j}\right\|_{L^{p_j}(\Omega_j)} &= \left\| \nabla u^{[\delta_0+\delta]}_j - \delta \partial_{y_1} u^{[\delta_0+\delta]}_j \nabla_{\mathbf{x}}\theta^{\delta} \right\|_{L^{p_j}(\Omega_j)} \\
&\leq (1 + \delta\|\nabla_{\mathbf{x}}\theta^{\delta}\|_{\infty}) \left\| \nabla u^{[\delta_0+\delta]}_j \right\|_{L^{p_j}(\Omega_j)} \qquad (69)
\end{aligned}$$

we get an estimate of the form

$$I_{2j} \geq (67) \geq -\delta C_{1j}\left(p_j, \mathfrak{s}, c\right) = -\delta C_{1j} \qquad (70)$$

since the norms of the minimizers are uniformly bounded by c according to lemma 4.5.

In the following, integral $I_{1j}, j = 1, 2,$ is estimated from below with the aid of estimate (28):

$$\begin{aligned}
I_{1j} &= \int\limits_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right)\right) \cdot \nabla(u^{[\delta_0+\delta]} - u_{[\delta_0]\,j})\,\mathrm{d}\mathbf{y} \\
&= \int\limits_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right)\right) \cdot (\nabla u^{[\delta_0+\delta]} - \delta\partial_{y_1}u^{[\delta_0+\delta]}\nabla_{\mathbf{x}}\theta^{\delta} - \nabla u_{[\delta_0]\,j})\,\mathrm{d}\mathbf{y} \\
&\quad + \int\limits_{\Omega_j} \left(\mathrm{D}W_{el\,j}\left(\mathfrak{T}^{\delta}_{u_j}\right) - \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right)\right) \cdot (\delta\partial_{y_1}u^{[\delta_0+\delta]})\nabla_{\mathbf{x}}\theta^{\delta}\,\mathrm{d}\mathbf{y} \\
&\overset{(28)}{\geq} \int\limits_{\Omega_j} \tilde{d}_j \left(\tilde{k}_j + |\mathfrak{T}^{\delta}_{u_j}| + |\nabla u_{[\delta_0]\,j}|\right)^{p_j-2} \left|\mathfrak{T}^{\delta}_{u_j} - \nabla u_{[\delta_0]\,j}\right|^2 \mathrm{d}\mathbf{y} + I_{4j} \\
&= I_{3j} + I_{4j}\ , \qquad (71)
\end{aligned}$$

where

$$I_{3j} = \int_{\Omega_j} \tilde{d}_j \left(\tilde{k}_j + |\mathfrak{T}^\delta_{u_j}| + |\nabla u_{[\delta_0]j}|\right)^{p_j-2} \left|\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}\right|^2 \mathrm{d}\mathbf{y}$$

and

$$I_{4j} = \int_{\Omega_j} \left(\mathrm{D}W_{elj}\left(\mathfrak{T}^\delta_{u_j}\right) - \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right) \cdot (\delta \partial_{y_1} u^{[\delta_0+\delta]}) \nabla_{\mathbf{x}} \theta^\delta \, \mathrm{d}\mathbf{y} \,. \tag{72}$$

For the estimation of I_{3j} we have to distinguish between two cases:
If $1 < p_j < 2$ we can use Hölder's inequality (104) with $p = \frac{p_j}{2}, q = \frac{\frac{p_j}{2}}{\frac{p_j}{2}-1} = \frac{p_j}{p_j-2}$:

$$\begin{aligned} I_{3j} &\overset{(104)}{\geq} \tilde{d}_j \left\|\tilde{k}_j + |\mathfrak{T}^\delta_{u_j}| + |\nabla u_{[\delta_0]j}|\right\|^{p_j-2}_{L^{p_j}(\Omega_j)} \left\|\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}\right\|^2_{L^{p_j}(\Omega_j)} \\ &\geq \tilde{d}_j \left\|\nabla u_{[\delta_0]j}\right\|^{p_j-2}_{L^{p_j}(\Omega_j)} \left\|\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}\right\|^2_{L^{p_j}(\Omega_j)} \,. \end{aligned} \tag{73}$$

In the case $p_j \geq 2$ we can use the triangle inequality, such that

$$I_{3j} \geq \tilde{d}_j \int_{\Omega_j} |\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}|^{p_j-2} |\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}|^2 \,\mathrm{d}\mathbf{y} = \tilde{d}_j \left\|\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}\right\|^{p_j}_{L^{p_j}(\Omega_j)} \,. \tag{74}$$

In both cases the norm $\left\|\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)}$ can be estimated in the following way:

$$\begin{aligned} \left\|\mathfrak{T}^\delta_{u_j} - \nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)} \geq& \left\|\nabla u_j^{[\delta_0+\delta]} - \nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)} - \left\|\delta \partial_{y_1} u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}} \theta^\delta\right\|_{L^{p_j}(\Omega_j)} \\ \geq& \left\|\nabla u_j^{[\delta_0+\delta]} - \nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)} - \delta \left\|\nabla u^{[\delta_0+\delta]}\right\|_{L^{p_j}(\Omega_j)} \|\nabla_{\mathbf{x}} \theta^\delta\|_\infty \end{aligned} \tag{75}$$

Integral I_{4j}, $j = 1, 2$, is made smaller using Hölder's inequality (103) and **(H2)**:

$$\begin{aligned} I_{4j} \geq& -\delta \int_{\Omega_j} \left|\mathrm{D}W_{elj}\left(\mathfrak{T}^\delta_{u_j}\right) - \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right| |\partial_{y_1} u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}} \theta^\delta| \,\mathrm{d}\mathbf{y} \\ \overset{(103)}{\geq}& -\delta \left\|\mathrm{D}W_{elj}\left(\mathfrak{T}^\delta_{u_j}\right) - \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right\|_{L^{q_j}(\Omega_j)} \left\|\partial_{y_1} u_j^{[\delta_0+\delta]}\right\|_{L^{p_j}(\Omega_j)} \|\nabla_{\mathbf{x}} \theta^\delta\|_\infty \\ \geq& -\delta \left(\left\|\mathrm{D}W_{elj}\left(\mathfrak{T}^\delta_{u_j}\right)\right\|_{L^{q_j}(\Omega_j)} + \left\|\mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right\|_{L^{q_j}(\Omega_j)}\right) \left\|\nabla u_j^{[\delta_0+\delta]}\right\|_{L^{p_j}(\Omega_j)} \|\nabla_{\mathbf{x}} \theta^\delta\|_\infty \\ \overset{\mathbf{(H2)}}{\geq}& -\delta c_j \left(\left\|1 + |\mathfrak{T}^\delta_{u_j}|^{p_j-1}\right\|_{L^{q_j}(\Omega_j)} + \left\|1 + |\nabla u_{[\delta_0]j}|^{p_j-1}\right\|_{L^{q_j}(\Omega_j)}\right) \left\|\nabla u_j^{[\delta_0+\delta]}\right\|_{L^{p_j}(\Omega_j)} \|\nabla_{\mathbf{x}} \theta^\delta\|_\infty . \end{aligned} \tag{76}$$

Taking into account estimates (68), (69) and the uniform boundedness of the minimizers by c we get the following estimate:

$$I_{4j} \geq (76) \geq -\delta C_2 \left(p_j, \mathcal{L}^2(\Omega_j), \theta^\delta, c\right) = -\delta C_{2j} \,. \tag{77}$$

Putting together (65), (70), (73), (76) in the case $1 < p_j < 2$ results in:

$$\begin{aligned} &\left\|\nabla u_j^{[\delta_0+\delta]} - \nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)} \\ &\leq \frac{\left\|\nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)}^{2-p_j}}{\tilde{d}_j} \left(\left\|u_{[\delta_0]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0})} \left(\left\|f^{[\delta_0+\delta]} - f\right\|_{L^{\vec{q}}(\Omega_{\delta_0})} + \delta\mathfrak{s}\|f\|_{L^{\vec{q}}(\Omega_{\delta_0})}\right) + \delta(C_{1j} + C_{2j})\right)^{\frac{1}{2}} \\ &\quad + \delta\left\|\nabla u^{[\delta_0+\delta]}\right\|_{L^{p_j}(\Omega_j)} \|\nabla_{\mathbf{x}}\theta^\delta\|_\infty \ . \end{aligned} \tag{78}$$

Evaluation of (65), (70), (74), (76) in the case $p_j > 2$ leads to:

$$\begin{aligned} &\left\|\nabla u_j^{[\delta_0+\delta]} - \nabla u_{[\delta_0]j}\right\|_{L^{p_j}(\Omega_j)} \\ &\leq \frac{1}{\tilde{d}_j} \left(\left\|u_{[\delta_0]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0})} \left(\left\|f^{[\delta_0+\delta]} - f\right\|_{L^{\vec{q}}(\Omega_{\delta_0})} + \delta\mathfrak{s}\|f\|_{L^{\vec{q}}(\Omega_{\delta_0})}\right) + \delta(C_{1j} + C_{2j})\right)^{\frac{1}{p_j}} \\ &\quad + \delta\left\|\nabla u^{[\delta_0+\delta]}\right\|_{L^{p_j}(\Omega_j)} \|\nabla_{\mathbf{x}}\theta^\delta\|_\infty \ . \end{aligned} \tag{79}$$

Estimates (78), (79) show that

$$\left\|u^{[\delta_0+\delta]} - u_{[\delta_0]}\right\|_{W^{1,\vec{p}}(\Omega_{\delta_0})} \to 0 \qquad \text{as} \quad \delta \to 0 \ ,$$

considering $\|f^{[\delta_0+\delta]} - f\|_{L^{\vec{q}}(\Omega_{\delta_0})} \to 0$ as $\delta \to 0$ due to theorem 4.2 and the uniform boundedness of the minimizers. ■

4.1.4 Derivation of Griffith formula by means of minimization problems

Now, $\lim\limits_{\delta\to 0} \frac{E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u^{[\delta_0+\delta]})}{\delta}$ will be calculated. This is not done directly but with the following considerations: Since the minimizers $u_{[\delta_0]} \in W^{1,\vec{p}}(\Omega_{\delta_0})$ and $u^{[\delta_0+\delta]} \in W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$ referring to the configurations Ω_{δ_0} and $\Omega_{\delta_0+\delta}$ are unique, we know that the transformed minimizers $u^{[\delta_0]} \in W^{1,\vec{p}}(\Omega_{\delta_0+\delta})$ and $u^{[\delta_0+\delta]} \in W^{1,\vec{p}}(\Omega_{\delta_0})$ are no solutions of the minimization problems on $\Omega_{\delta_0+\delta}$ and Ω_{δ_0} respectively, such that:

$$E(\Omega_{\delta_0}, u_{[\delta_0]}) \leq E\left(\Omega_{\delta_0}, u^{[\delta_0+\delta]}\right) \quad \text{and} \quad E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]}) \leq E\left(\Omega_{\delta_0+\delta}, u^{[\delta_0]}\right) \ .$$

Therefore the following chain of inequalities holds:

$$\frac{1}{\delta}(E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u^{[\delta_0]}) \tag{80}$$

$$\leq \quad \frac{1}{\delta}(E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]}) \tag{81}$$

$$\leq \quad \frac{1}{\delta}(E(\Omega_{\delta_0}, u^{[\delta_0+\delta]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]}) \ , \tag{82}$$

where

$$ERR(\Omega_{\delta_0}, u_{[\delta_0]}) = \lim_{\delta\to 0} (81) = \lim_{\delta\to 0} \frac{1}{\delta}(E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})$$

is the expression we are looking for. Instead, the limes inferior of expression (80) and the limes superior of expression (82) are calculated and both their boundedness and coincidence has to be shown:

$$\underbrace{\underline{\lim}_{\delta\to 0} \frac{E(\Omega_{\delta_0}, u_{[\delta_0]}) - E(\Omega_{\delta_0+\delta}, u^{[\delta_0]})}{\delta}}_{> -\infty} = \underbrace{\overline{\lim}_{\delta\to 0} \frac{E(\Omega_{\delta_0}, u^{[\delta_0+\delta]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})}{\delta}}_{< \infty} . \tag{83}$$

We start with the calculation of the limes superior.
Thereto the potential energy $E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})$ of the current configuration is transformed onto the reference configuration Ω_{δ_0}:

$$\begin{aligned} E_{[\delta_0+\delta]}\left(\Omega_{\delta_0}, u^{[\delta_0+\delta]}\right) &= \sum_{j=1}^{2} \int\limits_{\Omega_j} W_{elj}(\mathfrak{T}^\delta_{u_j}) \frac{1}{\det J^\delta_{T_\delta}} \,\mathrm{d}\mathbf{y} - \int\limits_{\Omega_j} f_j^{[\delta_0+\delta]} \frac{u_j^{[\delta_0+\delta]}}{\det J^\delta_{T_\delta}} \,\mathrm{d}\mathbf{y} - \int\limits_{\Gamma_{Nj}} h_j \frac{u_j^{[\delta_0+\delta]}}{1} \,\mathrm{d}\mathbf{s} \\ &= E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]}) . \end{aligned}$$

Expression (82) is now given by:

$$\begin{aligned} &\frac{1}{\delta}\left(E(\Omega_{\delta_0}, u^{[\delta_0+\delta]}) - E(\Omega_{\delta_0+\delta}, u_{[\delta_0+\delta]})\right) \\ =&\frac{1}{\delta}\left(E(\Omega_{\delta_0}, u^{[\delta_0+\delta]}) - E_{[\delta_0+\delta]}(\Omega_{\delta_0}, u^{[\delta_0+\delta]})\right) \\ =&\frac{1}{\delta}\sum_{j=1}^{2}\int\limits_{\Omega_j}\left(W_{elj}(\nabla u_j^{[\delta_0+\delta]}) - \frac{W_{elj}(\mathfrak{T}^\delta_{u_j})}{\det J^\delta_{T_\delta}}\right)\mathrm{d}\mathbf{y} - \int\limits_{\Omega_j}\left(f_j u_j^{[\delta_0+\delta]} - f_j^{[\delta_0+\delta]}\frac{u_j^{[\delta_0+\delta]}}{\det J^\delta_{T_\delta}}\right)\mathrm{d}\mathbf{y} \\ &\quad - \int\limits_{\Gamma_{Nj}} h_j\left(u_j^{[\delta_0+\delta]} - \frac{u_j^{[\delta_0+\delta]}}{1}\right)\mathrm{d}\mathbf{s} \quad = \frac{1}{\delta}\sum_{j=1}^{2} I_{1j} - I_{2j} - 0 , \end{aligned}$$

where

$$I_{1j} = \int\limits_{\Omega_j}\left(W_{elj}(\nabla u_j^{[\delta_0+\delta]}) - \frac{W_{elj}(\mathfrak{T}^\delta_{u_j})}{\det J^\delta_{T_\delta}}\right)\mathrm{d}\mathbf{y}$$

and

$$I_{2j} = \int\limits_{\Omega_j}\left(f_j u_j^{[\delta_0+\delta]} - f_j^{[\delta_0+\delta]}\frac{u_j^{[\delta_0+\delta]}}{\det J^\delta_{T_\delta}}\right)\mathrm{d}\mathbf{y} .$$

For the calculation of $\frac{1}{\delta} I_{1j}, j = 1, 2$, identity (63) is used first:

$$\begin{aligned} I_{1j} &= \int\limits_{\Omega_j}\left(W_{elj}(\nabla u_j^{[\delta_0+\delta]}) - W_{elj}(\mathfrak{T}^\delta_{u_j})\right)\mathrm{d}\mathbf{y} - \int\limits_{\Omega_j}\frac{\delta\partial_{x_1}\theta^\delta}{\det J^\delta_{T_\delta}} W_{elj}(\mathfrak{T}^\delta_{u_j})\,\mathrm{d}\mathbf{y} \\ &= I_{3j} - I_{4j} . \end{aligned} \tag{84}$$

Thereby I_{3j} denotes the difference of two elastic strain energy functionals defined on Ω_j. Since $J_{elj}(\Omega_j, \cdot)$ is Fréchet differentiable with respect to the argument, see theorem B.1 in the appendix, the mean value theorem for Fréchet differentiable functionals, see theorem B.2 in the appendix, gives us for a suitable $t_0 \in [0,1]$:

$$\begin{aligned}\frac{I_{3j}}{\delta} &= \frac{\left\langle \mathrm{D}J_{elj}(\nabla u_j^{[\delta_0+\delta]} - \delta t_0 \partial_{y_1} u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}}\theta^\delta), \delta \partial_{y_1} u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}}\theta^\delta \right\rangle}{\delta} \\ &= \int\limits_{\Omega_j} \mathrm{D}W_{elj}\left(\nabla u_j^{[\delta_0+\delta]} - \delta t_0 \partial_{y_1} \nabla_{\mathbf{x}}\theta^\delta\right)(\partial_{y_1} u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}}\theta^\delta)\,\mathrm{d}\mathbf{y}\end{aligned}$$

Taking into account **(H0)**, **(H2)** and theorem 4.4 we know that

$$\mathrm{D}W_{elj}\left(\nabla u_j^{[\delta_0+\delta]} - \delta t_0 \partial_{y_1} \nabla_{\mathbf{x}}\theta^\delta\right)(\partial_{y_1} u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}}\theta^\delta) \to \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)(\partial_{y_1} u_{[\delta_0]j} \nabla_{\mathbf{y}}\theta) \quad \text{in } L^1(\Omega_j)\,,$$

as $\delta \to 0$, and therefore

$$\frac{I_{3j}}{\delta} \to \int\limits_{\Omega_j} \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)(\partial_{y_1} u_{[\delta_0]j} \nabla_{\mathbf{y}}\theta)\,\mathrm{d}\mathbf{y} \quad \text{as } \delta \to 0\,. \tag{85}$$

Analogously, the application of **(H0)**, **(H1)** and theorem 4.4 shows that

$$\frac{I_{4j}}{\delta} = \int\limits_{\Omega_j} \frac{\partial_{x_1}\theta^\delta}{\det J_{T_\delta}^\delta} W_{elj}(\nabla u_j^{[\delta_0+\delta]} - \delta \partial y_1 u_j^{[\delta_0+\delta]} \nabla_{\mathbf{x}}\theta^\delta)\,\mathrm{d}\mathbf{y} \to \int\limits_{\Omega_j} \frac{\partial_{y_1}\theta}{1} W_{elj}(\nabla u_{[\delta_0]j})\,\mathrm{d}\mathbf{y} \tag{86}$$

as $\delta \to 0$.

Now it remains the calculation of $\frac{1}{\delta} I_{2j}$. Application of identity (63) leads to:

$$\begin{aligned} I_{2j} &= \int\limits_{\Omega_j} \left(f_j u_j^{[\delta_0+\delta]} - f_j^{[\delta_0+\delta]} \frac{u_j^{[\delta_0+\delta]}}{\det J_{T_\delta}^\delta} \right) \mathrm{d}\mathbf{y} \\ &= \int\limits_{\Omega_j} \left(f_j u_j^{[\delta_0+\delta]} - f_j^{[\delta_0+\delta]} u_j^{[\delta_0+\delta]} \right) \mathrm{d}\mathbf{y} - \int\limits_{\Omega_j} \frac{\delta \partial x_1 \theta^\delta}{\det J_{T_\delta}^\delta} f_j^{[\delta_0+\delta]} u_j^{[\delta_0+\delta]}\,\mathrm{d}\mathbf{y} \\ &= I_{5j} - I_{6j}\,. \end{aligned} \tag{87}$$

Thereby it can be proved that

$$\frac{I_{5j}}{\delta} \to -\int\limits_{\Omega_j} \partial_{y_1} f_j u_{[\delta_0]j} \theta\,\mathrm{d}\mathbf{y}\,, \tag{88}$$

using the fact that $C^\infty(\overline{\Omega_j}, \mathbb{R}^2)$ is dense in $L^{q_j}_{\partial_{y_1}}(\Omega_j)$, see [13] for the proof that

$$\left\| \frac{f_j^{[\delta_0+\delta]} - f_j}{\delta} - \theta \partial_{y_1} f_{[\delta_0]j} \right\|_{L^{q_j}(\Omega_j)} \to 0 \quad \text{as} \quad \delta \to 0\,.$$

Using theorem 4.2 and theorem 4.4 once more, we get that

$$\frac{I_{6j}}{\delta} = \int\limits_{\Omega_j} \frac{\delta \partial x_1 \theta^\delta}{\det J_{T_\delta}^\delta} f_j^{[\delta_0+\delta]} u_j^{[\delta_0+\delta]}\,\mathrm{d}\mathbf{y} \to \int\limits_{\Omega_j} \frac{\partial_{y_1}\theta}{1} f_j u_{[\delta_0]j}\,\mathrm{d}\mathbf{y} \quad \text{as} \quad \delta \to 0. \tag{89}$$

In total this leads to:

$$\overline{\lim}_{\delta\to 0}\frac{E(\Omega_{\delta_0},u^{[\delta_0+\delta]})-E(\Omega_{\delta_0+\delta},u_{[\delta_0+\delta]})}{\delta}=\overline{\lim}_{\delta\to 0}\frac{1}{\delta}\sum_{j=1}^{2}I_{1j}-I_{2j}$$

$$=\lim_{\delta\to 0}\frac{1}{\delta}\sum_{j=1}^{2}I_{3j}-I_{4j}-I_{5j}+I_{6j}$$

$$=\sum_{j=1}^{2}\left(\int\limits_{\Omega_j}\mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)(\partial_{y_1}u_{[\delta_0]j}\nabla_{\mathbf{y}}\theta)\,\mathrm{d}\mathbf{y}-\int\limits_{\Omega_j}\partial_{y_1}\theta W_{elj}(\nabla u_{[\delta_0]j})\,\mathrm{d}\mathbf{y}+\int\limits_{\Omega_j}\partial_{y_1}(f_j\theta)u_{[\delta_0]j}\,\mathrm{d}\mathbf{y}\right).\tag{90}$$

For the proof of the Griffith formula it remains the computation of

$$\underline{\lim}_{\delta\to 0}\frac{E(\Omega_{\delta_0},u_{[\delta_0]})-E(\Omega_{\delta_0+\delta},u^{[\delta_0]})}{\delta}.$$

Transformation of $E(\Omega_{\delta_0+\delta},u^{[\delta_0]})$ onto the reference configuration yields:

$$\begin{aligned}
&E(\Omega_{\delta_0},u_{[\delta_0]})-E(\Omega_{\delta_0+\delta},u^{[\delta_0]})=E(\Omega_{\delta_0},u_{[\delta_0]})-E^{[\delta_0]}(\Omega_{\delta_0},u_{[\delta_0]})\\
&=\sum_{j=1}^{2}\int\limits_{\Omega_j}\left(W_{elj}(\nabla u_{[\delta_0]j})-\frac{W_{elj}(\nabla u_{[\delta_0]j}-\delta\partial_{y_1}u_{[\delta_0]j}\nabla_{\mathbf{x}}\theta^{\delta})}{\det J^{\delta}_{T_\delta}}\right)\mathrm{d}\mathbf{y}\\
&\quad-\int\limits_{\Omega_j}\left(f_ju_{[\delta_0]j}-\frac{f_j^{[\delta_0+\delta]}u_{[\delta_0]j}}{\det J^{\delta}_{T_\delta}}\right)\mathrm{d}\mathbf{y}-\int\limits_{\Gamma_{Nj}}h_ju_{[\delta_0]j}-\frac{h_ju_{[\delta_0]j}}{1}\,\mathrm{d}\mathbf{s}\\
&=\sum_{j=1}^{2}I_{1j}-I_{2j}-0\,.
\end{aligned}$$

It is easy to understand that these integrals have the same limits as $\delta\to 0$ as those for the computation of the limes superior and therefore

$$\begin{aligned}
\underline{\lim}_{\delta\to 0}\frac{E(\Omega_{\delta_0},u_{[\delta_0]})-E(\Omega_{\delta_0+\delta},u^{[\delta_0]})}{\delta}&=\overline{\lim}_{\delta\to 0}\frac{E(\Omega_{\delta_0},u^{[\delta_0+\delta]})-E(\Omega_{\delta_0+\delta},u_{[\delta_0+\delta]})}{\delta}=(90)\\
&=ERR(\Omega_{\delta_0},u_{[\delta_0]})\,.
\end{aligned}$$

4.2 Numerical examples

In this section some numerical results for the behavior of the energy release rate obtained with the Griffith formula (32) are presented. The behavior of the energy release rate is studied in dependence of the material parameters and the crack length both for strainhardening compounds and for p-Laplacian compounds with arbitrary $p\in(1,\infty)$. The range of application of the constitutive law with arbitrary p with respect to the displacement gradients is investigated numerically.

4.2.1 Energy release rate versus crack length for strain-hardening compounds

In the following, the energy release rate for different strain-hardening compounds is investigated in dependence of the crack length. In this context, calculations are done on the domain $\Omega = (-5,5) \times (-5,5)$ with a variable crack tip $S_v = (s_1, 0)$. Thereby the coordinate s_1 has its values in the set $\{-5, -4.5, -3, -2, -1.5, -1, 0, 1, 2, 3, 4, 4.5, 4.7\}$.

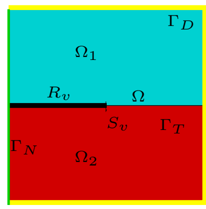

Figure 7: Domain with varying crack length

The following boundary transmission problems are posed on Ω: Find $u : \Omega \to \mathbb{R}$ with $u|_{\Omega_j} = u_j$, $j = 1, 2$, such that:

$$
\begin{aligned}
-\mathrm{div}\left((10^{-7} + |\nabla u_j|^2)^{\frac{p_j-2}{2}} \nabla u_j \right) &= 0 \quad \text{in } \Omega_j, \\
(10^{-7} + \nabla u_j)^{\frac{p_j-2}{2}} \nabla u_j \cdot \mathrm{n}_j &= 0 \quad \text{auf } (\Gamma_N \cap \partial\Omega_j) \cup R_{vj}, \\
u(\mathbf{x}) = \begin{cases} -0.02x_1 + x_2 + 5.1 & x_2 = -5, x_1 \in (-5,5) \\ 0 & x_1 = 5 \\ 0.02x_1 + x_2 - 5.1 & x_2 = 5, x_1 \in (-5,5) \end{cases} & \quad \text{on } \Gamma_D\ , \\
\text{with transmission conditions (7), (8)} & \quad \text{on } \Gamma_T\ .
\end{aligned}
\tag{91}
$$

The material parameters for the metal Cu (ann.) an the alloys brass (ann.) and stainless steel 14301 (ann.) are taken from table 1.

For all compounds fig. 8 shows the same characteristic curve progession with increasing crack length: When the crack tip is situated in $(-5, 0)$, which corresponds to a domain without any crack, then the energy release rate has the value 0. In this case the body is in an equilibrium state and there is no danger of crack initiation. For short cracks the energy release rate increases steeply because the crack tip is exposed to large stresses. This is due to the large displacements for small y_1, that act in opposite directions on opposite parts of the Dirichlet-boundary. The potential energy stored in the body is large, too, for a small crack length. The energy release rate reaches its maximum value for the crack tip situated in $(-1.5, 0)$. The subsequent decrease can be explained by the increasing independence of the crack lips for larger cracks. This has a lower potential energy as a result, since also the prescribed displacements on the Dirichlet boundary decrease with larger y_1.

The magnitude of the energy release rate depends on the toughness of the materials put together: The tougher the components in the compound, the larger the energy release rate.

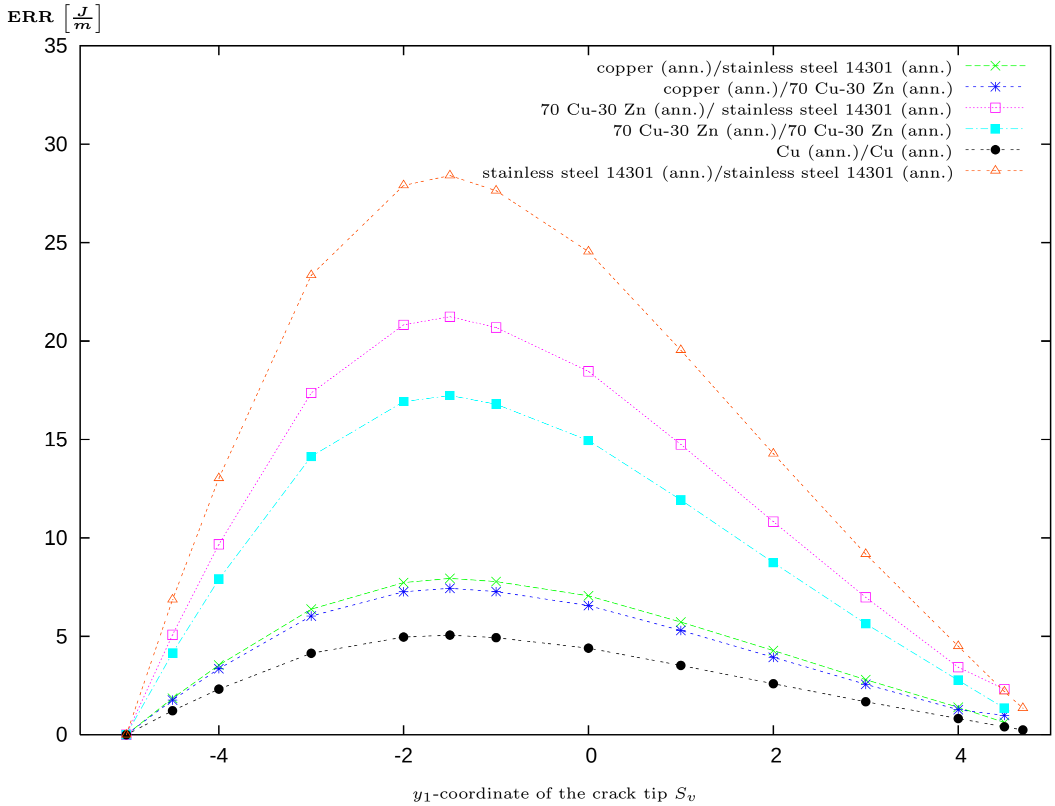

Figure 8: Energy release rate versus crack length for different strain-hardening compounds

4.2.2 The influence of larger boundary displacements and higher values of p

For the following computations the domain Ω and the transmission boundary value problems are chosen as in the previous section. Just the absolute value of the slope in the Dirichlet condition (91) is increased from 0.02 to 2, such that this condition reads now:

$$u(\mathbf{x}) = \begin{cases} -2x_1 + x_2 + 15 & x_2 = -5, x_1 \in (-5,5) \\ 0 & x_1 = 5 \\ 2x_1 + x_2 - 15 & x_2 = 5, x_1 \in (-5,5) \end{cases} \quad \text{on } \Gamma_D \, . \tag{92}$$

This does not affect the shape of the curves for increasing crack length by itself (see fig. 9). But the larger displacements on the boundary lead to larger displacement gradients in the domain and therefore larger stresses and larger potential energies appear. This has the effect that the magnitude of the energy release rate and even the relation between the curves for different p is changed significantly in comparison to fig. 8: Larger values of p now lead to larger values of the energy release rate. This opposite behaviour would imply that the constitution of the materials changes for large displacements, which shows that a constitutive law of p-Laplacian type combined with linearized strains is not applicable for praxis anymore in case of large displacements.

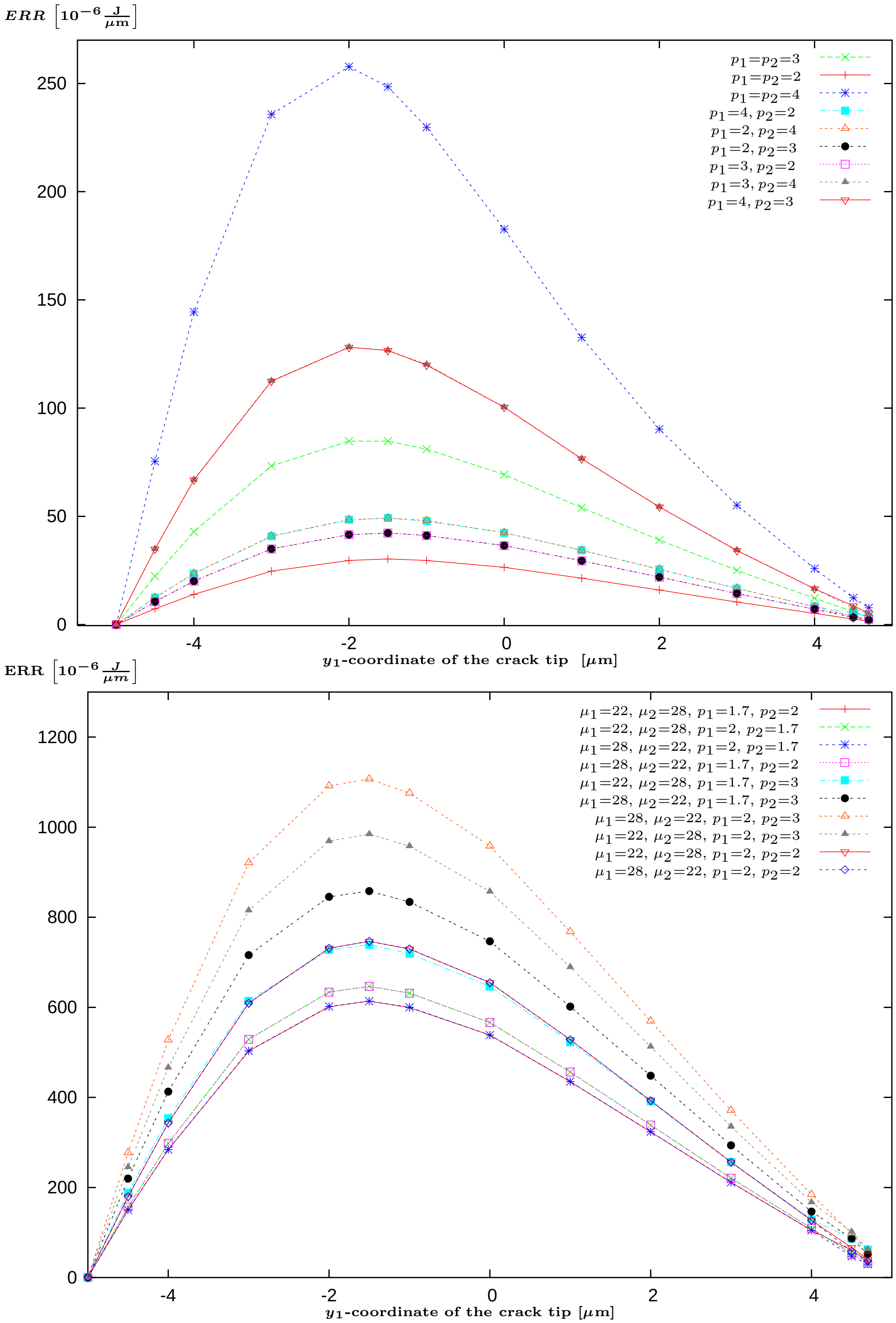

Figure 9: Energy release rate versus crack length for different p_1, p_2, μ_1, μ_2

5 The J-Integral for p-Laplacian compounds

Theorem 5.1 (J-Integral for the compound)
Let Ω_{δ_0} be a 2D domain, as in definition 2.1, containing the crack $R_{\delta_0} \subset \{\mathbf{x} \in \Omega_{\delta_0} : x_2 = 0\}$ with the crack propagation direction along the vector field $(1,0)^\top$ and the cracktip S_{δ_0}. Let $f \in L^{\vec{q}}_{\partial_{y_1}}(\Omega_{\delta_0})$, with $\partial_{y_1} f|_\Omega = 0$ a.e.. Then for any Lipschitz domain $B_1^\theta \Subset \Omega$ with $S_{\delta_0} \in B_1^\theta$ and $B^\theta_{1\,j} := B_1^\theta \cap \overline{\Omega_j}$, the J-integral for the cracked compound with a surface force density $h \in L^{\vec{q}}(\Gamma_N)$ and the minimizer $u_{[\delta_0]} \in V^{\vec{p}}_{(g)}(\Omega_{\delta_0})$ is calculated in the following way:

$$
\begin{aligned}
&ERR(\Omega_{\delta_0}, u_{[\delta_0]}) \\
&= -\sum_{j=1}^{2} \left\langle \partial_{y_1} u_{[\delta_0]\,j} \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) \cdot \mathbf{n}_j + (W_{el\,j}(\nabla u_{[\delta_0]\,j}) - u_{[\delta_0]\,j} f_j) n_{j\,1}, \theta_j^1 \right\rangle_{W^{\frac{1}{m_j}, \frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \setminus B^\theta_{1\,j}))},
\end{aligned}
\tag{93}
$$

where $1 < m_j < \frac{2}{1+\delta p_j}$ with $0 < \delta < \frac{1}{p_j}$ and

$$
\theta_j^1(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \partial B_1^\theta \cap \Omega_j \\ 0 & \text{else} \end{cases} .
$$

Furthermore, $\mathbf{n}_j$ denotes the outer unit normal vector of $\partial B_1^\theta \cap \Omega_j$ and n_{j_1} is the y_1-component.

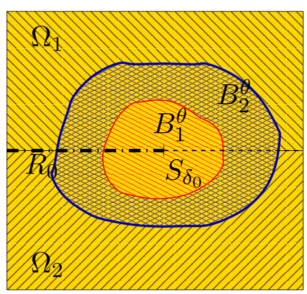

Figure 10: Support of the functions $\theta \in C_0^\infty(\Omega)$

Proof:
Starting from the Griffith formula for the cracked compound

$$
\begin{aligned}
ERR(\Omega_{\delta_0}, u_{[\delta_0]}) = &\sum_{j=1}^{2} \int_{\Omega_j} \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) \partial_{y_1} u_{[\delta_0]\,j} \cdot \nabla\theta \, \mathrm{d}\mathbf{y} \\
&- \sum_{j=1}^{2} \int_{\Omega_j} W_{el\,j}(\nabla u_{[\delta_0]\,j}) \partial_{y_1}\theta \, \mathrm{d}\mathbf{y} + \sum_{j=1}^{2} \int_{\Omega_j} u_{[\delta_0]\,j} \partial_{y_1}(f_j \theta) \, \mathrm{d}\mathbf{y} \, ,
\end{aligned}
\tag{94}
$$

where $\theta \in \mathcal{T} := \{\eta \in C_0^\infty(\Omega),\ \eta \equiv 1 \text{ in } B_1^\eta \text{ and } \eta \equiv 0 \text{ in } \Omega \backslash B_2^\eta\}$, see theorem 4.1, we introduce the functions

$$
F_j = \partial_{y_1} u_{[\delta_0]\,j} \mathrm{D}W_{el\,j}\left(\nabla u_{[\delta_0]\,j}\right) - W_{el\,j}(\nabla u_{[\delta_0]\,j}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_{[\delta_0]\,j} f_j \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{95}
$$

$j = 1, 2$, such that the energy release rate is given by:

$$ERR(\Omega_{\delta_0}, u_{[\delta_0]}) = \sum_{j=1}^{2} \int_{\Omega_j} F_j \cdot \nabla\theta \, \mathrm{d}\mathbf{y} = \sum_{j=1}^{2} \int_{B_{2\,j}^{\theta} \setminus B_{1\,j}^{\theta}} F_j \cdot \nabla\theta \, \mathrm{d}\mathbf{y} \quad \text{for } \theta \in \mathcal{T}\,. \tag{96}$$

At this point we make use of a regularity result for the minimizer $u_{[\delta_0]}$, compare [11] p. 147:

Lemma 5.1
In minimization problem (30) *with* $f \in L^{\vec{q}}(\Omega_{\delta_0})$ *and* $h \in L^{\vec{q}}(\Gamma_N)$ *the minimizer even satisfies*

$$u_{[\delta_0]\,j} \in \begin{cases} W^{1+\frac{1}{p_j}-\delta, p_j}(\Omega_j^{\varepsilon}) & \textit{if } p_j \geq 2 \\ W^{\frac{3}{2}-\delta, \frac{4p_j}{2+p_j}}(\Omega_j^{\varepsilon}) & \textit{if } 1 < p_j < 2 \end{cases} \quad \textit{for } \delta > 0\,,\ j = 1, 2 \tag{97}$$

on the domain $\Omega_j^{\varepsilon} := \Omega_j \backslash \{\mathbf{y},\, |S_{\delta_0} - \mathbf{y}| \leq \varepsilon\}$ *with arbitrarily small* $\varepsilon > 0$.

Thus we obtain for $i = 1, 2$

$$\partial_{y_i} u_{[\delta_0]\,j} \in \begin{cases} W^{\frac{1}{p_j}-\delta, p_j}(\Omega_j^{\varepsilon}) & \text{if } p_j \geq 2 \\ W^{\frac{1}{2}-\delta, \frac{4p_j}{2+p_j}}(\Omega_j^{\varepsilon}) & \text{if } 1 < p_j < 2 \end{cases} \quad \text{for } \delta > 0\,,\ j = 1, 2\,.$$

Since $\delta > 0$ arbitrarily we can regard these spaces as Sobolev spaces $W^{s,p}(\Omega)$ of fractional order $0 < s < 1$ and we apply the following imbedding property [1] p. 218, which also holds for bounded domains (see [6] theorem 1.4.4.1): *Let* $\Omega \subset \mathbb{R}^2$, $s > 0$ *and* $1 < p \leq q < \infty$. *Let*

$$\chi = s - \frac{2}{p} + \frac{2}{q}.$$

If $0 < \chi \notin \mathbb{N}$, *then*

$$W^{s,p}(\Omega) \subseteq W^{\chi,q}(\Omega)\ .$$

Thus, for $m_j > 1$ it has to be satisfied that $\chi = \frac{1}{p_j} - \delta - \frac{2}{p_j} + \frac{2}{p_j m_j} > 0$ in the case $p_j \geq 2$ and $\chi = \frac{1}{2} - \delta - \frac{2(2+p_j)}{4p_j} + \frac{2}{p_j m_j} > 0$ in the case $1 < p_j < 2$. Both conditions lead to

$$1 < m_j < \frac{2}{1 + \delta p_j} \quad \text{for } 0 < \delta < \frac{1}{p_j}\,,$$

which implies

$$W^{\frac{1}{p_j}-\delta, p_j}(\Omega_j^{\varepsilon}) \subseteq W^{\frac{2}{p_j m_j}-\frac{1}{p_j}-\delta, p_j m_j}(\Omega_j^{\varepsilon}) \subset L^{p_j m_j}(\Omega_j^{\varepsilon}) \quad \text{for } 0 < \delta < \frac{1}{p_j} \text{ and } p_j \geq 2 \quad \text{as well as}$$

$$W^{\frac{1}{2}-\delta, \frac{4p_j}{2+p_j}}(\Omega_j^{\varepsilon}) \subseteq W^{\frac{1}{2}-\delta-\frac{2+p_j}{2p_j}+\frac{2}{p_j m_j}, p_j m_j}(\Omega_j^{\varepsilon}) \subset L^{p_j m_j}(\Omega_j^{\varepsilon}) \quad \text{for } 0 < \delta < \frac{1}{p_j} \text{ and } 1 < p_j < 2\,.$$

We show now that $F_j \in L^{m_j}(\Omega_j^{\varepsilon})$, $m_j > 1$, for $\partial_{y_i} u_{[\delta_0]\,j} \in L^{p_j m_j}(\Omega_j^{\varepsilon})$, $i = 1, 2$.
For $W_{el\,j}(\cdot)$ we get under consideration of **(H1)**:

$$|W_{el\,j}(\nabla u_{[\delta_0]\,j})|^{m_j} \leq c_{2_j}(1 + |\nabla u_{[\delta_0]\,j}|^{p_j})^{m_j} \overset{(105)}{\leq} c_{2_j} 2^{m_j - 1}(1 + |\nabla u_{[\delta_0]\,j}|^{p_j m_j})\,, \tag{98}$$

where $1 < m_j \leq \frac{2}{1+\delta p_j}$ for $0 < \delta \leq \frac{1}{p_j}$. This shows that $W_{el\,j}(\cdot) \in L^{m_j}(\Omega_j^{\varepsilon})$.

Now we have to check whether $\left(\partial_{y_1} u_{[\delta_0]j} \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right) \in L^{m_j}(\Omega_j^\varepsilon)$:

$$\int\limits_{\Omega_j} |\partial_{y_1} u_{[\delta_0]j} \mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)|^{m_j}\,\mathrm{d}\mathbf{y}$$

$$\leq \left(\int\limits_{\Omega_j} |\partial_{y_1} u_{[\delta_0]j}|^{p_j m_j}\,\mathrm{d}\mathbf{y}\right)^{\frac{1}{p_j}} \left(\int\limits_{\Omega_j} \left|\mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right|^{\frac{p_j m_j}{p_j-1}}\,\mathrm{d}\mathbf{y}\right)^{\frac{p_j-1}{p_j}}.$$

Thereby

$$\begin{aligned}\left|\mathrm{D}W_{elj}\left(\nabla u_{[\delta_0]j}\right)\right|^{\frac{p_j m_j}{p_j-1}} &\overset{(\mathbf{H2})}{\leq} \left(c_j(1+|\nabla u_{[\delta_0]j}|^{p_j-1})\right)^{\frac{p_j m_j}{p_j-1}} \\ &\overset{(105)}{\leq} c_j^{\frac{p_j m_j}{p_j-1}} 2^{\frac{p_j(m_j-1)-1}{p_j-1}} (1+|\nabla u_{[\delta_0]j}|^{p_j mj})\end{aligned} \tag{99}$$

If we choose $f_j \in L^{q_j m_j}(\Omega_j^\varepsilon) \subset L^{q_j}(\Omega_j^\varepsilon)$, then

$$F_j \in L^{m_j}(\Omega_j^\varepsilon) \quad \text{for } 1 < m_j \leq \frac{2}{1+\delta p_j} \text{ and } 0 < \delta \leq \frac{1}{p_j}. \tag{100}$$

Now, we show that

$$\mathrm{div} F_j = 0 \;\; \mathcal{L}^2 - a.e. \text{ in } \Omega_j^\varepsilon\,, \tag{101}$$

which implies

$$F_j \in \mathcal{F}(\Omega_j^\varepsilon) := \{v \in (L^{m_j}(\Omega_j^\varepsilon))^2,\, \mathrm{div}\, v = 0\; \mathcal{L}^2 - a.e. \text{ in}\, \Omega_j^\varepsilon\}\,.$$

Use $\theta_1,\, \theta_2 \in \mathcal{T}$ with $B_1^{\theta_i} = B(S_{\delta_0}, \varepsilon) = \{\mathbf{y},\, |\mathbf{y} - S_{\delta_0}| < \varepsilon,\, \varepsilon > 0\}$ to calculate the energy release rate due to (96) and take the difference:

$$0 = \sum_{j=1}^{2} \int\limits_{\Omega_j} F_j \cdot (\nabla\theta_1 - \nabla\theta_2)\mathrm{d}\mathbf{y}\,,$$

which implies that

$$\sum_{j=1}^{2} \int\limits_{\Omega_j} F_j \cdot \nabla\tilde{\theta}\,\mathrm{d}\mathbf{y} = 0 \quad \text{for every } \tilde{\theta} \in C_0^\infty(\Omega^\varepsilon) \text{ and every } \varepsilon > 0\,.$$

For $j = 1, 2$, this leads especially to

$$\int\limits_{\Omega_j^\varepsilon} F_j \cdot \nabla\tilde{\theta}\mathrm{d}\mathbf{y} = \left\langle F_j, \nabla\tilde{\theta}\right\rangle_{C_0^\infty(\Omega_j^\varepsilon)} = \left\langle \mathrm{div} F_j, \tilde{\theta}\right\rangle_{C_0^\infty(\Omega_j^\varepsilon)} = 0 \quad \text{for every } \tilde{\theta} \in C_0^\infty(\Omega_j^\varepsilon) \subset C_0^\infty(\Omega^\varepsilon)\,,$$

which proves relation (101)

$$\mathrm{div} F_j = 0 \;\; \mathcal{L}^2 - a.e. \text{ in } \Omega_j^\varepsilon \text{ for every } \varepsilon > 0\,.$$

The next step is to prove that $\mathcal{M}(\Omega_j^\varepsilon) := \{\phi \in (C^\infty(\overline{\Omega_j^\varepsilon}))^2,\ \mathrm{div}\phi = 0\ \mathcal{L}^2 - a.e.\}$ is dense in $\mathcal{F}(\Omega_j^\varepsilon)$:
Let $F_j \in \mathcal{F}(\Omega_j^\varepsilon)$. Since $(C_0^\infty(\Omega_j^\varepsilon))^2 \subset (C^\infty(\overline{\Omega_j^\varepsilon}))^2$, we choose a function $\eta \in C_0^\infty(\Omega_j^\varepsilon)$ having the properties

1. $\eta(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq 1$,
2. $\int_{\mathbb{R}^2} \eta(\mathbf{x})\mathrm{d}\mathbf{x} = 1$,
3. $\eta_n(\mathbf{x}) := n^2\eta(n\mathbf{x}) \quad \text{for } n \in \mathbb{N}$.

Thus we construct a mollifying sequence $\{\psi_n\}_{n\in\mathbb{N}}$ via a convolution:

$$\psi_n(\mathbf{x}) := (\eta_n \star F_j)(\mathbf{x}) = \int_{\mathbb{R}^2} \eta_n(\mathbf{x}-\mathbf{y})F_j(\mathbf{y})\mathrm{d}\mathbf{y} \quad \text{for } F_j \equiv 0 \text{ in } \mathbb{R}^2\backslash\overline{\Omega_j^\varepsilon}\,.$$

See [1] p. 30 for $\{\psi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega_j^\varepsilon})$ and $\|\psi_n - F_j\|_{L^{m_j}(\Omega_j^\varepsilon)} \to 0$ as $n \to \infty$.
It has to be proved, that

$$\mathrm{div}_{\mathbf{x}}\psi_n(\mathbf{x}) = \sum_{i=1}^{2} \int_{\mathbb{R}^2} \partial_{x_i}\eta_n(\mathbf{x}-\mathbf{y}){F_j}_i(\mathbf{y})\mathrm{d}\mathbf{y} = 0\ \mathcal{L}^2 - a.e..$$

It is

$$\begin{aligned} 0 &= (\eta_n \star \mathrm{div}\, F_j)(\mathbf{x}) = \int_{\mathbb{R}^2} \eta_n(\mathbf{x}-\mathbf{y}) \sum_{i=1}^{2} \partial_{y_i}{F_j}_i(\mathbf{y})\mathrm{d}\mathbf{y} = -\sum_{i=1}^{2}\int_{\mathbb{R}^2} \partial_{y_i}\eta_n(\mathbf{x}-\mathbf{y})F_j(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \sum_{i=1}^{2}\int_{\mathbb{R}^2} \partial_{x_i}\eta_n(\mathbf{x}-\mathbf{y}){F_j}_i(\mathbf{y})\mathrm{d}\mathbf{y} = \mathrm{div}_{\mathbf{x}}\psi_n(\mathbf{x})\,. \end{aligned}$$

The last step is to apply Green's formula [5] p. 219 to expression (96):

Theorem 5.2
For $F_j \in L^{m_j}(\Omega_j^\varepsilon)$ and $\theta \in C_0^\infty(\Omega)$ the following Green's formula holds:

$$\int_{B^\theta_{2j}\backslash B^\theta_{1j}} F_j \cdot \nabla\theta \mathrm{d}\mathbf{y} + \int_{B^\theta_{2j}\backslash B^\theta_{1j}} \theta\, \mathrm{div}\, F_j \mathrm{d}\mathbf{y} = \langle F_j \cdot \tilde{\mathrm{n}}_j, \theta\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2j}\backslash B^\theta_{1j}))}\,, \qquad (102)$$

where $F_j|_{\partial(B^\theta_{2j}\backslash B^\theta_{1j})} \in W^{-\frac{1}{m_j},m_j}(\partial(B^\theta_{2j}\backslash B^\theta_{1j}))$ on the boundary and $\tilde{\mathrm{n}}_j$ denotes the outer unit normal vector of the domain $B^\theta_{2j}\backslash B^\theta_{1j}$.

In order to simplify the righthand side of expression (102), we introduce the functions

$$\theta_j^1(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \partial B_1^\theta \cap \Omega_j \\ 0 & \text{else} \end{cases}, \quad \theta^2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in R_{\delta_0} \cap B_2^\theta\backslash B_1^\theta \\ 0 & \text{else} \end{cases},$$
$$\theta^3(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Gamma_{T\delta_0} \cap B_2^\theta\backslash B_1^\theta \\ 0 & \text{else} \end{cases}.$$

Thus $\theta|_{\Omega_j} = \theta_j^1 + \theta_j^2 + \theta_j^3$. Therefore

$$\begin{aligned}
&\sum_{j=1}^{2} \langle F_j \cdot \tilde{\mathrm{n}}_j, \theta \rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} \\
&= \sum_{j=1}^{2} \left\langle F_j \cdot \tilde{\mathrm{n}}_j, \theta_j^1 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} + \sum_{j=1}^{2} \left\langle F_j \cdot \tilde{\mathrm{n}}_j, \theta^2 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} \\
&\quad + \left\langle F_1 \cdot \mathrm{n}_{12} - F_2 \cdot \mathrm{n}_{21}, \theta^3 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))}
\end{aligned}$$

It is $\tilde{\mathrm{n}}_j = (0, (-1)^j)^\top$ on $R_{\delta_0\,j}$ and thus

$$\left\langle F_j \cdot \tilde{\mathrm{n}}_j, \theta^2 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} = \left\langle \partial_{y_1} u_{[\delta_0]\,j} \mathrm{D}W_{el\,j} \left(\nabla u_{[\delta_0]\,j} \right) \cdot \tilde{\mathrm{n}}_j, \theta^2 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))},$$

which leads by simple calculation to

$$\begin{aligned}
\left\langle F_j \cdot \tilde{\mathrm{n}}_j, \theta^2 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} &= \left\langle \mathrm{D}W_{el\,j} \left(\nabla u_{[\delta_0]\,j} \right) \cdot \tilde{\mathrm{n}}_j, \theta^2 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))}, \\
&= 0
\end{aligned}$$

due to the homogeneous Neumann condition on the crack lips.
Since $u_{[\delta_0]\,1} = u_{[\delta_0]\,2}$ *a.e.* on the interface, we get

$$\left\langle F_1 \cdot \mathrm{n}_{12} - F_2 \cdot \mathrm{n}_{21}, \theta^3 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} = 0\,.$$

Thus

$$\begin{aligned}
\sum_{j=1}^{2} \langle F_j \cdot \tilde{\mathrm{n}}_j, \theta \rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} &= \sum_{j=1}^{2} \left\langle F_j \cdot \tilde{\mathrm{n}}_j, \theta_j^1 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))} \\
&= -\sum_{j=1}^{2} \left\langle F_j \cdot \mathrm{n}_j, \theta_j^1 \right\rangle_{W^{\frac{1}{m_j},\frac{m_j}{m_j-1}}(\partial(B^\theta_{2\,j} \backslash B^\theta_{1\,j}))},
\end{aligned}$$

where

$$\theta_j^1(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \partial B_1^\theta \cap \Omega_j \\ 0 & \text{else} \end{cases}.$$

∎

Appendix

A Inequalities

A.1 Hölder's inequality

Theorem A.1 (Hölder's inequality, [1] S.23, ff)
Let Ω be a subdomain of $\mathbb{R}^n, n \in \mathbb{N}$.

1. *Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $(uv) \in L^1(\Omega)$ and it holds:*

$$\int_\Omega |uv| \,\mathrm{d}\mathbf{x} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} \quad . \tag{103}$$

2. *Let $0 < p < 1$ and $q = \frac{p}{p-1} < 0$. For $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ we have $(uv) \in L^1(\Omega)$ and it holds:*

$$\int_\Omega |uv| \,\mathrm{d}\mathbf{x} \geq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} \quad . \tag{104}$$

A.2 Some inequalities in $\mathbb{R}^n$, $n = 1, 2$

Lemma A.1 ([14], p. 25)
For $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ with $a_i \geq 0$ and $1 \leq i \leq n$ it holds

$$\left(\sum_{i=1}^n a_i\right)^\alpha \leq n^{\alpha-1} \left(\sum_{i=1}^n a_i^\alpha\right) \text{ if } \alpha \geq 1 \text{ and} \tag{105}$$

$$\left(\sum_{i=1}^n a_i\right)^\alpha \geq n^{\alpha-1} \left(\sum_{i=1}^n a_i^\alpha\right) \text{ if } 0 \leq \alpha \leq 1 \,. \tag{106}$$

Lemma A.2 ([19] formula (2.20))
For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ with $|\mathbf{b}| \geq |\mathbf{a}|$ and $t \in [0, \frac{1}{4}]$ it holds:

$$|\mathbf{b} + t(\mathbf{a} - \mathbf{b})| \geq \frac{1}{4} \left(|\mathbf{a}| + |\mathbf{b}|\right) \,. \tag{107}$$

Lemma A.3 ([8] p.39)
Let x and y be positive and unequal, then it holds:

$$rx^{r-1}(x-y) > x^r - y^r > ry^{r-1}(x-y) \qquad \text{for } r < 0 \text{ or } r > 1\,, \text{ and} \tag{108}$$

$$rx^{r-1}(x-y) < x^r - y^r < ry^{r-1}(x-y) \qquad \text{for } 0 < r < 1. \tag{109}$$

A.3 An estimate from above

Theorem A.2
Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.
For $p > 2$ it holds

$$W_{el}(\mathbf{a} + \mathbf{b}) - W_{el}(\mathbf{a}) - \mathrm{D}W_{el}\,(\mathbf{a}) \cdot \mathbf{b} \leq c\,(1 + |\mathbf{a}| + |\mathbf{b}|)^{p-2}\,|\mathbf{b}|^2 \tag{110}$$

and for $1 < p \leq 2$ we have

$$W_{el}(\mathbf{a} + \mathbf{b}) - W_{el}(\mathbf{a}) - \mathrm{D}W_{el}\,(\mathbf{a}) \cdot \mathbf{b} \leq c|\mathbf{a}|^{p-2}|\mathbf{b}|^2 \,. \tag{111}$$

Proof:
It is

$$W_{el}(\mathbf{a} + \mathbf{b}) - W_{el}(\mathbf{a}) - \mathrm{D}W_{el}\,(\mathbf{a}) \cdot \mathbf{b} = \frac{\mu}{p}\left((\kappa + |\mathbf{a} + \mathbf{b}|^2)^{\frac{p}{2}} - (\kappa + |\mathbf{a}|^2)^{\frac{p}{2}} - p(\kappa + |\mathbf{a}|^2)^{\frac{p-2}{2}}\mathbf{a} \cdot \mathbf{b}\right) \tag{112}$$

Now theorem A.3 is applied with $x = (\kappa + |\mathbf{a}+\mathbf{b}|^2)$, $y = (\kappa + |\mathbf{a}|^2)$ and $r = \frac{p}{2}$.

For $p > 2$ it holds according to (108)

$$
\begin{aligned}
(112) \quad &\leq \frac{\mu}{p}\left(\frac{p}{2}(\kappa+|\mathbf{a}+\mathbf{b}|^2)^{\frac{p-2}{2}}\left(|\mathbf{a}+\mathbf{b}|^2-|\mathbf{a}|^2\right)-p(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\mathbf{a}\cdot\mathbf{b}\right)\\
&= \frac{\mu}{p}\left(\frac{p}{2}(\kappa+|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)^{\frac{p-2}{2}}(2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)-p(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\mathbf{a}\cdot\mathbf{b}\right) \qquad (113)
\end{aligned}
$$

$$
\begin{aligned}
(113) \quad &= \frac{\mu}{p}\left(\frac{p}{2}(\kappa+|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)^{\frac{p-2}{2}}|\mathbf{b}|^2\right.\\
&\qquad\left.+\underbrace{p\left((\kappa+|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)^{\frac{p-2}{2}}-(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\right)\mathbf{a}\cdot\mathbf{b}}_{A}\right)\\
&= \frac{\mu}{p}\left(\frac{p}{2}(\kappa+|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)^{\frac{p-2}{2}}|\mathbf{b}|^2+A\right)\\
&\leq \frac{\mu}{2}(1+|\mathbf{a}|+|\mathbf{b}|)^{p-2}|\mathbf{b}|^2+\frac{\mu}{p}A
\end{aligned}
$$

If $0 < \frac{p-2}{2} < 1$, application of (109) leads to

$$
\begin{aligned}
A &\leq p\,(\mathbf{a}\cdot\mathbf{b})\left(\frac{p-2}{2}(\kappa+|\mathbf{a}|^2)^{\frac{p-4}{2}}\left(2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2\right)\right)\\
&\leq \frac{p(p-2)}{2}(\kappa+|\mathbf{a}|^2)^{\frac{p-4}{2}}\left(2|\mathbf{a}|^2|\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})|\mathbf{b}|^2\right)\\
&\leq \frac{p(p-2)}{2}(\kappa+|\mathbf{a}|^2)^{\frac{p-4}{2}}\left(2(\kappa+|\mathbf{a}|^2)|\mathbf{b}|^2+(1+|\mathbf{a}|+|\mathbf{b}|)^2|\mathbf{b}|^2\right)\\
&\leq \frac{p(p-2)}{2}\left(2(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}|\mathbf{b}|^2+(1+|\mathbf{a}|+|\mathbf{b}|)^{p-2}|\mathbf{b}|^2\right)\leq\frac{3p(p-2)}{2}(1+|\mathbf{a}|+|\mathbf{b}|)^{p-2}|\mathbf{b}|^2\,.
\end{aligned}
$$

In the case $\frac{p-2}{2} > 1$ we have $p-4>0$ and (108) yields

$$
\begin{aligned}
A &\leq p\,(\mathbf{a}\cdot\mathbf{b})\,\frac{p-2}{2}\,(\kappa+|\mathbf{a}|^2+2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)^{\frac{p-4}{2}}(2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)\\
&\leq \frac{p(p-2)}{2}\,(1+|\mathbf{a}|+|\mathbf{b}|)^{p-4}(\mathbf{a}\cdot\mathbf{b})(2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2)\\
&\leq \frac{p(p-2)}{2}\,(1+|\mathbf{a}|+\mathbf{b}|)^{p-4}\left(2(1+|\mathbf{a}|+|\mathbf{b}|)^2|\mathbf{b}|^2+(\mathbf{a}\cdot\mathbf{b})|\mathbf{b}|^2\right)\\
&\leq \frac{p(p-2)}{2}\,(1+|\mathbf{a}|+\mathbf{b}|)^{p-4}\left(2(1+|\mathbf{a}|+|\mathbf{b}|)^2|\mathbf{b}|^2+(1+|\mathbf{a}|+|\mathbf{b}|)^2|\mathbf{b}|^2\right)\\
&= \frac{3p(p-2)}{2}\,(1+|\mathbf{a}|+|\mathbf{b}|)^{p-2}|\mathbf{b}|^2\,.
\end{aligned}
$$

Therefore

$$W_{el}(\mathbf{a}+\mathbf{b})-W_{el}(\mathbf{a})-\mathrm{D}W_{el}\,(\mathbf{a})\cdot\mathbf{b}\leq c\,(1+|\mathbf{a}|+|\mathbf{b}|)^{p-2}\,|\mathbf{b}|^2$$

with $c = \frac{\mu}{2}(3p-5)$.

In the case $1 < p \leq 2$ application of (109) leads to

$$\begin{aligned}
(112) \quad \leq \quad & \frac{\mu}{p}\left(\frac{p}{2}(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\left(|\mathbf{a}+\mathbf{b}|^2-|\mathbf{a}|^2\right)-p(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\mathbf{a}\cdot\mathbf{b}\right)\\
\leq \quad & \frac{\mu}{p}\left(\frac{p}{2}(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\left(2(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2\right)-p(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}\mathbf{a}\cdot\mathbf{b}\right)\\
= \quad & \frac{\mu}{2}(\kappa+|\mathbf{a}|^2)^{\frac{p-2}{2}}|\mathbf{b}|^2 \leq \frac{\mu}{2}|\mathbf{a}|^{p-2}|\mathbf{b}|^2 \,.
\end{aligned}$$

Here $c = \frac{\mu}{2}$. ■

B Fréchet differentiability

Definition B.1 (Fréchet differentiability)
Let X, Y be normed vector spaces and $\mathcal{L}(X,Y) = \{A : X \to Y,\ A \text{ is linear and continuous}\}$. Let $F : Z \subset X \to Y$ be an operator defined on a subset $Z \subset X$. Then F is Fréchet differentiable iff there exists a functional $DF \in \mathcal{L}(X,Y)$, such that:

$$F(a+h) = F(a) + \langle DF(a), h\rangle + o(h) \qquad \text{for every } h \in X \text{ with } (a+h) \in Z. \tag{114}$$

thereby

$$o(h) = \|h\|_X\,\varepsilon(h) \quad \text{with } \lim_{h\to 0}\varepsilon(h) = 0 \text{ in } Y\,. \tag{115}$$

B.1 Fréchet differentiability of the elastic strain energy

Theorem B.1
For the elastic strain energy $J_{el}(\Omega_\delta,\cdot)$ of the compound Ω_δ the Fréchet derivative $\mathrm{D}J_{el}(\Omega_\delta,\cdot)$ is given by

$$\mathrm{D}J_{el}(\Omega_\delta,\cdot) : V^{\vec{p}}_{(g)}(\Omega_\delta) \to \left(V^{\vec{p}}_{(0)}(\Omega_\delta)\right)'$$

$$\langle \mathrm{D}J_{el}(\Omega_\delta,a), v\rangle = \sum_{j=1}^{2}\left\langle \mathrm{D}J_{el\,j}(\Omega_j,a_j),h_j\right\rangle = \sum_{j=1}^{2}\int_{\Omega_j}\mathrm{D}W_{el\,j}\left(\nabla a_j\right)\cdot\nabla h_j\,\mathrm{d}\mathbf{x}$$

for every $a \in V^{\vec{p}}_{(g)}(\Omega_\delta)$ and $h \in V^{\vec{p}}_{(0)}(\Omega_\delta)$.

Proof:
Let $a \in V^{\vec{p}}_{(g)}(\Omega_\delta)$ and $h \in V^{\vec{p}}_{(0)}(\Omega_\delta)$. According to definition B.1 we have to show that

$$J_{el\,j}(\Omega_j, a_j+h_j) - J_{el\,j}(\Omega_j,a_j) - \left\langle \mathrm{D}J_{el\,j}(\Omega_j,a_j),h_j\right\rangle = o(h_j)\,, \tag{116}$$

then summation over $j = 1,2$ leads to the assertion. It is

$$\begin{aligned}
&\left|J_{el\,j}(\Omega_j,a_j+h_j) - J_{el\,j}(\Omega_j,a_j) - \left\langle \mathrm{D}J_{el\,j}(\Omega_j,a_j),h_j\right\rangle\right|\\
&\leq \int_{\Omega_j}\left|W_{el\,j}(\nabla(a_j+h_j)) - W_{el\,j}(\nabla a_j) - \mathrm{D}W_{el\,j}\left(\nabla a_j\right)\cdot\nabla h_j\right|\mathrm{d}\mathbf{x}\,.
\end{aligned}\tag{117}$$

$\underline{p_j \geq 2:}$
Application of theorem A.2 leads to:

$$(117) \leq c \int_{\Omega_j} |(1 + |\nabla a_j| + |\nabla h_j|)|^{p_j - 2} \, |\nabla h_j|^2 \, \mathrm{d}\mathbf{x} \qquad (118)$$

Now Hölder's inequality (104) with $p = \frac{p_j}{2}$ und $q = \frac{p_j/2}{(p_j/2)-1} = \frac{p_j}{p_j-2}$ is applied and we get:

$$(118) \leq c \|1 + |\nabla a_j| + |\nabla h_j|\|^{p_j-2}_{L^{p_j}(\Omega_j)} \|\nabla h_j\|^2_{L^{p_j}(\Omega_j)} \leq a_{p_j} \|\nabla h_j\|^2_{L^{p_j}(\Omega_j)} \, .$$

$\underline{1 < p_j < 2:}$
If $1 < p_j < 2$, it holds according to theorem A.2

$$(117) \leq c \int_{\Omega_j} |\nabla a_j|^{p_j - 2} |\nabla h_j|^2 \, \mathrm{d}\mathbf{x} \, . \qquad (119)$$

As before, Hölder's inequality (104) is applied:

$$(119) \leq c \|\nabla a_j\|^{p_j-2}_{L^{p_j}(\Omega_j)} \|\nabla h_j\|^2_{L^{p_j}(\Omega_j)} \leq a_{p_j} \|\nabla h_j\|^2_{L^{p_j}(\Omega_j)} \, .$$

Since $\frac{a_{p_j} \|\nabla h_j\|^2_{L^{p_j}(\Omega_j)}}{\|\nabla h_j\|_{L^{p_j}(\Omega_j)}} \to 0$ as $\|\nabla h_j\|_{L^{p_j}(\Omega_j)} \to 0$ for every $p_j \in (1, \infty)$ we have under consideration of (115):

$$J_{elj}(\Omega_j, a_j + h_j) - J_{elj}(\Omega_j, a_j) - \left\langle \mathrm{D}J_{elj}(\Omega_j, a_j), h_j \right\rangle = o(h_j) \qquad \text{as } h_j \to 0.$$

■

B.2 Mean value theorem for Fréchet differentiable functionals

Theorem B.2 (Mean value theorem, [11] p. 169)
Let X be a Banach space and $I : X \to \mathbb{R}$ a functional which is Fréchet differentiable with derivative $\mathrm{D}I \in X'$. For every u and $h \in X$ there exists a constant $t_0 = t_0(u, h) \in [0, 1]$ such that

$$I(u + h) - I(u) = \langle \mathrm{D}I(u + t_0 h), h \rangle_{(X', X)}$$

Acknowledgements

The authors were partially supported by SFB 404 *Multifield Problems in Continuum Mechanics*, project C5, and the Research Training Group 1128 *Analysis, Numerics and Optimization of Multiphase Problems* of the German Research Foundation.
The authors express their gratitude to D. Knees for many fruitful discussions.

References

[1] ADAMS, R. A.: *Sobolev Spaces,* v.65 of Pure and applied mathematics series. Academic Press, New York, 1975.

[2] BROBERG, K.B.: *Cracks and Fracture.* Academic Press, San Diego, London, 1999.

[3] CHAROENPHAN, S.: *Computer Methods for Modeling the Progressive Damage of Composite Material Plates and Tubes,* A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Engieering Mechanics). University of Wisconsin-Madison, 2002.

[4] EVANS, L. C., GARIEPY, R. F.: *Measure Theory and Fine Properites of Functions.* CRC Press, 1992.

[5] GEYMONAT, G., SUQUET, P.: *Functional Spaces for Norton-Hoff Materials.* Math. Meth. in the Appl. Sci. 8 (1986) 206-222.

[6] GRISVARD, P.: *Elliptic Problems in Nonsmooth Domains.* Pitman Advanced Publishing Program 1985.

[7] GROSS, D., SEELIG, TH.: *Bruchmechanik mit einer Einführung in die Mikromechanik*, 3. Auflage. Springer, 2001.

[8] HARDY, G. H., LITTLEWOOD, J. E., PÓLYA, G.: *Inequalities.* Cambridge University Press, 1934.

[9] KHLUDNEV, A. M., SOKOLOWSKI, J.: *The Griffith formula and the Rice-Cherepanov integral for crack problems with unilateral conditions in nonsmooth domains.* Eur. J. Appl. Math. 10, 4 (1999), 379-394.

[10] KHLUDNEV, A. M., SOKOLOWSKI, J.: *Griffith formulae for elasticity systems with unilateral conditions in domais with cracks.* Eur. J. Mech., A/Solids 19 (2000), 105-119.

[11] KNEES, D.: *Regularity results for quasilinear elliptic systems of power-law growth in nonsmooth domains -Boundary, transmission and crack problems-*.Universität Stuttgart, 2004.

[12] KNEES, D.: *Griffith-formula and J-integral for a crack in a power-law hardening material.* Mathematical Models and Methods in Applied Sciences (M3AS), vol. 16, no. 11, p. 1723-1749, 2006.

[13] KNEES, D., MIELKE, A.: *Energy release rate for cracks in finite-strain elasticity.* WIAS Preprint No. 1100, Berlin 2006.

[14] KOROWKIN, P. P.: *Ungleichungen,* vierte, berichtigte Auflage. Deutscher Verlag der Wissenschaften, Berlin, 1965.

[15] LIU, W. B.: *Degenerate quasilinear elliptic equations arising from bimaterial problems in elastic-plastic mechanics.* Nonlinear Anal., Theory Methods Appl. 35, (1999), 517-529.

[16] POINCARÉ, H.: *Sur les équations aux dérivées partialles de la physique mathematique.* Amer. J. Math. 12 (1890), 211-294.

[17] Roubicek, T.: Nonlinear Partial Differential Equations with Applications. Birkhäuser 2005.

[18] Shackelford, J. F.: *Introduction to Materials Science For Engineers.* Sixth Edition. Pearson Education International, 2005.

[19] Tolksdorf, P.: *Everywhere-regularity for some quasilinear systems with a lack of ellipticity.* Ann. Mat. Pura Appl. 134 (1983), 241-266.

[20] Werner, D.: *Funktionalanalysis.* Springer Verlag, 1997.

[21] Wloka, J.: *Partielle Differentialgleichungen.* B. G. Teubner, Stuttgart, 1982.

[22] Zeidler, E.: *Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization.* Springer Verlag, 1984.

Erschienene Preprints ab No. 2004/01
Komplette Liste: *http://www.sfb404.uni-stuttgart.de/sfb404/*

2004/01	Lamichhane, B.P. / Wohlmuth, B.I.: Mortar finite elements with dual Lagrange multipliers: Some applications.
2004/02	Hüeber, S. / Mair, M. / Wohlmuth, B.I.: A priori error estimates and an inexact primal-dual active set strategy for linear and quadratic finite elements applied to multibody contact problems.
2004/03	Fischer, M. / Gaul, L.: Fast BEM-FEM mortar coupling for acoustic-structure interaction.
2004/04	Fischer, M. / Perfahl, H / Gaul, L.: Approximate inverse preconditioning for the fast multipole BEM in acoustics.
2004/05	Lamichhane, B.P. / Reddy, B.D. / Wohlmuth, B. I.: Convergence in the incompressible limit of finite element approximations based on the Hu-Washizu formulation.
2004/06	Mielke, A.: Evolution of rate-independent systems.
2004/07	Gravemeier, V. / Wall, W.A. / Ramm, E.: Large eddy simulation of turbulent incompressible flows by a three--level finite element method.
2004/08	Of, G. / Steinbach, O. / Wendland, W.L.: The fast multipole method for the symmetric boundary integral formulation.
2004/09	Of, G. / Steinbach, O. / Wendland, W.L.: Applications of a fast multipole Galerkin boundary element method in linear elastostatics.
2004/10	Neumann, M. / Tiyyagura, S.R. / Wall , W.A. / Ramm, E.: Robustness and efficiency aspects for computational fluid structure interaction.
2005/01	Mielke, A. / Müller, S.: Lower semi-continuity and existence of minimizers in incremental finite-strain elastoplasticity.
2005/02	Lamichhane, B.P. / Wohlmuth, B.I.: Biorthogonal bases with local support and approximation properties.
2005/03	Flemisch, B. / Wohlmuth, B.I.: Nonconforming methods for nonlinear elasticity problems.
2005/04	Knees, D.: Global stress regularity for convex and some nonconvex variational problems on Lipschitz domains.
2005/05	Förster, C. / Wall, W.A. / Ramm, E.: On the geometric conservation law in transient flow calculations on deforming domains.
2005/06	Langer, U. / Of, G. / Steinbach, O. / Zulehner, W.: Inexact data-sparse boundary element tearing and interconnecting methods.
2005/07	Hüeber, S. / Matei, A. / Wohlmuth, B.I.: Efficient algorithms for problems with friction.
2005/08	Knees, D.: Global regularity of the elastic fields of a power-law model on Lipschitz domains.
2005/09	Hüeber, S. / Wohlmuth, B.I.: Mortar methods for contact problems.

2005/10 Flemisch, B. / Kaltenbacher, M. / Wohlmuth, B.I.: Elasto-acoustic and acoustic-acoustic coupling on nonmatching grids.

2005/11 Knees, D. / Sändig, A.-M. (Hrsg.): Regularity for nonlinear and linear PDEs in nonsmooth domains. International Workshop in Hirschegg, September 04-07, 2005.

2005/12 Knees, D.: Griffith-formula and J-integral for a crack in a power-law hardening material.

2005/13 Chavan, K.S. Lamichhane, B.P. / Wohlmuth, B.I.: Locking-free finite element methods for linear and nonlinear elasticity in 2D and 3D.

2006/01 Efendiev, M.A. / Wendland, W.L.: On nonlinear Riemann-Hilbert problems with discontinuous coefficients.

2006/02 Niessner, J. / Helmig, R.: Multi-scale modeling of two-phase - two-component processes in heterogeneous porous media.

2006/03 Kohr, M. / Wendland, W.L.: The application of a fast multipole Galerkin boundary element method for the Stokes system.

2006/04 Mainik, A.: A rate-independent model for phase transformations in shape-memory alloys.

2006/05 Ölmann, U. / Aavatsmark, I. / Helmig, R.: Buckley-Leverett heterogen - Konstruktion der Lösungen mit der Charakteristikenmethode.

2006/06 Shutov, A. / Sändig, A.-M.: Mathematical analysis of fully coupled approach to creep damage.

2006/07 Utzmann, J. / Schwartzkopff, T. / Dumbser, M. / Munz, C.-D.: Heterogeneous domain decomposition for CAA.

2006/08 Muth, B. / Of, G. / Eberhard, P. / Steinbach, O.: Collision detection for complicated polyhedra using the fast multipole method or ray crossing.

2006/09 Förster, C. / Wall, W.A. / Ramm, E.: Artificial added mass instabilities in sequential staggered coupling of nonlinear structures and incompressible viscous flow.

2006/10 Berres, S. / Bürger, R.: On Riemann problems for non-genuinely nonlinear systems of conservation laws with an application to a sedimentation model.

2006/11 Hager, C. / Hüeber, S. / Wohlmuth, B.I.: A stable energy conserving approach for frictional contact problems based on quadrature formulas.

2006/12 Salomon, J. / Weiß, A. / Wohlmuth, B.I.: Energy conservative algorithms for a co-rotational formulation.

2006/13 Thomas, M. / Sändig, A.-M.: Energy release rate for interface-cracks in compounds of p-Laplacian type – Griffith formula and J-integral.