Rate-independent Damage Processes in Nonlinearly Elastic Materials

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Vorwort

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Zusammenfassung

Die vorliegende Dissertation befasst sich mit der analytischen Untersuchung ratenunabhängiger Schädigungs- und Delaminationsprozesse in physikalisch nichtlinear elastischen Materialien. Dies geschieht mithilfe ihrer sogenannten energetischen Formulierung, welche die Prozesse durch ein Energiefunktional $\mathcal{E}$ und ein Dissipationspotential $\mathcal{R}$ charakterisiert. Ersteres enthält die gespeicherte elastische Energie sowie den Energieanteil, den die äußeren Kräfte erzeugen. Das Dissipationspotential beschreibt den Energieanteil, der beim Übergang von einem Schädigungsstadium in ein fortgeschritteneres dissipiert wird. Der Lösungsbegriff im Rahmen der energetischen Formulierung ist die sogenannte energetische Lösung, welche durch die Erfüllung zweier mittels $\mathcal{E}$ und $\mathcal{R}$ beschriebener globaler Bedingungen, der Stabilitätsbedingung sowie der Energieerhaltungsgleichung, definiert ist.

Die Modellierung der Schädigung eines Festkörpers $\Omega \subset \mathbb{R}^d$ erfolgt mit den Methoden der Kontinuums-Schädigungs-Mechanik. Ähnlich der Behandlung von Plastizität wird auch hier eine innere Variable, die sogenannte Schädigungsvariable $z : [0,T] \times \Omega \to [0,1]$, ins Materialgesetz eingeführt, wo sie die Veränderung des elastischen Materialverhaltens aufgrund der Zunahme von Schädigung beschreibt. In analoger Weise kann auch die Delamination eines Verbundkörpers entlang eines Interfaces erfasst werden.

Die Hauptresultate dieser Arbeit sind:

- Existenz energetischer Lösungen für partielle, isotrope Schädigung:
  Die Existenz wird sowohl für kleine, als auch für finite Verzerrungen untersucht. Die hier bewiesenen Existenzresultate stellen eine Erweiterung der Ergebnisse in [MR06] dar, wo die Existenz energetischer Lösungen unter der Annahme $z \in W^{1,r}(\Omega)$ mit $r > d$ gezeigt wurde. Durch eine neue Methode zur Konstruktion gemeinsamer Wiederherstellungsfolgen (joint recovery sequences) ist es in dieser Arbeit gelungen, das Existenzresultat auf $r \in (1, \infty)$ auszudehnen.
- Ein Delaminationsmodell als Γ-Limes partieller, isotroper Schädigungsmodelle:
  Für den Grenzübergang wird ein Verbundkörper aus drei Schichten betrachtet, in dessen mittlerer Schicht partielle Schädigung auftritt. Mit verschwindender Dicke dieser Schicht entsteht das Interface zwischen den beiden übrigen Komponenten, wo nun Delamination möglich ist. Das Grenzmodell beschreibt die Transmissionsbedingung an den nichtdelaminierten Stellen im Interface sowie die Nichtdurchdringungsbedingung an den Rissufern in korrekter Weise.
- Aussagen zur Regularität energetischer Lösungen bezüglich der Zeit:
  Diese Ergebnisse bilden eine Verallgemeinerung der Resultate in [MT04], die u. a. die zeitliche Lipschitzstetigkeit energetischer Lösungen für gleichmäßig konvexe Energiefunktionale sichern. In der vorliegenden Arbeit wird jedoch gezeigt, dass nichtquadratische Energiefunktionale eventuell allgemeinere gleichmäßige Konvexitätsungleichungen erfüllen, als in [MT04] angenommen wurde, und dass in solchen Fällen zeitliche Hölderstetigkeit der energetischen Lösungen möglich ist. Außerdem wird erläutert, wie die zeitliche Regularität durch die geschickte Wahl der zugrundeliegenden Zustandsräume verbessert werden kann.

Abstract

This thesis is devoted to the analytical study of rate-independent damage and delamination processes in physically nonlinearly elastic materials. The analysis is done using their so-called energetic formulation, which characterizes the processes by an energy functional $\mathcal{E}$ and a dissipation potential $\mathcal{R}$. The first comprises the stored elastic energy and the amount of energy generated by the external loadings. The dissipation potential describes the amount of energy dissipated when changing from a damage stage to a more proceeded one. The notion of solution in the framework of the energetic formulation is the so-called energetic solution which is defined by satisfying two conditions given in terms of $\mathcal{E}$ and $\mathcal{R}$, namely the global stability condition and the global energy balance.

The damage of a body $\Omega \subset \mathbb{R}^d$ is modelled by the tools of continuum damage mechanics. Similarly to the treatment of plasticity an inner variable, the so-called damage variable $z : [0, T] \times \Omega \to [0, 1]$ is incorporated into the constitutive law, where it reflects the changes of the elastic behavior due to the increase of damage. The delamination of a compound along an interface can be described analogously.

The main results of this thesis are the following:

- Existence of energetic solutions for partial, isotropic damage:
  The existence is analyzed both at small and finite strains. The results obtained here are a generalization of those in [MR06], where the existence of energetic solutions was proven under the assumption $z \in W^{1,r}(\Omega)$ with $r > d$. Using a new technique for the construction of joint recovery sequences it is possible to extend this existence result to all $r \in (1, \infty)$.

- A delamination model as the Γ-limit of partial, isotropic damage models:
  For the limit passage a three-specimen-sandwich-structure is considered, where the middle constituent experiences partial damage. As the thickness of this middle layer tends to zero the interface between the two other components is formed, where delamination may occur. The limit model correctly captures the transmission conditions in nondelaminated regions of the interface as well as the noninterpenetration conditions at the crack lips.

- Results on the temporal regularity of energetic solutions:
  These results state a generalization of those in [MT04] which ensure the temporal Lipschitz continuity of energetic solutions in the case of uniformly convex energy functionals. In the present work it is shown that nonquadratic energy functionals may satisfy more general uniform convexity inequalities as the one assumed in [MT04] and that these inequalities allow it to prove the temporal Hölder continuity of energetic solutions. Moreover it is explained how to improve the temporal regularity by an effective use of the underlying state spaces.

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Chapter 1

Motivation

To understand the failure of solids under the influence of external loadings is of great interest ever since the technical progress drives mankind to create more and more complex structures. Experimental studies enable engineers to develop mechanisms and models to describe failure processes [Gri21, CNF+06, DMSE92, All02]. In order to forecast the durability of specimen it is important to have models and criteria at hand which allow it to predict – e.g. by numerical simulations – under which loading conditions a crack will form and propagate.

One branch of solid mechanics, which is concerned with these questions is so-called fracture mechanics. Within this theory a crack in a solid is viewed as a surface which can be noticed macroscopically, so that the crack is modelled as a part of the boundary and boundary conditions have to be imposed. In other words, the formation of cracks can be understood as the creation of new surfaces in the solid, see Fig. 1.1a). A.A. Griffith expressed exactly this coherence in [Gri21] when he explained the formation of cracks by an extention of the principle of minimum energy:

> *In an elastic solid body deformed by specified forces applied at its surface, the sum of the potential energy of the applied forces and the strain energy of the body is diminished or unaltered by the introduction of a crack whose surfaces are traction-free.*

Griffith pointed out that for *the formation of a crack in a body composed of molecules which attract one another, work must be done against the cohesive forces of the molecules on either side of the crack. This work appears as potential surface energy* ... and *the energy per unit area is a constant of the material, namely, its surface tension* γ. Since a crack consists of two crack surfaces this determines the fracture toughness $\mathcal{G}_c = 2\gamma$. The potential surface energy due to a crack of length l_0 is then given by $\mathcal{G}_c l_0$. For a body Ω_{l_0} containing a crack of length l_0 the sum of the strain energy and the potential energy due to the applied forces is denoted by $\mathcal{E}(\Omega_{l_0})$. Therewith Griffith deduced the following condition for crack extension:

$$-\left.\frac{\mathrm{d}\mathcal{E}(\Omega_{l_0+l})}{\mathrm{d}l}\right|_{l=0} = \mathcal{G}_c \left.\frac{\mathrm{d}(l_0+l)}{\mathrm{d}l}\right|_{l=0}, \tag{1.1}$$

i.e. the pre-existing crack grows if the so-called energy release rate $ERR(l_0) := -\frac{\mathrm{d}\mathcal{E}(\Omega_{l_0+l})}{\mathrm{d}l}\Big|_{l=0}$ attains the critical value $\mathcal{G}_c$. This condition is nowadays known as the Griffith' fracture criterion and it has become an important and reliable tool in fracture mechanics to predict whether a pre-existing crack will propagate.

Both engineering and mathematical literature is concerned with the development of formulas that allow it to compute the energy release rate in an effective way, such as by means of singular expansions and stress intensity factors in linear elasticity [DO87, DO88, BS99, BS01, NS07, LSS08] or by Griffith- and J-integral formulas [Ric68, KS99, KS00, Kne06, TS06, Tho08], which can also be applied to physically nonlinearly elastic materials.

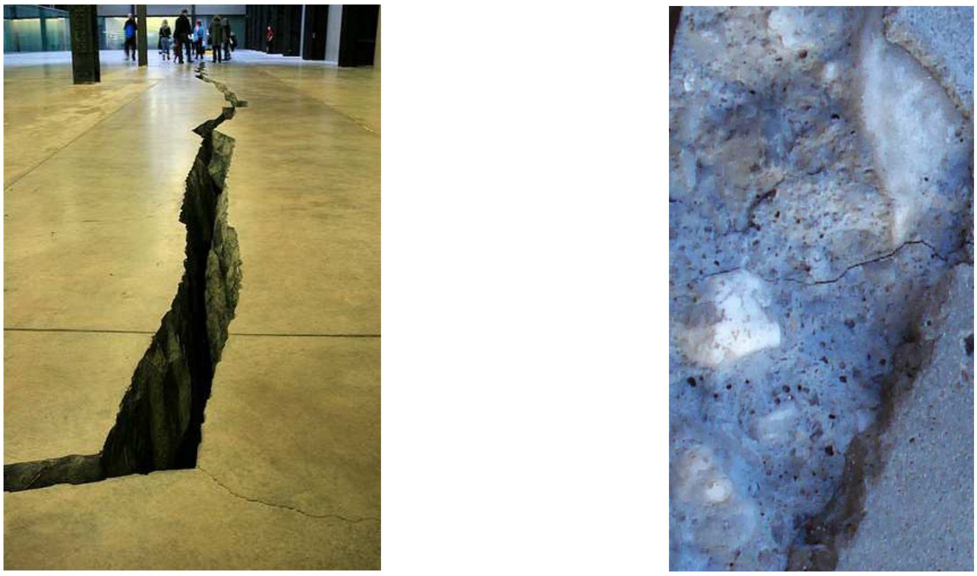
An alternative method to study the failure of bodies is so-called continuum damage mechanics. Starting in 1958 with works of L.M. Kachanov [Kac58, Kac60, Kac90] and Yu.N. Rabotnov [Rab69] on the creep failure of metals, continuum damage mechanics has become a rapidly developing branch of engineering fracture mechanics within the last thirty years. In contrast to the concept based on Griffith' fracture criterion, where a crack in the body is a macroscopically visible surface, this alternative approach takes into account that the failure of a solid already starts on its micro-level, when the material macroscopically still seems to be intact. J.L. Lemaitre and R. Desmorat specify the main ideas behind continuum damage mechanics as follows [LD05]:

> *Damage in its mechanical sense in solid materials is the creation and growth of microvoids or microcracks which are discontinuities in a medium considered as continuous at a larger scale.*

This means that an increase of damage on the microlevel of a solid is macroscopically noticed as a change of the material properties. In particular, experiments document a decrease of the material's hardness and a change of its stiffness and strength. Therefore damage is modelled with the aid of an inner variable, the damage variable, which is incorporated to the constitutive law, where it reflects the change of the material's elastic behavior due to damage. This ansatz enables us to apply the usual tools of continuum mechanics on the larger scale in order to describe the deformations of the solid (bearing microdefects) under the influence of external loadings. Such an approach may be better known from the field of plasticity, where the plastic strain is introduced to the constitutive law as an inner variable. Similar to this application also the evolution of the damage variable is described by an evolutionary law or flow rule, which is an ordinary differential equation or inclusion.

The microscale of a material is specified with the aid of a so-called representative volume element (RVE), which has to be of such a size that all the characteristic ingredients of the material are contained. Assuming a uniform distribution of the microvoids in the material, the damage variable $z(t, x)$ at a point x in a body $\Omega \subset \mathbb{R}^3$ at time $t \in [0, T]$ can be defined as the volume fraction of undamaged material in the representative volume element with its center in x. Hence the damage variable attains values between 0 and 1, where the value 1 stands for a purely undamaged RVE and the value 0 means the total disintegration of the material in the RVE. The microcracks and microvoids are then defined by the set $C(t) = \{x \in \Omega \,|\, z(t,x) = 0\}$ and a macrocrack is visible if the measure of the boundary $\partial C(t)$ is positive.

Thus, continuum damage mechanics models the prestage of macrocracking and therewith indeed captures (macro)crack initiation. Moreover, by tracking $\partial C(t)$, it can be used for crack path prediction.



a) Macrocrack: Shibboleth by D. Salcedo, Tate Gallery

b) Microcrack in a pillar of the Holocaust Memorial, Berlin

Figure 1.1: Macrocracks versus microcracks in concrete

Using the ideas of continuum damage mechanics also a delamination process can be modelled via a delamination variable $z : [0,T] \times \Gamma_{\mathrm{C}} \to [0,1]$, where $\Gamma_{\mathrm{C}} \subset \mathbb{R}^2$ denotes an interface along which two constituents of a compound will fall apart. Thereby delamination along an interface can macroscopically be noticed as crack growth on a prescribed maximal crack path (the full interface) and the size of the crack is given by the measure of the set $N_z(t) := \{x \in \Gamma_{\mathrm{C}} \,|\, z(x) = 0\}$. Of course, such a delamination model has to reflect the properties of cracks stated by Griffith, i.e. on $N_z(t)$ the model must supply boundary conditions which identify $N_z(t)$ as a part of the boundary, whereas on $\Gamma_{\mathrm{C}} \backslash N_z(t)$ it has to provide transmission conditions expressing that the two constituents are bonded there.

This thesis is devoted to the analytical study of rate-independent damage and delamination processes in physically nonlinearly elastic materials.

The required tools from continuum mechanics and continuum damage mechanics are provided in Chapter 2, Sections 2.1-2.2.

The analysis is done using the energetic formulation for rate-independent systems. This approach is solely based on an energy functional $\mathcal{E}$ and a dissipation potential $\mathcal{R}$. In this framework one is interested in so-called energetic solutions which are defined by satisfying a global stability condition and a global energy balance. This notion of solution and the relation to other concepts are explained in Chapter 2, Sections 2.3-2.4.

In Chapter 3 the existence of energetic solutions of partial, isotropic damage processes is proven both for small and finite strains. In contrast to previous work [MR06], where the existence could only be proven for damage variables $z \in W^{1,r}(\Omega)$ with r larger than the space dimension d, the existence result obtained in this thesis holds for all $r \in (1, \infty)$.

This result is used in Chapter 4 to study a delamination process as the Γ-limit of damage processes: The delamination on the interface between two unbreakable structures is understood as a process allowing for complete damage on a domain with zero-thickness. It is approximated by processes describing the partial damage of the middle constituent of three-specimen-sandwich-structures, when the thickness of the middle component tends to 0. The delamination model, which is obtained in the limit, is the one analyzed in [RSZ09] and it reflects both the transmission and boundary conditions on the interface.

In Chapter 5 the temporal regularity of energetic solutions is studied. Settings that lead to temporal continuity, such as jointly strictly convex energy functionals, are discussed. The thesis pays special attention to nonlinear, jointly uniformly convex energy functionals combined with time-dependent Dirichlet boundary conditions. In this setting temporal Lipschitz continuity can be verified for energy functionals with (sub-)quadratic growth, whereas a super-quadratic growth only leads to Hölder continuity with respect to time. This is an extension of the results obtained in [MT04] where temporal Lipschitz continuity was proven for jointly uniformly convex energy functionals in the case of time-independent Dirichlet conditions or quadratic functionals in combination with time-dependent Dirichlet conditions.

Finally, Chapter 6 provides a summary of the results obtained in this thesis and gives an outlook on prospective work.

Chapter 2

The Theoretical Background

This chapter provides the main tools for the mechanical and mathematical modeling of damage in nonlinearly elastic materials and the delamination of sandwich-structures. The respective models analyzed in Chapters 3–5 treat these processes as quasistatic ones. This means that kinetic effects are neglected, so that the process is given by a chronology of static equilibria. Section 2.1 summarizes the required tools of the continuum mechanics of solids for quasistatic settings according to [Cia88]. An introduction to the corresponding kinetic theory can be found e.g. in [MH83, Hau02, TN65]. Continuum mechanics in its quasistatic form is the basis to study damage and delamination by means of so-called continuum damage mechanics. This approach is explained in Section 2.2. Section 2.3 presents different mathematical models describing quasistatic processes and analyzes their relation. Since the results in this thesis are obtained using the so-called energetic formulation of the respective processes Section 2.4 is concerned with this theory. With this theoretical background Section 2.5 states the outline of this thesis.

2.1 Tools from the Continuum Mechanics of Solids

This section introduces the basic notations from the continuum mechanics of solids. A detailed deduction of this matter can be found e.g. in [Cia88].

The continuum mechanics of solids can be used to study the deformation of bodies. This approach does not consider a solid as a particle system of finitely many atoms, but idealizes it to a continuum with uncountably many material points: The solid is described by a domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and the points $x \in \overline{\Omega}$ are understood as the material points.

2.1.1 Deformation, Displacement and Strain Tensors

The deformation of the solid during the time interval $[0, T]$, $T > 0$, is described by a mapping φ, which transforms it from the undeformed state occupying the reference configuration Ω at time $t_0 = 0$ to a deformed state occupying the current configuration $\varphi(t, \Omega)$, see Fig. 2.1.

To reflect the physical properties of a deformation appropriately it is claimed that

$$\left.\begin{array}{l} \varphi : [0,T] \times \Omega \to \mathbb{R}^d \text{ is sufficiently smooth,} \\ \varphi(t,\cdot) : \overline{\Omega} \to \varphi(t,\overline{\Omega}) \text{ is a bijection for all } t \in [0,T], \\ \det \nabla\varphi(t,x) > 0, \text{ i.e. } \nabla\varphi(t,x) \in \mathrm{GL}_+(d) \text{ for all } x \in \Omega \text{ and } t \in [0,T], \end{array}\right\} \tag{2.1}$$

where $\mathrm{GL}_+(d) := \{F \in \mathbb{R}^{d\times d} \mid \det F > 0\}$. The last claim means that φ is orientation preserving, which ensures the non-interpenetration of matter. The variable $\varphi(t,x) \in \varphi(t,\overline{\Omega})$ of the current configuration is called Euler variable, whereas $x \in \Omega$ of the reference configuration is the so-called Lagrange variable.

For all $t \in [0,T]$ the deformation of the body can be characterized by a displacement field $u : [0,T] \times \Omega \to \mathbb{R}^d$, which determines the displacement due to the deformation $\varphi : [0,T] \times \Omega \to \mathbb{R}^d$ for each material point with respect to the reference configuration

$$u(t,\cdot) : \Omega \to \mathbb{R}^d,\ u(t,\cdot) := \varphi(t,\cdot) - \mathrm{id},\ u(t,x) = \varphi(t,x) - x \text{ for all } x \in \Omega\,, \tag{2.2}$$

where $\mathrm{id} : \mathbb{R}^d \to \mathbb{R}^d$ is the identity mapping.

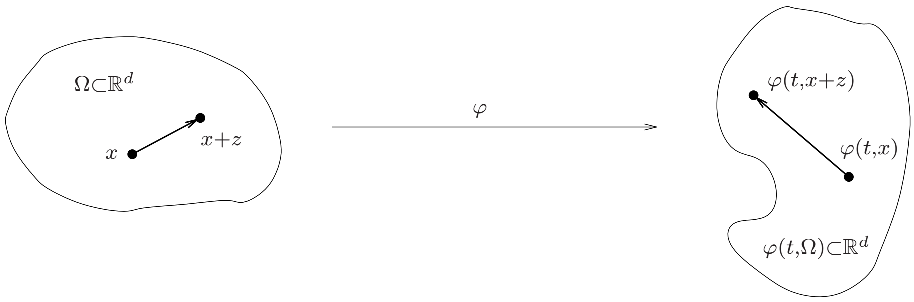


Figure 2.1: Deformation $\varphi : [0,T] \times \Omega \to \mathbb{R}^d$

To determine the strain induced by the deformation φ one may consider two points in the reference configuration $x, (x+z) \in \Omega$ and their images $\varphi(t,x), \varphi(t,x+z) \in \varphi(t,\Omega)$ in the current configuration. By Taylor's expansion one obtains for their Euclidean distance

$$\begin{aligned} |\varphi(t,x+z) - \varphi(t,x)|^2 &= |\nabla\varphi(t,x)z + o(z)|^2 = |(\nabla\varphi(t,x))z|^2 + 2o(z)^\top \nabla\varphi(t,x)z + |o(z)|^2 \\ &= z^\top \nabla\varphi(t,x)^\top \nabla\varphi(t,x) z + o(|z|^2)\,, \end{aligned}$$

where $A^\top$ denotes the transpose of the matrix A. The symmetric matrix $\nabla\varphi(t,x)^\top\nabla\varphi(t,x)$ provides a local measure for the strains induced by φ and this suggests to define the following tensors

$$\text{left Cauchy-Green tensor:} \quad B = \nabla\varphi\nabla\varphi^\top : [0,T] \times \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}\,, \tag{2.3}$$

$$\text{right Cauchy-Green tensor:} \quad C = \nabla\varphi^\top\nabla\varphi : [0,T] \times \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}\,, \tag{2.4}$$

$$\text{Green-St. Venant strain tensor:} \quad E = \tfrac{1}{2}(C - \mathrm{Id}) : [0,T] \times \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}\,. \tag{2.5}$$

Expressing these tensors in terms of the displacement field u yields

$$C = \mathrm{Id} + \nabla u + \nabla u^\top + \nabla u^\top \nabla u,\ B = \mathrm{Id} + \nabla u + \nabla u^\top + \nabla u \nabla u^\top,$$
$$E = \tfrac{1}{2}(\nabla u + \nabla u^\top + \nabla u^\top \nabla u)\,. \tag{2.6}$$

Thus, if the displacement gradient ∇u, i.e. $\sup_{x\in\Omega} \max_{i,j=1,\dots,d} |\partial_{x_i} u_j(x)|$, is small, the quadratic terms can be neglected and one may introduce the

$$\text{linearized (Green-St. Venant) strain tensor: } e = e(u) := \tfrac{1}{2}(\nabla u + \nabla u^\top)\,. \tag{2.7}$$

The assumption of small displacement gradients and the use of e determines the so-called *small-strain setting*. On the contrary one speaks of the *finite-strain setting* if quadratic terms are not neglected, so that one has to take into account the strain tensors C, B, E or equivalently the deformation gradient $\nabla\varphi$. Because of the nonlinear terms solely due to large deformation of the geometry one also speaks of geometric nonlinearities.

2.1.2 Stress Tensors and Equations of Equilibrium

In this section the equations of equilibrium of solids are introduced in the kinetic, the static and the quasistatic setting. They state a balance between the external and the internal forces. The internal forces resisting the external ones, are expressed by the stresses. They are given by stress tensors (of second order), which are different for the Eulerian and the Lagrangian concept. In the following we will describe their relation in the static setting, which means that the deformation is considered to be time-independent, i.e. $\varphi : \overline{\Omega} \to \varphi(\overline{\Omega})$. For a clearer notation we put $\varphi(\Omega) = \Omega^\varphi$ and $\varphi(x) = x^\varphi$ in the following.

A fundamental axiom of continuum mechanics is the stress principle of Euler and Cauchy. For a body occupying the deformed configuration $\varphi(\overline{\Omega})$ and subjected to applied loadings it states the existence of the so-called Cauchy stress vector $\tau^\varphi : \overline{\Omega^\varphi} \times \mathbb{S}^{(d-1)}$, where $\mathbb{S}^{(d-1)}$ is the unit sphere in $\mathbb{R}^d$, i.e. $\mathbb{S}^{(d-1)} := \{y \in \mathbb{R}^d \,|\, |y| = 1\}$. If the body is in static equilibrium the Cauchy stress vector is the local reaction of the body to the external loadings and it satisfies the axiom of force balance as well as the axiom of moment balance [Cia88, p. 60]. For all $x^\varphi \in \Omega^\varphi$ the stress vector $\tau^\varphi(x^\varphi, \mathrm{n}^\varphi)$ also depends on the outer unit normal vector n^φ of an area element $\omega^\varphi \subset \overline{\Omega^\varphi}$, which is necessary for its definition. Cauchy's theorem now postulates the existence of the symmetric

$$\text{Cauchy stress tensor: } \mathrm{T}^\varphi : \overline{\Omega^\varphi} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}\,, \tag{2.8}$$

such that $\tau^\varphi(x^\varphi, \mathrm{n}) = \mathrm{T}^\varphi(x^\varphi)\mathrm{n}$ for all $x^\varphi \in \overline{\Omega^\varphi}$ and all $\mathrm{n} \in \mathbb{S}^{(d-1)}$ and moreover such that the equations of equilibrium in the current configuration are satisfied

$$-\operatorname{div}_{x^\varphi} \mathrm{T}^\varphi(x^\varphi) = f^\varphi(x^\varphi) \ \text{ for all } x^\varphi \in \Omega^\varphi\,, \tag{2.9}$$
$$\mathrm{T}^\varphi(x^\varphi)^\top = T^\varphi(x^\varphi) \ \text{ for all } x^\varphi \in \Omega^\varphi\,, \tag{2.10}$$
$$\mathrm{T}^\varphi(x^\varphi)\mathrm{n}^\varphi = h^\varphi(x^\varphi) \ \text{ for all } x^\varphi \in \Gamma_N^\varphi\,, \tag{2.11}$$

where $f^\varphi : \Omega^\varphi \to \mathbb{R}^d$ is a given volume force density and $h^\varphi : \Gamma_N^\varphi \to \mathbb{R}^d$ is a given surface force density with n^φ as the outer unit normal vector to the Neumann boundary $\Gamma_N^\varphi \subset \partial\Omega^\varphi$ [Cia88, p. 62].

Since the equations of equilibrium are formulated in terms of the unknown Euler variable $x^\varphi = \varphi(x)$ it may be useful to transform the Cauchy stress and equations (2.9)–(2.11) to the reference configuration Ω. For this, the Piola transform is applied to the Cauchy stress tensor, which results in the

$$\text{first Piola-Kirchhoff stress tensor: } \mathrm{T}(x) = (\det \nabla\varphi(x))\mathrm{T}^\varphi(\varphi(x))\nabla\varphi(x)^{-\top}\,, \tag{2.12}$$

where $A^{-\top} := (A^{-1})^\top$ is the transposed of the inverse A^{-1} of the matrix $A \in \mathbb{R}^{d\times d}$ [Cia88, p. 71]. The equations of equilibrium in the reference configuration read as follows

$$-\operatorname{div} \mathrm{T}(x) = f(x) \ \text{ for all } x \in \Omega\,, \tag{2.13}$$

$$\mathrm{T}(x)^\top = \nabla\varphi(x)^{-1}\mathrm{T}(x)\nabla\varphi(x)^\top \ \text{ for all } x \in \Omega\,, \tag{2.14}$$

$$\mathrm{T}(x)\mathrm{n} = h(x) \ \text{ for all } x \in \Gamma_N\,. \tag{2.15}$$

Relation (2.14) shows that $\mathrm{T}(x)$ is in general unsymmetric. Thus, in order to work with a symmetric stress tensor in the reference configuration one defines the symmetric

$$\text{second Piola-Kirchhoff stress tensor: } \Sigma : \overline{\Omega} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}\,,\ \ \Sigma(x) = \nabla\varphi(x)^{-1}\mathrm{T}(x)\,, \tag{2.16}$$

[Cia88, p. 72], which satisfies the equations of equilibrium

$$-\operatorname{div}(\nabla\varphi(x)\Sigma(x)) = f(x) \ \text{ for all } x \in \Omega\,, \tag{2.17}$$

$$\Sigma(x)^\top = \Sigma(x) \ \text{ for all } x \in \Omega\,, \tag{2.18}$$

$$\nabla\varphi(x)\Sigma(x)\mathrm{n} = h(x) \ \text{ for all } x \in \Gamma_N\,. \tag{2.19}$$

The quasistatic setting is characterized by a chronology of static equilibria along $[0,T]$ with slowly varying time-dependent loads $f : [0,T]\times\Omega \to \mathbb{R}^d$ and $h : [0,T]\times\Gamma_N \to \mathbb{R}^d$. Thus, the equations of quasistatic equilibrium with respect to the reference configuration Ω are also given by (2.17)–(2.19) and (2.17)–(2.19) with $\varphi(t,x)$, $f(t,x)$, $h(t,x)$ and $\mathrm{T}(t,x)$ or $\Sigma(t,x)$ respectively.

In the kinetic setting the balance equations read:

$$\operatorname{div}(\nabla\varphi(x)\Sigma(x)) + f(x) = \rho(t,x)\partial_{tt}\varphi(t,x) \ \text{ for all } x \in \Omega\,, \tag{2.20}$$

$$\Sigma(x)^\top = \Sigma(x) \ \text{ for all } x \in \Omega\,, \tag{2.21}$$

$$\nabla\varphi(x)\Sigma(x)\mathrm{n} = h(x) \ \text{ for all } x \in \Gamma_N\,, \tag{2.22}$$

where (2.20) is the equation of motion with $\rho : [0,T]\times\Omega \to (0,\infty)$ being the mass density. Hence, in the quasistatic setting it is assumed that the kinetic term on the right-hand side of (2.20) is negligible.

2.1.3 Constitutive Equations of Elastic Materials

Relation (2.13) consists of d equations and relation (2.14) provides $d(d-1)/2$ equations for in total $d+d^2$ unknowns given by the deformation φ and the stress tensor T. The $d(d-1)/2$ missing equations are gained from assumptions concerning the nature of the material under consideration. Such an assumption is that a material has an elastic response to external loadings. This has to be understood as follows: The external loading causes a deformation φ of the material and strains characterized by $\nabla\varphi$. They induce certain stresses T. If the material is elastic, removing the external loadings makes the material return immediately to its undeformed configuration so that no additional strains remain. This implies that the stress tensor T only depends on the material points $x \in \Omega$, the deformation φ and its gradient $F = \nabla\varphi(x)$, i.e. $\mathrm{T}(x) = \overline{\mathrm{T}}(x, \varphi, F)$.

A further constitutive assumption is the so-called hyperelasticity. A material is hyperelastic if there exists a stored energy density $\hat{W} : \overline{\Omega} \times \mathrm{GL}_+(d)$ such that

$$\overline{\mathrm{T}}(x, F) = \partial_F \hat{W}(x, F) \quad \text{for all } x \in \overline{\Omega},\ F \in \mathrm{GL}_+(d)\,, \tag{2.23}$$

where $\partial_F \hat{W}(x, F) = \frac{\partial \hat{W}(x,F)}{\partial F} = \big(\frac{\partial \hat{W}(x,F)}{\partial F_{ij}}\big)_{i,j=1}^{d}$ denotes the partial derivative of $\hat{W}$ with respect to F [Cia88, Chap. 4].

As an assumption arising from physics the constitutive equation has to be independent of the choice of the coordinate system:

(N1) Independence of the constitutive law of constant translations φ_0:

$$\hat{W}(x, \varphi(x)+\varphi_0, \nabla\varphi(x)) = \hat{W}(x, \varphi(x), \nabla\varphi(x)) \tag{2.24}$$

for all $x \in \Omega$ and deformations $\varphi : \Omega \to \mathbb{R}^d$.

Choosing $\varphi_0 = \varphi(x)$ for a fixed $x \in \Omega$ shows that $\hat{W}$ cannot depend explicitly on φ, so that $\hat{W} = \hat{W}(x, \nabla\varphi(x))$.

(N2) Material frame indifference (objectivity): Independence of the constitutive law of rotations $Q \in \mathrm{SO}(d) := \{R \in \mathbb{R}^{d\times d} \,|\, R^{-1} = R^\top,\, \det R = 1\}$

$$\hat{W}(x, QF) = \hat{W}(x, F) \quad \text{for all } x \in \Omega,\ F \in \mathrm{GL}_+(d)\,. \tag{2.25}$$

We say that $\hat{W}$ is objective, if it respects (2.25).

The axiom of material frame indifference implies that a material is hyperelastic if and only if there exist densities $\tilde{W}, \bar{W}$ such that $\hat{W}(x, F) = \tilde{W}(x, F^\top F) = \bar{W}(x, \frac{1}{2}(F^\top F - \mathrm{Id}))$ for all $x \in \Omega$, $F \in \mathrm{GL}_+(d)$, i.e. $\tilde{W}$ depends on the left Cauchy-Green tensor C and $\bar{W}$ on the Green-St. Venant strain tensor E, so that Σ is characterized by a constitutive equation of the form

$$\Sigma(x) = 2\partial_C \tilde{W}(x, C) = \partial_E \bar{W}(x, E) \tag{2.26}$$

for all $x \in \overline{\Omega}$ and all $\mathrm{Id} + 2E = C = F^\top F$ with $F \in \mathrm{GL}_+(d)$ [Cia88, Th. 4.2-1, Th. 4.2-2].

Note that (N1), (N2) are axioms induced by physical aspects. Additionally, further material features can influence the mathematical properties of $\hat{W}$.

Such material features are

(M1) Homogeneity: A material is homogeneous if translations of the coordinate system in the material do not change the material properties, i.e.

$$\hat{W}(x,F) = \hat{W}(F) \quad \text{for all } F \in \mathrm{GL}_+(d)\,. \tag{2.27}$$

(M2) Isotropy: A material is isotropic, if rotations of the coordinate system in the material do not change the material properties. This means that $\hat{W}$ has to satisfy isotropy

$$\hat{W}(FQ) = \hat{W}(F) \quad \text{for all } F \in \mathrm{GL}_+(d) \text{ and all } Q \in \mathrm{SO}(d)\,. \tag{2.28}$$

Remark 2.1.1 *Since objective, isotropic densities $\hat{W}$ are invariant under orthogonal transformations by matrices $Q \in \mathrm{SO}(d)$ one can transform $C = F^\top F$ to a diagonal matrix. Thus it suffices to consider functions of either the eigenvalues or the invariants of C.*

A well-known example for hyperelastic, homogeneous, objective, isotropic materials is the St. Venant-Kirchhoff material.

Example 2.1.2 (St. Venant-Kirchhoff materials) *A St. Venant-Kirchhoff material satisfies the constitutive equations*

$$\Sigma = \lambda(\operatorname{tr} E)\operatorname{Id} + 2\mu E$$

and its stored energy density is given by

$$\bar{W}(E) = \tfrac{\lambda}{2}(\operatorname{tr} E)^2 + \mu|E|^2, \tag{2.29}$$

where $\lambda, \mu > 0$ are the Lamé-constants of the material and $|E|^2 := \sum_{i,j=1}^d E_{ij}^2$. Here, Σ depends linearly on E so that one speaks of a linearly elastic material.

If one operates close to the undeformed configuration, i.e. in the small-strain setting, then E may be replaced by the linearized strain tensor e, see (2.7). This leads to the constitutive relations of linearized elasticity.

Example 2.1.3 (Linear elasticity at small strains) *In the small-strain setting the constitutive relations of St. Venant-Kirchhoff materials yield*

$$\sigma = \lambda(\operatorname{tr} e)\operatorname{Id} + 2\mu e\,, \quad \bar{W}(e) = \tfrac{\lambda}{2}(\operatorname{tr} e)^2 + \mu|e|^2\,. \tag{2.30}$$

Again relations (2.30) characterize a linearly elastic material. Here the linearized tensors σ and $e = 1/2(\nabla u + \nabla u^\top)$ are approximate stresses or strains, which can be used if the displacement gradient ∇u is small enough. The symmetric tensor $\sigma \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is called the linearized stress tensor.

This constitutive relation can be generalized to Hooke's law.

Example 2.1.4 (Hooke's law) *Let $\mathbb{B} \in \mathbb{R}^{(d\times d)\times(d\times d)}$ be a symmetric, positive definite fourth order tensor, i.e. $\mathbb{B}_{ijkl} = \mathbb{B}_{jikl} = \mathbb{B}_{ijlk} = \mathbb{B}_{klij}$ for all $i,j,k,l \in \{1,\dots,d\}$ and there are constants $c_1^{\mathbb{B}}, c_2^{\mathbb{B}} > 0$ such that $c_1^{\mathbb{B}}|e|^2 \le e : \mathbb{B} : e \le c_2^{\mathbb{B}}|e|^2$ for all $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Then it is*

$$\sigma = \mathbb{B} : e\,, \quad \bar{W}(e) = \tfrac{1}{2}\, e : \mathbb{B} : e\,, \tag{2.31}$$

where $\mathbb{B} : e = \sum_{k,l=1}^{d} \mathbb{B}_{ijkl}e_{kl}$ and $e : \mathbb{B} : e = \sum_{i,j,k,l=1}^{d} \mathbb{B}_{ijkl}e_{ij}e_{kl}$.

The relations (2.30) *for St. Venant Kirchhoff materials can be written in the form* (2.31) *with $\mathbb{B} = \mathbb{B}^{\mathrm{SVK}}$, where*

$$\mathbb{B}^{\mathrm{SVK}}_{ijkl} := \begin{cases} \lambda{+}2\mu & \textit{if } i=j=k=l\,, \\ \lambda & \textit{if } i=j\neq k=l\,, \\ \mu & \textit{if } i=k\neq j=l\,, \\ 0 & \textit{else}\,. \end{cases}$$

We speak of a nonlinearly elastic material, if the stress tensor depends nonlinearly on the displacement gradient and the deformation gradient. To be more precise these materials are called physically nonlinearly elastic to indicate that the nonlinearity is due to the constitutive relation and hence to distinguish it from possible geometric nonlinearities due to large deformation gradients.

Example 2.1.5 (Nonlinearly elastic material at small strain) *As an example for a nonlinearly elastic material we define for fixed $p \in (1,2)\cup(2,\infty)$, for all $e \in \mathbb{R}^{d\times d}$*

$$\sigma = (1{+}|e|^2)^{\frac{p-2}{2}} e\,,\ \ \bar{W}(e) = \tfrac{1}{p}(1{+}|e|^2)^{\frac{p}{2}}\,. \tag{2.32}$$

In the finite-strain setting, where large deformation gradients are admissible, apart from (N1), (N2) the following further natural assumptions are made

(N3) The strain energy in the undeformed state is 0 and hence $\hat{W}(x,\mathrm{Id}) = 0$, whereas the strain energy of any deformed state is positive, i.e. $\hat{W}(x,F) > 0$ for all $F \in \mathrm{GL}_+(d)$.

(N4) Extreme deformations lead to extremely large values of stored elastic energy, so that

$$\hat{W}(x,F) \to \infty \quad \text{if either } |F| \to \infty \ \text{ or } \det F \to 0\,. \tag{2.33}$$

The first property in (N4) prevents an infinite slope of deformation, the second property inhibits the compression to zero-volume and preserves orientation.

Examples for constitutive relations for homogeneous, isotropic, objective materials in the finite-strain setting provides the class of Ogden's materials.

Example 2.1.6 (Ogden's materials [Cia88]) *For $F \in \mathrm{GL}_+(d)$ let*

$$\hat{W}(F) := \sum_{i=1}^{M} a_i(\mathrm{tr}(F^\top F))^{\frac{\gamma_i}{2}} + \sum_{j=1}^{N} b_j(\mathrm{tr}\,\mathrm{Cof}(F^\top F))^{\frac{\delta_j}{2}} + \Gamma(\det F) \tag{2.34}$$

where $\operatorname{Cof} A := (\det A)A^{-\top}$ *and* $M, N \in \mathbb{N}$, a_i, $b_i > 0$, γ_i, $\delta_i > d$, *and* $\Gamma : (0,\infty) \to (0,\infty)$ *is a convex function satisfying* $\Gamma(h) \to +\infty$ *as* $h \to 0_+$. *This means that the homogeneous, isotropic, objective density* $\hat{W}$ *depends on the invariants of* $C = F^\top F$, *namely* $\operatorname{tr} C$, $\operatorname{tr}\operatorname{Cof} C$, $\det C = (\det F)^2$.

2.1.4 Energy Balance

For an isothermal body the energy principle states that, in any time interval $[0,T]$, the work done by the external forces has to be equal to the change of kinetic and stored energy. For a hyperelastic material in the quasistatic setting, where the kinetic term is neglected, we introduce the energy functional

$$\mathcal{E}(t,u(t)) = \int_\Omega \bar{W}(e(u(t,x)))\,\mathrm{d}x - \int_\Omega f(t,x)\cdot u(t,x)\,\mathrm{d}x - \int_{\Gamma_N} h(t,s)\cdot u(t,s)\,\mathrm{d}s \tag{2.35}$$

for sufficiently smooth external loadings f, h. For $u : [0,T]\times\Omega \to \mathbb{R}^d$ being a (weak) solution to the Euler-Lagrange equations (2.17)-(2.19) the energy balance reads as follows

$$\mathcal{E}(t,u(t)) = \mathcal{E}(0,u(0)) + \int_0^t \partial_\xi \mathcal{E}(\xi, u(\xi))\,\mathrm{d}\xi\,, \tag{2.36}$$

where $\partial_\xi \mathcal{E}(\xi,u(\xi)) = -\int_\Omega \dot{f}(\xi,x)\cdot u(\xi,x)\,\mathrm{d}x - \int_{\Gamma_N} \dot{h}(\xi,s)\cdot u(\xi,s)\,\mathrm{d}s$ is the partial derivative of $\mathcal{E}(\xi,u)$ with respect to the component ξ. Hence the energy balance (2.36) expresses that the energy at time t is equal to the energy at time 0 plus the work done by the external loadings up to time t. Here (2.35) and (2.36) are formulated for small strains. In the finite-strain setting an analogous relation holds.

2.2 Modeling of Damage and Delamination

Modern engineering materials subjected to unfavourable mechanical and environmental conditions undergo microstructural changes which decrease their strength. Since the changes impair the mechanical properties of these materials, the term damage is used. Thus in engineering literature, damage means the creation and growth of cracks and voids on the micro-level of a solid material. Similarly, the delamination of a bonded structure, which macroscopically can be noticed as crack growth along an interface, can be understood as a process on the micro-level of the material, namely the damage of the adhesive.

Subsections 2.2.1 and 2.2.2 are concerned with the mechanical modeling of damage and delamination by means of so-called continuum damage mechanics.

2.2.1 Continuum Damage Mechanics

Continuum damage mechanics goes back on L.M. Kachanov in 1958 [Kac58, Kac60, Kac90] and Yu.N. Rabotnov in 1969 [Rab69], since they suggested to model the phenomena of damage with the aid of an internal variable. They used this ansatz to describe the creep damage of metals, which occurs at high temperatures and under action of stresses. This phenomenon is characterized by the accumulation and growth of both micro-voids in grains (ductile transgranular creep fracture) and micro-cracks on intergranular boundaries (brittle intergranular creep fracture) [Kac90]. For this type of damage the inner variable is linked with the irreversible creep strain. The original model for creep damage contains two coupled flow rules for these two variables and the creep strain enters in the constitutive law to model the influence of creep damage on the material behavior.

In [Kac90, Cha88, LD05] their approach is sketched for a simpler relation. The main idea is to introduce a scalar damage variable d, which is incorporated to the constitutive law (2.31), where it models the influence of damage on the elastic behavior of the material:

$$\sigma^{\mathrm{eff}} = \frac{\sigma}{1-\mathrm{d}} = \frac{1}{1-\mathrm{d}}\,\mathbb{B} : e\,. \tag{2.37}$$

This ansatz is also called the concept of effective stresses and strain equivalence, since it is assumed that "a damaged volume of material under the applied stress σ shows the same strain response as the undamaged one submitted to the effective stress σ^{eff}" [Cha88], which is the stress acting on the resisting area [LD05].

In other words, the damage variable is introduced to describe the change of material response due to defects on the micro-scale of a material, so that it can be analyzed by means of continuum mechanics on a larger scale. To specify these scales engineers use a representative volume element (RVE), which has to be chosen of such a size that all the characteristic ingredients of a material are contained, see Tab. 2.1. Thus, on any scale larger or equal than the size of the RVE, the material is considered to be homogeneous.

Table 2.1: Orders of magnitude of RVEs [LC90] p. 71

Materials	Type of Inhomogeneities	RVE
Metals, alloys, ceramics	Crystals 1μm-0.1mm	(0.5mm)^3
Polymers	Molecules 10μm-0.05mm	(1mm)^3
Wood	Fibres 0.1mm-1mm	(1cm)^3
Concrete	Granulates ca. 1cm	(10cm)^3

The damage discontinuities, i.e. the microvoids and -cracks, are considered to be small with respect to the size of the RVE. In the works [Kac90, LD05] this was used to introduce the scalar damage variable d as the surface density of micro-cracks and intersections of micro-voids lying on a plane cutting the RVE of cross section S_0, i.e.

$$\mathrm{d} = \frac{S_{\mathrm{d}}}{S_0}\,, \tag{2.38}$$

where S_{d} is the surface density of the cracks and voids in the RVE, which cut the plane. Since d is independent of the choice of the plane, it is defined under the assumption that all the defects are distributed uniformly in the material. This is called isotropic damage. If d depends on the choice of the plane, one speaks of anisotropic damage. In more general cases of anisotropic damage a scalar damage variable is not sufficient and vector or tensor valued damage variables have to be used, see e.g. [Mur88, LDS00]. The evolution of the damage variable is described by an evolution equation, sometimes also called flow rule,

$$\dot{\mathrm{d}} = F(t, \mathrm{d}, e)\,. \tag{2.39}$$

This thesis focusses on isotropic damage and uses a scalar, local damage variable

$$z : [0, T] \times \Omega \to [0, 1]\,, \tag{2.40}$$

which for all $(t, x) \in [0, T] \times \Omega$ can be defined as the volume fraction of undamaged material in a RVE with center in x. Hence $z(t, x) = 1$ means that the RVE around x is totally undamaged at time t, whereas $z(t, x) = 0$ stands for the complete damage of the RVE, i.e. all the material in the RVE is disintegrated. This ansatz has been used in [FN96, Fré02, MR06] with application to concrete. As in (2.37) z is incorporated to the constitutive law so that an energy density of Section 2.1.3 changes to

$$\overline{W}(z, e, \nabla z) = f_1(z)\bar{W}(e) + f_2(z) + f_3(\nabla z)\,. \tag{2.41}$$

Thus, the energy functional from (2.35) changes to

$$\mathcal{E}(t, u(t), z(t)) = \int_\Omega \overline{W}(e(u(t, x)), z(t, x), \nabla z(t, x))\,\mathrm{d}x - \int_\Omega f(t, x)\cdot u(t, x)\,\mathrm{d}x - \int_{\Gamma_N} h(t, s)\cdot u(t, s)\,\mathrm{d}s \tag{2.42}$$

for external loadings $f : [0, T] \times \Omega \to \mathbb{R}^d$ and $h : [0, T] \times \Gamma_N \to \mathbb{R}^d$.

The damage gradient ∇z takes into account microscopic interactions. Its use goes back on [FN96] and it has also found acceptance in many engineering works such as [HSS03, LB05], since it works well in numerical simulations [FN96]. A suitable evolution law for z will be discussed in Example 2.3.9 and Chapter 3. It will model damage as a unidirectional process, i.e. it prevents healing by ensuring that $\partial_t z(t, x) \le 0$ for a.e. $(t, x) \in [0, T] \times \Omega$.

2.2.2 The Modeling of Delamination

On the microscale, delamination (or debonding) is one main reason for the macroscopic failure of compounds besides transverse matrix micro-cracking and fiber breakage. Opposite, sometimes delamination is an intentional mechanism in engineering constructions designed for efficient absorption of energy during impacts. The study of delamination is very important for practice since the global behavior of composite materials often is strongly influenced by the quality of the adhesion between the different components when material degradation occurs. Thus, the reliable modeling of delamination has recently received a considerable attention both in engineering and in mathematical communities.

On the macro-level of compounds the evolution of delamination can be noticed as the propagation of cracks along the interfaces between the different components [Kac88]. The behavior of such macro-cracks can be analyzed by means of fracture mechanics. Within this theory one main tool is Griffith' fracture criterion (1.1), which uses the energy release rate as the decisive quantity to predict whether a pre-existing crack will grow under prescribed loadings. The quasistatic evolution of such a Griffith-crack is then described by the following conditions [BFM08]:

- a crack is only allowed to grow, not to heal,
- the energy release rate $ERR(l) := -\frac{\mathrm{d}\mathcal{E}(\Omega_{l+\tilde{l}})}{\mathrm{d}\tilde{l}}\Big|_{\tilde{l}=0}$ is always bounded from above by the fracture toughness $\mathcal{G}_c$ of the material:
$$ERR(l) \leq \mathcal{G}_c\,,$$
- the crack cannot grow unless the energy release rate is critical:
$$(ERR(l) - \mathcal{G}_c)\dot{l}(t) = 0\,.$$

Here $\mathcal{E}(\Omega_l)$ denotes the sum of the energy due to the external loadings and the strain energy of the cracked body with reference configuration Ω_l and crack-length l and $\Omega_{l+\tilde{l}} \subset \Omega_l$. This model has been widely analyzed both at small and finite strains under consideration of various types of materials [DT02, DFT05, BFM08, DMGP09].

In contrast to the macroscopic approach by fracture mechanics, many engineering contributions view delamination as a process occurring on the meso- or micro-level of a compound [Lad92, DBS02, AC96, All02]. In this context delamination is interpreted as the damage of interfaces and the ideas of continuum damage mechanics are applied. This means that the delamination along an interface $\Gamma_{\mathrm{C}} \subset \mathbb{R}^{d-1}$ between two constituents located in the domains Ω_-, $\Omega_+ \subset \mathbb{R}^d$ is modelled by an inner variable, the delamination variable

$$z : [0,T] \times \Gamma_{\mathrm{C}} \to [0,1]\,, \tag{2.43}$$

which reflects the current state of the bonding along the interface, i.e. $z(t,x) = 1$ means that the bonding is fully intact at $x \subset \Gamma_{\mathrm{C}}$ at time $t \in [0,T]$, whereas $z(t,x) = 0$ expresses that the bonding is completely broken. This ansatz was introduced in [Fré88] to model contact with adhesion between a body and a support. In this work z is the ratio of active bonds, called adhesion intensity. There are two possible types of contact: For $z(t,x) > 0$ the two structures are bonded at x on the interface Γ_{C}, which must be expressed by transmission conditions for the displacement, whereas for $z(t,x) = 0$ the two structures only touch each other at x, so that their displacements are independent form each other, i.e. in the first case displacement jumps across Γ_{C} are interdicted, whereas the second case allows for jumps. In both cases interpenetration of the structures is impossible. These conditions are expressed by the following constraints, which couple the delamination variable z and the displacements $u_- := u|_{\Omega_-}$, $u_+ := u|_{\Omega_+}$ of the two structures with each other:

$$z(u_+ - u_-) = 0 \quad \text{a.e. on } \Gamma_{\mathrm{C}}, \tag{2.44}$$
$$(u_+ - u_-) \cdot \mathrm{n} \geq 0 \quad \text{a.e. on } \Gamma_{\mathrm{C}}\,, \tag{2.45}$$

where n denotes the unit normal vector along Γ_{C}.

As in Section 2.2.1 the energy functional $\mathcal{E}$ depends on the delamination variable, see (2.42), and its evolution can be described by a flow rule like (2.39).

In Chapter 4 the micro-mechanical approach by formulas (2.43)-(2.45) will be used to model the delamination of a compound.

2.3 Different Mathematical Concepts

In this section three different mathematical formulations for quasistatic, rate-independent processes are introduced and their relations are clarified. Additionally their relation to the formulation via the force balance (2.17) and the flow rule (2.39), which is usual in engineering, is explained. As it was already mentioned, a process is called quasistatic, if kinetic effects are negligible, so that it is given by a chronology of static equilibria. This is the case, if the process is rate-independent, which means that the time-scales imposed to the system from the exterior are much larger than the intrinsic ones, i.e. if the external loadings evolve much slower than the internal variables. The rate-independence of a system along the time interval $[s, T]$ with the initial condition $q(s) = q_0 \in \mathcal{Q}$, the given external loadings $l \in \mathrm{C}^1([s, T], \mathcal{Q}^*)$ and the solution process $q : [s, T] \to \mathcal{Q}$ can be defined using an input-output operator

$$\mathcal{H}_{[s,T]} : \mathcal{Q} \times \mathrm{C}^1([s, T], \mathcal{Q}^*) \to B([s, T], \mathcal{Q}),\ (q_0, l) \mapsto q\,, \tag{2.46}$$

i.e. $\mathcal{H}_{[s,T]}$ maps the given data onto a solution of the problem. Here $\mathcal{Q}$ is the state space, which is assumed to be a Banach space, and $\mathcal{Q}^*$ is its dual space. Moreover, $B([s, T], \mathcal{Q})$ denotes the space of functions $\tilde{q} : [s, T] \to \mathcal{Q}$ which are measurable, bounded and defined everywhere on $[s, T]$. Here, the external loading $l \in \mathrm{C}^1([s, T], \mathcal{Q}^*)$ comprises both volume and surface forces. The rate-independence of a system can be characterized as follows:

Definition 2.3.1 (Abstract definition of rate-independence) *A system, which can be expressed by* (2.46)*, is called rate-independent if for all $s_\star < T_\star$ and all $\alpha \in \mathrm{C}^1([s_\star, T_\star])$ with $\dot{\alpha} > 0$ and $\alpha(s_\star) = s$, $\alpha(T_\star) = T$ the following holds:*

$$\mathcal{H}_{[s_\star,T_\star]}(q_0, l \circ \alpha) = \mathcal{H}_{[s,T]}(q_0, l) \circ \alpha\,. \tag{2.47}$$

This means that reparametrizations of the time-scale do not change the behavior of the system, since a solution is also just reparametrized in time. If for example the external loadings act twice as fast, i.e. $\dot{\alpha} = 2$, then a solution of the rate-independent problem responds also twice as fast.

Unlike [BS96], where hysteresis operators are used to model a rate-independent process, we focus on the so-called energetic formulation of the process. Within this approach a rate-independent process is characterized by an energy functional $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$, $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, and a dissipation potential $\mathcal{R} : \mathcal{Q} \to [0, \infty]$. The first specifies the energy of the system due to strain energy and the potential energy of the external loadings and the latter accounts for the energy dissipated when changing the system from one state into another. From now on a rate-independent system will be characterized by the triple $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$.

To reflect rate-independence one assumes that the dissipation potential

$$\mathcal{R} : \mathcal{Q} \to [0, \infty] \text{ is convex, positively 1-homogeneous and lower semi-continuous.} \tag{2.48}$$

Convexity of $\mathcal{R}$ is defined by $\mathcal{R}(\alpha v_1 + (1-\alpha)v_0) \leq \alpha\mathcal{R}(v_1) + (1-\alpha)\mathcal{R}(v_0)$ for all $\alpha \in [0, 1]$, all $v_0, v_1 \in \mathcal{Q}$ and the positive 1-homogeneity of $\mathcal{R}$ means that $\mathcal{R}(0) = 0$ and $\mathcal{R}(\alpha v) = \alpha\mathcal{R}(v)$ for all $\alpha > 0$, $v \in \mathcal{Q}$.

Due to convexity and positive 1-homogeneity claimed in (2.48) the dissipation potential satisfies a triangle inequality, i.e. for all $q_1, q_2, q_3 \in \mathcal{Q}$ it holds

$$\begin{aligned}\mathcal{R}(q_1-q_2) = 2\mathcal{R}(\tfrac{1}{2}(q_1-q_3) + \tfrac{1}{2}(q_3-q_2)) &\leq 2\big(\tfrac{1}{2}\mathcal{R}(q_1-q_3) + \tfrac{1}{2}\mathcal{R}(q_3-q_2)\big)\\ &= \mathcal{R}(q_1-q_3) + \mathcal{R}(q_3-q_2)\,.\end{aligned}$$

Hence the dissipation potential generates a dissipation distance

$$\mathcal{D}(q, \tilde{q}) = \mathcal{R}(\tilde{q} - q)\,, \tag{2.49}$$

which is an extended pseudo-distance on the state space $\mathcal{Q}$. This means that $\mathcal{D}$ satisfies the axioms of a metric (positivity, triangle inequality), except symmetry and it may attain the value ∞, as it will be the case in the Chapters 3 and 4.

With these properties at hand we may now introduce the mathematical formulation of the rate-independent evolutionary problem, which will be used throughout this thesis:

Definition 2.3.2 (Energetic formulation) *For the initial datum $q_0 \in \mathcal{Q}$ find $q : [s, T] \to \mathcal{Q}$ such that for all $t \in [s, T]$ the global stability* (S) *and the global energy balance* (E) *hold*

$$\text{Stability}: \quad \textit{for all } \tilde{q} \in \mathcal{Q}: \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{R}(\tilde{q}-q(t)), \tag{S}$$

$$\text{Energy balance}: \quad \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(q, [s, t]) = \mathcal{E}(s, q(s)) + \int_s^t \partial_\xi \mathcal{E}(\xi, q(\xi))\,\mathrm{d}\xi \tag{E}$$

with $\mathrm{Diss}_{\mathcal{R}}(q, [s, t]) := \sup\big\{ \sum_{j=1}^N \mathcal{R}(q(\xi_j)-q(\xi_{j-1})) \,|\, s = \xi_0 < \ldots < \xi_N = t,\ N \in \mathbb{N}\big\}$.

The claim that (S) & (E) have to hold for all $t \in [s, T]$ entails that the energetic formulation is only solvable for initial data q_0 which satisfy (S) for $t = s$. A solution in terms of the energetic formulation is called an energetic solution.

In general the dissipation potential is given as an integral over a bounded domain $\Omega \subset \mathbb{R}^d$ representing the reference configuration of the body experiencing the rate-independent process, i.e. with the convex, positively 1-homogeneous, lower semicontinuous density $R : \mathbb{R}^m \to [0, \infty]$ (for some $m \in \mathbb{N}$) it holds $\mathcal{R}(v) = \int_\Omega R(v(x))\,\mathrm{d}x$. If additionally there are constants $c, \tilde{c}$, such that for all $v \in \mathbb{R}^m$ with $R(v) < \infty$ it holds $c|v| \leq R(v) \leq \tilde{c}|v|$, then any function $q : [s, T] \to \mathcal{Q}$ with $\mathrm{Diss}_{\mathcal{R}}(q, [s, T]) < \infty$ satisfies

$$c\,\mathrm{Var}_{L^1}(q, [s, t]) \leq \mathrm{Diss}_{\mathcal{R}}(q, [s, t]) \leq \tilde{c}\,\mathrm{Var}_{L^1}(q, [s, t]) \quad \text{for all } t \in [s, T] \tag{2.50}$$

with $\mathrm{Var}_{L^1}(q, [s, t]) := \sup\{\sum_{j=1}^N \|q(\xi_j)-q(\xi_{j-1})\|_{L^1(\Omega,\mathbb{R}^m)} \,|\, s = \xi_0 < \xi_1 < \ldots < \xi_N = t,\ N \in \mathbb{N}\}$.

As we have seen in Sections 2.1.4 and 2.2 the energy functional $\mathcal{E}(t,\cdot)$ consists of the strain energy functional, which we denote by $\mathcal{I}(\cdot)$, and a term comprising the potential energy due to the external loadings, denoted by $\langle l(t), q\rangle$, so that $\mathcal{E}(t,q)=\mathcal{I}(q)-\langle l(t),q\rangle$.

Coming back to the input-output operator from (2.46) we can specify in view of Definition 2.3.2 that $\mathcal{H}_{[s,T]}:\mathcal{Q}\times\mathrm{C}^1([s,T],\mathcal{Q}^*)\to B([s,T],\mathcal{Q})$ maps the given data (q_0,l) onto an energetic solution $q:[s,T]\to\mathcal{Q}$.

In the following we show that the positive 1-homogeneity of $\mathcal{R}$ indeed implies the rate-independence of $(\mathcal{Q},\mathcal{E},\mathcal{R})$, i.e. (2.47) holds. For this, we prove that the energy dissipated during $[s,t]$ can be written as an integral provided $q:[s,T]\to\mathcal{Q}$ is sufficiently smooth:

Proposition 2.3.3 *Let $q\in W^{1,1}([s,T],\mathcal{Q})$ with $\mathrm{Diss}_{\mathcal{R}}(q,[s,T])<\infty$. Let (2.48) and (2.50) hold. Then*

$$\mathrm{Diss}_{\mathcal{R}}(q,[s,t])=\int_s^t \mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi \quad \textit{for all } t\in[s,T]\,. \tag{2.51}$$

Proof: It is $\mathrm{Var}_{L^1}(q,[s,t])=\int_s^t\|\dot q(\xi)\|_{L^1(\Omega,\mathbb{R}^m)}\,\mathrm{d}\xi$ for $q\in W^{1,1}([s,T],\mathcal{Q})$ with finite dissipation. By definition, the Lebesgue-integral $\int_s^t\|\dot q(\xi)\|_{L^1(\Omega,\mathbb{R}^m)}\,\mathrm{d}\xi$ is the supremum over the integrals of all positive, simple functions of the form $\sum_{j=1}^N\|\dot q(s_j)\|_{L^1(\Omega,\mathbb{R}^m)}I_{[\xi_{j-1},\xi_j]}(\xi)$ satisfying $\sum_{j=1}^N\|\dot q(s_j)\|_{L^1(\Omega,\mathbb{R}^m)}I_{[\xi_{j-1},\xi_j]}(\xi)\le\|\dot q(\xi)\|_{L^1(\Omega,\mathbb{R}^m)}$ for a.e. $\xi\in[s,t]$. Here, $s_j\in[\xi_{j-1},\xi_j]$ are suitable nodes, $s=\xi_0<\xi_1<\ldots<\xi_N=t$ for $N\in\mathbb{N}$ are partitions of $[s,t]$ and $I_{[\xi_{j-1},\xi_j]}(\xi)\in\{0,1\}$ is the indicator function of $[\xi_{j-1},\xi_j]$. The positive 1-homogeneity of $\mathcal{R}$ implies that $\sum_{j=1}^N\mathcal{R}(\dot q(s_j))I_{[\xi_{j-1},\xi_j]}(\xi)\le\mathcal{R}(\dot q(\xi))$ a.e. for these simple functions and hence $\int_s^t\mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi=\sup\big\{\sum_{j=1}^N\mathcal{R}(\dot q(s_j))(\xi_j-\xi_{j-1})\,|\,N\in\mathbb{N}\big\}$. Thus, $\int_s^t\mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi\le\mathrm{Diss}_{\mathcal{R}}(q,[s,t])$. On the other hand the convexity of $\mathcal{R}$ and Jensen's inequality yield

$$\sum_{j=1}^N\mathcal{R}(q(\xi_j)-q(\xi_{j-1}))=\sum_{j=1}^N\mathcal{R}\Big(\int_{\xi_{j-1}}^{\xi_j}\dot q(\xi)\Big)\,\mathrm{d}\xi\le\sum_{j=1}^N\int_{\xi_{j-1}}^{\xi_j}\mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi=\int_s^t\mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi$$

for any partition. Thus, $\mathrm{Diss}_{\mathcal{R}}(q,[s,t])\le\int_s^t\mathcal{R}(\dot q(\xi))\,\mathrm{d}\xi$ and the statement is proven. ■

Conditions (S) & (E) yield that an energetic solution satisfies $\mathrm{Diss}_{\mathcal{R}}(q,[s,T])<\infty$, so that $q\in BV([s,T],L^1(\Omega,\mathbb{R}^m))$ by (2.50).

In view of [AFP05, Th. 3.9] we may state the following approximation result:

Proposition 2.3.4 *Let $q\in BV([s,T],L^1(\Omega,\mathbb{R}^m))$ with $\mathrm{Diss}_{\mathcal{R}}(q,[s,T])<\infty$. Then there is a sequence $(q_n)_{n\in\mathbb{N}}\subset\mathrm{C}^\infty([s,T]\times\Omega,\mathbb{R}^m)$ so that $q_n\to q$ in $L^1([s,T]\times\Omega,\mathbb{R}^m)$ as well as $\mathrm{Diss}_{\mathcal{R}}(q_n,[s,t])\le C$ for all $t\in[s,T]$. In particular, for all $t\in[s,T]$ it holds that $\mathrm{Diss}_{\mathcal{R}}(q_n,[s,t])\to\mathrm{Diss}_{\mathcal{R}}(q,[s,t])$.*

This is used to verify that relation (2.47) holds true.

Proposition 2.3.5 *Let $\mathcal{H}_{[s,T]}:\mathcal{Q}\times\mathrm{C}^1([s,T],\mathcal{Q}^*)\to B([s,T],\mathcal{Q})\cap BV([s,T],L^1(\Omega,\mathbb{R}^m))$, $(q_0,l)\mapsto q$, be the input-output-operator for the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{R})$, where $\mathcal{E}(t,q)=\mathcal{I}(q)-\langle l(t),q\rangle$ for all $(t,q)\in[0,T]\times\mathcal{Q}$ and where $\mathcal{R}$ is convex and positively 1-homogeneous. Then (2.47) holds true.*

Proof: To indicate the dependence of $\mathcal{E}$ on the external loading l we write $\mathcal{E}_l$ in the following.

Let $s_\star < T_\star$ and $\alpha \in \mathrm{C}^1([s_\star, T_\star])$ with $\dot{\alpha} > 0$ and $\alpha(s_\star) = s$, $\alpha(T_\star) = T$. In particular it holds $s_\star = \alpha^{-1}(s)$, $T_\star = \alpha^{-1}(T)$ and $(\alpha^{-1})' > 0$. Assume that $q : [s,T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_l, \mathcal{R})$ satisfying $q(s) = q_0$. Hence (S) & (E) are satisfied for all $t \in [s,T]$. Now the time interval is rescaled, i.e. $t = \alpha(t_\star)$ for all $t \in [s,T]$. Then (S) implies that $\mathcal{I}(q\circ\alpha(t_\star)) - \langle l\circ\alpha(t_\star), q\circ\alpha(t_\star)\rangle \le \mathcal{I}(\tilde{q}) - \langle l\circ\alpha(t_\star), \tilde{q}\rangle + \mathcal{R}(\tilde{q} - q\circ\alpha(t_\star))$ for all $\tilde{q} \in \mathcal{Q}$, i.e. (S) holds true for all $t_\star \in [s_\star, T_\star]$ for $q\circ\alpha : [s_\star, T_\star] \to \mathcal{Q}$ and the system $(\mathcal{Q}, \mathcal{E}_{l\circ\alpha}, \mathcal{R})$.

Due to Proposition 2.3.4 there is a sequence $(q_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty([s,T]\times\Omega, \mathbb{R}^m)$ with $q_n \to q$ in $L^1([0,T]\times\Omega, \mathbb{R}^m)$, such that $\mathrm{Diss}_{\mathcal{R}}(q_n, [s,t]) \to \mathrm{Diss}_{\mathcal{R}}(q, [s,t]) < \infty$ for all $t \in [s,T]$.

Then, for $s = \alpha(s_\star)$ and $t = \alpha(t_\star)$, using Proposition 2.3.3 and the chain rule on $q_n(\alpha(t_\star))$ together with the positive 1-homogeneity of $\mathcal{R}$ implies that

$$\int_s^t \mathcal{R}(\dot{q}_n(\xi))\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \mathcal{R}(\partial_\alpha q_n(\alpha(\xi)))\dot{\alpha}(\xi)\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \mathcal{R}(\partial_\alpha q_n(\alpha(\xi))\dot{\alpha}(\xi))\,\mathrm{d}\xi = \int_{s_\star}^{t_\star} \mathcal{R}(\partial_\xi q_n\circ\alpha(\xi))\,\mathrm{d}\xi,$$

which proves that $\mathrm{Diss}_{\mathcal{R}}(q_n, [s,t]) = \mathrm{Diss}_{\mathcal{R}}(q_n\circ\alpha, [s_\star, t_\star])$ for all $n \in \mathbb{N}$ and due to the convergence also $\mathrm{Diss}_{\mathcal{R}}(q, [s,t]) = \mathrm{Diss}_{\mathcal{R}}(q\circ\alpha, [s_\star, t_\star])$.

Again by the chain rule we calculate that

$$\int_s^t \partial_\xi \mathcal{E}_l(\xi, q(\xi))\,\mathrm{d}\xi = -\int_{s_\star}^{t_\star} \langle \partial_\alpha l\circ\alpha(\xi), q(\xi)\rangle \dot{\alpha}(\xi)\,\mathrm{d}\xi = -\int_{s_\star}^{t_\star} \langle \partial_\xi l\circ\alpha(\xi), q(\xi)\rangle\,\mathrm{d}\xi$$

and hence (E) is verified for all $t_\star \in [s_\star, T_\star]$ for $q\circ\alpha$ and $(\mathcal{Q}, \mathcal{E}_{l\circ\alpha}, \mathcal{R})$. Moreover the initial condition is satisfied since $q_0 = q(s) = q\circ\alpha(s_\star)$.

With the same arguments we can verify for an energetic solution $q_\star : [s_\star, T_\star] \to \mathcal{Q}$ of $(\mathcal{Q}, \mathcal{E}_{l\circ\alpha}, \mathcal{R})$ with $q_\star(s_\star) = q_0$ that $q_\star\circ\alpha^{-1}$ satisfies (S) & (E) with $(\mathcal{Q}, \mathcal{E}_l, \mathcal{R})$ for all $t \in [s,T]$ and with $q_0 = q_\star(s_\star) = q_\star\circ\alpha^{-1}(s)$. Thus, (2.47) is proven. ■

In the following we discuss the relation of the energetic formulation to other mathematical concepts.

A mathematical formulation of a rate-independent process, which is widely used both in mathematical and engineering communities, involves the subdifferential of the dissipation potential (2.53) and the Gâteaux-derivative of the energy functional. Thereby $\mathcal{E} : [0,T]\times\mathcal{Q}$ is called Gâteaux-differentiable, if for all $t \in [0,T]$, all $q, \tilde{q} \in \mathcal{Q}$ the following limit exists

$$\mathrm{D}_q\mathcal{E}(t,q)[\tilde{q}] = \lim_{\alpha\to 0} \frac{\mathcal{E}(t, q+\alpha\tilde{q}) - \mathcal{E}(t,q)}{\alpha} \tag{2.52}$$

and the linear functional $\mathrm{D}_q\mathcal{E}(t,q) \in \mathcal{Q}^*$ is called the Gâteaux-derivative of $\mathcal{E}(t,\cdot)$ in $q \in \mathcal{Q}$. In contrast to the energy functional $\mathcal{E}$ the dissipation potential $\mathcal{R} : \mathcal{Q} \to [0,\infty]$ not necessarily has to be Gâteaux-differentiable. The subdifferential is defined as follows

$$\partial_v\mathcal{R}(v) := \{q^* \in \mathcal{Q}^* \mid \mathcal{R}(w) \ge \mathcal{R}(v) + \langle q^*, w-v\rangle \text{ for all } w \in \mathcal{Q}\}\,. \tag{2.53}$$

Thus, the subdifferential formulation of the evolutionary problem reads as follows.

Definition 2.3.6 (Subdifferential formulation) *For the given initial datum $q_0 \in \mathcal{Q}$ find $q : [0,T] \to \mathcal{Q}$ such that for a.e. $t \in [0,T]$ it holds*

$$0 \in \partial\mathcal{R}(\dot{q}(t)) + \mathrm{D}_q\mathcal{E}(t, q(t)) \subset \mathcal{Q}^* \quad \text{and } q(0) = q_0 \in \mathcal{Q}\,. \tag{SDF}$$

Moreover (SDF) is equivalent to $-\mathrm{D}_q\mathcal{E}(t,q) \in \partial\mathcal{R}(\dot{q})$ and by exploiting the definition of the subdifferential (2.53) we may equivalently formulate the rate-independent evolutionary problem in terms of a variational inequality.

Definition 2.3.7 (Variational inequality) *For the initial datum $q_0 \in \mathcal{Q}$ find $q : [0,T] \to \mathcal{Q}$ such that for a.e. $t \in [0,T]$ and for all $v \in \mathcal{Q}$ it holds*

$$\langle \mathrm{D}_q\mathcal{E}(t,q), v - \dot{q}\rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \geq 0 \quad \text{and } q(0) = q_0 \in \mathcal{Q}\,. \tag{VI}$$

The next result states the relations between the formulations (S) & (E), (SDF) and (VI).

Proposition 2.3.8 *Let $\mathcal{Q}$ be a Banach space and $q_0 \in \mathcal{Q}$. Assume that $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ is Gâteaux-differentiable on $\mathcal{Q}$ for all $t \in [0,T]$ and continuous on $[0,T] \times \mathcal{Q}$. Moreover, let $\mathcal{R} : \mathcal{Q} \to [0,\infty]$ satisfy* (2.48) *as well as* (2.50). *Assume that for a solution $q : [0,T] \to \mathcal{Q}$ of* (S) & (E) *holds $q \in W^{1,1}([0,T], \mathcal{Q})$. Then q is also a solution in the sense of* (SDF) *and* (VI). *Furthermore, if $\mathcal{E}(t,\cdot)$ is convex on $\mathcal{Q}$ for all $t \in [0,T]$ and if q_0 satisfies* (S) *at $t = 0$, then also the reverse implication holds true.*

Proof: Due to the assumption $q \in W^{1,1}([0,T], \mathcal{Q})$ and Proposition 2.3.3 the energy balance (E) can be written as $\mathcal{E}(t, q(t)) + \int_0^t \mathcal{R}(\dot{q}(\xi))\,\mathrm{d}\xi = \mathcal{E}(0, q(0)) + \int_0^t \partial_t\mathcal{E}(\xi, q(\xi))\,\mathrm{d}\xi$. Application of $\frac{\mathrm{d}}{\mathrm{d}t}$ to (E) leads to $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) + \mathcal{R}(\dot{q}(t)) = \partial_t\mathcal{E}(t, q(t))$ for almost all $t \in [0,T]$. This yields

$$\langle \mathrm{D}_q\mathcal{E}(t, q(t)), \dot{q}(t)\rangle + \mathcal{R}(\dot{q}(t)) = 0\,. \tag{$\mathrm{E}_{\mathrm{loc}}$}$$

Furthermore, inserting $q(t) + hv$ for $v \in \mathcal{Q}$ in (S) results in

$$\langle \mathrm{D}_q\mathcal{E}(t, q(t)), v\rangle + \mathcal{R}(v) \geq 0 \quad \text{for all } v \in \mathcal{Q} \tag{$\mathrm{S}_{\mathrm{loc}}$}$$

due to the positive 1-homogeneity of $\mathcal{R}$ and the Gâteaux-differentiability of $\mathcal{E}(t,\cdot)$. Subtracting ($\mathrm{E}_{\mathrm{loc}}$) from ($\mathrm{S}_{\mathrm{loc}}$) finally yields (VI), which is equivalent to (SDF).

Assume now that q solves (VI) as well as (SDF) for a.e. $t \in [0,T]$ and additionally that $\mathcal{E}(t,\cdot)$ is convex for all $t \in [0,T]$. Multiply (VI) by $h > 0$ and put $v = \frac{\tilde{q}}{h}$. For $h \to 0$ one obtains ($\mathrm{S}_{\mathrm{loc}}$). Due to the convexity and the Gâteaux-differentiability of $\mathcal{E}(t,\cdot)$ for all $q \in \mathcal{Q}$ we find from ($\mathrm{S}_{\mathrm{loc}}$) with $v = \tilde{q} - q(t)$ that for a.e. $t \in [0,T]$ it holds $0 \leq \langle \mathrm{D}_q\mathcal{E}(t,q(t)), \tilde{q} - q(t)\rangle + \mathcal{R}(\tilde{q} - q(t)) \leq \mathcal{E}(t,\tilde{q}) - \mathcal{E}(t, q(t)) + \mathcal{R}(\tilde{q} - q(t))$. Now (E) has to be proven. Choosing $v = \dot{q}(t)$ in ($\mathrm{S}_{\mathrm{loc}}$) gives $\langle \mathrm{D}_q\mathcal{E}(t,q(t)), \dot{q}(t)\rangle + \mathcal{R}(\dot{q}(t)) \geq 0$ and $v = 0$ in (VI) yields $\langle \mathrm{D}_q\mathcal{E}(t,q(t)), -\dot{q}(t)\rangle - \mathcal{R}(\dot{q}(t)) \geq 0$, which proves ($\mathrm{E}_{\mathrm{loc}}$). By integrating ($\mathrm{E}_{\mathrm{loc}}$) over $[0,t]$ we verify that (E) holds for all $t \in [0,T]$.

We finally have to make sure that (S) holds for all, not only for a.e. $t \in [0,T]$. For this we verify the closedness of the stable sets, i.e. if $t_j \to t$ in $[0,T]$, $q(t_j) \to q(t)$ in $\mathcal{Q}$ and if $q(t_j)$ satisfies (S) at t_j, $j \in \mathbb{N}$, then also $q(t)$ satisfies (S) at t. W.l.o.g. assume that $t_j > t_{j-1}$ for all $t \in [0,T]$, since $q(0) = q_0$ satisfies (S) in $t = 0$ by assumption. By continuity of $\mathcal{E}$ on $[0,T] \times \mathcal{Q}$ we immediately know that $\mathcal{E}(t_j, q(t_j)) \to \mathcal{E}(t, q(t))$. Choose now $\tilde{q} \in \mathcal{Q}$. Then we have to find a sequence $(\tilde{q}_j)_{j\in\mathbb{N}}$ satisfying $\mathcal{E}(t_j, \tilde{q}_j)+\mathcal{R}(\tilde{q}_j-q(t_j)) \to \mathcal{E}(t,\tilde{q})+\mathcal{R}(\tilde{q}-q(t))$. A good candidate is $\tilde{q}_j = \tilde{q}+(q(t_j) - q(t))$, since $\tilde{q}_j \to \tilde{q}$ in $\mathcal{Q}$ by this construction, so that $\mathcal{E}(t_j, \tilde{q}_j) \to \mathcal{E}(t, \tilde{q})$ by continuity and $\mathcal{R}$ is constant along this sequence. This implies that (S) holds for all $t \in [0,T]$. ∎

As it can be seen from Section 2.2 in the setting of damage and delamination the states $q = (u,z) \in \mathcal{Q}$ are given by the displacement field $u \in \mathcal{U}$ and the internal (damage or delamination) variable $z \in \mathcal{Z}$. In this framework it is assumed that the dissipation potential only depends on the inner variable z and not on the full state $q = (u,z)$. Then $\partial\mathcal{R}(\dot{q}) = \partial_{\dot{u}}\widetilde{\mathcal{R}}(\dot{z}) \times \partial_{\dot{z}}\widetilde{\mathcal{R}}(\dot{z}) = \{0\} \times \partial\widetilde{\mathcal{R}}(\dot{z})$ and from now on $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ will denote a dissipation potential which only depends on the inner variable.

For such a potential $\mathcal{R}$ the subdifferential formulation (SDF) may be rewritten as

$$\mathcal{U}^* \ni \quad \mathrm{D}_u\mathcal{E}(t,q(t)) = 0\,, \tag{2.54}$$
$$\mathcal{Z}^* \ni -\mathrm{D}_z\mathcal{E}(t,q(t)) \in \partial\mathcal{R}(\dot{q}(t))\,, \tag{2.55}$$

where $\mathcal{U}^*$ and $\mathcal{Z}^*$ are the duals of the Banach spaces $\mathcal{U}$ and $\mathcal{Z}$.

If $\mathcal{E}(t,u,z) = \mathcal{I}(u,z) - \langle l(t), u\rangle$ and if $\mathcal{I} : \mathcal{Q} \to \mathbb{R}_\infty$ is the integral functional from (2.42), i.e. $\mathcal{I}(u,z) = \int_\Omega \overline{W}(e(u), z, \nabla z)\,\mathrm{d}x$, then (2.54) means in particular that

$$\int_\Omega \partial_e\overline{W}(e(u), z, \nabla z) : e(\tilde{u})\,\mathrm{d}x = \langle l(t), \tilde{u}\rangle \quad \text{for all } \tilde{u} \in \{v\in\mathcal{U} \,|\, v = 0 \text{ on } \partial\Omega\}\,, \tag{2.56}$$

which is the weak formulation of the force balance of a hyperelastic material, see (2.17).

Using the duality theory of functionals one can establish a relation between (2.55) and the flow rule (2.39) under the assumption that $\mathcal{Z}$ is a reflexive Banach space. In view of the definition of the subdifferential

$$\partial\mathcal{R}(z) = \{z^* \in \mathcal{Z}^* \,|\, \mathcal{R}(\tilde{z})-\mathcal{R}(z) \geq \langle z^*, \tilde{z}-z\rangle \text{ for all } \tilde{z} \in \mathcal{Z}\} \tag{2.57}$$

the direct calculation of the Legendre-Fenchel transform of the positively 1-homogeneous dissipation potential $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ yields that its dual functional is given as the indicator function of $\partial\mathcal{R}(0)$, i.e.

$$\mathcal{R}^*(z^*) = \sup_{z\in\mathcal{Z}} \big(\langle z^*, z\rangle - \mathcal{R}(z)\big) = I_{\partial\mathcal{R}(0)}(z^*) \text{ for all } z^* \in \mathcal{Z}^* \text{ with } I_{\partial\mathcal{R}(0)}(z^*) = \begin{cases} 0 & \text{if } z \in \partial\mathcal{R}(0)\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Since $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ is assumed to be convex and lower semicontinuous on the reflexive Banach space $\mathcal{Z}$ the theorem of Fenchel-Moreau implies that $\mathcal{R} = (\mathcal{R}^*)^*$, see [IT79].

Assume now that the dissipation potential is an integral functional, i.e. for all $z \in \mathcal{Z}$ it is $\mathcal{R}(z) = \int_\Omega R(z(x))\,\mathrm{d}x$, where $R : \mathbb{R} \to [0,\infty]$ is positively 1-homogeneous, convex and lower semicontinuous and $\Omega \subset \mathbb{R}^d$ is a d-dimensional, bounded domain. Then [IT79, p. 296, Th. 1] states that $\mathcal{R}^*(\cdot) = \Big(\int_\Omega R(\cdot)\,\mathrm{d}x\Big)^* = \int_\Omega R^*(\cdot)\,\mathrm{d}x$, i.e. for the density R holds the analogous relation to its Legendre-Fenchel transformed: $R(z) = R^{**}(z)$ for all $z \in \mathbb{R}$. In view of (2.41), the following relation between the subdifferential formulation (SDF) of Definition 2.3.6 and the flow rule (2.39) has been verified

$$\dot z \in F(e,z) = \partial R^*\big(-\partial_z \overline{W}(e,z,\nabla z) + \operatorname{div} \partial_{\nabla z} \overline{W}(e,z,\nabla z)\big)\,, \tag{2.58}$$

where R^* is the Legendre-Fenchel transformed of the density R of the positively 1-homogeneous dissipation potential $\mathcal{R}$.

Example 2.3.9 *Throughout this work the dissipation potential $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ for damage and delamination will take the form*

$$\mathcal{R}(v) = \int_\Omega R(v)\,\mathrm{d}x \quad \textit{with } R : \mathbb{R} \to [0,\infty],\ R(v) = \begin{cases} \varrho|v| & \textit{if } v \le 0\,, \\ \infty & \textit{otherwise} \end{cases} \tag{2.59}$$

with the constant $\varrho > 0$. The subdifferential is given by

$$\partial R(v) = \begin{cases} -\varrho & \textit{if } v < 0\,, \\ [-\varrho,\infty) & \textit{if } v = 0\,, \\ \emptyset & \textit{if } v > 0\,. \end{cases}$$

Thus $\partial R(0) = [-\varrho,\infty)$ and hence the flow rule reads

$$\dot z \in F(e,z) = \partial I_{[-\varrho,\infty)}\big(-\partial_z \overline{W}(e,z,\nabla z) + \operatorname{div} \partial_{\nabla z} \overline{W}(e,z,\nabla z)\big).$$

Remark 2.3.10 *In Proposition 2.3.8 it was stated that the three formulations* (S) & (E), (SDF) *and* (VI) *are equivalent under the assumptions that it holds $q \in W^{1,1}([s,T],\mathcal{Q})$ for solutions and that the energy functional $\mathcal{E}(t,\cdot,\cdot)$ is jointly convex for all $(u,z) \in \mathcal{Q}$.*

If $\mathcal{E}(t,\cdot,\cdot)$ is not jointly convex in (u,z) but only separately convex in u and z, as it will explicitly be the case in Chapter 4, the three formulations are no longer equivalent. The energetic formulation then represents a proper generalized formulation based on the minimum-energy principle competing with the maximum-dissipation principle or rather with Levitas' realizability principle [Mie05, MT04, MTL02]. *In* [MT04] *further relations for non-convex energy functionals are discussed.*

Moreover, the assumption on the solution $q \in W^{1,1}([0,T],\mathcal{Q})$ cannot be guaranteed for non-convex energy functionals. In particular already for not strictly convex energy functionals the solutions may have jumps with respect to time. As formula (2.50) *already indicates it will only be possible to guarantee that an energetic solution is of bounded variation with respect to time. This provides that the time-derivative $\dot q$ exists only as a Radon-measure. Relations between the three formulations in this case are also provided in* [MT04].

2.4 General Theory for the Energetic Formulation

Sections 3 – 5 will use the energetic formulation of the rate-independent evolutionary problems. This section provides general results of this theory.

In order to apply the energetic formulation to a rate-independent evolutionary problem we fix a state space $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$, which is assumed to be a weakly closed subset of a reflexive Banach space. Our approach is based on the energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ and the dissipation potential $\mathcal{R} : \mathcal{Z} \to \mathbb{R}_\infty$, i.e. we assume that $\mathcal{R}$ only depends on the inner variable, not on the full state. We search for an energetic solution $q : [0,T] \to \mathcal{Q}$ for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$, which is supposed to satisfy the global stability condition (2.60(S)) and the global energy balance (2.60(E)).

Definition 2.4.1 (Energetic solution) *A function $q = (u, z) : [0,T] \to \mathcal{Q}$ is called an energetic solution for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$, if $t \mapsto \partial_t \mathcal{E}(t,q) \in L^1((0,T))$ and if for all $s, t \in [0,T]$ we have $\mathcal{E}(t, q(t)) < \infty$, global stability* (2.60(S)) *and global energy balance* (2.60(E))*:*

$$\text{for all } \tilde{q} = (\tilde{u}, \tilde{z}) \in \mathcal{Q} \text{ holds}: \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{R}(\tilde{z} - z(t)), \tag{2.60(S)}$$

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(z, [s,t]) = \mathcal{E}(s, q(s)) + \int_s^t \partial_\xi \mathcal{E}(\xi, q(\xi))\, \mathrm{d}\xi \tag{2.60(E)}$$

with $\mathrm{Diss}_{\mathcal{R}}(z, [s,t]) := \sup \big\{ \sum_{j=1}^N \mathcal{R}(z(\xi_j) - z(\xi_{j-1})) \,|\, s = \xi_0 < \ldots < \xi_N = t,\ N \in \mathbb{N} \big\}$.

In view of (2.60(S)) we introduce the sets of stable states and stable sequences.

Definition 2.4.2 *The set of stable states at time $t \in [0,T]$ is defined by:*

$$\mathcal{S}(t) := \{ q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) < \infty,\ \forall \tilde{q} \in \mathcal{Q} : \mathcal{E}(t,q) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{R}(\tilde{z} - z) \}\ .$$

A sequence $(t_k, q_k)_{k \in \mathbb{N}} \subset [0,T] \times \mathcal{Q}$ is called a stable sequence if (i) and (ii) hold:

(i) $\sup_{k \in \mathbb{N}} \{ \mathcal{E}(t_k, q_k) \} < \infty$ *, i.e. there is a constant $E \in \mathbb{R}$ such that*

$$q_k \in L_E(t_k) := \{ q \in \mathcal{Q} \,|\, \mathcal{E}(t_k, q) \leq E \}\ , \tag{2.61}$$

(ii) $q_k \in \mathcal{S}(t_k)$ for every $k \in \mathbb{N}$.

2.4.1 The Main Existence Theorem

In order to guarantee the existence of an energetic solution, certain general assumptions have to be made on $\mathcal{E}$ and $\mathcal{R}$, see also [MM05, MRS08].

The energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ has to fulfill the following conditions:

Compactness of energy sublevels: $\forall\, t \in [0,T]\ \forall\, E \in \mathbb{R}$:

$$L_E(t) := \{ q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \leq E \} \text{ is weakly seq. compact.} \tag{2.62(E1)}$$

Uniform control of the power:

$$\exists\, c_0 \in \mathbb{R}\ \exists\, c_1 > 0\ \forall\, (t_q, q) \in [0,T] \times \mathcal{Q} \text{ with } \mathcal{E}(t_q, q) < \infty : \\ \mathcal{E}(\cdot, q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t \mathcal{E}(t,q)| \leq c_1 (c_0 + \mathcal{E}(t,q)) \text{ for all } t \in [0,T]. \tag{2.62(E2)}$$

Condition (2.62(E2)) enables us to apply Gronwall's lemma in order to derive a Lipschitz estimate for $\mathcal{E}$ with respect to time:

$$|\mathcal{E}(t,q)-\mathcal{E}(s,q)| \le \left(\mathrm{e}^{c_1|t-s|}-1\right)(\mathcal{E}(t,q)+c_0) \le \mathrm{e}^{c_1 T}(\mathcal{E}(t,q)+c_0)|t-s|\,. \tag{2.63}$$

Hence, if $\mathcal{E}(t,q)<E$ for $E\in\mathbb{R}$, then, for $c_E:=\mathrm{e}^{c_1T}(E+c_0)$, estimate (2.63) implies

$$|\mathcal{E}(t,q)-\mathcal{E}(s,q)| \le c_E|t-s|\,. \tag{2.64}$$

The abstract existence theory requires the following general assumptions on the dissipation distance $\mathcal{D}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ with $\mathcal{D}(z,\tilde z):=\mathcal{R}(\tilde z-z)$ for all $z,\tilde z\in\mathcal{Z}$:

Quasi-distance: $\forall\, z_1,z_2,z_3\in\mathcal{Z}:$ $\mathcal{D}(z_1,z_2)=0 \;\Leftrightarrow z_1=z_2$ and $\mathcal{D}(z_1,z_3)\le\mathcal{D}(z_1,z_2)+\mathcal{D}(z_2,z_3)$; (2.65(D1))

Semi-continuity:
$\mathcal{D}:\;\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ is weakly sequentially lower semi-continuous. (2.65(D2))

Remark 2.4.3 *$\mathcal{D}$ is an extended quasi-distance on $\mathcal{Z}$, since all metric axioms except symmetry are satisfied and since the value ∞ is allowed. $\mathcal{D}$ on $\mathcal{Q}$ is a pseudo-distance or semi-distance, because for $q_1=(u_1,z_1)$, $q_2=(u_2,z_2)$ the property $\mathcal{D}(z_1,z_2)=0$ not necessarily implies $q_1=q_2$.*

Conditions (2.62) and (2.65) are useful to state an abstract existence result for the energetic formulation of rate-independent problems. This abstract version of the main existence theorem was developed within the works [MM05, FM06, MRS08, Mie09].

Theorem 2.4.4 (Abstract main existence theorem [Mie09]) *Let $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfy conditions* (2.62) *and* (2.65)*. Moreover, let the following compatibility conditions hold: For every stable sequence $(t_k,q_k)_{k\in\mathbb{N}}$ with $t_k\to t$, $q_k\rightharpoonup q$ in $[0,T]\times\mathcal{Q}$ we have*

$$\partial_t\mathcal{E}(t,q_k)\to\partial_t\mathcal{E}(t,q)\,, \tag{2.66(C1)}$$
$$q\in\mathcal{S}(t)\,. \tag{2.66(C2)}$$

Then, for each $q_0\in\mathcal{S}(0)$ there exists an energetic solution $q:[0,T]\to\mathcal{Q}$ for $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfying $q(0)=q_0$.

The proof of Theorem 2.4.4 is based on a time-discretization, where (2.62(E1)) and (2.65(D2)) ensure the existence of a minimizer for the time-incremental minimization problem at each time-step. For a given partition $\Pi:=\{0=t_0<t_1<\ldots<t_M=T\}$, for every $k=1,\ldots,M$ we have to

$$\text{find } q_k\in\operatorname{Argmin}\{\mathcal{E}(t_k,\tilde q)+\mathcal{D}(z_{k-1},\tilde z)\,|\,\tilde q=(\tilde u,\tilde z)\in\mathcal{Q}\}\,. \tag{2.67}$$

One then defines a piecewise constant interpolant q^Π with $q^\Pi(t):=q_{k-1}$ for $t\in[t_{k-1},t_k)$ and $q^\Pi(T)=q_M$. Choosing a sequence $(\Pi_m)_{m\in\mathbb{N}}$ of partitions, where the fineness of Π_m tends to 0 as $m\to\infty$, it is possible to apply Helly's selection principle to the sequence

$(q^{\Pi_m})_{m\in\mathbb{N}}$, see [MM05]. Then, it is shown that the limit function fulfills the properties (2.60(S)) and (2.60(E)) of an energetic solution. As a consequence of stability and energy balance (2.60) one obtains the temporal boundedness and BV-regularity of the energetic solution, since $\mathrm{Diss}_{\mathcal{R}}(z,[0,T])$ is equivalent to the total variation of z in time with respect to the L^1-norm in space due to the positive 1-homogeneity of $\mathcal{R}$. The measurability of $q:[0,T]\to\mathcal{Q}$ with respect to time stated in (2.46), is due to results in set-valued analysis, which allow it to extract a measurable function as the limit of the measurable piecewise constant interpolants $(q^{\Pi_m})_{m\in\mathbb{N}}$, see e.g. [AF90, Th. 8.2.5 & Th. 8.2.10]. A detailed proof of Theorem 2.4.4 can be found in [Mie09].

2.4.2 Γ-Convergence of Rate-independent Systems

In Section 4 we will deduce a model for Griffith-type delamination from models describing partial damage. For this we will apply the theory on the Γ-convergence of rate-independent systems, which was developed in [MRS08] and which is based on the original notion of Γ-convergence of functionals.

However Γ-convergence introduced by De Giorgi in [DG77] is a method to gain a static variational problem and its minimizers as a limit of a sequence of static variational problems and their minimizers. A sequence of functionals $(\mathcal{G}_j)_{j\in\mathbb{N}}$ with $\mathcal{G}_j:\mathcal{X}\to\mathbb{R}_\infty=\mathbb{R}\cup\{\infty\}$, where $\mathcal{X}$ is a metric space, is said to Γ-converge to a functional $\mathcal{G}:\mathcal{X}\to\mathbb{R}_\infty$ if for every $w\in\mathcal{X}$ the following two conditions are satisfied:

$$\Gamma\text{-}\liminf\text{ inequality:}\quad \forall\,(w_j)_{j\in\mathbb{N}},\ w_j\overset{\mathcal{X}}{\to}w:\ \mathcal{G}(w)\le\liminf_{j\to\infty}\mathcal{G}_j(w_j)\,,\tag{2.68a}$$

$$\text{Recovery sequence:}\quad \exists\,(\hat{w}_j)_{j\in\mathbb{N}},\ \hat{w}_j\overset{\mathcal{X}}{\to}w:\ \mathcal{G}(w)\ge\limsup_{j\to\infty}\mathcal{G}_j(\hat{w}_j)\,.\tag{2.68b}$$

For rate-independent systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{D}_j)_{j\in\mathbb{N}}$ having time-dependent (energetic) solutions $q_k:[0,T]\to\mathcal{Q}$ it is easy to see that

$$\mathcal{E}_\infty=\Gamma\text{-}\lim\mathcal{E}_j\quad\text{and}\quad\mathcal{D}_\infty=\Gamma\text{-}\lim\mathcal{D}_j$$

is not sufficient to ensure (2.60(S)) and (2.60(E)). Hence a modified concept of Γ-convergence has to be used for the quasistatic setting and the energetic formulation of rate-independent processes. It is desired that energetic solutions $q_j:[0,T]\to\mathcal{Q}$ of the approximating systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)$ converge to an energetic solution $q:[0,T]\to\mathcal{Q}$ of the limit system $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{R}_\infty)$. This means that conditions (2.60) must be maintained under convergence, so that the interplay of $\mathcal{E}_j$ and $\mathcal{R}_j$ is important. In [MRS08] the theory of Γ-convergence was adapted to the framework of the energetic formulation of rate-independent processes. In the following we introduce sufficient conditions guaranteeing that a subsequence of energetic solutions of the approximating systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)$ converges to an energetic solution of the limit system $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{R}_\infty)$. The topology for the convergence of the energetic solutions is denoted by $\mathcal{T}$ in the following, i.e. we want to obtain that $q_j(t)\overset{\mathcal{T}}{\to}q(t)$ for all $t\in[0,T]$. For all $j\in\mathbb{N}_\infty=\mathbb{N}\cup\{\infty\}$ we introduce the stable sets $\mathcal{S}_j(t):=\{q\in\mathcal{Q}\,|\,\mathcal{E}_j(t,q)<\infty,\ \forall\tilde{q}=(\tilde{u},\tilde{z}):\mathcal{E}_j(t,q_j)\le\mathcal{E}_j(t,\tilde{q})+\mathcal{R}_j(\tilde{z}-z_j)\}$.

In order to ensure the Γ-convergence of the systems $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)_{j\in\mathbb{N}}$ the following conditions have to be satisfied by the energy functionals $\mathcal{E}_j : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ for all $j \in \mathbb{N}_\infty$.

Compactness of energy sublevels: $\forall\, t{\in}[0,T]\ \forall\, E{\in}\mathbb{R}$:

$$\forall j \in \mathbb{N}_\infty :\ L_E^j(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}_j(t,q) \le E\} \text{ is compact wrt. } \mathcal{T}, \qquad \text{(2.69(E1))}$$
$$\textstyle\bigcup_{j=1}^\infty L_E^j(t) \text{ is relatively compact wrt. } \mathcal{T}\,,$$

Uniform control of the power:

$$\exists\, c_0{\in}\mathbb{R}\ \exists\, c_1{>}0\ \forall j \in \mathbb{N}_\infty \forall\,(t_q,q){\in}[0,T]{\times}\mathcal{Q} \text{ with } \mathcal{E}(t_q,q) < \infty : \qquad \text{(2.69(E2))}$$
$$\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t\mathcal{E}(t,q)| \le c_1(c_0{+}\mathcal{E}(t,q)) \text{ for all } t{\in}[0,T]\,,$$

Uniform time-continuity of $\partial_t\mathcal{E}_\infty$:

$$\forall\varepsilon > 0\,\forall E \in \mathbb{R}\ \exists\,\delta > 0\,\forall q \in \mathcal{Q} \text{ with } \mathcal{E}(0,q) < E : \qquad \text{(2.69(E3))}$$
$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |\partial_t\mathcal{E}_\infty(t_1,q) - \partial_t\mathcal{E}_\infty(t_2,q)| < \varepsilon\,.$$

Furthermore the dissipation distances $\mathcal{D}_j : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ with $\mathcal{D}_j(z,\tilde z) = \mathcal{R}_j(\tilde z{-}z)$ for all $z, \tilde z \in \mathcal{Z}$ must fulfill for all $j \in \mathbb{N}_\infty$:

Quasi-distance:

$$\forall j \in \mathbb{N}_\infty\ \forall\, z_1,z_2,z_3 \in \mathcal{Z} : \quad \mathcal{D}_j(z_1,z_2) = 0 \ \Leftrightarrow z_1 = z_2 \ \text{ and} \qquad \text{(2.70(D1))}$$
$$\mathcal{D}_j(z_1,z_3) \le \mathcal{D}_j(z_1,z_2) + \mathcal{D}_j(z_2,z_3)\,,$$

Semi-continuity:

$$\forall j \in \mathbb{N}_\infty :\ \mathcal{D}_j :\ \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is lower semi-continuous wrt. } \mathcal{T}\,, \qquad \text{(2.70(D2))}$$

Positivity of $\mathcal{D}_\infty$:

$$\forall \text{ compact } A \subset \mathcal{Z}\,,\ \forall (z_j)_{j\in\mathbb{N}} \subset A :$$
$$\min\{\mathcal{D}_j(z_j,z), \mathcal{D}_j(z,z_j)\} \to 0 \quad \Rightarrow \quad z_j \xrightarrow{\mathcal{T}_\mathcal{Z}} z\,, \qquad \text{(2.70(D3))}$$

where $\mathcal{T}_\mathcal{Z}$ is the restriction of $\mathcal{T}$ to the z-component of $q = (u,z)$.

Additionally the following compatibility conditions have to be satisfied:
For all $t_j \to t$ in $[0,T]$, $q_j = (u_j,z_j) \xrightarrow{\mathcal{T}} q = (u,z)$ with $q_j \in \mathcal{S}_j(t_j)$ for all $j \in \mathbb{N}$ it holds

Conditioned continuous convergence of $\partial_t\mathcal{E}_j$:

$$\partial_t\mathcal{E}_j(t_j,q_k) \to \partial_t\mathcal{E}(t,q)\,, \qquad \text{(2.71(C1))}$$

Conditioned upper semi-continuity of stable sets:

$$q \in \mathcal{S}_\infty(t)\,, \qquad \text{(2.71(C2))}$$

Lower Γ*-limit for* $\mathcal{E}_j$:

$$\mathcal{E}(t,q) \le \liminf_{j\to\infty} \mathcal{E}_j(t_j,q_j)\,, \qquad \text{(2.71(C3))}$$

Lower Γ*-limit for* $\mathcal{D}_j$: Let additionally $\hat q_j = (\hat u_j, \hat z_j) \xrightarrow{\mathcal{T}} \hat q = (\hat u, \hat z)$ with $\hat q_j \in \mathcal{S}_j(t_j), j \in \mathbb{N}$, then $\mathcal{D}(z,\hat z) \le \liminf_{j\to\infty} \mathcal{D}_j(z_j,\hat z_j)$. (2.71(C4))

Conditions (2.69(E1)), (2.69(E2)) and (2.70(D1)), (2.70(D2)) are analoga to the conditions (2.62) and (2.65), which are needed when establishing the existence of energetic solutions for all $j \in \mathbb{N}$ via time-discretizations. Moreover, for fixed $t \in [0,T]$ condition (2.69(E1)) enables us to extract subsequences $q_j(t)$ which converge with respect to $\mathcal{T}$ to a limit $q(t) \in \mathcal{Q}$. Then the compatibility conditions (2.71) make sure that the properties (2.60(S)) and (2.60(E)) satisfied by the energetic solutions q_j of the approximating systems $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)$ can be maintained during convergence, so that it is indeed possible to extract a (measurable) energetic solution $q : [0,T] \to \mathcal{Q}$ for the limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{R}_\infty)$ from the pointwise accumulation points obtained from (2.69(E1)).

The theorem below states the convergence result. A proof is given in [MRS08, Th. 3.1].

Theorem 2.4.5 (Γ-convergence of $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)_{j\in\mathbb{N}}$) *Let the conditions* (2.69), (2.70) *and* (2.71) *hold and for all $j \in \mathbb{N}$ let $q_j : [0,T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)$ in the sense of Definition 2.4.1. If $q_j(t) \xrightarrow{\mathcal{T}} q(t)$ for all $t \in [0,T]$ and if $\mathcal{E}_j(0, q_j(0)) \to \mathcal{E}_\infty(0, q(0))$ then $q : [0,T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{R}_\infty)$, i.e. for all $t \in [0,T]$ it holds*

$$q(t) \in \mathcal{S}_\infty(t) \quad \textit{and} \quad \mathcal{E}_\infty(t) + \mathrm{Diss}_{\mathcal{R}}(q, [0,t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_\xi \mathcal{E}(\xi, q(\xi))\, \mathrm{d}\xi\,. \tag{2.72}$$

Moreover, for all $t \in [0,T]$ it holds $\mathcal{E}_j(t, q_j(t)) \to \mathcal{E}(t, q(t))$, $\mathrm{Diss}_{\mathcal{R}_j}(q_j, [0,t]) \to \mathrm{Diss}_{\mathcal{R}}(q, [0,t])$ and $\partial_t \mathcal{E}_j(t, q_j(t)) \to \partial_t \mathcal{E}(t, q(t))$ for a.a. $t \in [0,T]$. Furthermore, for $\mathcal{Q}$ being a separable, reflexive Banach space, the energetic solution q is measurable with respect to time.

In particular the conditioned upper semicontinuity of stable sets (2.71(C2)) is important, but very difficult to verify for specific applications. Thus in [MRS08, Condition (2.16)] it is elaborated that this relation can be guaranteed by the so-called joint recovery condition.

Lemma 2.4.6 (Joint recovery condition) *Let $t_j \to t$ in $[0,T]$ and $q_j \xrightarrow{\mathcal{T}} q$ with $q_j \in \mathcal{S}_j(t_j)$ for all $j \in \mathbb{N}$. Then, $q \in \mathcal{S}_\infty(t)$ is satisfied if the joint recovery condition holds, i.e. for any $\hat{q} \in \mathcal{Q}$, it must be possible to construct a joint recovery sequence $(\hat{q}_j)_{j\in\mathbb{N}}$ with $\hat{q}_j \xrightarrow{\mathcal{T}} \hat{q}$ so that*

$$\limsup_{j\to\infty} \left(\mathcal{E}_j(t_j, \hat{q}_j) + \mathcal{D}_j(z_j, \hat{z}_j) - \mathcal{E}_j(t_j, q_j)\right) \le \mathcal{E}_\infty(t, \hat{q}) + \mathcal{D}_\infty(z, \hat{z}) - \mathcal{E}_\infty(t, q). \tag{2.73}$$

Proof: Assume that the joint recovery condition holds so that (2.73) can be verified for all $\hat{q} \in \mathcal{Q}$. Then (2.73) implies that

$$\begin{aligned}\mathcal{E}_\infty(t, q) &\le \mathcal{E}_\infty(t, \hat{q}) + \mathcal{D}_\infty(z, \hat{z}) - \limsup_{j\to\infty} \left(\mathcal{E}_j(t_j, \hat{q}_j) + \mathcal{D}_j(z_j, \hat{z}_j) - \mathcal{E}_j(t_j, q_j)\right) \\ &\le \mathcal{E}_\infty(t, \hat{q}) + \mathcal{D}_\infty(z, \hat{z})\,,\end{aligned}$$

since $(\mathcal{E}_j(t_j, \hat{q}_j) + \mathcal{D}_j(z_j, \hat{z}_j) - \mathcal{E}_j(t_j, q_j)) \ge 0$ for all $j \in \mathbb{N}$ due to $q_j \in \mathcal{S}_j(t_j)$. ■

The term *recovery sequence* shows the close relation of this tool to the original definition of Γ-convergence, but *joint* expresses that this new kind of recovery sequence has to supply more: It has to recover the limit state *jointly* in all components and hence, in many applications the sequence recovering the displacement cannot be constructed independently from the one for the inner variable, since these quantities may be linked with each other by the energy functional and the dissipation potential. This will particularly be the case in Section 4.

2.5 Outline of the Thesis

In Chapters 3– 5 the results of this thesis are demonstrated. Chapter 3 establishes the existence of energetic solutions for isotropic damage processes both for small and finite strains. The energy $\mathcal{E}$ will be characterized by a functional of the form (2.42) for general stored energy densities $\overline{W}$, so that the result also applies to physically nonlinearly elastic materials such as in Examples 2.1.5, 2.1.6. The dissipation potential $\mathcal{R} : \mathcal{Z} \to [0, \infty]$ will be given as in (2.59), where $\mathcal{R}(v) = \infty$ if $v > 0$ accounts for the unidirectionality of the damage process, i.e. healing is forbidden. The existence of energetic solutions for the damage problem $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$, where $\mathcal{Q}$ is a suitable state space will be proven by verifying the assumptions (2.62), (2.65) and (2.66) of the main existence theorem 2.4.4. The main difficulty lies in the verification of the closedness of the stable sets (2.66(C2)), which is due to the discontinuity of the dissipation potential. Since it is even not weakly continuous on the set of admissible damage variables $\mathcal{Z}$, one cannot obtain the stability of the limit by simply passing to the limit with stable sequences in stability condition (2.60(S)). As already exploited in [MR06] the joint recovery condition (Lemma 2.4.6) has to be applied to overcome the lack of continuity. The technique which is used in Chapter 3 to construct a joint recovery sequence for this problem is new and applies to a larger class of damage models than the one presented in [MR06]. In particular the existence result established with this new technique also covers the engineering models proposed in [HSS03, LB05].

For a joint recovery sequence there does not exist a general ansatz. The way to find it is individual for each problem since the construction of the sequence particularly depends on the form of the functionals. This can be seen in Chapter 4, where the existence result of Chapter 3 is used to deduce a delamination model from models describing the damage of three-specimen-sandwich-structures by means of Γ-convergence when flattening their middle component to the thickness 0. The Γ-convergence of the damage problems will be obtained by verifying the assumptions (2.69), (2.70) and (2.71) of the convergence theorem 2.4.5. Since the limit delamination model shall include the transmission and non-interpenetration conditions (2.44), (2.45) one has to use a larger state space than for the damage models and hence has to be very careful with the choice of the topology $\mathcal{T}$ of convergence. In particular, here the recovery sequences have to be constructed in such a way that conditions (2.44), (2.45) can be recovered, i.e. the construction has to preserve the link between the displacements and the delamination variable in the limit. Hence one indeed has to construct a *joint* recovery sequence for this application, since the recovery sequence for the displacements has to interact with the one for the inner variables. In order to obtain the transmission conditions for the limit model it is of particular importance that the gradient of the delamination variable is kept in the limit, since it supplies better properties for the limit functions, namely continuity due to a compact embedding. Since this term is quite artificial for any delamination model describing a Griffith-crack, the Γ-limit is performed via a double limit: In a first limit passage the thickness of the middle component of the sandwich-structure is flattened to 0, so that the $(d-1)$-dimensional interface Γ_{C} experiencing possible delamination is approximated by d-dimensional damageable domains. In this limit passage the transmission and non-interpenetration conditions (2.44), (2.45) are gained from the damage models, see thereto Section 4.2. In a second step, carried out in Section 4.3, the gradient of the delamination variable is suppressed by a further Γ-limit, so that the final limit model coincides with the one discussed in [RSZ09].

Chapter 5 is concerned with the temporal regularity of energetic solutions. As already indicated in Section 2.3 an energetic solution $q = (u, z) : [0, T] \to \mathcal{Q}$ has in general only a quite low temporal regularity. Stability condition (2.60(S)) provides that $q \in L^\infty([0, T], \mathcal{Q})$, whereas the boundedness of $\mathrm{Diss}_{\mathcal{R}}(z, [0, T])$, which can be concluded from the energy balance (2.60(E)), supplies that $z \in BV([0, T], L^1(\Omega))$. Thus, in general energetic solution may have jumps in time and the time-derivative $\dot{z}$ of the inner variable is only a Radon measure. In view of the other possible formulations of evolutionary problems by subdifferentials (SDF), a variational inequality (VI), see Definitions 2.3.6 and 2.3.7, or even by the classical formulation by force balance (2.17) and evolution equation (2.39), which all involve the time derivative $\dot{z}$ at least in suitable Sobolev spaces, one is interested in specifying settings that lead to a higher temporal regularity of energetic solutions. In Chapter 5 it is established that the strict convexity of the energy functional $\mathcal{E}(t, \cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ for all $t \in [0, T]$ implies the continuity of energetic solutions in time. This result is due to the fact that strict convexity of the functional provides the uniqueness of minimizers.

Moreover Chapter 5 generalizes the results obtained in [MT04] on the temporal Lipschitz continuity of energetic solutions. In [MT04] it was developed that an energetic solution is Lipschitz continuous with respect to time, if the energy functional satisfies for all $q_0, q_1 \in \mathcal{Q}$ and all $\theta \in [0, 1]$ a uniform convexity inequality of the form

$$\mathcal{E}(t, \theta q_1 + (1-\theta) q_0) \leq \theta \mathcal{E}(t, q_1) + (1-\theta)\mathcal{E}(t, q_0) - \theta(1-\theta) c \|q_1 - q_0\|_{\mathcal{Q}}^\alpha \tag{2.74}$$

with $\alpha = 2$. In other words, it was claimed that (2.74) has to hold globally on $\mathcal{Q}$ for the exponent $\alpha = 2$. In this thesis it is shown that this assumption is too restrictive, since it is in general only satisfied by quadratic energy functionals, i.e. for linear elasticity. But it turns out that energy functionals referring to nonlinearly elastic materials may satisfy a kind of uniform convexity restricted on energy sublevels for exponents $\alpha > 2$, i.e. the constant c in (2.74) depends on the energy sublevel. Since energetic solutions have bounded energy this observation allows it to prove at least their Hölder continuity in time. Furthermore Section 5 demonstrates that the temporal regularity can be improved when using a bigger space $\mathcal{V} \supset \mathcal{Q}$ to establish the uniform convexity estimates, so that temporal Hölder continuity with respect to $\mathcal{Q}$ may even improve to temporal Lipschitz continuity with respect to $\mathcal{V}$.

Finally Chapter 6 summarizes the results of this thesis and gives an outlook on future work on this field.

Chapter 3

Rate-independent Damage Processes

In this Chapter the existence of energetic solutions of rate-independent models for isotropic damage in physically nonlinearly elastic materials is investigated both in the small-strain and in the finite-strain setting. See Sect. 2.1, p. 7 for the definition of these settings and Sect. 2.2.1 for more information about the mechanical modeling of damage processes. The focus lies on partial damage, which means that the damage variable $z : [0,T] \times \Omega \to [z_\star, 1]$, with $z_\star > 0$, cannot attain the value 0, i.e. $z(t,x) = 1$ means that the material is undamaged in the point $x \in \Omega$ at time $t \in [0,T]$, whereas $z(t,x) = z_\star$ stands for maximal damage in x. In other words, partial damage means that the material cannot completely disintegrate and for $z = z_\star$ it is still able to support arbitrary stresses without further damage.

The model that is analyzed in this Chapter is based on one proposed by Frémond and Nedjar to describe the damage of concrete, see [Fré02, Chap. 12] or [FN96]. It consists of an energy functional representing the strain energy plus the amount of energy generated by the external loadings, and a dissipation potential accounting for the energy dissipated by the damage process. The dissipation potential is given similar to (2.59), i.e. by

$$\mathcal{R}(\dot z) := \int_\Omega R(x, \dot z)\,\mathrm{d}x\,, \quad \text{where } R(x,v) := \begin{cases} \varrho(x)|v| & \text{if } v \in (-\infty, 0] \\ \infty & \text{if } v > 0 \end{cases} \tag{3.1}$$

with $\varrho \in L^\infty(\Omega)$ satisfying $0 < \varrho_0 \le \varrho(x)$ for a.e. $x \in \Omega$.

This definition of the dissipation potential accounts for the unidirectionality of the damage process: Only those damage variables, that describe an increase of damage, lead to finite dissipation. Moreover, the dissipation potential defined via (3.1) is rate-independent, since it is homogeneous of degree one, i.e.: $R(x, \alpha w) = \alpha R(x,w)$ for every $\alpha > 0$ and every $w \in \mathbb{R}$. Thus, it generates a so-called dissipation distance, see Section 2.3 for more details:

$$\mathcal{D}(z_0, z_1) = \mathcal{R}(z_1 - z_0)\,. \tag{3.2}$$

The energy functional depends on time $t \in [0,T]$, the damage variable $z \in [z_\star, 1]$ and – at small strains – on the linearized Green-St. Venant strain tensor $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$, where $u : \Omega \to \mathbb{R}^d$ is the displacement field. It is defined via three different energy terms:

$$\mathcal{E}(t,u,z) := \int_\Omega W(x, e(u), z)\,\mathrm{d}x + \int_\Omega \Big(\frac{\kappa}{r}\,|\nabla z|^r + \delta_{[z_\star,1]}(z)\Big)\,\mathrm{d}x - \langle l(t), u\rangle\,. \tag{3.3}$$

The third term in formula (3.3) represents the work of external loadings, which may comprise both volume and surface forces. Here, $\langle\cdot,\cdot\rangle$ denotes the dual pairing in suitable Sobolev spaces.

The second term in (3.3) includes the indicator function $\delta_{[z_\star,1]}(\cdot)$, which ensures that $z(x)\in[z_\star,1]$ for a.e. $x\in\Omega$ by $\delta_{[z_\star,1]}(z(x))=0$ if $z(x)\in[z_\star,1]$ and $\delta_{[z_\star,1]}(z(x))=\infty$ else. Furthermore it involves the gradient of damage and takes into account nonlocal, microscopic interactions, i.e. it considers the influence of damage in a point x on its neighborhood. Here, $\kappa>0$ denotes the so-called factor of influence of damage. The ansatz involving the damage gradient is quite often used in engineering for the exponent $r=2$, see [HS03, LB05], and since previous work [MR06] only deals with $r>d$ it is the aim in this chapter to cover all exponents $r\in(1,\infty)$.

The first term in (3.3) denotes the stored elastic energy, which is determined by the stored elastic energy density $W:\Omega\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times[z_\star,1]\to\mathbb{R}_\infty$ with $\mathbb{R}_\infty:=\mathbb{R}\cup\{\infty\}$.

Since $e(u)$ is no sufficient measure for the strains when large deformation gradients occur, one has to consider the deformation φ and its gradient $\nabla\varphi$ instead of u and $e(u)$ in the finite-strain setting, see Section 2.1.1. Hence at finite strains the energy changes to $\mathcal{E}(t,\varphi,z)$ and the only obvious difference in the model given by $\mathcal{E}$ and $\mathcal{R}$ is that u and $e(u)$ in (3.3) are replaced by φ and $\nabla\varphi$. But the mathematical properties of the stored energy density $W:\Omega\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times[z_\star,1]\to\mathbb{R}_\infty$ are different in these two settings. A common assumption in the small-strain setting to guarantee the existence of minimizers is convexity. At finite strains the stored energy density has to satisfy the natural requirements (N1)-(N4) explained in Section 2.1.3, i.e. in particular material frame indifference (2.25) and the noninterpenetration condition (2.33). But these two assumptions are incompatible with convexity. Thus, it is required to use the notion of polyconvexity, which was introduced by J.M. Ball in [Bal76] to model finite-strain elasticity.

Moreover the two settings need a different treatment of inhomogeneous Dirichlet conditions. While they are realized at small strains by an additive split of a given extention of the Dirichlet datum into the domain and an unknown displacement satisfying homogeneous Dirichlet conditions, it is necessary in the finite-strain setting to model them by a composition of the unknown function y and the given Dirichlet datum g, so that the unknown deformation is given by $\varphi(t,x)=g(t,y(x))$. This assumption makes the test of the noninterpenetration condition (2.33) easier, since the determinant acts multiplicatively. Under the assumption that $\det\nabla_y g(t,y)>0$ for all $y\in\mathbb{R}^d$ one only has to check whether $\det\nabla_x y(x)>0$, since $\det\nabla_x\varphi(t,x)=\det\nabla_y g(t,y)\det\nabla_x y(x)$ due to the chain rule and the multiplicativity of the determinant.

Due to these differences the existence analysis for the damage model given by (3.1) and (3.3) has to be done separately for the two settings. In Section 3.1 the existence of energetic solutions at small strains is established by verification of the assumptions of the abstract main existence theorem 2.4.4. Using the same approach an equivalent existence result is deduced for the finite-strain setting in Section 3.2. Both sections discuss examples on stored energy densities W, which are used in engineering and which fit to the mathematical framework.

3.1 Isotropic Damage at Small Strains

Main goal of this section is to show the existence of energetic solutions $(u, z) : [0, T] \to \mathcal{Q}$ for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ at small strains, where $\mathcal{E}$ and $\mathcal{R}$ are defined by (3.3) and (3.1) respectively and where $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ denotes a suitable state space for the displacements $u \in \mathcal{U}$ and the damage variables $z \in \mathcal{Z}$. Energetic solutions are characterized by energy balance (2.60(E)) and stability condition (2.60(S)). Section 3.1.1 provides the assumptions that are made on the setting of the damage process at small strains and it contains the existence result. Its proof is carried out in Section 3.1.2. It consists of an application of the main existence theorem 2.4.4. Finally, Section 3.1.3 discusses classes of energy functionals known in engineering, which fit into the framework of our setting.

One main difference to previous works on the existence analysis of rate-independent processes [FM06, MP07, MPP09] is, that we do not claim a growth property on the stored elastic energy density of the form $c_1|e|^p - C \le W(x, e, z) \le c_2|e|^p + \tilde{C}$ for constants $c_1, c_2, C, \tilde{C} > 0$ and $1 < p < \infty$, which would lead under the assumption of convexity to a growth condition on the stresses of the form

$$\textbf{(H4*)} \quad |\partial_e W(x, e, z)| \le c(|e|^{p-1} + \tilde{c}) \quad \text{for constants } c, \tilde{c} > 0.$$

This condition is not applicable for our purposes, since we want to allow for stored elastic energy densities used in literature [Ser93] to describe strain hardening, such as e.g.:

$$W(e, z) := \tfrac{1+z}{4}(\operatorname{tr} e)^2 + |e^D|^q, \tag{3.4}$$

for a constant $3 < q < \infty$ and the deviator $e^D := e - \frac{\operatorname{tr} e}{d}\mathrm{Id}$. Coercivity is only obtained for the exponent 2, i.e. $\frac{1}{4}(|e|^2 - 1) \le W(e, z)$, whereas (H4*) can only be verified with q, namely $|\partial_e W(e, z)| = |\frac{1+z}{2}(\operatorname{tr} e)\,\mathrm{Id} + q\,|e^D|^{q-2} 2e^D| \le |\operatorname{tr} e| + 2q\,|e^D|^{q-1} \le \max\{2q, 2^{\frac{q-3}{2}}\}(1 + |e|^{q-1})$. Hence we use the alternative stress control:

$$\textbf{(H4)} \quad |\partial_e W(x, e, z)| \le c\,(W(x, e, z) + \tilde{c}) \quad \text{for constants } c, \tilde{c} > 0.$$

The main challenge of the existence analysis lies in the discontinuity of the dissipation distance $\mathcal{D}$ arising from the unidirectionality of the damage process. Compared to [FM06, MP07, MPP09], where the dissipation distance was assumed to be (weakly) continuous, another method is required for proving the stability of limit states, see Condition (2.66(C2)) and Section 3.1.2. The possibly infinitely valued dissipation distance does not allow to pass to the limit along a stable sequence in stability condition (2.60(S)), see Definitions 2.4.1 and 2.4.2 as well as Section 3.1.2. To overcome this problem the joint recovery condition, which was explained in Section 2.4.2 for sequences of rate-independent systems $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)$, has to be applied with $\mathcal{E}_j = \mathcal{E}$ in (3.3) and $\mathcal{R}_j = \mathcal{R}$ in (3.1). This method was already applied in previous work [MR06] to prove the existence of an energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ defined by (3.3), (3.2) under the assumption of $r > d$. This restriction was necessary because the construction of the joint recovery sequence required the compact embedding of $W^{1,r}(\Omega) \Subset \mathrm{C}(\overline{\Omega})$. In Section (3.1.2) we provide a more delicate construction for the joint recovery sequence that allows us to handle weak convergence in $W^{1,r}(\Omega)$ with $r \in (1, \infty)$.

The results of this section will appear as a part of [TM10].

3.1.1 Assumptions and the Existence Result

We consider a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$ modeling the reference configuration of a nonlinearly elastic material. This body undergoes a damage process driven by exterior forces $l(t)$, which may change with time. Furthermore, the body is assumed to be fixed at one part Γ_{Dir} of its boundary $\partial\Omega$ with positive $(d-1)$-dimensional measure $\mathcal{L}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0$, such that the displacement field $\tilde{u} : \Omega \to \mathbb{R}^d$ is prescribed there: $\tilde{u} = g(t)$ on Γ_{Dir} for $t \in [0,T]$. This means that we allow for time-dependent Dirichlet conditions, where the Dirichlet boundary Γ_{Dir} itself is fixed in time. From now on we write $g(t)$ also for the given extention into the domain Ω of the function g specifying the Dirichlet condition on the boundary. Hence, using the splitting $\tilde{u} = u + g(t)$, we define the state $q = (u,z)$ and the energy functional

$$\mathcal{E}(t,u,z)=\int_\Omega W(x,e(u)+e_D(t),z)\,\mathrm{d}x + \int_\Omega \big(\tfrac{\kappa}{r}\,|\nabla z|^r + \delta_{[z_\star,1]}(z)\big)\,\mathrm{d}x - \langle l(t), u+g(t)\rangle\,, \quad (3.5)$$

where $u = 0$ on Γ_{Dir}, such that $u+g(t) = g(t)$ on Γ_{Dir}. Moreover, $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ and similarly $e_D(t) := \frac{1}{2}(\nabla g(t) + \nabla g(t)^\top)$ denote the linearized strain tensor of u and $g(t)$.

We make the following general assumptions on the domain Ω and the given data g, l :

(3.6(A1)) Ω is a bounded Lipschitz domain, $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ with $\mathcal{L}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0$,
(3.6(A2)) $g \in \mathrm{C}^1([0,T], W^{1,\infty}(\Omega,\mathbb{R}^d))$ with $c_g := \max\{1, \mathcal{L}^d(\Omega)\}\|g\|_{\mathrm{C}^1([0,T],W^{1,\infty}(\Omega,\mathbb{R}^d))}$,
(3.6(A3)) $l \in \mathrm{C}^1([0,T], W^{-1,p'}(\Omega,\mathbb{R}^d))$ with $c_l := \|l\|_{\mathrm{C}^1([0,T],W^{-1,p'}(\Omega,\mathbb{R}^d))}$.

Here $p' = p/(p-1)$, where $p \in (1,\infty)$ will be fixed in (3.7(H3)) below. Recall that $\mathcal{L}^m(A)$ denotes the m-dimensional Lebesgue-measure of a set $A \subset \mathbb{R}^m$ with $m \in \{(d-1), d\}$.

Furthermore, we claim the following hypotheses on the stored elastic energy density:

(3.7(H1)) Carathéodory-function:
$W(x,\cdot,\cdot) \in \mathrm{C}^0(\mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z_\star,1])$ for a.e. $x \in \Omega$ and $W(\cdot,e,z)$ is measurable in Ω.

(3.7(H2)) Convexity:
For every $(x,z) \in \Omega \times [z_\star,1]$ the function $W(x,\cdot,z)$ is convex.

(3.7(H3)) Coercivity:
There are constants $c_1,\ C > 0,$ and $1 < p < \infty$ such that for all $(x,e,z) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z_\star,1]$ we have $c_1|e|^p - C \le W(x,e,z)$.

(3.7(H4)) Stress control:
For all $(x,z) \in \Omega \times [z_\star,1]$ we have $W(x,\cdot,z) \in \mathrm{C}^1(\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and there exist constants $c > 0, \tilde{c} \ge 0$ such that for all $(x,e,z) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z_\star,1]$ we have

$$|\partial_e W(x,e,z)| \le c(W(x,e,z) + \tilde{c})\,.$$

(3.7(H5)) Uniform continuity of the stresses:
There is a modulus of continuity $\omega : [0,\infty] \to [0,\infty]$ so that for all $(x,e,z),\ (x,\hat{e},z) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z_\star,1]$ we have

$$|\partial_e W(x,e,z) - \partial_e W(x,\hat{e},z)| \le \omega\big(|e-\hat{e}|\big)\big(W(x,e,z) + W(x,\hat{e},z) + \tilde{c}\big)\,.$$

(3.7(H6)) Monotonicity:
There are constants $K>0, \widetilde{K}\geq 0$ so that for all $(x,e,z),(x,e,\tilde{z})\in\Omega\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times[z_\star,1]$ with $z\leq\tilde{z}$ we have

$$W(x,e,z)\leq W(x,e,\tilde{z})\leq K(W(x,e,z)+\widetilde{K})\,.$$

Hypotheses (3.7(H1))-(3.7(H3)) will ensure condition (2.62(E1)). Hypothesis (3.7(H4)) is the basis to prove Lipschitz estimate (2.64) and (2.62(E1)). Hypothesis (3.7(H5)) is required to establish condition (2.66(C1)). The first estimate in assumption (3.7(H6)) reflects the physical property of damage, that an increase of damage decreases the stored elastic energy. The second estimate in (3.7(H6)) states that the remaining elastic properties after all damage has occurred are still comparable to the undamaged material. This assumption is reasonable, because we only treat partial damage in our analysis. Total damage would neither allow for the second inequality in (3.7(H6)) nor for coercivity (3.7(H4)), since for a completely disintegrated body the displacement field has no meaning any longer.

In view of hypothesis (3.7(H3)) we choose the space of admissible displacements as

$$\mathcal{U}:=\{u\in W^{1,p}(\Omega,\mathbb{R}^d)\,|\,u=0\text{ on }\Gamma_{\mathrm{Dir}}\}. \tag{3.8}$$

Under consideration of formula (3.5) we put the set of admissible damage variables

$$\mathcal{Z}:=W^{1,r}(\Omega) \tag{3.9}$$

and $\mathcal{Q}:=\mathcal{U}\times\mathcal{Z}$ indicates the set of admissible states.

For the analysis we will consider the convergence of sequences $(q_k)_{k\in\mathbb{N}}\subset\mathcal{Q}$ to a limit q in the weak topology of $\mathcal{Q}$ and we will indicate the weak convergence in $\mathcal{Q}$ by $q_k\rightharpoonup q$ in $\mathcal{Q}$.

With these tools at hand we state the existence theorem for the damage problem.

Theorem 3.1.1 (Existence theorem for the damage problem) *Let $\mathcal{Q}=\mathcal{U}\times\mathcal{Z}$ be given as above. Let $\mathcal{E}$ be defined via* (3.5) *such that the assumptions* (3.6) *and* (3.7) *hold. Let $\mathcal{R}$ be given by* (3.1). *Then, for the rate-independent damage process defined by* $(\mathcal{Q},\mathcal{E},\mathcal{R})$ *there exists an energetic solution for any initial state* $q_0\in\mathcal{S}(0)$.

The proof of Theorem 3.1.1 is carried out in Section 3.1.2. The main difficulty lies in the missing weak continuity of the dissipation distance, which especially complicates the proof of the compatibility conditions (2.66).

3.1.2 Proof of the Existence Theorem

In this subsection the assumptions (2.62), (2.65) and (2.66) of main existence theorem 2.4.4 are checked. An analysis similar to ours is given in [MP07, MM09, MPP09]. As our model allows for more general assumptions in (3.7) we repeat all steps for the readers convenience. In particular, previous work (e.g. [MPP09]) assumes (3.7(H2)) and (H4*), where (H4*) ensures that $\partial_A W(x,A,z)\in L^{p'}(\Omega,\mathbb{R}^{d\times d})$, which is not guaranteed by (3.7(H4)).

For a shorter notation in the proofs we introduce the following abbreviations:

$$\left.\begin{aligned}\mathcal{I}(t,u,z) &:= \int_\Omega \big(W(x,e(u)+e_D(t),z)+\delta_{[z_\star,1]}(z)\big)\,\mathrm{d}x,\\ \mathcal{C}(z) &:= \int_\Omega \tfrac{\kappa}{r}\,|\nabla z|^r\,\mathrm{d}x,\\ \mathcal{J}(t,u,z) &:= \mathcal{I}(t,u,z)-\langle l(t),u+g(t)\rangle,\end{aligned}\right\}\tag{3.10}$$

such that

$$\mathcal{E}(t,u,z)=\mathcal{I}(t,u,z)+\mathcal{C}(z)-\langle l(t),u+g(t)\rangle=\mathcal{J}(t,u,z)+\mathcal{C}(z)\,.\tag{3.11}$$

An important tool in the proofs will be Korn's inequality, which holds for functions $u\in\mathcal{U}\subset W^{1,p}(\Omega,\mathbb{R}^d)$ for $\mathcal{U}$ defined by (3.8). A proof of Korn's inequality for $p\in(1,\infty)$ can e.g. be found in [KO88, Pom03] and we also refer to Theorem 4.1.5 in Chapter 4.

Theorem 3.1.2 (Korn's inequality) *Let* $\Omega\subset\mathbb{R}^d$ *and* $\Gamma_{\mathrm{Dir}}\subset\partial\Omega$ *satisfy* (3.6(A1)) *and let* $1<p<\infty$. *There is a constant* $C_K=C_K(\Omega,p)$ *such that for every* $v\in\mathcal{U}$ *the following estimate holds:*

$$\|v\|_{W^{1,p}(\Omega,\mathbb{R}^d)}\le C_K\|e(v)\|_{L^p(\Omega,\mathbb{R}^{d\times d})}\,.\tag{3.12}$$

Compactness of the energy sublevels (2.62(E1))

In the following, the weak sequential compactness of the energy sublevels is established using the standard approach in the direct method of the calculus of variations.

Lemma 3.1.3 *Let assumptions* (3.6) *and* (3.7) *hold. Then there are constants* $c_3, C_3>0$ *such that* $\mathcal{E}(t,\cdot,\cdot):\mathcal{U}\times\mathcal{Z}\to\mathbb{R}_\infty$ *satisfies a growth estimate of the form*

$$\mathcal{E}(t,u,z)\ge c_3\Big(\|u\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)}+\|z\|^r_{W^{1,r}(\Omega)}\Big)-C_3\ \ \textit{for all}\ (u,z)\in\mathcal{U}\times\mathcal{Z}\,.\tag{3.13}$$

Proof: For $(x,e,z,A)\in\Omega\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times[z_\star,1]\times\mathbb{R}^d$ we set

$$\overline{W}(x,e,z,A):=W(x,e,z)+\tfrac{\kappa}{r}\,|A|^r\,.$$

Let $(u,z)\in\mathcal{Q}$. If $\delta_{[z_\star,1]}(z)=\infty$ on a set of positive measure then (3.13) trivially holds. Hence assume that z satisfies $\delta_{[z_\star,1]}(z)=0$ a.e.. Using hypotheses (3.6(A2)), (3.6(A3)), (3.7(H3)), Young's and Korn's inequality we get

$$\begin{aligned}\mathcal{E}(t,u,z)&=\int_\Omega \overline{W}(x,e(u){+}e_D(t),z,\nabla z)\,\mathrm{d}x-\langle l(t),u{+}g(t)\rangle\\ &\ge c_1(\|e(u)\|_{L^p}{-}c_g)^p{-}(C{+}\tfrac{\kappa}{r})\mathcal{L}^d(\Omega){-}c_l(\|u\|_{W^{1,p}}{+}c_g){+}\tfrac{\kappa}{r}\|z\|^r_{W^{1,r}}\\ &\ge c_1(2^{1-p}\|e(u)\|^p_{L^p}{-}c_g^p){-}(C{+}\tfrac{\kappa}{r})\mathcal{L}^d(\Omega){-}c_l(\|u\|_{W^{1,p}}{+}c_g){+}\tfrac{\kappa}{r}\|z\|^r_{W^{1,r}}\\ &\ge \tfrac{2^{1-p}c_1}{C_K^p}\|u\|^p_{W^{1,p}}{-}(C{+}\tfrac{\kappa}{r})\mathcal{L}^d(\Omega){-}c_1c_g^p{-}\tfrac{1}{p'}\left(\tfrac{c_l}{\varepsilon}\right)^{p'}-\tfrac{(\varepsilon\|u\|_{W^{1,p}})^p}{p}{-}c_lc_g{+}\tfrac{\kappa}{r}\|z\|^r_{W^{1,r}}\\ &\ge \tfrac{2^{-p}c_1}{C_K^p}\|u\|^p_{W^{1,p}}{+}\tfrac{\kappa}{r}\|z\|^r_{W^{1,r}}{-}(C{+}\tfrac{\kappa}{r})\mathcal{L}^d(\Omega){-}c_1c_g^p{-}c_lc_g{-}\tfrac{1}{p'}\left(\tfrac{c_l}{\varepsilon}\right)^{p'}\,,\end{aligned}\tag{3.14}$$

where Young's inequality with $\varepsilon := \left(\frac{2^{-p}c_1 p}{C_K^p}\right)^{\frac{1}{p}}$ leads to the third inequality of (3.14).

This proves (3.13) with suitable c_3 and C_3. ■

Coercivity estimate (3.13) together with weak lower semicontinuity yields the compactness of the energy sublevels.

Proposition 3.1.4 *Let assumptions* (3.6) *as well as* (3.7) *hold. Then* $\mathcal{E}(t,\cdot,\cdot)$ *is sequentially lower semicontinuous with respect to the weak topology of* $\mathcal{Q}$ *and its sublevels* $L_E(t)$ *are weakly sequentially compact in* $\mathcal{Q}$.

Proof: First, we obtain that $\mathcal{C}(\cdot) : W^{1,r}(\Omega) \to \mathbb{R}$ is bounded from below by 0 and lower semicontinuous, since every L^r-converging sequence contains a subsequence that converges pointwise a.e. by Riesz' convergence theorem. Moreover, $\mathcal{C}(\cdot)$ is convex and hence weakly sequentially lower semicontinuous by [Dac89, p. 49, Th. 1.2.]. Furthermore, [Dac89, p. 74] states the weak sequential lower semicontinuity of $J(\xi,\eta) = \int_\Omega \widetilde{W}(x,\xi(x),\eta(x))\,\mathrm{d}x$ for $\widetilde{W}(x,\xi(x),\eta(x)) = W(x,\xi(x),\eta(x)) + \delta_{[z_\star,1]}(\eta)$, $\eta = z$, $\xi = e(u)$ on $W^{1,p}(\Omega,\mathbb{R}^d) \times L^r(\Omega)$ if hypotheses (3.7(H1))-(3.7(H3)) hold true, since the compact embedding $W^{1,r}(\Omega) \Subset L^r(\Omega)$ by Rellich's embedding theorem implies the strong L^r-convergence of a sequence converging weakly in $W^{1,r}(\Omega)$. Hence, $\mathcal{E}$ is weakly sequentially lower semicontinuous on $\mathcal{Q}$.

Let now $(u_k,z_k)_{k\in\mathbb{N}} \subset L_E(t) \subset \mathcal{Q}$. Then estimate (3.13) yields

$$\|e(u_k)\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + \|z_k\|_{W^{1,r}(\Omega)} \le \left(\tfrac{E+C_3}{c_3}\right)^{\frac{1}{p}} + \left(\tfrac{E+C_3}{c_3}\right)^{\frac{1}{r}}. \tag{3.15}$$

Since the spaces $W^{1,p}(\Omega,\mathbb{R}^d)$, $W^{1,r}(\Omega)$ are real, reflexive Banach spaces for $1 < p, r < \infty$, the sequence $(u_k,z_k)_{k\in\mathbb{N}}$ contains a subsequence converging weakly in $\mathcal{Q}$. In particular, due to the compact embedding of $\mathcal{Q}$ into $L^p(\Omega,\mathbb{R}^d) \times L^r(\Omega)$ and Riesz' convergence theorem we find a further subsequence converging pointwise a.e. in Ω with their limits $z \in \mathcal{Z}$, $u \in W^{1,p}(\Omega,\mathbb{R}^d)$ satisfying $z_\star \le z \le 1$ a.e. in Ω and $u = 0$ on Γ_{Dir}. Due to the weak sequential lower semicontinuity of $\mathcal{E}(t,\cdot)$ on $\mathcal{Q}$ the limit (u,z) of the subsequence is an element of $L_E(t)$. This shows that the sublevels are weakly sequentially compact, so that (2.62(E1)) is proven. ■

Remark 3.1.5 (Existence, uniqueness of minimizers) *A direct consequence of Proposition 3.1.4 is the existence of minimizers for the minimization problems*

$$\min_{(\tilde{u},\tilde{z})\in\mathcal{Q}} (\mathcal{E}(t,\tilde{u},\tilde{z}) + \mathcal{D}(z,\tilde{z})),\ \min_{\tilde{u}\in\mathcal{U}} \mathcal{J}(t,\tilde{u},z) \ \text{and}\ \min_{\tilde{q}\in\mathcal{Q}} \mathcal{J}(t,\tilde{q})$$

for all $t\in[0,T]$ *and all* $z\in\mathcal{Z}$, *as well as for the time-incremental problems* (2.67) *in every time step. This implies that the stable sets* $\mathcal{S}(t)$ *are non-empty for every* $t\in[0,T]$. *If strict convexity is claimed in* (3.7(H2)), *then the minimizers* $u\in\mathcal{U}$ *of* $\mathcal{J}(t,\cdot,z)$ *are even unique.*

Control of the power of the energy (2.62(E2))

In the sequel, condition (2.62(E2)) is shown using assumptions (3.6) and (3.7(H1))-(3.7(H4)).

As a first step we derive a Lipschitz estimate for the stored elastic energy density.

Lemma 3.1.6 (Lipschitz estimate for W) *Let* (3.7(H2)) *as well as* (3.7(H4)) *be satisfied. Then for every* $(x,z)\in\Omega\times[z_\star,1]$ *and any* $e,\tilde e\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ *we have*

$$|W(x,\tilde e,z)-W(x,e,z)|\le\frac{c}{2}(W(x,e,z)+W(x,\tilde e,z)+2\tilde c)|\tilde e-e|\,. \tag{3.16}$$

Proof: Under consideration of (3.7(H2)) and (3.7(H4)) we obtain for $\alpha\in[0,1]$:

$$\begin{aligned}|W(x,\tilde e,z)-W(x,e,z)|&=\left|\int_0^1\partial_e W(x,(e+\alpha(\tilde e-e)),z):(\tilde e-e)\mathrm{d}\alpha\right|\\&\le\int_0^1 c(\alpha(W(x,\tilde e,z)+\tilde c)+(1-\alpha)(W(x,e,z)+\tilde c))|\tilde e-e|\,\mathrm{d}\alpha\\&=\frac{c}{2}(W(x,\tilde e,z)+\tilde c)|\tilde e-e|+\frac{c}{2}(W(x,e,z)+\tilde c)|\tilde e-e|\,,\end{aligned}$$

which gives the result. ■

Now, we are in a position to prove condition (2.62(E2)).

Theorem 3.1.7 *Let* (3.7(H2))-(3.7(H4)) *and* (3.6) *be satisfied. Then there exist constants* $c_0\ge 0$, $c_1>0$ *such that for every* $(t_q,q)\in[0,T]\times\mathcal{Q}$ *with* $\mathcal{E}(t_q,q)<\infty$ *it holds*

$$\mathcal{E}(\cdot,q)\in\mathrm{C}^1([0,T]),\ \ \textit{where}$$

$$\partial_t\mathcal{E}(t,q)=\int_\Omega\partial_e W(x,e(u)+e_D(t),z):\dot e_D(t)\,\mathrm{d}x-\langle\dot l(t),u+g(t)\rangle-\langle l(t),\dot g(t)\rangle \tag{3.17}$$

$$\textit{and}\quad|\partial_t\mathcal{E}(t,q)|\le c_1(\mathcal{E}(t,q)+c_0)\quad\textit{for every } t\in[0,T]\,. \tag{3.18}$$

Proof: Let $q=(u,z)\in\mathcal{Q}$. Note that (3.7(H4)) together with Gronwall's lemma implies that $W(x,e+\tilde e,z)\le\exp(c|\tilde e|)(W(x,e,z)+c\tilde c|\tilde e|)$ for all (x,e,z), $(x,\tilde e,z)\in\Omega\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times[z_\star,1]$. Hence the assumption $\mathcal{E}(t_q,q)=:E_q<\infty$ for some $t_q\in[0,T]$ together with (3.6(A2)) and (3.6(A3)) yields $\mathcal{E}(t,q)<\tilde E_q<\infty$ for every t in a sufficiently small neighborhood $\mathrm{N}(t_q)\subset[0,T]$ of t_q, since obviously $\delta_{[z_\star,1]}(z)=0$. Thus, $\mathcal{E}(\cdot,q)$ as the sum and composition of the continuous functions $l(\cdot),g(\cdot)$, $W(x,\cdot,z)$, $\langle\cdot,\cdot\rangle$ and $\int_\Omega(\cdot)\mathrm{d}x$ is continuous itself.

In a first step, we prove that the time-derivative $\partial_t\mathcal{E}(\cdot,q)$ exists in $\mathrm{N}(t_q)$. In this neighborhood the estimate (3.18) can be derived as a second step. We will obtain that the constants are independent of t_q and $\mathrm{N}(t_q)$. This allows us to apply Gronwall's lemma and Lipschitz estimate (2.64) uniformly in each neighborhood of any time t_q with finite energy. Thus, $\mathcal{E}(\cdot,q)\in\mathrm{C}^1([0,T])$ follows.

Now, we prove the existence of $\partial_t\mathcal{E}(t,q)$ for $t\in\mathrm{N}(t_q)$. For this we define for $t\in\mathrm{N}(t_q)$

$$h(x,t,\alpha):=\begin{cases}\frac{1}{\alpha}\left(W(x,e(u)+e_D(t+\alpha),z)-W(x,e(u)+e_D(t),z)\right) & \text{if } \alpha\neq 0\\ \partial_e W(x,e(u)+e_D(t),z):\dot e_D(t) & \text{if } \alpha=0\end{cases}$$

and we must show that $h(x,t,\cdot)\in\mathrm{C}^0([-\alpha_t,\alpha_t])$ for α_t suitably. By the mean value theorem of differentiability, we know the existence of $\tilde\alpha=\tilde\alpha(\alpha)$ for every $\alpha\in[-\alpha_t,\alpha_t]$, such that

$$\begin{aligned}
&\frac{1}{\alpha}\left(W(x,e(u)+e_D(t+\alpha),z)-W(x,e(u)+e_D(t),z)\right)\\
&= \partial_e W(x,e(u)+e_D(t+\tilde\alpha),z) : \dot e_D(t+\tilde\alpha) =: F \qquad (3.19)\\
&\qquad\to \partial_e W(x,e(u)+e_D(t),z) : \dot e_D(t) \quad \text{as } \alpha,\,\tilde\alpha\to 0 \text{ by (3.7(H4)) and (3.6(A2))}\,.
\end{aligned}$$

In order to show that the integrals converge as well, we are going to apply the dominated convergence theorem. For this we obtain by (3.6(A2)), (3.7(H4)) and the Gronwall estimate

$$|F| \le c_g c\,(W(x,e(u)+e_D(t+\tilde\alpha),z)+\tilde c) \le c_g c\,\exp(2cc_g)(W(x,e(u)+e_D(t),z)+2c_g c\tilde c+\tilde c)$$

By Lipschitz estimate (3.16), (3.6(A2)) and (3.6(A3)) we have

$$\begin{aligned}
&\left|\textstyle\int_\Omega W(x,e(u)+e_D(t+\tilde\alpha),z)-W(x,e(u)+e_D(t),z)\,\mathrm{d}x\right|\\
&\le \|e_D(t+\tilde\alpha)-e_D(t)\|_{L^\infty(\Omega,\mathbb{R}^{d\times d})}\left(2c\tilde c\mathcal{L}^d(\Omega)+\mathcal{E}(t,u,z)+\mathcal{E}(t+\tilde\alpha,u,z)+2c_lc_g\right) \overset{\tilde\alpha\to 0}{\longrightarrow} 0,
\end{aligned} \qquad (3.20)$$

since $\mathcal{E}(t+\tilde\alpha,u,z)<\tilde E_q$ for every $t+\tilde\alpha\in\mathrm{N}(t_q)$. The differentiability of $\langle l(t),u+g(t)\rangle$ is ensured by (3.6(A2)), (3.6(A3)). Thus we have proven the existence of $\partial_t\mathcal{E}(\cdot,q)$ in $\mathrm{N}(t_q)$.

By (3.13) we find an upper estimate for $\|e(u)+e_D(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})}$ in terms of $\mathcal{E}(t,q)$:

$$\begin{aligned}
\|e(u)+e_D(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})} &\le 2^{p-1}\left(\|e(u)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})}+c_g^p\right)\\
&\le 2^{p-1}\left(\tfrac{\mathcal{E}(t,q)+C_3}{c_3}+c_g^p\right) =: A_1\mathcal{E}(t,q)+B_1
\end{aligned} \qquad (3.21)$$

This estimate will be used in the following to get (3.18). We have

$$|\partial_t\mathcal{E}(t,q)| \le \left|\int_\Omega \partial_e W(x,e(u)+e_D(t),z) : \dot e_D(t)\,\mathrm{d}x\right| + |\langle \dot l(t),u+g(t)\rangle| + |\langle l(t),\dot g(t)\rangle|\,,$$

where the loading terms are treated with Korn's and Young's inequality as in the proof of (3.13), such that one obtains an estimate of the form

$$|\langle \dot l(t),u+g(t)\rangle| + |\langle l(t),\dot g(t)\rangle| \le A_2\mathcal{E}(t,q)+B_2\,. \qquad (3.22)$$

Application of (3.7(H4)) to the stored elastic energy term yields

$$\begin{aligned}
&\left|\int_\Omega \partial_e W(x,e(u)+e_D(t),z) : \dot e_D(t)\,\mathrm{d}x\right| \le c_g c\,(\mathcal{I}(t,q)+\tilde c\mathcal{L}^d(\Omega))\\
&\le c_g c\left(\mathcal{E}(t,q)+c_l\|u\|_{W^{1,p}(\Omega,\mathbb{R}^{d\times d})}+c_lc_g+\tilde c\mathcal{L}^d(\Omega)\right)\\
&= A_3\left(\mathcal{E}(t,q)+\|u\|_{W^{1,p}(\Omega,\mathbb{R}^{d\times d})}\right)+B_3 =: G\,.
\end{aligned} \qquad (3.23)$$

Applying Korn's inequality (3.12) to $\|u\|_{W^{1,p}(\Omega,\mathbb{R}^{d\times d})}$ leads to the estimate

$$
\begin{aligned}
&\left|\int_\Omega \partial_e W(x, e(u){+}e_D(t), z) : \dot{e}_D(t)\,\mathrm{d}x\right| \le G \\
&\le A_3\left(\mathcal{E}(t,q) + C_K\|e(u){+}e_D(t)\|_{L^p(\Omega,\mathbb{R}^{d\times d})}\right) + A_3C_Kc_g + B_3 \\
&\le A_4(1 + \|e(u){+}e_D(t)\|_{L^p(\Omega,\mathbb{R}^{d\times d})})^p + A_3\,\mathcal{E}(t,q) + B_3 \\
&\le A_4 2^{p-1}(1 + \|e(u){+}e_D(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})}) + A_3\,\mathcal{E}(t,q) + B_3 \\
&\le A_4 2^{p-1}(1 + A_1\mathcal{E}(t,q) + B_1) + A_3\,\mathcal{E}(t,q) + B_3 = A_5\,\mathcal{E}(t,q) + B_5\,,
\end{aligned}
\tag{3.24}
$$

where (3.21) has been applied to obtain the last inequality. Combining (3.22), (3.24) yields the desired estimate (3.18). ∎

Proof of the abstract assumptions on the dissipation distance

Now, we show that the dissipation distance that refers to a rate-independent damage process satisfies the assumptions (2.65(D1)) and (2.65(D2)).

Theorem 3.1.8 *The dissipation distance $\mathcal{D}$ on $\mathcal{Z}$ given by* (3.1), (3.2) *satisfies* (2.65).

Proof: **Ad (2.65(D1)):** By (3.1) we have $\mathcal{D}(z_1,z_2) \ge \varrho_0\|z_2{-}z_1\|_{L^1(\Omega)}$. Hence, $\mathcal{D}(z_1,z_2)=0$ implies $z_1=z_2$. Let now $z_1,z_2,z_3\in\mathcal{Z}$ to show that the triangle-inequality holds. If its right-hand side is infinite, then the inequality is satisfied trivially. For a finite right-hand side $z_1\ge z_2\ge z_3$ a.e. in Ω is necessary and hence we even obtain equality.
Ad (2.65(D2)): To show sequential lower semicontinuity, let $z_{0_k}\rightharpoonup z_0$, $z_{1_k}\rightharpoonup z_1$ in $W^{1,r}(\Omega)$ and put $w_k := z_{1_k}-z_{0_k}$, $w := z_1-z_0$. Hence $0\ge w_k\to w$ pointwise a.e. for a subsequence so that also $w\le 0$ a.e.. Thus $\mathcal{D}(z_0,z_1)=\infty$ can be excluded. Assume now that $\liminf_{k\to\infty}\mathcal{D}(z_{0_k},z_{1_k})<\infty$, otherwise the inequality trivially holds. For a subsequence that attains the limit inferior, i.e. $w_k\le 0$ for all $k\in\mathbb{N}$, we obtain that

$$|\mathcal{D}(z_{0_k},z_{1_k})-\mathcal{D}(z_0,z_1)| \le \|\varrho\|_{L^\infty(\Omega)}\|w_k-w\|_{L^1(\Omega)} \to 0 \text{ as } k\to\infty$$

due to the compact embedding $W^{1,r}(\Omega)\Subset L^1(\Omega)$. Thus $\mathcal{D}(z_0,z_1)\le\liminf_{k\to\infty}\mathcal{D}(z_{0_k},z_{1_k})$. ∎

Convergence of the time-derivative of the energy functional (2.66(C1))

The aim in this subsection is to prove the first compatibility condition.

Theorem 3.1.9 *Let hypotheses* (3.7), (3.6) *and* (2.65) *hold true. Then, for every stable sequence $(t_k,q_k)_{k\in\mathbb{N}}\subset[0,T]\times\mathcal{Q}$ with $t_k\to t$ and $q_k\rightharpoonup q$ in $\mathcal{Q}$ we have*

$$\partial_t\mathcal{E}(t,q_k)\to\partial_t\mathcal{E}(t,q)\,. \tag{3.25(C1)}$$

Proof: Since $\mathcal{C}(z):=\int_\Omega \frac{\kappa}{r}|\nabla z|^r\,\mathrm{d}x$ does not depend on time t we have $\partial_t\mathcal{E}(t,q)=\partial_t\mathcal{J}(t,q)$, where $\mathcal{J}(t,q)=\mathcal{I}(t,q)-\langle l(t),u+g(t)\rangle$. As the last term is linear, it is sufficient to prove Theorem 3.1.9 for $\mathcal{I}$.

The following two properties, shown in separate lemmata later on, are utilized to obtain the convergence result:

(**P1**) It holds $\mathcal{I}(t,u_k,z_k)\to\mathcal{I}(t,u,z)$ for every stable sequence $(t_k,u_k,z_k)_{k\in\mathbb{N}}$, where $t_k\to t$, $(u_k,z_k)\rightharpoonup(u,z)$ in $\mathcal{Q}$, see Lemma 3.1.10 .

(**P2**) For $q\in L_E(0)$ the derivatives $\partial_t\mathcal{I}(\cdot,q)$ are uniformly continuous, see Lemma 3.1.11 .

If the properties (P1) and (P2) hold, then we are able to apply Proposition 3.3 of [FM06] to $\mathcal{I}$, which states that $\partial_t\mathcal{I}(t,q_k)\to\partial_t\mathcal{I}(t,q)$. Thus, (3.25(C1)) is established. ∎

It remains to verify the properties (P1) and (P2) from the proof of Theorem 3.1.9.

Property (P1) is a consequence of

Lemma 3.1.10 *Let $(t_k,u_k,z_k)_{k\in\mathbb{N}}$ be a stable sequence with $t_k\to t$, $(u_k,z_k)\rightharpoonup(u,z)$ in $\mathcal{Q}$ as $k\to\infty$ and let (3.6) as well as (3.7) hold. Then*

$$\mathcal{J}(t,u,z_k)\to\mathcal{J}(t,u,z)\quad\textit{and}\quad\mathcal{J}(t,u_k,z_k)\to\mathcal{J}(t,u,z)\quad\textit{as } k\to\infty\,.$$

Proof: As a first step, we show that $\mathcal{J}(t,u,z_k)\to\mathcal{J}(t,u,z)$.

We have $W(\cdot,e(u)+e_D(t),z_k)\to W(\cdot,e(u)+e_D(t),z)$ in measure, since each subsequence $(W(\cdot,e(u)+e_D(t),z_{k_l}))_{l\in\mathbb{N}}$ contains a further subsequence that converges pointwise a.e.. This is due to the continuity of W with respect to z and Riesz' convergence theorem. By (3.7(H6)) we obtain for every $k\in\mathbb{N}$ that

$$W(x,e(u)+e_D(t),z_k)\le K(W(x,e(u)+e_D(t),z_\star)+\widetilde{K})\le K(W(x,e(u)+e_D(t),z)+\widetilde{K})\,.$$

Moreover, we have

$$\begin{aligned}&\int_\Omega(W(x,e(u)+e_D(t),z)+\widetilde{K})\,\mathrm{d}x\le\mathcal{E}(t,u,z)+\widetilde{K}\mathcal{L}^d(\Omega)+c_l(\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)}+c_g)\\&\le\liminf_{k\to\infty}(\mathcal{E}(t_k,u_k,z_k)+c_E|t-t_k|)+\widetilde{K}\mathcal{L}^d(\Omega)+c_l(\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)}+c_g)<\infty\end{aligned}$$

by lower semicontinuity, (2.61) and (2.64). The dominated convergence theorem now yields $\mathcal{J}(t,u,z_k)\to\mathcal{J}(t,u,z)$. Since u_k minimizes $\mathcal{J}(t_k,\cdot,z_k)$ and since (2.61), (2.64) hold, we infer

$$\mathcal{J}(t,u_k,z_k)-c_E|t_k-t|\le\mathcal{J}(t_k,u_k,z_k)\le\mathcal{J}(t,u,z_k)+c_E|t_k-t|\to\mathcal{J}(t,u,z)$$

We conclude $\mathcal{J}(t_k,u_k,z_k)\to\mathcal{J}(t,u,z)$ exploiting the weak sequential lower semicontinuity

$$\begin{aligned}\mathcal{J}(t,u,z)&\le\liminf_{k\to\infty}\big(\mathcal{J}(t,u_k,z_k)-c_E|t_k-t|\big)\le\liminf_{k\to\infty}\mathcal{J}(t_k,u_k,z_k)\\&\le\limsup_{k\to\infty}\mathcal{J}(t_k,u_k,z_k)\le\limsup_{k\to\infty}\big(\mathcal{J}(t,u,z_k)+c_E|t_k-t|\big)=\mathcal{J}(t,u,z)\,.\end{aligned}$$

∎

The next lemma refers to property (P2) from the proof of Theorem 3.1.9. It is based on the fact that the given data are continuously differentiable on the compact time interval $[0,T]$ by (A2), (A3) in (3.6), and hence they and their time-derivatives are uniformly continuous.

Lemma 3.1.11 (Equi-uniform continuity of $\partial_t\mathcal{I}(\cdot,q)$) *Let* (3.6) *and* (3.7) *be satisfied. Then, for each $E,\varepsilon>0$ there exists $\delta>0$ such that for every $q\in\mathcal{Q}$ with $\mathcal{E}(0,q)<E$ it holds:*

$$\text{If } |t-s|<\delta \text{ then } |\partial_t\mathcal{I}(t,q)-\partial_t\mathcal{I}(s,q)|<\varepsilon\,.$$

Proof: Due to (3.6(A2)) and (3.6(A3)) we find for every $\tilde\varepsilon>0$ a $\tilde\delta>0$ such that for all $s,t\in[0,T]$ with $|s-t|<\tilde\delta$ we have $\|g(s)-g(t)\|_{W^{1,\infty}(\Omega,\mathbb{R}^d)}+\|\dot g(s)-\dot g(t)\|_{W^{1,\infty}(\Omega,\mathbb{R}^d)}<\tilde\varepsilon$. Choose now $\varepsilon,E>0$ and let $(u,z)\in L_E(0)$. By estimate (3.13) we obtain for $t=0$:

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)}\leq\left(\tfrac{\mathcal{E}(0,u,z)+C_3}{c_3}\right)^{\frac1p}\leq\left(\tfrac{E+C_3}{c_3}\right)^{\frac1p}=:\tilde B\,.$$

This shows that $u+g(t)$ with $(u,z)\in L_E(0)$ are uniformly bounded for every $t\in[0,T]$, since $\|u+g(t)\|_{W^{1,p}(\Omega,\mathbb{R}^d)}\leq\|u\|_{W^{1,p}(\Omega,\mathbb{R}^d)}+\|g(t)\|_{W^{1,p}(\Omega,\mathbb{R}^d)}\leq\tilde B+c_g=:B$.

Furthermore we estimate

$$\begin{aligned}
&|\partial_t\mathcal{I}(t,q)-\partial_t\mathcal{I}(s,q)|\\
&\leq\left|\int_\Omega\partial_eW(x,e(u){+}e_D(t),z):(\dot e_D(t)-\dot e_D(s))\,\mathrm{d}x\right| &&(3.26)\\
&\quad+\left|\int_\Omega(\partial_eW(x,e(u){+}e_D(t),z)-\partial_eW(x,e(u){+}e_D(s),z)):\dot e_D(s)\,\mathrm{d}x\right|\,. &&(3.27)
\end{aligned}$$

In view of (3.7(H3)), (3.7(H4)) and Lipschitz estimate (2.64) we see that

$$\begin{aligned}
(3.26)&\leq\|\partial_eW(\cdot,e(u){+}e_D(t),z)\|_{L^1(\Omega)}\|\dot e_D(t)-\dot e_D(s)\|_{L^\infty(\Omega,\mathbb{R}^{d\times d})}\\
&\leq(\mathcal{E}(0,q)+C\mathcal{L}^d(\Omega)+c_ET+c_lB)\|\nabla\dot g(t)-\nabla\dot g(s)\|_{L^\infty(\Omega,\mathbb{R}^{d\times d})}<\frac{\varepsilon}{2}\,,
\end{aligned}$$

if $|t{-}s|{<}\tilde\delta_1$ is sufficiently small. In view of (3.7(H5)) and the Gronwall estimate we find

$$(3.27)\leq c_g\omega\big(\|e_D(t){-}e_D(s)\|_{L^\infty}\big)\big(\|W(\cdot,e(u){+}e_D(t),z)\|_{L^1}\big(1+\exp(2cc_g)\big)+C\big)<\tfrac{\varepsilon}{2}$$

for $|s-t|<\tilde\delta_2$ sufficiently small, where we used $C:=(1{+}\exp(2cc_g)cc_g)\tilde c\mathcal{L}^d(\Omega)$. Hence we obtain $(3.27)<\frac{\varepsilon}{2}$ if $|s{-}t|<\tilde\delta_2$. Altogether we conclude that $|\partial_t\mathcal{I}(s,q){-}\partial_t\mathcal{I}(t,q)|<\varepsilon$ if $|s{-}t|<\delta:=\min\{\tilde\delta_1,\tilde\delta_2\}$. ■

Closedness of the stable sets (2.66(C2)) and joint recovery condition

In the framework of damage we have to cope with a dissipation distance that is not weakly continuous on $W^{1,r}(\Omega)$. Hence it is not possible to show (2.66(C2)) directly as in [FM06, MP07], where weak continuity is essential. Like in [MR06, MRS08] we get (2.66(C2)) via the so-called joint recovery condition.

Definition 3.1.12 (Joint recovery condition) *The rate-independent system* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *satisfies the joint recovery condition if for all stable sequences* $(t_k, q_k)_{k\in\mathbb{N}} = (t_k, u_k, z_k)_{k\in\mathbb{N}}$ *with* $(t_k, q_k) \rightharpoonup (t, q)$ *in* $[0, T] \times \mathcal{Q}$ *and for every* $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ *there is a sequence* $(\hat{q}_k)_{k\in\mathbb{N}} = (\hat{u}_k, \hat{z}_k)_{k\in\mathbb{N}}$ *with* $\hat{q}_k \rightharpoonup \hat{q}$ *in* $\mathcal{Q}$ *and*

$$\limsup_{k\to\infty} \left(\mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(z_k, \hat{z}_k) - \mathcal{E}(t_k, q_k)\right) \le \mathcal{E}(t, \hat{q}) + \mathcal{D}(z, \hat{z}) - \mathcal{E}(t, q)\,. \tag{3.28}$$

As it was proven in Lemma 2.4.6 the joint recovery condition implies (2.66(C2)).

For our purpose, if $\mathcal{D}(z, \hat{z}) < \infty$, the joint recovery sequence has to be constructed in such a manner that also $\mathcal{D}(z_k, \hat{z}_k) < \infty$ is satisfied for every $k \in \mathbb{N}$. Otherwise the left-hand side in (3.28) is too big. In fact, we will enforce $\mathcal{D}(z_k, \hat{z}_k) \to \mathcal{D}(z, \hat{z})$, which follows from $z_k \rightharpoonup z$ and $\hat{z}_k \rightharpoonup \hat{z}$ if the additional constraint $\hat{z}_k \le z_k$ holds.

To this end, the case $1 < r \le d$ requires substantially new ideas compared to [MR06]. There the compact embedding $W^{1,r}(\Omega) \Subset \mathrm{C}(\overline{\Omega})$ permitted to gain the finiteness of the dissipation distance by choosing $\hat{z}_k := (z_k - \|z_k - z\|_{\mathrm{C}(\overline{\Omega})})^+$ with $(f)^+ := \max\{0, f\}$.

In the following, the result of [MR06] is extended to the case of $1 < r < \infty$ by constructing the joint recovery sequence in such a manner that the compact embedding $W^{1,r}(\Omega) \Subset \mathrm{C}(\overline{\Omega})$ is not needed for the proof of estimate (3.28).

For the construction of a joint recovery sequence we will entirely use that the superposition of a $W^{1,r}$-function with the Lipschitz continuous function $\max\{0, f\} : \mathbb{R} \to \mathbb{R}$ again gives a $W^{1,r}$-function:

Lemma 3.1.13 (Superposition lemma, [MM72]) *Let* $g : \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous and* $v \in W^{1,r}(\Omega)$*. Then* $g \circ v \in W^{1,r}(\Omega)$ *and*

$$\nabla(g \circ v)(x) = g'(v(x))\nabla v(x) \quad \textit{for a.a. } x \in \Omega\,.$$

The following result establishes the compatibility condition (2.66(C2)).

Theorem 3.1.14 (Joint recovery condition for $r \in (1, \infty)$) *Let* (3.7) *hold. Then, the rate-independent system* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *satisfies the joint recovery condition. Hence, if* $(t_k, q_k)_{k\in\mathbb{N}}$ *is a stable sequence with* $t_k \to t$, $q_k \rightharpoonup q$ *in* $\mathcal{Q}$*, then* $q \in \mathcal{S}(t)$*, i.e.* (2.66(C2)) *holds.*

Proof: Let $(t_k, u_k, z_k)_{k\in\mathbb{N}} \subset [0, T] \times \mathcal{U} \times \mathcal{Z}$ with $t_k \to t$, $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$ and $z_k \rightharpoonup z$ in $W^{1,r}(\Omega)$. Choose $\hat{q} \in \mathcal{Q}$ such that $\hat{q} \in L_E(t)$ for some $E \in \mathbb{R}$, otherwise (3.28) trivially holds. Now we distinguish between the following two cases:

Case A: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ be such that there exists a $\mathcal{L}^d$-measurable set $B \subset \Omega$ with $\mathcal{L}^d(B) > 0$ and $\hat{z} > z$ on B. Then $\mathcal{D}(z, \hat{z}) = \infty$ and (3.28) holds.

Case B: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ be such that $\hat{z} \le z$ a.e. in Ω. Then, $\mathcal{D}(z, \hat{z}) = \int_\Omega \varrho(z - \hat{z})\,\mathrm{d}x < \infty$. To construct a joint recovery sequence we put $\hat{u}_k := \hat{u}$ for every $k \in \mathbb{N}$ and

$$\hat{z}_k := \min\left\{\max\left\{(\hat{z} - \delta_k), z_\star\right\},\, z_k\right\} = \begin{cases} (\hat{z} - \delta_k) & \text{if } z_\star < (\hat{z} - \delta_k) \le z_k, \\ z_\star & \text{if } (\hat{z} - \delta_k) \le z_\star \le z_k, \\ z_k & \text{if } \max\left\{(\hat{z} - \delta_k), z_\star\right\} > z_k \end{cases} \tag{3.29}$$

where $0<\delta_k \overset{\mathbb{R}}{\to} 0$ will be chosen suitably in Step 2. Thus, $\hat{z}_k \le z_k$ a.e. and $\mathcal{D}(z_k, \hat{z}_k)<\infty$ for every $k\in\mathbb{N}$. Besides, it holds $\hat{z}_k(x)<\hat{z}(x)\le z(x)$ for a.e. $x\in\Omega$ with $\hat{z}(x)\neq z_\star$. Again we have $\hat{z}_k = z_k - \max\{0, z_k-(z_\star+\max\{0,(\hat{z}-\delta_k)-z_\star\})\}\in W^{1,r}(\Omega)$ by Lemma 3.1.13. See also Fig. 3.1 for the construction of $(\hat{z}_k)_{k\in\mathbb{N}}$.

For the joint recovery sequence constructed by (3.29) we can in general only prove weak convergence in $W^{1,r}(\Omega)$. This can be seen from Example 3.1.16 below the proof.

It holds $\mathcal{E}(t_k,\hat{q}_k)\le\mathcal{E}(t_k,\hat{q})+\mathcal{C}(\hat{z}_k)\le\hat{c}$ due to $\hat{q}\in L_E(t)$ and estimate (2.64) for $\hat{q}$. Furthermore, (2.64) provides a uniform Lipschitz constant for $(\hat{q}_k)_{k\in\mathbb{N}}$ such that

$$\mathcal{E}(t_k,\hat{q}_k)+\mathcal{D}(z_k,\hat{z}_k)-\mathcal{E}(t_k,q_k)\le\mathcal{E}(t,\hat{q}_k)+\mathcal{D}(z_k,\hat{z}_k)-\mathcal{E}(t,q_k)+2L|t_k-t|\,, \tag{3.30}$$

where L is the maximum of the uniform Lipschitz constants for $(q_k)_{k\in\mathbb{N}}$ and $(\hat{q}_k)_{k\in\mathbb{N}}$. Since $|t_k-t|\to 0$, inequality (3.28) holds if we can prove

$$\limsup_{k\to\infty}\left(\mathcal{E}(t,\hat{q}_k)+\mathcal{D}(z_k,\hat{z}_k)-\mathcal{E}(t,q_k)\right)\le\mathcal{E}(t,\hat{q})+\mathcal{D}(z,\hat{z})-\mathcal{E}(t,q)\,. \tag{3.31}$$

In order to show (3.31) we take into account that

$$\begin{aligned}&\limsup_{k\to\infty}(\mathcal{E}(t,\hat{q}_k)+\mathcal{D}(z_k,\hat{z}_k)-\mathcal{E}(t,q_k))\\ &\le\limsup_{k\to\infty}\mathcal{I}(t,\hat{q}_k)-\liminf_{k\to\infty}\mathcal{I}(t,q_k)+\limsup_{k\to\infty}\mathcal{D}(z_k,\hat{z}_k)+\limsup_{k\to\infty}(\mathcal{C}(\hat{z}_k)-\mathcal{C}(z_k))-\langle l(t),\hat{u}-u\rangle\end{aligned} \tag{3.32}$$

and estimate these limits in separate steps.

For a shorter notation we introduce the abbreviation $[f<g]:=\{x\in\Omega\,|\,f(x)<g(x)\}$ with an analogous meaning for $\le,\ge$ and $>$.

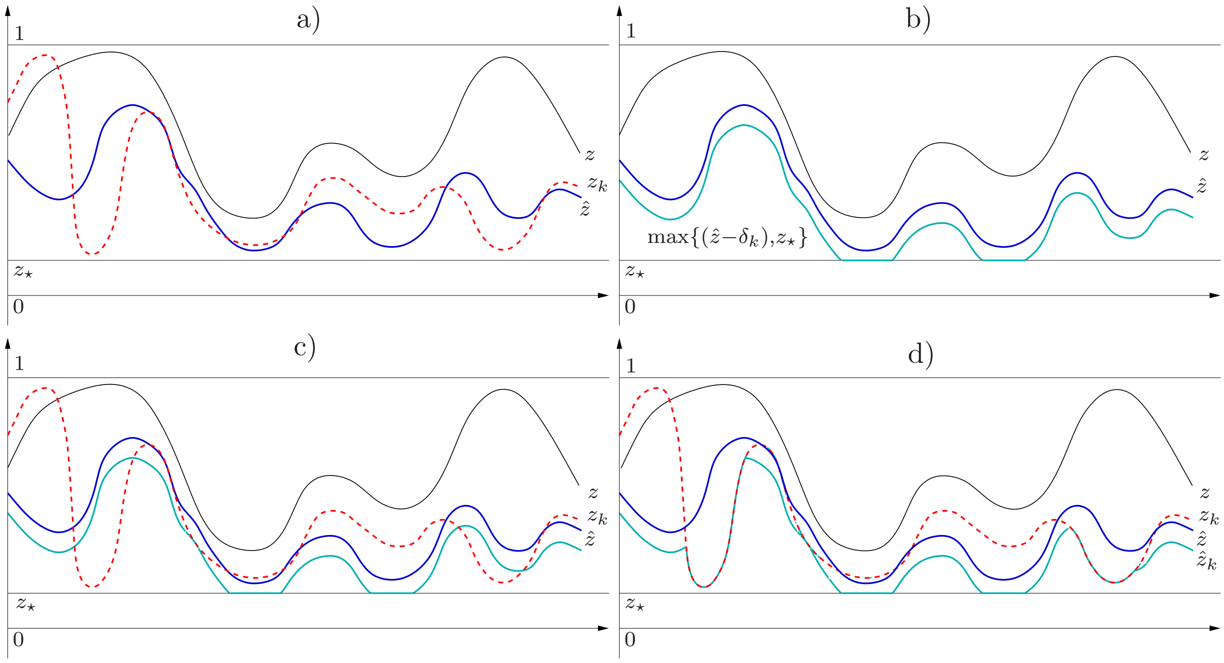
Step 1: We prove that $\hat{z}_k\rightharpoonup\hat{z}$ in $W^{1,r}(\Omega)$ as $k\to\infty$.
By construction the sequence $(\hat{z}_k)_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,r}(\Omega)$. Thus, there is a weakly convergent subsequence $\hat{z}_{k_l}\rightharpoonup\tilde{z}\in W^{1,r}(\Omega)$. By the compact embedding this subsequence converges strongly in $L^r(\Omega)$ and by Riesz' convergence theorem it has a further subsequence converging pointwise a.e. in Ω. This last subsequence has to converge $\hat{z}_{k_{l_m}}\to\hat{z}$ a.e. in Ω by definition of $\hat{z}_k$. Hence, we obtain $\tilde{z}=\hat{z}$ and thus, $\hat{z}_k\to\hat{z}$ in $L^r(\Omega)$. Since $(\hat{z}_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,r}(\Omega)$, the same arguments also yield $\hat{z}_k\rightharpoonup\hat{z}$ in $W^{1,r}(\Omega)$.

Step 2: We show that $\limsup_{k\to\infty}(\mathcal{C}(\hat{z}_k)-\mathcal{C}(z_k))\le\mathcal{C}(\hat{z})-\mathcal{C}(z)$:
For the calculation of the limit, the domain Ω is decomposed as follows:

$$\Omega=A_k\cup B_k \text{ with } B_k=[\max\{z_\star,(\hat{z}-\delta_k)\}>z_k]=[(\hat{z}-\delta_k)>z_k] \text{ and } A_k=\Omega\backslash B_k.$$

It holds $B_k=[(\hat{z}-\delta_k)>z_k]\subset[(z-\delta_k)>z_k]\subset[|z-z_k|\ge\delta_k]$. By application of Markov's inequality in estimate $(\mathbf{M})$ we can now determine $(\delta_k)_{k\in\mathbb{N}}$ in such a way that $\mathcal{L}^d(B_k)\to 0$ as $k\to\infty$:

$$\mathcal{L}^d(B_k)\le\mathcal{L}^d([|z-z_k|\ge\delta_k])\overset{(\mathbf{M})}{\le}\frac{1}{\delta_k^r}\int_\Omega|z-z_k|^r\,\mathrm{d}x\quad\to 0\,,$$



a) limit z, element of the sequence z_k and $\hat z$ b) construction of $\max\{(\hat z-\delta_k),z_\star\}$
c) the functions z, $\hat z$, z_k and $\max\{(\hat z-\delta_k),z_\star\}$ d) $\hat z_k=\min\{z_k,\max\{(\hat z-\delta_k),z_\star\}\}$

Figure 3.1: Construction of the joint recovery sequence

if, for instance, $\delta_k := \|z_k - z\|_{L^r(\Omega)}^{\frac{1}{r}}$. Note that Markov's inequality is only applicable if $\delta_k > 0$. But $\|z_k - z\|_{L^r(\Omega)} = 0$ implies $\mathcal{L}^d([|z_k - z| > 0]) = 0$ and hence $\mathcal{L}^d(B_k) \to 0$ as $k \to \infty$ is guaranteed. For $A_k = \Omega\backslash B_k$ we have $\mathcal{L}^d(A_k) \to \mathcal{L}^d(\Omega)$ as $k \to \infty$. Using the characteristic functions of these sets

$$I_{A_k}(x) := \begin{cases} 1 & \text{if } x \in A_k \\ 0 & \text{if } x \in B_k \end{cases}$$

and Lemma 3.1.15 from below we find $I_{A_k}\nabla z_k \rightharpoonup \nabla z$ in $L^r(\Omega,\mathbb{R}^d)$. By Lemma 3.1.13 and the weak sequential lower semicontinuity we conclude

$$\begin{aligned}\limsup_{k\to\infty}(\mathcal{C}(\hat z_k) - \mathcal{C}(z_k)) &= \limsup_{k\to\infty}\int_{A_k}\tfrac{\kappa}{r}\big(|\nabla\max\{z_\star,(\hat z-\delta_k)\}|^r - |\nabla z_k|^r\big)\,\mathrm{d}x\\ &\le \int_\Omega \tfrac{\kappa}{r}|\nabla\hat z|^r\,\mathrm{d}x - \liminf_{k\to\infty}\int_\Omega\tfrac{\kappa}{r}|I_{A_k}\nabla z_k|^r\,\mathrm{d}x \le \mathcal{C}(\hat z) - \mathcal{C}(z)\,.\end{aligned}$$

Step 3: Estimation of the remaining terms in line (3.32):
To calculate $\limsup_{k\to\infty}\mathcal{I}(t,\hat u,\hat z_k)$ we choose a subsequence $(\hat z_{k_l})_{l\in\mathbb{N}} \subset (\hat z_k)_{k\in\mathbb{N}}$ such that $\hat z_{k_l} \to \hat z\ \mathcal{L}^d$-a.e.. By (3.7(H1)) we have $W(x,e,\cdot) \in \mathrm{C}^0([z_\star,1])$ and hence we obtain that $W(\cdot,e(\hat u){+}e_D(t),\hat z_{k_l}) \to W(\cdot,e(\hat u){+}e_D(t),\hat z)\ \mathcal{L}^d$-a.e.. Furthermore, by (3.7(H6)) we infer that $W(x,e(\hat u){+}e_D(t),\hat z_{k_l}) \le K(W(x,e(\hat u){+}e_D(t),\hat z){+}\widetilde{K}) \in L^1(\Omega)$. Then, the dominated convergence theorem gives $\mathcal{I}(t,\hat u,\hat z_{k_l}) \to \mathcal{I}(t,\hat u,\hat z)$.

The estimate $-\liminf_{k\to\infty}\mathcal{I}(t,q_k)\le -\mathcal{I}(t,q)$ is obvious by the weak sequential lower semicontinuity of $\mathcal{I}(t,\cdot)$.

By construction it holds $\hat{z}_k\le z_k$ for every $k\in\mathbb{N}$ and thus

$$\lim_{k\to\infty}\mathcal{D}(z_k,\hat{z}_k) \;=\; \lim_{k\to\infty}\int_\Omega R(x,\hat{z}_k-z_k)\,\mathrm{d}x=\int_\Omega R(x,\hat{z}-z)\,\mathrm{d}x=\mathcal{D}(z,\hat{z})\ ,$$

due to the continuity of R, since both $z_k\to z$ and $\hat{z}_k\to\hat{z}$ in $L^1(\Omega)$ as $k\to\infty$.

Hence inequality (3.31) is proven. ∎

It remains to verify the lemma applied in Step 2 of the above proof.

Lemma 3.1.15 *Let $A_k\subset\Omega$, $\mathcal{L}^d(A_k)\to\mathcal{L}^d(\Omega)$ and $f_k\rightharpoonup f$ in $L^r(\Omega,\mathbb{R}^d)$. Then $I_{A_k}f_k\rightharpoonup f$.*

Proof: Let $\varphi\in L^{r'}(\Omega,\mathbb{R}^d)$ and put $\varphi_k:=I_{A_k}\varphi$. Since $\mathcal{L}^d(\Omega\backslash A_k)\to 0$ it holds that $\|\varphi_k-\varphi\|^{r'}_{L^{r'}(\Omega,\mathbb{R}^d)}=\int_{\Omega\backslash A_k}|\varphi|^{r'}\,\mathrm{d}x\to 0$. Together with $f_k\rightharpoonup f$ in $L^r(\Omega,\mathbb{R}^d)$ this implies $\int_\Omega I_{A_k}f_k\cdot\varphi\,\mathrm{d}x=\int_\Omega f_k\cdot\varphi_k\,\mathrm{d}x\to\int_\Omega f\cdot\varphi\,\mathrm{d}x$ for any $\varphi\in L^{r'}(\Omega,\mathbb{R}^d)$. ∎

Now, we give an example of a weakly converging sequence, where the method (3.29) generates a weakly converging recovery sequence, that does not converge strongly.

Example 3.1.16 *Consider $z_\star=0$, $\Omega=\{(r,\phi)\,|\,0\le r<1,\ 0\le\phi\le 2\pi\}\subset\mathbb{R}^2$ and*

$$z_k(r):=\begin{cases} kr & \text{for } 0\le r\le\frac{1}{2k},\\ \frac{1}{2} & \text{for } \frac{1}{2k}<r<1,\end{cases}\qquad k\in\mathbb{N}\ . \tag{3.33}$$

Then $z_k\rightharpoonup z=\frac{1}{2}$ in $H^1(\Omega)$. For $\hat{z}:=\frac{1}{4}$ and $(\hat{z}_k)_{k\in\mathbb{N}}$ as in (3.29) it holds $\hat{z}_k\rightharpoonup\hat{z}$ in $H^1(\Omega)$, but $\|\hat{z}_k-\hat{z}\|^2_{H^1(\Omega)}\to\frac{\pi}{16}$.

However, the sequence in (3.33) may not be stable. Thus, it still might be possible to prove strong convergence of a recovery sequence (3.29) if a stable sequence is used.

3.1.3 Examples

In the following we discuss some examples, which are well-known in engineering, and we show that they satisfy the hypotheses (3.7).

Damage of concrete

In the style of [Fré02, p. 319], where a model describing the damage of concrete is introduced, we consider here a stored elastic energy density of the form

$$W(e,z):=\mu|e|^2+\varphi_-\big(\operatorname{tr}(-e)^+\big)+z\varphi_+\big(\operatorname{tr}(e)^+\big), \tag{3.34}$$

where $\mu>0$ is the shear modulus. The functions $\varphi_\pm:[0,\infty)\to[0,\infty)$ only see the volume changes. They are convex and continuously differentiable with $\varphi_\pm(0)=0$ and

$|\varphi'_\pm(x)| \le c(\varphi_\pm(x)+\hat{c})$ for constants $c, \hat{c} > 0$. Since damage mostly occurs under extension and since compression corresponds to $\mathrm{tr}(e) < 0$, the function φ_- is not coupled to damage. However, φ_+ is premultiplied by z, since tension forces in concrete easily produce damage.

It is obvious that $W : \mathbb{R}^{d\times d}_{\mathrm{sym}}\times[z_\star, 1] \to \mathbb{R}$ satisfies (3.7(H1)), (3.7(H3)) and (3.7(H6)). Convexity condition (3.7(H2)) holds, since $\mathrm{tr}(\cdot)$ is linear, $\varphi_\pm$ are monotone and convex and $(\pm(\cdot))^+$ are convex as well. To demonstrate (3.7(H4)) we use $\partial_e(\pm\,\mathrm{tr}(e)^+){:}\tilde{e} = \mathrm{sgn}(\pm\,\mathrm{tr}(e)^+)\mathrm{Id}{:}\tilde{e}$. Applying the chain rule on $\varphi_\pm\big(\,\mathrm{tr}(\pm e(u))^+\big)$ we conclude that

$$\begin{aligned}|\partial_e W(e,z)| &= |2\mu e + \varphi'_-\big(\mathrm{tr}(-e)^+\big)\,\mathrm{sgn}(-\,\mathrm{tr}(e)^+)\mathrm{Id} + z\varphi'_+\big(\,\mathrm{tr}(e)^+\big)\,\mathrm{sgn}(\mathrm{tr}(e)^+)\mathrm{Id}|\\ &\le 4\mu(|e|^2{+}1) + dc\big(\varphi_-\big(\,\mathrm{tr}(-e)^+\big){+}\hat{c}\big) + zdc_+\big(\varphi_+\big(\,\mathrm{tr}(e)^+\big){+}\hat{c}\big)\\ &\le \max\{1, dc\}\big(W(e,z) + \max\{1,\hat{c}\}\big)\,.\end{aligned}$$

Assume now that there is a modulus of continuity $\omega_\pm : [0,\infty) \to [0,\infty)$, such that $|\varphi'_\pm(x) - \varphi'_\pm(x)| \le \omega_\pm(|x-y|)\big(\varphi_\pm(x) + \varphi_\pm(y) + 1\big)$ for all $x,\, y \in [0,\infty)$ and additionally that $\varphi'(0) = 0$. Then we may verify (3.7(H5)). For $e,\, \hat{e} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we obtain

$$\begin{aligned}|\partial_e W(e,z) - \partial_e W(\hat{e},z)| &\le 2\mu(|e|^2 + |\hat{e}|^2 + 1)|e-\hat{e}|\\ &+ d\big|\varphi'_-\big(\mathrm{tr}(-e)^+\big) - \varphi'_-\big(\mathrm{tr}(-\hat{e})^+\big)\big||\mathrm{sgn}(\mathrm{tr}(-\hat{e})^+)|\\ &+ d\big|\varphi'_-\big(\mathrm{tr}(-\hat{e})^+\big)\big||\mathrm{sgn}(\mathrm{tr}(-\hat{e})^+) - \mathrm{sgn}(\mathrm{tr}(-e)^+)|\\ &+ zd\big|\varphi'_+\big(\mathrm{tr}(e)^+\big) - \varphi'_+\big(\mathrm{tr}(\hat{e})^+\big)\big||\mathrm{sgn}(\mathrm{tr}(e)^+)|\\ &+ zd\big|\varphi'_+\big(\mathrm{tr}(\hat{e})^+\big)\big||\mathrm{sgn}(\mathrm{tr}(\hat{e})^+) - \mathrm{sgn}(\mathrm{tr}(e)^+)|\,.\end{aligned} \tag{3.35}$$

Consider now $e_1,\, e_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. If $\mathrm{tr}(e_1),\, \mathrm{tr}(e_2) > 0$ we set $e = e_1$ and $\hat{e} = e_2$ in (3.35) and we find $|\mathrm{sgn}(\mathrm{tr}(-\hat{e})^+)| = |\mathrm{sgn}(\mathrm{tr}(-\hat{e})^+) - \mathrm{sgn}(\mathrm{tr}(-e)^+)| = |\mathrm{sgn}(\mathrm{tr}(\hat{e})^+) - \mathrm{sgn}(\mathrm{tr}(e)^+)| = 0$ and $|\mathrm{sgn}(\mathrm{tr}(e)^+)| = 1$. With the additional assumption for φ'_+ and $C := \max\{2\mu, dc, dc\hat{c}, 3d\}$ as well as $E := e_1 - e_2$ we conclude that

$$|\partial_e W(e_1,z) - \partial_e W(e_2,z)| \le C\big(|E| + \omega_+(|E|) + \omega_-(|E|)\big)\big(W(e_1,z) + W(e_2,z) + 1\big)\,. \tag{3.36}$$

Here we used that $|a^+ - b^+| \le |a-b|$ for all $a,\, b \in \mathbb{R}$ and that $|\,\mathrm{tr}\, e| \le \sqrt{d}\,|e|$ for all $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. If $\mathrm{tr}(e_1),\, \mathrm{tr}(e_2) < 0$ we obtain (3.36) in the same way. If $\mathrm{tr}(e_1) > 0$ and $\mathrm{tr}(e_2) \le 0$ it is important to set $e = e_1$ and $\hat{e} = e_2$, since $|\mathrm{sgn}(\pm\,\mathrm{tr}\, e)^+ - \mathrm{sgn}(\pm\,\mathrm{tr}\,\hat{e})^+| = 1$ is cancelled out by the prefactor $\varphi'_-\big(\,\mathrm{tr}(-e_1)^+\big) = \varphi'_+\big(\,\mathrm{tr}(e_2)^+\big) = 0$, which then allows it to prove (3.36). To exploit a similar relation if $\mathrm{tr}(e_1) \le 0$ and $\mathrm{tr}(e_2) > 0$ one has to choose $e = e_2$ and $\hat{e} = e_1$ in (3.35) in this case. Hence (3.7(H5)) is verified.

Ramberg-Osgood materials

This section deals with Ramberg-Osgood materials, which are defined by energy densities composed similarly to (3.4), but formulated in terms of the complementary energy density depending on the stresses instead of the strains. Anyhow, in the following it is explained that the corresponding stored energy density of Ramberg-Osgood materials cannot be controlled by (3.7(H3)) together with (H4*) but does satisfy (3.7(H3)) together with (3.7(H4)).

As introduced in [OR43], Ramberg-Osgood materials can be described by a constitutive relation of a power-law type formulated in terms of the complementary energy density

$$W_{\mathrm{cp}} : \mathbb{R}^{d\times d} \to \mathbb{R},\ \sigma \mapsto \tfrac{1}{2}\sigma{:}\mathbb{A} : \sigma + \tfrac{a}{p'}|\sigma^D|^{p'}, \tag{3.37}$$

which depends on the linearized 2nd Piola-Kirchhoff stress tensor σ and its deviatoric part $\sigma^D := \sigma - \frac{1}{d}\operatorname{tr}\sigma\,\mathrm{Id}$. Here, $a \in \mathbb{R}^+$, $2 < p' < \infty$, and $\mathbb{A} \in \mathbb{R}^{(d\times d)\times(d\times d)}$ is symmetric, positive definite with constants $0 < c_1^{\mathbb{A}} < c_2^{\mathbb{A}}$ such that $c_1^{\mathbb{A}}|\sigma|^2 \le \sigma{:}\mathbb{A}{:}\sigma \le c_2^{\mathbb{A}}|\sigma|^2$ for all $\sigma \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. The complementary energy and the stored elastic energy, which depends on the strain tensor $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, are linked by a Legendre transform, i.e.:

$$W(e) = \sup_{\sigma\in\mathbb{R}^{d\times d}_{\mathrm{sym}}} \{\sigma : e - W_{\mathrm{cp}}(\sigma)\} \quad \text{so that} \quad \partial_e W(e) = \sigma \text{ and } \partial_\sigma W_{\mathrm{cp}}(\sigma) = e. \tag{3.38}$$

See [Zei85, Chap. 51] and [ET76, Prop. IX 2.1.] for more details. This relation together with (3.37) yields $e = \partial_\sigma W_{\mathrm{cp}}(\sigma) = \mathbb{A}(x) : \sigma + a|\sigma^D|^{p'-2}\sigma^D$, which is used to check the hypotheses (3.7(H2))-(3.7(H5)). In view of the first relation in (3.38), convexity is easily obtained for $W(\cdot)$. Furthermore, we derive the coercivity inequality:

$$\begin{aligned}
W(e) &\ge \sup\nolimits_{\sigma\in\mathbb{R}^{d\times d}_{\mathrm{sym}}} \left\{\sigma : e - \tfrac{c_2^{\mathbb{A}}}{2}|\sigma|^2 - \tfrac{a}{p'}|\sigma^D|^{p'}\right\}\\
&= \sup\nolimits_{\sigma\in\mathbb{R}^{d\times d}_{\mathrm{sym}}} \left\{\sigma^D : e^D - \tfrac{c_2^{\mathbb{A}}}{2}|\sigma^D|^2 - \tfrac{a}{p'}|\sigma^D|^{p'} + \tfrac{1}{d^2}\operatorname{tr}\sigma\operatorname{tr}e - \tfrac{c_2^{\mathbb{A}}}{2}(\operatorname{tr}\sigma)^2\right\}\\
&= \sup\nolimits_{t\in\mathbb{R}} \left\{\tfrac{t}{d^2}\operatorname{tr}e - \tfrac{c_2^{\mathbb{A}}}{2}t^2\right\} + \sup\nolimits_{\tau\in\mathbb{R}^{d\times d}_{\mathrm{dev}}} \left\{\tau : e^D - \tfrac{c_2^{\mathbb{A}}}{2}|\tau|^2 - \tfrac{a}{p'}|\tau|^{p'}\right\}\\
&= \tfrac{1}{2d^4c_2^{\mathbb{A}}}(\operatorname{tr}e)^2 + \sup\nolimits_{t\ge0}\left\{t|e^D| - \tfrac{c_2^{\mathbb{A}}}{2}t^2 - \tfrac{a}{p'}t^{p'}\right\}\\
&\ge \tfrac{1}{2d^4c_2^{\mathbb{A}}}(\operatorname{tr}e)^2 + \sup\nolimits_{t\ge0}\left\{t|e^D| - t^{p'}(\tfrac{2a}{p'}) + C_1\right\}\\
&= \tfrac{1}{2d^4c_2^{\mathbb{A}}}(\operatorname{tr}e)^2 + \tfrac{|e^D|^p}{p(2a)^{p-1}} - C_1 \ \ge\ \min\left\{\tfrac{1}{2d^4c_2^{\mathbb{A}}}, \tfrac{1}{p(c_2^{\mathbb{A}}+a)^{p-1}}\right\}|e|^p - C_2,
\end{aligned} \tag{3.39}$$

where Young's inequality $t^2 \le bt^{p'} + C_b$ has been used for the second estimate. The last inequality results from $1 < p \le 2$. Hence, (3.7(H3)) holds for the exponent $p = \frac{p'}{p'-1}$. On the other hand we obtain with the same technique

$$\begin{aligned}
W(e) \ &\le \sup_{\sigma\in\mathbb{R}^{d\times d}_{\mathrm{sym}}} \left\{\sigma : e - \tfrac{c_1^{\mathbb{A}}}{2}|\sigma|^2 - \tfrac{a}{p'}|\sigma^D|^{p'}\right\} \le \frac{(\operatorname{tr}e)^2}{2c_1^{\mathbb{A}}d^4} + \frac{|e^D|^2}{8c_1^{\mathbb{A}}} + \frac{(p'-1)|e^D|^p}{2p'(2a)^{p-1}}\\
&\le\ 3\max\left\{\tfrac{1}{2c_1^{\mathbb{A}}d^4}, \tfrac{1}{8c_1^{\mathbb{A}}}, \tfrac{(p'-1)}{2p'(2a)^{p-1}}\right\}(|e|^2+2)\,,
\end{aligned}$$

which yields $|\partial_e W(e)| \le c(|e| + \tilde{c})$ due to convexity. Thus, (H3) and (H4*) are not satisfied for the same exponent. But (3.7(H3)) in combination with (3.7(H4)) holds, since (3.39) gives

$$|\partial_e W(e)| \le c(|e|+\tilde{c}) \le c\Big(\big(\tfrac{1}{c_0}(W(e)+C_2)\big)^{1/p} + \tilde{c}\Big) \le c_1(W(e)+\tilde{c}_1).$$

In order to verify hypothesis (3.7(H5)) we once more exploit relation (3.38) and obtain

$$|\partial_e W(e) - \partial_e W(\hat{e})| = |\sigma - \hat{\sigma}| = \frac{|e-\hat{e}|\,|\sigma-\hat{\sigma}|}{|\partial_\sigma W_{\mathrm{cp}}(\sigma) - \partial_\sigma W_{\mathrm{cp}}(\hat{\sigma})|}\,. \tag{3.40}$$

By the triangle inequality we deduce that $|\partial_\sigma W_{\mathrm{cp}}(\sigma) - \partial_\sigma W_{\mathrm{cp}}(\hat{\sigma})| \geq |S-D|$, where we introduced $S := |\mathbb{A} : (\sigma-\hat{\sigma})|$ and $D := a\big||\sigma^D|^{p'-2}\sigma^D - |\hat{\sigma}^D|^{p'-2}\hat{\sigma}^D\big|$. If $D \geq 2S$ we obtain that $|S-D| \geq S \geq c_1^{\mathbb{A}}|\sigma - \hat{\sigma}|$. Similarly we get $|S-D| \geq |S-2S| = S \geq c_1^{\mathbb{A}}|\sigma-\hat{\sigma}|$ if $D < 2S$. Thus we conclude

$$|\partial_e W(e) - \partial_e W(\hat{e})| \leq \tfrac{1}{c_1^{\mathbb{A}}}|e-\hat{e}|\,, \tag{3.41}$$

which proves (3.7(H5)).

3.2 Isotropic Damage at Finite Strains

Finite-strain elasticity is a geometrically nonlinear material model. This means that also such deformations can be considered, whose gradients $\nabla\varphi$ are large, so that the right Cauchy-Green tensor $C := \nabla\varphi^\top\nabla\varphi$ has to be considered. Hence, one does not formulate the problem in terms of the displacement field $u(x) = \varphi(x) - x$ but directly in terms of the deformation φ and the deformation gradient $F = \nabla\varphi$. Hence, in the finite-strain setting the energy functional $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ of the damage model is given by

$$\mathcal{E}(t,\varphi,z) := \int_\Omega W(x,\nabla\varphi,z)\,\mathrm{d}x + \int_\Omega \Big(\tfrac{\kappa}{r}\,|\nabla z|^r + \delta_{[z_\star,1]}(z)\Big)\,\mathrm{d}x - \langle l(t),\varphi\rangle \tag{3.42}$$

with constants $r \in (1,\infty)$, $\kappa > 0$, $z_\star \in (0,1)$ and the indicator function $\delta_{[z_\star,1]}(z) = 0$ if $z \in [z_\star, 1]$ and $\delta_{[z_\star,1]}(z) = \infty$ otherwise. Thus, in comparison to (3.3), here u and $e(u)$ are replaced by φ and $\nabla\varphi$. Since the dissipation potential $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ from (3.1) solely depends on the damage variable, it remains the same also at finite strains and hence both $\mathcal{R}$ and $\mathcal{D}$ are given as in (3.1) and (3.2) respectively.

A physically reasonable deformation preserves orientation, which is ensured by

$$\nabla\varphi \in \mathrm{GL}_+(d) = \{A \in \mathbb{R}^{d\times d} \mid \det A > 0\}\,.$$

Further natural requirements on the constitutive relations were already discussed in Section 2.1.3 and of particular importance are the material frame indifference (N2) and the non-interpenetration condition (N4), which were introduced in (2.25) and (2.33):

$$\text{(N2)} \quad \hat{W}(RF) = \hat{W}(F) \quad \text{for } R \in \mathrm{SO}(d),\ F \in \mathbb{R}^{d\times d}, \tag{3.43}$$

$$\text{(N4)} \begin{cases} \hat{W}(F) = +\infty & \text{for } \det F \leq 0, \\ \hat{W}(F) \to +\infty & \text{for } \det F \to 0_+, \end{cases} \tag{3.44}$$

since they are not compatible with convexity, which is a convenient claim in the setting of small strains as in Section 3.1. To see the incompatibility with convexity consider $P, Q \in \mathrm{SO}(d)$, $\lambda \in (0,1)$, such that $(\lambda P + (1-\lambda)Q) \notin \mathrm{SO}(d)$, which conforms to a strain. Then convexity together with material frame indifference yields the following contradiction:

$$0 < \hat{W}(\lambda P + (1-\lambda)Q) \leq \lambda \hat{W}(P) + (1-\lambda)\hat{W}(Q) = \lambda \hat{W}(I) + (1-\lambda)\hat{W}(I) = 0.$$

The class of energy densities which fit to these natural requirements and which admit to prove existence are the polyconvex energy densities. They were introduced by J.M. Ball in [Bal76].

Definition 3.2.1 (Polyconvexity) *The function $\hat{W} : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ is called polyconvex if there exists a convex function $\tilde{W} : \mathbb{R}^{\mu_d} \to \mathbb{R}_\infty$, such that $\hat{W}(F) = \tilde{W}(\mathbb{M}(F))$ for all $F \in \mathbb{R}^{d\times d}$, where*

$$\mathbb{M} : \mathbb{R}^{d\times d} \to \mathbb{R}^{\mu_d} \quad \textit{with} \quad \mu_d = \sum_{s=1}^{d} \binom{d}{s}^2 \tag{3.45}$$

is the function, which maps a matrix to all its minors.

In [Bal76, p. 362] it was established that the polyconvexity of $\hat{W} : \mathbb{R}^{d\times d} \to \mathbb{R}$ implies its quasiconvexity. By C.B. Morrey in [Mor52] it was proven that quasiconvexity is the notion of convexity which is necessary and sufficient for the lower semicontinuity of the corresponding integral functionals, so that quasiconvexity together with other technical assumptions ensures the existence of minimizers. But quasiconvexity does not admit infinitely valued functions, i.e. $\hat{W} : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$. However in [Bal76, Th. 7.3, p. 376] it was shown that the polyconvexity of the density $\hat{W} : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ together with other technical assumptions is sufficient for the existence of minimizers of infinitely valued functionals.

The aim of this section is to transfer the hypotheses (3.7), which guarantee the existence of energetic solutions at small strains, by the ideas of polyconvexity to the finite-strain setting. Our procedure is strongly oriented at [MM09]. The adapted hypotheses will be used to prove the existence of energetic solutions for the damage process given by (3.42) and (3.1) at finite strains in Subsection 3.2.2.

3.2.1 Assumptions and the Existence Result

In this section we introduce analytical requirements that have to be made to describe damage in the context of finite strains. The fact that large strains are admissible within this setting implies that Dirichlet data have to be treated differently from the method applied for small strains. Hence we first state the general assumptions on the domain and the given data and prove some helpful consequences. Secondly we introduce hypotheses on the stored elastic energy density, that are on the one hand needed for analytical reasons and on the other hand reflect physical properties of both damage and finite strains. With these tool at hand the existence result for the damage process is then formulated.

General assumptions

As in the small-strain setting we consider a body $\Omega \subset \mathbb{R}^d$ consisting of a nonlinearly elastic material, with a Lipschitz boundary $\partial\Omega$. This body undergoes a damage process driven by exterior forces $l(t)$, which may change with time. Moreover, the body is assumed to be clamped at one part Γ_{Dir} of its boundary $\partial\Omega$ with positive $(d-1)$-dimensional Lebesgue-measure $\mathcal{L}^{d-1}(\Gamma_{\text{Dir}}) > 0$, so that the deformation is prescribed there: $\varphi(t) = g(t)$ on Γ_{Dir}.

Thus, the set of admissible deformations at time $t \in [0,T]$ is given by

$$\mathcal{F}(t) := \{\phi \in W^{1,p}(\Omega, \mathbb{R}^d) \,|\, \phi = g(t) \text{ on } \Gamma_{\text{Dir}}\} \quad \text{for } d < p < \infty \tag{3.46}$$

with the weak $W^{1,p}$-topology. The assumption $d < p < \infty$ implies the compact embedding $W^{1,p}(\Omega, \mathbb{R}^d) \Subset \mathrm{C}(\overline{\Omega})$. Using the ideas of [FM06] we assume that the Dirichlet datum can be extended to $\mathbb{R}^d$ in the following way:

$$g \in \mathrm{C}^1([0,T] \times \mathbb{R}^d, \mathbb{R}^d),\ \nabla g \in \mathrm{BC}^1([0,T] \times \mathbb{R}^d, \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^d))\,, \tag{3.47}$$

$$|(\nabla g(t,y))^{-1}| \le \tilde{C}_g \text{ for all } (t,y) \in [0,T] \times \mathbb{R}^d. \tag{3.48}$$

To handle the time-dependent Dirichlet conditions at finite strains one assumes that the deformation is of the form

$$\varphi(t,x) = g(t, y(x)) \text{ with } y \in \mathcal{Y}, \text{ where} \tag{3.49}$$

$$\mathcal{Y} := \{y \in W^{1,p}(\Omega, \mathbb{R}^d) \,|\, y = \mathrm{id} \text{ on } \Gamma_{\text{Dir}}\} \quad \text{for } d < p < \infty \tag{3.50}$$

with the weak $W^{1,p}$-topology. By the chain rule, this composition leads to a multiplicative split of the deformation gradient:

$$\nabla\varphi(t,x) = \nabla_x g(t, y(x)) = \nabla_y g(t, y(x)) \nabla_x y(x) = \nabla g(t,y) \nabla y\,.$$

Additionally we require a growth restriction on the Dirichlet datum, i.e. there is a constant $c_g > 0$ such that:

$$|g(t,y)| \le c_g(1 + |y|) \quad \text{for all } (t,y) \in [0,T] \times \mathbb{R}^d\,. \tag{3.51}$$

Furthermore, we introduce the space

$$\mathcal{Y}_0 := \mathcal{Y} - \{\mathrm{id}\}\,. \tag{3.52}$$

Under consideration of formulas (3.42) and (3.1) we choose the set of admissible damage variables equal to (3.9) as

$$\mathcal{Z} := W^{1,r}(\Omega) \quad \text{with } 1 < r < \infty \tag{3.53}$$

equipped with the weak $W^{1,r}$-topology. The sets $\mathcal{Y}$ and $\mathcal{Z}$ form the state space $\mathcal{Q} := \mathcal{Y} \times \mathcal{Z}$, which is endowed with the weak topology of the product space $W^{1,p}(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega)$.

We now list all these general assumptions that are made throughout this section:

(3.54(A1)) Ω is a bounded Lipschitz domain, $\Gamma_{\text{Dir}} \subset \partial\Omega$ with $\Gamma_{\text{Dir}} \neq \emptyset$,
(3.54(A2)) $g \in \mathrm{C}^1([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and $|g(t,y)| \leq c_g(1+|y|)$ for all $(t,y) \in [0,T] \times \mathbb{R}^d$,
$\nabla g \in \mathrm{BC}^1([0,T] \times \mathbb{R}^d, \mathbb{R}^{d\times d})$ with $C_g := \sup_{t\in[0,T],y\in\mathbb{R}^d}(|\nabla g(t,y)|+|\partial_t \nabla g(t,y)|)$,
$|\nabla g(t,y)^{-1}| \leq \tilde{C}_g$ for all $(t,y) \in [0,T] \times \mathbb{R}^d$,
(3.54(A3)) $l \in \mathrm{C}^1([0,T], W^{-1,p'}(\Omega,\mathbb{R}^d))$ with $c_l := \|l\|_{\mathrm{C}^1([0,T],W^{-1,p'}(\Omega,\mathbb{R}^d))}$ for $p' = \frac{p}{p-1}$.

Remark 3.2.2 *Writing $y_k \rightharpoonup y$ in $\mathcal{Y}$, $\varphi_k \rightharpoonup \varphi$ in $\mathcal{F}$, $z_k \rightharpoonup z$ in $\mathcal{Z}$ and $q_k \rightharpoonup q$ in $\mathcal{Q}$ stands for the convergence in the respective weak topologies.*

For the closed subspace $\mathcal{Y}_0 \subset W^{1,p}(\Omega,\mathbb{R}^d)$ one can prove Friedrich's inequality by contradiction using that the embedding $W^{1,p}(\Omega,\mathbb{R}^d) \Subset L^p(\Omega,\mathbb{R}^{d\times d})$ is compact.

Theorem 3.2.3 (Friedrich's inequality) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain with Dirichlet conditions on $\Gamma_{\text{Dir}} \subset \partial\Omega$, where $\Gamma_{\text{Dir}} \neq \emptyset$. Let $1 < p < \infty$. There is a constant $C_F = C_F(\Omega, p)$ such that the following estimate holds for every $y_0 \in \mathcal{Y}_0$:*

$$\|y_0\|_{W^{1,p}(\Omega,\mathbb{R}^d)} \leq C_F \|\nabla y_0\|_{L^p(\Omega,\mathbb{R}^{d\times d})} \,. \tag{3.55}$$

The lemma below is a consequence of the growth restriction (3.51) claimed in (3.54(A2)).

Lemma 3.2.4 *Let* (3.54(A1)), (3.54(A2)) *as well as* (3.49) *hold. For every $y \in \mathcal{Y}$ and $\varphi(t) = g(t,y)$ it holds $\|\varphi(t)\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} \leq \hat{C}_g(\|y\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} + 1)$*

Proof: By the growth restriction (3.51) claimed in (3.54(A2)) one directly obtains

$$\|\varphi(t)\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} \leq 2^{p-1} c_g^p (\mathcal{L}^d(\Omega) + \|y\|^p_{L^p(\Omega,\mathbb{R}^d)}) + C_g^p \|\nabla y\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})} \,.$$

Hence $\hat{C}_g := \max\{2^{p-1}c_g^p, C_g^p, 2^{p-1}c_g^p\mathcal{L}^d(\Omega)\}$. ∎

Lemma 3.2.5 *Let* (3.54(A2)), (3.49) *and* (3.50) *hold. Consider a sequence $(y_k)_{k\in\mathbb{N}} \subset \mathcal{Y}$ such that $y_k \rightharpoonup y$ in $\mathcal{Y}$. Then $\varphi_k(t) = g(t,y_k) \rightharpoonup g(t,y) = \varphi(t)$ in $\mathcal{F}$ for all $t \in [0,T]$.*

Proof: Due to the claim $d<p<\infty$ we have the compact embedding $W^{1,p}(\Omega,\mathbb{R}^d) \Subset \mathrm{C}(\bar{\Omega},\mathbb{R}^d)$ and hence $y_k \to y$ uniformly in Ω. The continuity of g on $\mathbb{R}^d$ now yields $g(y_k) \to g(y)$ uniformly in Ω. Furthermore, we obtain $\nabla\varphi_k(t) \rightharpoonup \nabla\varphi(t)$ in $L^p(\Omega,\mathbb{R}^{d\times d})$, since $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega,\mathbb{R}^{d\times d})$ and $\nabla g(t,y_k) \to \nabla g(t,y)$ uniformly in Ω by the same arguments. ∎

Assumptions on the stored elastic energy density

In the following, we are going to state the hypotheses on the stored elastic energy density introduced in (3.42). In order to transfer the hypothesis of convexity (3.7(H2)) from the small strain to the finite-strain setting we use the ideas of [MM09] to apply the Definition 3.2.1 of polyconvexity. Hence we assume that there is a function $\widetilde{W}:\Omega\times\mathbb{R}^{\mu_d}\times[z_\star,1]\to\mathbb{R}_\infty$ such that

$$W(x,F,z)=\widetilde{W}(x,\mathbb{M}(F),z)\quad\text{for all }(x,F,z)\in\Omega\times\mathbb{R}^{d\times d}\times[z_\star,1]\tag{3.56}$$

with $\mathbb{M}:\mathbb{R}^{d\times d}\to\mathbb{R}^{\mu_d}$ as in (3.45) and by claiming that $\widetilde{W}(x,\cdot,z):\mathbb{R}^{\mu_d}\to\mathbb{R}_\infty$ is convex we have adapted W to the definition of polyconvexity.

With the aid of (3.56) we state the following hypotheses on $W:\Omega\times\mathbb{R}^{d\times d}\times[z_\star,1]\to\mathbb{R}_\infty$ by transferring the remaining hypotheses (3.7) to the finite-strain setting:

(3.57(H1)) **Carathéodory-function:**
$\widetilde{W}(x,\cdot,\cdot)\in \mathrm{C}^0(\mathbb{R}^{\mu_d}\times[z_\star,1],\mathbb{R}_\infty)$ for a.e. $x\in\Omega$,
$\widetilde{W}(\cdot,\mathbb{M}(F),z)$ is measurable in Ω for all $(F,z)\in\mathbb{R}^d\times[z_\star,1]$.

(3.57(H2)) **Polyconvexity:**
$\widetilde{W}(x,\cdot,z):\mathbb{R}^{\mu_d}\to\mathbb{R}_\infty$ is convex.

(3.57(H3)) **Coercivity:**
There are constants $p>d, c_1, C>0$ so that it holds for all $(x,F,z)\in\Omega\times\mathbb{R}^{d\times d}\times[z_\star,1]:\quad c_1|F|^p-C\le W(x,F,z)$.

(3.57(H4)) **Stress control:**
For all $(x,z)\in\Omega\times[z_\star,1]$ we have $W(x,\cdot,z)\in \mathrm{C}^1(\mathrm{GL}_+(d),\mathbb{R})$ and there are constants $c>0,\tilde c\ge0$ such that for all $(x,F,z)\in\Omega\times\mathbb{R}^{d\times d}\times[z_\star,1]$ it holds

$$|\partial_F W(x,F,z)F^\top|\le c(W(x,F,z)+\tilde c)\ .$$

(3.57(H5)) **Uniform continuity of the stresses:**
There is a modulus of continuity $\omega:[0,\infty]\to[0,\infty]$, $\gamma>0$ so that for all $(x,F,z)\in\Omega\times\mathbb{R}^{d\times d}\times[z_\star,1]$ and all $C\in\mathrm{GL}_+(d)$ with $|C-\mathrm{Id}\,|\le\gamma$ we have
$|\partial_F W(x,CF,z)(CF)^\top-\partial_F W(x,F,z)F^\top|\le\omega(|C-\mathrm{Id}\,|)(W(x,F,z)+\tilde c)$.

(3.57(H6)) **Monotonicity:**
There are constants $K>0$, $\tilde K\ge0$ so that for all $(x,F,z),(x,F,\tilde z)\in\Omega\times\mathbb{R}^{d\times d}\times[z_\star,1]$ with $z\le\tilde z$ we have

$$W(x,F,z)\le W(x,F,\tilde z)\le W(x,F,1)\le K(W(x,F,z)+\tilde K)\ .$$

The next lemma goes back on [Bal76]. A proof of the version below is given in [FM06].

Lemma 3.2.6 *Let* (3.57(H4)) *be satisfied. Then there is* $\gamma>0$ *so that for all* $C\in\mathrm{GL}_+(d)$ *with* $|C-\mathrm{Id}|\le\gamma$ *we have*

$$W(x,CF,z)+\tilde c\le\frac{d}{d-1}(W(x,F,z)+\tilde c)\tag{3.58}$$

$$|\partial_F W(x,CF,z)F^\top|\le dc\,(W(x,F,z)+\tilde c)\,.\tag{3.59}$$

Existence result for the rate-independent damage process

The general assumptions stated in (3.54) and the hypotheses (3.57) made on the stored elastic energy density will ensure that the key properties on the energy functional (2.62) as well as the compatibility conditions (2.66) hold. Hence, we are now in a position to formulate the existence result for the rate-independent damage process.

Theorem 3.2.7 (Existence theorem for the damage problem) *Let the assumptions* (3.54) *as well as* (3.57) *hold and let* $p>d$. *Then for the damage problem* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *defined by formulas* (3.2), (3.42), (3.46) *and* (3.53) *there exists an energetic solution* $q : [0,T] \to \mathcal{Q}$ *for any initial state* $q_0 \in \mathcal{S}(0)$.

The proof will be carried out in Section 3.2.2. Similar to the small-strain setting it is an application of the abstract existence theorem 2.4.4, so that the steps of the proof are the same as in Section 3.1.2. But the methods to verify the conditions (2.62), (2.65) and (2.66) differ from Section 3.1.2 due to the assumption of polyconvexity and the different treatment of the time-dependent Dirichlet data.

3.2.2 Proof of the Existence Theorem

In the following we verify existence theorem 3.2.7 as an application of the abstract existence theorem 2.4.4. Since $\mathcal{R} : \mathcal{Z} \to [0, \infty]$ is defined independently of small or finite strains, cf. (3.1), the conditions (2.65) verified in Theorem 3.1.8 are still valid. It remains to carry the proofs of (2.62) and (2.66) over to the finite-strain setting.

Conditions on the Energy Functional (2.62)

We first show that the energy functional associated to the damage process satisfies property (2.62(E1)) of the abstract main existence theorem 2.4.4:

Theorem 3.2.8 *Let* (3.54) *as well as* (3.57) *hold. Then for every* $t \in [0,T]$ *the sublevels* $L_E(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \leq E\}$ *for* $\mathcal{E}$ *given by* (3.42) *are weakly sequentially compact in* $\mathcal{Q}$.

As it can be concluded from [Dac89, Th. 2.1] the weak sequential compactness of the sublevels is guaranteed if $\mathcal{E}(t, \cdot)$ is weakly sequentially lower semicontinuous and coercive on $\mathcal{Q}$. In order to establish weak sequential lower semicontinuity for the polyconvex functional $\mathcal{E}(t, \cdot)$ we use the following result on the convergence of minors of gradients, which goes back on [Res67, Bal76]. A proof for the d-dimensional case can be found in [MM09].

Proposition 3.2.9 (Convergence of minors of gradients) *Let* $y_k \rightharpoonup y$ *in* $W^{1,p}(\Omega, \mathbb{R}^d)$. *Then* $\mathbb{M}(\nabla g(t, y_k)) \rightharpoonup \mathbb{M}(\nabla g(t, y))$ *in* $\prod_{s=1}^{d} L^{p/s}(\Omega, \mathbb{R}^{\tau(d,s)})$, *where* $\tau(d,s) = \binom{d}{s}^2$.

With this at hand we now establish weak sequential lower semicontinuity and coercivity.

Lemma 3.2.10 *1. Let* (3.54) *as well as* (3.57(H1))–(3.57(H3)) *hold. Then* $\mathcal{E}(t,\cdot)$ *is weakly sequentially lower semicontinuous on* $\mathcal{Q}$ *for all* $t \in [0,T]$.

2. Let (3.54) *as well as* (3.57(H3)) *hold. Then* $\mathcal{E}(t,\cdot)$ *is coercive on* $\mathcal{Q}$ *for all* $t \in [0,T]$.

Proof: Ad 1.: In the proof of Proposition 3.1.4 it was already verified that the functional $\mathcal{C}(\cdot) : W^{1,r}(\Omega) \to \mathbb{R}, z \mapsto \int_\Omega \frac{\kappa}{r}|\nabla z|^r \,\mathrm{d}x$ is weakly sequentially lower semicontinuous.

Consider a sequence $(y_k, z_k)_{k\in\mathbb{N}} \subset \mathcal{Q}$ with $y_k \rightharpoonup y$ in $W^{1,p}(\Omega,\mathbb{R}^d)$ and $z_k \rightharpoonup z$ in $W^{1,r}(\Omega)$. Assumption (3.54(A3)) and Lemma 3.2.5 ensure that $\langle l(t), g(t,y_k)\rangle \to \langle l(t), g(t,y)\rangle$. Taking into account Proposition 3.2.9 and hypotheses (3.57(H1)), (3.57(H2)) and (3.57(H3)), which state that $\widetilde{W}$ is a Carathéodory-function, polyconvex and bounded from below for every $F \in \mathrm{GL}_+(d)$, the weak sequential lower semicontinuity of $\mathcal{I}(t,\cdot)$ can be obtained by applying classical lower semicontinuity results for multiple integrals as in [Eis79] on $\widetilde{W}$.

Ad 2.: Let $(q_k)_{k\in\mathbb{N}} = (y_k, z_k)_{k\in\mathbb{N}} \subset \mathcal{Q}$ with
$N_k := \|y_k\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + \|z_k\|_{W^{1,r}(\Omega)} \le \|y_k\|_{W^{1,p}(\Omega,\mathbb{R}^d)} + (\mathcal{L}^d(\Omega) + \|\nabla z_k\|^r_{L^r(\Omega,\mathbb{R}^d)})^{\frac{1}{r}} \to \infty$ as $k \to \infty$. By (3.57(H3)), (3.54(A2)), (3.54(A3)), Young's inequality with $\varepsilon = (\frac{c_1 p}{2C_g^p C_F^p})^{\frac{1}{p}}$, Lemma 3.2.4 and Friedrich's inequality it is:

$$\begin{aligned}
\frac{\mathcal{E}(t,q_k)}{N_k} &\ge \tfrac{1}{N_k}\Big(c_1\|\nabla\varphi_k(t)\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})} - C\mathcal{L}^d(\Omega) + \mathcal{C}(z_k) - \Big(\tfrac{c_l}{p'\varepsilon}\Big)^{p'} - \tfrac{\varepsilon^p}{p}\|\varphi_k(t)\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)}\Big)\\
&\ge \tfrac{1}{N_k}\Big(\tfrac{c_1}{C_g^p}(2^{1-p}\|\nabla(y_k - \mathrm{id})\|^p_{L^p(\Omega,\mathbb{R}^{d\times d})} - d^{\frac{1}{p}}\mathcal{L}^d(\Omega)) + \mathcal{C}(z_k) - \tfrac{\varepsilon^p}{p}(\|y_k\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} + 1) - \tilde{B}\Big)\\
&\ge \tfrac{1}{N_k}\left(\left(\tfrac{c_1}{C_g^p C_F^p} - \tfrac{\varepsilon^p}{p}\right)\|y_k\|^p_{W^{1,p}(\Omega,\mathbb{R}^d)} + \tfrac{\kappa}{r}\|\nabla z_k\|^r_{L^r(\Omega,\mathbb{R}^d)} - B\right) \overset{k\to\infty}{\longrightarrow} \infty.
\end{aligned} \tag{3.60}$$

■

In the following condition (2.62(E2)) is verified under the assumption (3.54) and (3.57). The proof requires the same steps as the one of Theorem 3.1.7 for the small-strain setting. An analogous result was first obtained in [FM06, Lemma 5.5].

Theorem 3.2.11 *Let* (3.57(H2))-(3.57(H4)) *and* (3.54) *be satisfied. Then there exist constants* $c_0 \ge 0, c_1 > 0$ *such that for all* $(t_q, q) \in [0,T] \times \mathcal{Q}$ *with* $\mathcal{E}(t_q,q) < \infty$ *it holds:* $\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T])$ *with*

$$\partial_t\mathcal{E}(t,q) = \int_\Omega \partial_F W(x,F(t),z)F^\top : G(t)\,\mathrm{d}x - \langle \dot{l}(t), \varphi(t)\rangle - \langle l(t), \partial_t\varphi(t)\rangle \tag{3.61}$$

for $F(t) := \nabla\varphi(t)$ *and* $G(t) := (\nabla g(t,y))^{-1}\partial_t\nabla g(t,y)$ *and*

$$|\partial_t\mathcal{E}(t,q)| \le c_1(\mathcal{E}(t,q) + c_0) \quad \textit{for every } t \in [0,T]\,. \tag{3.62}$$

Proof: We confine ourselves to prove the existence of $\partial_t\mathcal{E}(\cdot,q)$ and estimate (3.62) in a neighborhood $\mathrm{N}(t_q)$ of $t_q \in [0,T]$. Similarly to the small-strain setting this is basically done with the mean value theorem of differentiability and the dominated convergence theorem.

But the different treatment of the inhomogeneous Dirichlet condition requires different estimates, which will be carried out here. The existence of $\partial_t\mathcal{E}(\cdot,q)$ and the validity of (3.62) on the whole interval $[0,T]$ can then be concluded with the same arguments as in the proof of Theorem 3.1.7.

Since $\partial_t\langle l(t),\varphi(t)\rangle$ exists by (3.54(A2)), (3.54(A3)), it remains to show the existence of $\partial_t\mathcal{I}(\cdot,q)$ in $\mathrm{N}(t_q)$. As in the small-strain setting we define for $t\in\mathrm{N}(t_q)$

$$h(x,t,\alpha):=\begin{cases}\frac{1}{\alpha}\left(W(x,\nabla\varphi(t{+}\alpha),z)-W(x,\nabla\varphi(t),z)\right) & \text{if } \alpha\neq 0\\ \partial_F W(x,\nabla\varphi(t),z)(\nabla\varphi(t))^\top{:}(\nabla g(t,y))^{-1}\partial_t\nabla g(t,y) & \text{if } \alpha=0\end{cases}$$

and we have to show that $h(x,t,\cdot)\in\mathrm{C}^0([-\alpha_t,\alpha_t])$ for α_t suitably. By the mean value theorem of differentiability we find $\tilde\alpha=\tilde\alpha(\alpha)$ such that it holds for every $\alpha\in[-\alpha_t,\alpha_t]$

$$\begin{aligned}&\frac{1}{\alpha}\left(W(x,\nabla\varphi(t{+}\alpha),z){-}W(x,\nabla\varphi(t),z)\right)\\&=\partial_F W(x,\nabla\varphi(t{+}\tilde\alpha),z)(\nabla\varphi(t{+}\tilde\alpha))^\top{:}(\nabla g(t{+}\tilde\alpha,y))^{-1}\partial_t\nabla g(t{+}\tilde\alpha,y)\\&\to\partial_F W(x,\nabla\varphi(t),z)(\nabla\varphi(t))^\top{:}(\nabla g(t,y))^{-1}\partial_t\nabla g(t,y)\end{aligned}\tag{3.63}$$

as $\alpha,\tilde\alpha\to 0$ by (3.57(H4)) and (3.54(A2)). In order to show that the integrals converge as well, we are going to apply the dominated convergence theorem. For this, we have to construct an integrable majorant for expression (3.63). Again by the mean value theorem of differentiability we first obtain $\hat\alpha$ such that

$$\nabla\varphi(t{+}\tilde\alpha){=}\nabla(\varphi(t){+}\partial_t\varphi(t{+}\hat\alpha)\tilde\alpha){=}\big(\mathrm{Id}+\tilde\alpha\partial_t\nabla g(t{+}\hat\alpha,y)(\nabla g(t,y))^{-1}\big)\nabla\varphi(t){=}C(\tilde\alpha)\nabla\varphi(t)$$

with $C(\tilde\alpha)\to\mathrm{Id}$ as $\tilde\alpha\to 0$. Hence we conclude by (3.59) and (3.54(A2)):

$$\begin{aligned}|(3.63)|&\le\tilde C_g C_g|\partial_F W(x,C(\tilde\alpha),z)(\nabla\varphi(t))^\top C(\tilde\alpha)^\top|\\&\le\tilde C_g C_g dc(W(x,\nabla\varphi(t),z)+\tilde c)(\sqrt{d}{+}\tilde\alpha C_g\tilde C_g)\,.\end{aligned}\tag{3.64}$$

Now, estimate (3.18) is derived under consideration of

$$|\partial_t\mathcal{E}(t,q)|\le\Big|\int_\Omega h(x,t,0)\,\mathrm{d}x\Big|+|\langle\dot l(t),\varphi(t)\rangle|+|\langle l(t),\partial_t\varphi(t)\rangle|\,.\tag{3.65}$$

In view of (3.54(A2)), (3.54(A3)), Lemma 3.2.4, Friedrich's inequality (3.55), Young's inequality and (3.57(H3)) we derive for the loading terms in (3.65) an estimate of the form

$$|\langle\dot l(t),\varphi(t)\rangle|+|\langle l(t),\partial_t\varphi(t)\rangle|\le A_1\mathcal{E}(t,q)+B_1.$$

For the elastic energy term in (3.65) estimate (3.64) and (3.54(A2)), (3.54(A3)) lead to

$$\Big|\int_\Omega h(x,t,0)\,\mathrm{d}x\Big|\le(3.64)\le A_2\mathcal{E}(t,q)+B_2,$$

so that inequality (3.18) is obtained. ∎

Compatibility Conditions (2.66)

It remains to verify the compatibility conditions (2.66).

Theorem 3.2.12 *Let assumptions* (3.57), (3.54) *and* (2.65) *hold true. Then, for every stable sequence* $(t_k, q_k)_{k\in\mathbb{N}} \subset [0,T]\times\mathcal{Q}$ *with* $t_k \to t$ *and* $q_k \rightharpoonup q$ *in* $\mathcal{Q}$ *we have*

$$\partial_t\mathcal{E}(t,q_k) \to \partial_t\mathcal{E}(t,q)\,. \tag{3.66}$$

Proof: The proof uses exactly the same arguments as the one of Theorem 3.1.9. The only difference is that in the proof of Lemma 3.1.10 the application of the dominated convergence theorem is now possible due to (3.57(H6)), Lemma 3.2.4 and (3.57(H5)). ∎

The closedness of the stable sets (2.66(C2)) is also in the finite-strain setting proven by means of the joint recovery condition, recall Definition 3.1.12.

Theorem 3.2.13 (Joint recovery condition for $r \in (1,\infty)$) *Let* (3.57) *hold. Then, the rate-independent system* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *satisfies the joint recovery condition. Hence, if* $(t_k, q_k)_{k\in\mathbb{N}}$ *is a stable sequence with* $t_k \to t$, $q_k \rightharpoonup q$ *in* $\mathcal{Q}$, *then* $q \in \mathcal{S}(t)$, *i.e.* (2.66(C2)) *holds.*

Proof: The proof of Theorem 3.1.14 can directly be adopted. For a stable sequence $(t, q_k)_{k\in\mathbb{N}}$ satisfying $t_k \to t$, $y_k \rightharpoonup y$ in $\mathcal{Y}$, $z_k \rightharpoonup z$ in $\mathcal{Z}$ and a state $\hat{q} = (\hat{y},\hat{z})$ we obtain a joint recovery sequence by choosing $\hat{y}_k = \hat{y}$ and $\hat{z}_k$ as in (3.29) for all $k \in \mathbb{N}$. ∎

3.2.3 Examples

We now verify that the class of Ogden's materials coupled with damage satisfy the hypotheses (3.57).

Ogden's materials coupled with damage

Ogden's materials, see Example 2.1.6, provide a typical example for polyconvex energy densities, since they are defined as the sum of convex functions depending on the minors of a matrix. We couple the energy density $\hat{W}$ from Example 2.1.6 for $d = 3$ with damage in the following way: For $(F,z) \in \mathrm{GL}_+(3)\times[z_\star,1]$ let

$$W(F,z) := \frac{1}{\eta - z}\Big(\sum_{i=1}^{M} a_i(\mathrm{tr}(F^\top F))^{\frac{\gamma_i}{2}} + \sum_{j=1}^{N} b_j(\mathrm{tr}\,\mathrm{Cof}(F^\top F))^{\frac{\delta_j}{2}} + \det F^{-2\alpha}\Big) \tag{3.67}$$

with $\eta > 1, M, N \in \mathbb{N}, a_i > 0, \gamma_i > d, b_j > 0, \delta_j > d$ and $\alpha > d/2$.

Note that $\mathrm{tr}(F^\top F) = |F|^2$ and $\mathrm{tr}\,\mathrm{Cof}(F^\top F) = |(\det F)F^{-1}|^2$. We set $f_1(F) := |F|^2$, $f_2(F) := |(\det F)F^{-1}|^2$ and $f_3(F) := \det F$.

Clearly each of the individual terms in (3.67) is continuous and convex with respect to its argument $\operatorname{tr}(F^\top F)$, $\operatorname{tr}\operatorname{Cof}(F^\top F)$ or $\det F^2$, which are the invariants of the matrix $C = F^\top F$. Hence both (3.57(H1)) and (3.57(H2)) hold. Coercivity (3.57(H3)) immediately follows from $\eta > 1$ for all exponents γ_i, since $\operatorname{tr}(F^\top F) = |F|^2$. Monotonicity (3.57(H6)) follows from

$$\frac{1}{\eta - z} \le \frac{1}{\eta - \tilde{z}} \le \frac{1}{\eta - 1} \le \frac{\eta - z_\star}{\eta - 1}\,\frac{1}{\eta - z} \quad \text{for all } z \le \tilde{z} \in [z_\star, 1]\,.$$

We now verify the stress control (3.57(H4)). For the terms depending on $f_1(F)$ we calculate that $\partial_F f_1(F)^{\frac{\gamma_i}{2}} = \partial_F |F|^{\gamma_i} = \gamma_i |F|^{\gamma_i - 2} F$ and thus we obtain

$$|\partial_F f_1(F)^{\frac{\gamma_i}{2}} F^\top| = \gamma_i |F|^{\gamma_i - 2} |F|^2 = \gamma_i f_1(F)^{\frac{\gamma_i}{2}}\,.$$

In order to estimate the terms depending on $f_2(F)$ and $f_3(F)$ we conclude from [Cia88, p. 10 ff] that their Fréchet derivatives are given as follows:

$$\partial_F f_3(F)[H] = \operatorname{Cof} F : H\,, \tag{3.68}$$
$$\partial_F f_2(F)[H] = 2 f_3(F) \tilde{f}(F)^2 \partial_F f_3(F)[H] + 2 f_3(F) \tilde{f}(F) : f_3(F) \partial_F \tilde{f}(F)[H]\,, \tag{3.69}$$

where $\tilde{f}(F) := F^{-1}$ and $\partial_F \tilde{f}(F)[H] = -F^{-1} H F^{-1}$. With the chain rule we conclude from (3.68) that

$$\begin{aligned} |(\partial_F f_3(F)^{-2\alpha}) F^\top| &= |(\partial_{f_3} f_3(F)^{-2\alpha} (\partial_F f_3(F))) F^\top| \\ &= |2\alpha (f_3(F)^{-2\alpha - 1} \operatorname{Cof} F) F^\top| \\ &= 2\alpha f_3(F)^{-2\alpha - 1} \det F |\operatorname{Id}| = 6\alpha f_3(F)^{-2\alpha}. \end{aligned}$$

In (3.69) we use (3.68) and $f_3(F)\tilde{f}(F) : f_3(F)\partial_F \tilde{f}(F)[H] = -f_3(F)^2 F^{-\top} F^{-1} F^{-\top} : H$. Applying the chain rule on $f_2(F)^{\delta_j/2}$ we obtain that

$$\begin{aligned} |(\partial_F f_2(F)^{\frac{\delta_j}{2}}) F^\top| &= \big|\big(\partial_{f_2} (f_2(F)^{\frac{\delta_j}{2}}) (\partial_F f_2(F)\big) F^\top| \\ &= \delta_j |(\det F) F^{-1}|^{\delta_j - 2} (\det F)^2 |(F^{-1})^2 - F^{-\top} F^{-1}| \\ &\le 2\delta_j |(\det F) F^{-1}|^{\delta_j} = 2\delta_j f_2(F)^{\delta_j/2}\,. \end{aligned}$$

Thus, with $K := \max\{\gamma_i, 2\delta_j, 6\alpha \,|\, i = 1, \ldots, M,\ j = 1, \ldots, N\}$ it holds that

$$|\partial_F W(F, z) F^\top| \le K W(F, z)\,.$$

Finally, it remains to verify hypothesis (3.7(H5)). Consider $G \in \mathrm{GL}_+(3)$ with $|G – \operatorname{Id}| < \varepsilon$. Hence,

$$3 - \varepsilon < |G| < 3 + \varepsilon\,. \tag{3.70}$$

Since G can be transformed into a Jordan matrix with the eigenvalues G_1, G_2 and G_3, we conclude from $|G - \mathrm{Id}\,| < \varepsilon$ that also $|G_i - 1| < \varepsilon$ and hence $1 - \varepsilon < |G_i| < 1 + \varepsilon$ for $i \in \{1, 2, 3\}$. This implies

$$(1-\varepsilon)^3 < |\det G| < (1+\varepsilon)^3\,, \tag{3.71}$$

$$|1 - \det G| \le |1 - (1+\varepsilon)^3| = \varepsilon(3 + 3\varepsilon + \varepsilon^2) =: \omega_3\big(|G - \mathrm{Id}\,|\big) \tag{3.72}$$

Moreover, we will apply the following estimate:

$$\big||A|^q - |B|^q\big| \le 2^{q-1}q\big(|A-B|\,|B|^{q-1} + |A-B|^q\big)\,, \tag{3.73}$$

which holds for all $A, B \in \mathbb{R}^{3\times3}$ and a fixed $q > 1$. This estimate can be proven with the aid of the function $\widetilde{W}(t) := |B + t(A-B)|^q$ with $t \in [0,1]$. By the chain rule we obtain that $\partial_t\widetilde{W}(t) = q|B + t(A-B)|^{q-2}(B + t(A-B)) : (A-B)$. Hence we find

$$\begin{aligned}
|\widetilde{W}(1) - \widetilde{W}(0)| &= \Big|\int_0^1 \partial_t\widetilde{W}(t)\,\mathrm{d}t\Big| \le q|A-B|\int_0^1 |B + t(A-B)|^{q-1}\,\mathrm{d}t \\
&\le 2^{q-2}q|A-B|\int_0^1 \big(|B|^{q-1} + t^{q-1}|(A-B)|^{q-1}\big)\,\mathrm{d}t = 2^{q-1}q\big(|A-B|\,|B|^{q-1} + |A-B|^q\big)\,.
\end{aligned}$$

We now prove (3.7(H5)) for the terms involving f_1. By adding a telescope sum and by the triangle inequality we infer

$$\begin{aligned}
&\big|\partial_{(GF)}(f_1(GF)^{\frac{\gamma_i}{2}})(GF)^\top - \partial_F(f_1(F)^{\frac{\gamma_i}{2}})F^\top\big| \\
&\le \gamma_i|G - \mathrm{Id}\,||F|^{\gamma_i}\big(|G|^{\gamma_i-1} + |G|^{\gamma_i-2}\big) + \gamma_i\big||GF|^{\gamma_i-2} - |F|^{\gamma_i-2}\big|\,|F|^2 \\
&\le \gamma_i f_1(F)^{\frac{\gamma_i}{2}}|G - \mathrm{Id}\,|\big((3+\varepsilon)^{\gamma_i-1} + (3+\varepsilon)^{\gamma_i-2} + 2^{q-1}(1 + |G - \mathrm{Id}\,|^{\gamma_i-2})\big)\,,
\end{aligned}$$

where we applied (3.70) and (3.73) with $q = \gamma_i - 2 > 1$, $A = GF$ and $B = F$ in the last estimate.

With the same techique we deduce for the terms involving f_3

$$\begin{aligned}
&\big|\partial_{(GF)}(f_3(GF)^{-\alpha})(GF)^\top - \partial_F(f_3(F)^{-\alpha})F^\top\big| \\
&= 2\alpha\big|(\det(GF))^{-2\alpha-1}\big(\operatorname{Cof}(GF)\big)(GF)^\top - (\det F)^{-2\alpha-1}\big(\operatorname{Cof} F\big)F^\top\big| \\
&= 2\alpha(\det F)^{-2\alpha}(\det G)^{-2\alpha}|(\det G)^{2\alpha} - 1| \\
&\le 2\alpha(\det F)^{-2\alpha}(\det G)^{-2\alpha}|\det G - 1|\big(2 + |\det G|^{2\alpha-1}\big) \\
&\le 2\alpha(\det F)^{-2\alpha}(1-\varepsilon)^{-6\alpha}2^{2\alpha-1}\big(2 + (1+\varepsilon)^{6\alpha-3}\big)\omega_3\big(|G - \mathrm{Id}\,|\big)\,.
\end{aligned}$$

Here we applied (3.73) with $A = \det G$, $B = 1$ and $q = 2\alpha$ for the first estimate and (3.71) as well as (3.72) for the second estimate.

Finally, we verify (3.7(H5)) for the terms involving f_2. We introduce the abbreviation $a(F) = |(\det F)F^{-1}|^{\delta_j-2}(\det F)^2$. Then we find

$$\begin{aligned}
&\big|\partial_{GF}\big(f_2(GF)^{\frac{\delta_j}{2}}\big)(GF)^\top - \partial_F\big(f_2(F)^{\frac{\delta_j}{2}}\big)F^\top\big| \\
&= \delta_j a(F)\big|a(G)F^{-1}G^{-1}F^{-1}G^{-1} - F^{-1}F^{-1} - a(G)G^{-\top}F^{-\top}F^{-1}G^{-1} + F^{-\top}F^{-1}\big|\,.
\end{aligned} \tag{3.74}$$

By adding a telescope sum we obtain

$$(3.74) \leq 2f_2(F)^{\frac{\delta_j}{2}}\big(|G-\mathrm{Id}|f_2(G)^{\frac{\delta_j}{2}}(1+|G|) + |a(G)-1|\big) \tag{3.75}$$

We use that

$$\begin{aligned} &|a(F)||a(G)-1| \\ &\leq |\det(GF)|^{\delta_j}\big||(GF)^{-1}|^{\delta_j-2} - |F^{-1}|^{\delta_j-2}\big| + |F^{-1}|^{\delta_j-2}\big||\det(GF)|^{\delta_j} - |\det F|^{\delta_j}\big|, \end{aligned} \tag{3.76}$$

where $\big||(GF)^{-1}|^{\delta_j-2} - |F^{-1}|^{\delta_j-2}\big| \leq 2^{\delta_j-2}\big(|G^{-1}|^{\delta_j-2}|G-\mathrm{Id}\,|^{\delta_j-2} + |G^{-1}||G-\mathrm{Id}\,|\big)|F^{-1}|^{\delta_j-2}$ by (3.73) for $q = \delta_j - 2$, $A = (GF)^{-1}$ and $B = F^{-1}$. Moreover, we apply (3.72) on the difference of the determinants. Furthermore, it is $|(\det G)^{\delta_j}| \leq (1+\varepsilon)^{3\delta_j}$ by (3.71) and $|G^{-1}| \leq \sum_{i=1}^{3}(G_i^{-1})^2 + 3 \leq 3((1-\varepsilon)^{-1}+1)$. Hence, we conclude that

$$(3.74) \leq 2f_2(F)^{\frac{\delta_j}{2}}\big(c_1(\varepsilon)|G-\mathrm{Id}| + c_2(\varepsilon)|G-\mathrm{Id}|^{\delta_j-2}\big)$$

which proves that (3.7(H5)) holds true for the terms involving f_2.

Alltogether we have now verified the uniform continuity of the stresses for the density W introduced in (3.67)

Chapter 4

From Damage to Delamination

On the microscale, *delamination* (or *debonding*) is one main reason for the macroscopic failure of compounds. Opposite, sometimes delamination is an intentional mechanism in engineering constructions designed for efficient absorption of energy during impacts.

On the macro-level of compounds the evolution of delamination appears as the propagation of cracks along the interfaces between the different components [Kac88]. The behavior of such macro-cracks can be analyzed by means of fracture mechanics, using Griffith' fracture criterion, which considers the energy release rate as the decisive quantity to predict whether a pre-existing crack will grow under prescribed loadings, see equation (1.1).

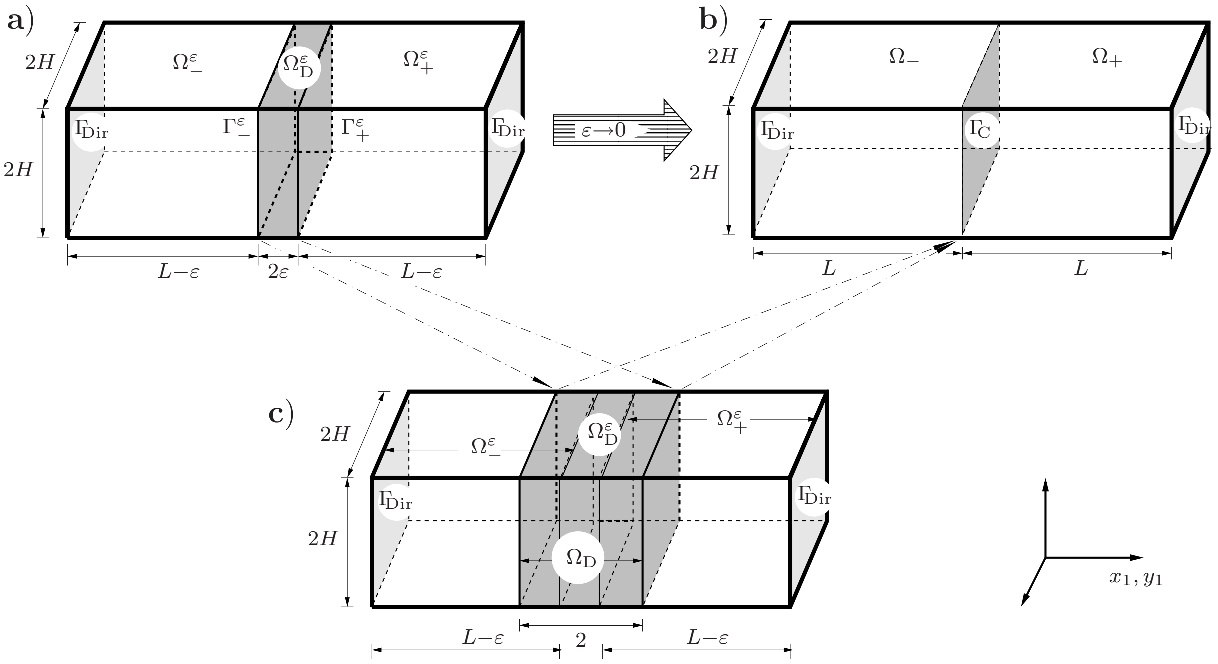
In contrast to that many engineering contributions view delamination as a process occurring on the meso- or micro-level of a compound [All02, AC96, DBS02, Lad92]. Here it is interpreted as the *damage* of interfaces and the ideas of continuum damage mechanics are applied. This means that the delamination along an interface, denoted by Γ_{C}, is modelled by an inner variable, the delamination variable $z : [0, T] \times \Gamma_{\mathrm{C}} \to [0, 1]$, which reflects the current state of the bonding along the interface; see also Section 2.2.2 for more details.

In [All02] it is suggested to understand interfaces as the limit of a thin medium, which links two constituents and which follows its own constitutive law. Such interface models have been exploited in [PS96a, PS96b] to study delamination in the framework of the adhesion models of Frémond, see e.g. [Fré88].

The aim of this chapter is to rigorously perform this limit: Starting from a sandwich-structure composed of three constituents of non-zero thickness, where the middle component is exposed to partial, isotropic damage, the delamination of two perfectly unbreakable specimen glued together with a breakable adhesive of thickness 0 is gained when flattening the thickness of the middle component to 0, see Fig. 4.1. The damage model applied for this purpose was analyzed in Section 3.1. The limit passage is mathematically performed via a double limit. The first limit models describe delamination using the transmission- and noninterpenetration conditions (2.44), (2.45), but their energy functionals involve the delamination gradient, which is also the case e.g. in [BBR08, BBR09]. Nevertheless, in the sense of [Gri21] this surface energy term is objectionable. Thus, the gradient is suppressed in a second limit, so that the delamination model discussed in [RSZ09] is obtained.

4.1 Setup and Outline

For all $\varepsilon \in (0, \varepsilon_0]$ we consider a domain $\Omega := (-L, L) \times (-H, H)^{d-1}$, which is the union of the three cuboid-type Lipschitz domains $\Omega_-^\varepsilon := (-L, -\varepsilon) \times \Gamma_C$, $\Omega_+^\varepsilon := (\varepsilon, L) \times \Gamma_C$ for $L > 1$, $\Omega_D^\varepsilon := (-\varepsilon, \varepsilon) \times \Gamma_C \subset \mathbb{R}^d$ with the interfaces $\Gamma_\pm^\varepsilon := \{\pm\varepsilon\} \times \Gamma_C \subset \mathbb{R}^{d-1}$ and $\Gamma_C := (-H, H)^{d-1}$, see also Fig. 4.1a). We assume that the domains $\Omega_\pm^\varepsilon$ are occupied by a nonlinearly elastic material which is damage-resistive, whereas Ω_D^ε refers to a material undergoing a rate-independent damage process that only leads to partial damage of that specimen. This damage process is assumed to be driven by slow, time-dependent external loadings induced by time-dependent Dirichlet conditions on parts of the outer boundary $\Gamma_{Dir} = \{L, -L\} \times \Gamma_C$, i.e. in particular $\mathcal{L}^{d-1}(\Gamma_{Dir}) > 0$. Thereby $\mathcal{L}^m(A)$ denotes the m-dimensional Lebesgue-measure of the set $A \subset \mathbb{R}^m$ with $m \in \{(d-1), d\}$.



a) *Domain with a thin subdomain Ω_D^ε undergoing possible damage. Loading is realized through Dirichlet boundary conditions prescribed on the sides Γ_{Dir}.*

b) *Domain obtained for ε=0 with an interface Γ_C undergoing possible delamination with a subsequent unilateral Signorini condition.*

c) *Setup for the analysis: the original, ε-dependent domains Ω_-^ε, Ω_+^ε and Ω_D^ε are used for the displacements, whereas the auxiliary transformed damageable domain Ω_D of fixed size is used for the damage/delamination variable.*

Figure 4.1: Geometry and notation of the cuboid-type domains and surfaces used

For $q = (u, z)$ the energy of the compound Ω, see Fig. 1a), is given by:

$$\widetilde{\mathcal{E}}_\varepsilon^\kappa(t, u, z) := \int_{\Omega_-^\varepsilon \cup \Omega_+^\varepsilon} W(e(u+g(t)))\,\mathrm{d}x + \int_{\Omega_D^\varepsilon} \Big(W_D(e(u+g(t)), z) + \tfrac{\kappa}{r\varepsilon} |\nabla z|^r + \delta_{[\varepsilon^\gamma, 1]}(z) \Big)\,\mathrm{d}x, \tag{4.1}$$

where $r > d$ and $\varepsilon, \kappa > 0$. Since we are going to perform the limit passages $\varepsilon, \kappa \to 0$, we restrict our analysis to small values $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in (0, \kappa_0]$ for constants $0 < \varepsilon_0 \ll 1$, $0 < \kappa_0 \ll 1$. For the stored elastic energy density $W_{\mathrm{D}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0,1] \to \mathbb{R}$ of the damageable region we make a specific ansatz for all $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $z \in [0,1]$, namely

$$W_{\mathrm{D}}(e,z) := z\widetilde{W}(e) + |(-e_{11})^+|^p \tag{4.2}$$

where $1 < p < \infty$ is fixed, $(f)^+ := \max\{0, f\}$ and e_{11} is the 11-component of $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Hence, the term $|(-e_{11})^+|^p$ prevents strong compression in x_1-direction. The properties of the densities W and $\widetilde{W}$ are explained in detail in Section 4.1.1. Basically they have to satisfy the hypotheses introduced in Section 3.1.1, which ensure the existence of energetic solutions for partial, isotropic damage processes.

The function $u : \Omega \to \mathbb{R}^d$ denotes the unknown displacement and $e(w) := \frac{1}{2}(\nabla w + \nabla w^\top)$ the linearized strain tensor for all $w : \Omega \to \mathbb{R}^d$. Here u satisfies homogeneous Dirichlet conditions on Γ_{Dir} and the given displacement $g(t) = g(t,\cdot) : \Omega \to \mathbb{R}^d$ with $t \in [0,T]$ incorporates the time-dependent Dirichlet condition. The properties of g are specified more precisely in Section 4.1.1. As in Chapter 3 the function $z : [0,T] \times \Omega^\varepsilon_{\mathrm{D}} \to [0,1]$ denotes the damage variable. The energy functional defined in (4.1) allows for partial damage only, which is ensured by the indicator function $\delta_{[\varepsilon^\gamma,1]}$ of the interval $[\varepsilon^\gamma, 1]$ for $\gamma > 0$. This means that we specify the lower bound $z_\star$ of the indicator function $\delta_{[z_\star,1]}$ used in formula (3.3) here by $z_\star = \varepsilon^\gamma$. More precisely, $\delta_{[\varepsilon^\gamma,1]}(z) = 0$ if $\varepsilon^\gamma \le z(x) \le 1$ for a.e. $x \in \Omega^\varepsilon_{\mathrm{D}}$ and $\delta_{[\varepsilon^\gamma,1]}(z) = \infty$ otherwise. However $\delta_{[\varepsilon^\gamma,1]}$ prevents total damage for each $\varepsilon \in (0, \varepsilon_0]$, but it will allow for complete delamination in the limit $\varepsilon = 0$.

Similar to Chapter 3 we assume that the damage process is unidirectional, i.e. that healing of the material is impossible, meaning $\dot z \le 0$, where $\dot z = \partial_t z$ is the partial time-derivative. The evolution of the damage variable is described by the dissipation potential

$$\widetilde{\mathcal{R}}_\varepsilon(v) := \begin{cases} \int_{\Omega^\varepsilon_{\mathrm{D}}} -\frac{\varrho}{\varepsilon} v \,\mathrm{d}x & \text{if } v \le 0 \text{ a.e. on } \Omega^\varepsilon_{\mathrm{D}}, \\ \infty & \text{otherwise,} \end{cases} \tag{4.3}$$

for a constant $\varrho > 0$ and $v = \dot z$. The scaling by ε^{-1} indicates that the amount of dissipated energy due to damage is independent of the thickness of the medium. Note that $\widetilde{\mathcal{R}}_\varepsilon(\cdot)$ is degree-1 homogeneous and thus models a rate-independent process.

The aim of this chapter is to deduce a model for Griffith-type delamination from the damage models given by $\widetilde{\mathcal{E}}^\kappa_\varepsilon$ and $\widetilde{\mathcal{R}}_\varepsilon$ as both $\varepsilon \to 0$ and $\kappa \to 0$ and to show that energetic solutions of the approximating models converge to energetic solutions of the limit model in a suitable topology. In particular, the limit model describing Griffith-type delamination has to include the transmission condition (2.44). Due to $z_\star = \varepsilon^\gamma \to 0$ and (4.2) the passage from partial damage to complete delamination entails a loss of coercivity on the damageable domains $\Omega^\varepsilon_{\mathrm{D}}$. Thus the information about the displacements on the interface Γ_{C} is lost in the limit $\varepsilon = 0$. This seems to be in conflict with the transmission condition, which is formulated in terms of the displacements. In order to extract the transmission condition despite of the loss of coercivity, it is crucial to keep the gradient term $\int_{\Gamma_{\mathrm{C}}} |\nabla z|^r \,\mathrm{d}s$. Assuming

$r > d$, the compact embedding of $W^{1,r}(\Omega_{\mathrm{D}}^{\varepsilon}) \Subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}^{\varepsilon}})$ ensures the uniform convergence of (sub-) sequences with equibounded energies and the continuity of their limit. These properties are needed to obtain the transmission condition, see Theorem 4.2.2. Due to the presence of the gradient term in the energy functional after the limit passage $\varepsilon \to 0$ we call such models gradient delamination models. The limit $\varepsilon \to 0$ is performed in Section 4.2.

Classical Griffith-crack or -delamination models do not presume surface energy terms on the crack lips. Usually the crack surfaces are either assumed to be traction-free, as it was already postulated by Griffith in [Gri21], or the noninterpenetration condition (2.45) is prescribed there. The special structure of the energy density W_{D}, see (4.2), will allow us to prove the latter, see Theorems 4.2.2 and 4.3.5.

In order to gain a proper delamination model of Griffith-type we will suppress the delamination gradient in a second limit passage $\kappa \to 0$. This is developed in Section 4.3.

Since the double limit passage was necessary only for mathematical reasons to extract the transmission condition we implicitly prove in Section 4.4 that the two limit passages can be performed simultaneously, i.e. that there exists a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that a subsequence of energetic solutions $(u_{\varepsilon}^{\kappa}(t), z_{\varepsilon}^{\kappa}(t))_{\varepsilon\in(0,\varepsilon_0],\kappa\in(0,\kappa_0],\varepsilon\leq G(\kappa)}$ of the approximating problems converges in a suitable topology to an energetic solution of the limit Griffith-type delamination problem.

The subsequent Sections 4.1.1, 4.1.2 and 4.1.3 are concerned with the preparation of the limit passages $\varepsilon \to 0$ and $\kappa \to 0$. First of all, Section 4.1.1 comprises the assumptions on the given data and the energy densities W and $\widetilde{W}$, see (4.1) and (4.2), which are needed to ensure the existence of energetic solutions of the partial damage models given by $\widetilde{\mathcal{E}}_{\varepsilon}^{\kappa}$ and $\widetilde{\mathcal{R}}_{\varepsilon}$. For this, the results of Chapter 3 are used.

Section 4.1.2 introduces a bigger state space $\mathcal{Q}$ which is admissible likewise for all $\varepsilon \in (0, \varepsilon_0]$ and all $\kappa \in (0, \kappa_0]$. This requires to transform the energy functionals $\widetilde{\mathcal{E}}_{\kappa}^{\varepsilon} : \mathcal{Q}_{\varepsilon} \to \mathbb{R}_{\infty}$ suitably and to extend them to the bigger state space $\mathcal{Q}$. Moreover the existence result from Section 4.1.1 has to be carried over to this new setting. Furthermore, Section 4.1.3 specifies a topology, which allows us to prove the convergence of (sub-)sequences of states with equibounded energies both as $\varepsilon \to 0$ and as $\kappa \to 0$.

4.1.1 General Assumptions and an Existence Result

In the following, we state the general assumptions on the given data and the energy densities $W, \widetilde{W}$, which allow us to deduce the existence of energetic solutions for the partial damage models given by $\widetilde{\mathcal{E}}_{\varepsilon}^{\kappa}$ and $\widetilde{\mathcal{R}}_{\varepsilon}$ from the existence theorem 3.1.1.

We claim that the Dirichlet data satisfy

$$\left.\begin{array}{l} g \in \mathrm{C}^1([0,T], W^{1,p}(\Omega,\mathbb{R}^d)),\\ \|g\|_{\mathrm{C}^1([0,T],W^{1,p}(\Omega,\mathbb{R}^d))} =: \hat{c}_g,\\ \operatorname{supp} g(t) \cap \overline{\Omega_{\mathrm{D}}^{\varepsilon_0}} = \emptyset \text{ for } 0<\varepsilon_0 \ll 1 \text{ and for all } t\in[0,T]. \end{array}\right\} \tag{4.4}$$

Thereby the third assumption in (4.4) leads to $\operatorname{supp} g(t) \cap \overline{\Omega_{\mathrm{D}}^{\varepsilon}} = \emptyset$ also for all $\varepsilon \in (0, \varepsilon_0]$.

Furthermore we make the following hypotheses on the energy densities $W : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$, $\widetilde{W} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ of the unbreakable and of the damageable regions respectively:

(4.5(H1)) **Convexity:** the densities $W, \widetilde{W}$ are strictly convex.

(4.5(H2)) **Coercivity:** there are constants $1 < p < \infty$, $c, \tilde{c}, \widetilde{C} > 0$ such that
$$c|e|^p \le W(e) \le \tilde{c}(|e|^p + \widetilde{C})\,, \quad c|e|^p \le \widetilde{W}(e) \le \tilde{c}(|e|^p + \widetilde{C})$$
for all $e,\, \hat{e} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

(4.5(H3)) **Continuity of the stresses:** There are constants $c, C > 0$ such that
$$|\partial_e W(e) - \partial_e W(\hat{e})| \le c(C + |e|^{p-1} + |\hat{e}|^{p-1})\, |e - \hat{e}|$$
for all $e,\, \hat{e} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

As a direct consequence of (4.5(H1), (H2)) one obtains, see [Dac89, Th. 2.31],

(4.5(C1)) **Continuity:** the densities W, $\widetilde{W}$ are continuous on $\mathbb{R}^{d\times d}$.

Moreover, (4.5(H1), (H2)) imply the following stress control for the densities

(4.5(C2)) **Stress control:** There are constants c, $C > 0$ such that
$$|\partial_e W(e)| \le c(|\partial_e W(e)|^{p-1} + C)\,, \quad |\partial_e \widetilde{W}(e)| \le c(|\partial_e \widetilde{W}(e)|^{p-1} + C)$$
for all $e,\, \hat{e} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

In view of (4.2) we realize that the composed density

$$\overline{W}(x, e, z) := \begin{cases} W(e) & \text{if } x \in \Omega^\varepsilon_- \cup \Omega^\varepsilon_+ \\ W_{\mathrm{D}}(e, z) & \text{if } x \in \Omega_{\mathrm{D}} \end{cases} \tag{4.6}$$

also satisfies (4.5(H1)-(H3)) and (4.5(C1), (C2)) with constants that depend on ε and

(4.5(C3)) **Monotonicity:** for all $\varepsilon \in (0, \varepsilon_0]$ there are constants $K > 0, \widetilde{K} \ge 0$ such that for all $e \in \mathbb{R}^{d\times d}$ and all $\varepsilon^\gamma \le z \le \tilde{z} \le 1$ it holds
$$\overline{W}(e, z) \le \overline{W}(e, \tilde{z}) \le K(\overline{W}(e, z) + \widetilde{K}).$$

This is a property of partial damage. In view of (4.5(H2)) we introduce the spaces

$$\begin{aligned} &\mathcal{U}_{\mathrm{D}} := \{u \in W^{1,p}(\Omega, \mathbb{R}^d) \,|\, u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\}\,, \\ &\mathcal{Z}_\varepsilon := W^{1,r}(\Omega^\varepsilon_{\mathrm{D}}) \text{ with } r > d\,, \\ &\mathcal{Q}_\varepsilon := \mathcal{U}_{\mathrm{D}} \times \mathcal{Z}_\varepsilon \end{aligned} \tag{4.7}$$

and $\widetilde{\mathcal{S}}^\kappa_\varepsilon(t) := \{q \in \mathcal{Q}_\varepsilon \,|\, \widetilde{\mathcal{E}}^\kappa_\varepsilon(t, q) < \infty,\ \widetilde{\mathcal{E}}^\kappa_\varepsilon(t, q) \le \widetilde{\mathcal{E}}^\kappa_\varepsilon(t, \tilde{q}) + \widetilde{\mathcal{R}}_\varepsilon(\tilde{z} - z) \text{ for all } \tilde{q} \in \mathcal{Q}_\varepsilon\}$ denote the stable sets at time t. Due to the assumption $r > d$ we have $\mathcal{Z}_\varepsilon \Subset \mathrm{C}(\overline{\Omega^\varepsilon_{\mathrm{D}}})$ and therefore $z|_{\Gamma_{\mathrm{C}}} \in \mathrm{C}(\overline{\Gamma_{\mathrm{C}}})$ with $0 < z(x) \le 1$ for all $x \in \Gamma_{\mathrm{C}}$ if $z \in \mathcal{Z}_\varepsilon$ for any $\varepsilon \in (0, \varepsilon_0]$.

For all fixed $\varepsilon \in (0, \varepsilon_0]$, $\kappa \in (0, \kappa_0]$ the systems $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ thus fit to the setting studied in Chapter 3 so that the existence of energetic solutions is guaranteed by Theorem 3.1.1.

Proposition 4.1.1 (Energetic solutions of $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$) *For all fixed $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in (0, \kappa_0]$, let the rate-independent system $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ be defined via (4.1)-(4.7). Then, for $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ and for any initial state $q_0 \in \widetilde{\mathcal{S}}^\kappa_\varepsilon(0)$, there exists an energetic solution q of the initial-value problem $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon, q_0)$.*

4.1.2 The Damage Models in a Common State Space

As $\varepsilon \to 0$ the d-dimensional domains $\Omega_{\mathrm{D}}^{\varepsilon}$ flatten to the $(d-1)$-dimensional interface Γ_{C} between the domains $\Omega_{\pm}$, see Fig. 1a), b). Moreover, the lower bound on the damage variable ε^{γ} tends to 0. This means that the displacements may jump across the interface Γ_{C} for $\varepsilon = 0$, whereas jumps are prevented as long as $\varepsilon > 0$, see (4.7). Furthermore, as $\kappa \to 0$ the delamination gradient vanishes, so that the only requirement on the delamination variable is $0 \leq z \leq 1$ a.e. on Γ_{C}. Hence, in order to show that $(\mathcal{Q}_{\varepsilon}, \widetilde{\mathcal{E}}_{\varepsilon}^{\kappa}, \widetilde{\mathcal{R}}_{\varepsilon})$ approximate a rate-independent system describing delamination along the interface it is necessary to reformulate the approximating problems in a common state space $\mathcal{Q}$, which is large enough to study both limits $\varepsilon \to 0$ and $\kappa \to 0$.

In particular, for all $\varepsilon \in (0, \varepsilon_0]$, we have to use damage variables that are defined on a common domain $\Omega_{\mathrm{D}} = (-1, 1) \times \Gamma_{\mathrm{C}}$, see Fig. 1a), c), i.e. we have to consider $z : \Omega_{\mathrm{D}} \to [0, 1]$ from now on. Therefore the energy functionals $\widetilde{\mathcal{E}}_{\varepsilon}^{\kappa}$ have to be adapted. This is realized with the following mappings:

$$T_{\varepsilon} : \Omega_{\mathrm{D}} \to \Omega_{\mathrm{D}}^{\varepsilon},\ x = T_{\varepsilon} y = (\varepsilon y_1, s) \in \Omega_{\mathrm{D}}^{\varepsilon} \quad \text{for } y = (y_1, s) \in \Omega_{\mathrm{D}}, \tag{4.8}$$

$$T^{\varepsilon} : \Omega_{\mathrm{D}}^{\varepsilon} \to \Omega_{\mathrm{D}},\ y = T^{\varepsilon} x = (x_1/\varepsilon, s) \in \Omega_{\mathrm{D}} \text{ for } x = (x_1, s) \in \Omega_{\mathrm{D}}^{\varepsilon}, \tag{4.9}$$

with $s = (x_2, \ldots, x_d) \in \Gamma_{\mathrm{C}}$. For all $\varepsilon \in (0, \varepsilon_0]$ these transformations are welldefined and continuous and clearly $T^{\varepsilon} = T_{\varepsilon}^{-1}$. Then we introduce the following transformations:

$$\Pi_{\varepsilon}\tilde{z} : \Omega_{\mathrm{D}} \to [0,1],\ \Pi_{\varepsilon}\tilde{z}(y) = \tilde{z}(T_{\varepsilon} y) \ \text{ for } \tilde{z} : \Omega_{\mathrm{D}}^{\varepsilon} \to [0,1]\,, \tag{4.10}$$

$$\Pi^{\varepsilon} z : \Omega_{\mathrm{D}}^{\varepsilon} \to [0,1],\ \Pi^{\varepsilon} z(x) = z(T^{\varepsilon} x) \ \text{ for } z : \Omega_{\mathrm{D}} \to [0,1]\,. \tag{4.11}$$

In view of (4.8) and (4.10) the gradient of $\tilde{z}$ transforms as follows:

$$\nabla_x \tilde{z}(x) = \nabla_y \Pi_{\varepsilon}\tilde{z}(y) \nabla_x y = \left(\tfrac{1}{\varepsilon}\partial_{y_1}\Pi_{\varepsilon}\tilde{z}(y), (\nabla_s \Pi_{\varepsilon}\tilde{z}(y))^{\top}\right)^{\top} =: \nabla_{\varepsilon}\Pi_{\varepsilon} z(y)\,, \tag{4.12}$$

where we used $y = T^{\varepsilon} x$ and $\nabla_s := (\partial_{y_2}, \ldots, \partial_{y_d})^{\top}$.

We are now in a position to define a common state space by

$$\mathcal{U} := \{u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \,|\, u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\}\,, \tag{4.13}$$

$$\mathcal{Z} := L^{\infty}(\Omega_{\mathrm{D}})\,, \tag{4.14}$$

$$\mathcal{Q} := \mathcal{U} \times \mathcal{Z}\,. \tag{4.15}$$

Thereby $\mathcal{U}$ allows the displacements to jump across the interface Γ_{C} and $\mathcal{Z}$ is the correct space if the delamination gradient is suppressed. Hence, the state space for all problems with $\varepsilon, \kappa > 0$ is a subspace of $\mathcal{Q}$. With $\mathcal{U}_{\mathrm{D}}$ as in (4.7) it is given by

$$\mathcal{Z}_{\mathrm{D}} := W^{1,r}(\Omega_{\mathrm{D}}) \text{ with } r > d\,, \tag{4.16}$$

$$\mathcal{Q}_{\mathrm{D}} := \mathcal{U}_{\mathrm{D}} \times \mathcal{Z}_{\mathrm{D}}\,. \tag{4.17}$$

Therewith we introduce the extended energy functionals $\mathcal{E}_\varepsilon^\kappa : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$

$$\mathcal{E}_\varepsilon^\kappa(t,q) := \begin{cases} \Pi\mathcal{E}_\varepsilon^\kappa(t,q) & \text{if } q=(u,z)\in\mathcal{Q}_{\mathrm{D}}, \\ \infty & \text{if } q\in\mathcal{Q}\backslash\mathcal{Q}_{\mathrm{D}}, \end{cases} \qquad \text{where}$$
$$\Pi\mathcal{E}_\varepsilon^\kappa(t,u,z) := \int_{\Omega_-^\varepsilon\cup\Omega_+^\varepsilon} W(e(u+g(t))\mathrm{d}x + \int_{\Omega_{\mathrm{D}}^\varepsilon} W_{\mathrm{D}}(e(u),\Pi^\varepsilon z)\mathrm{d}x + \int_{\Omega_{\mathrm{D}}} \Big(\tfrac{\kappa}{r}|\nabla_\varepsilon z|^r + \delta_{[\varepsilon^\gamma,1]}(z)\Big)\mathrm{d}y. \tag{4.18}$$

Thereby we used that $\operatorname{supp} g(t)\cap\Omega_{\mathrm{D}}^\varepsilon = 0$ for all $\varepsilon\in(0,\varepsilon_0]$ and all $t\in[0,T]$. Compared to $\widetilde{\mathcal{E}}_\varepsilon^\kappa$ in (4.1) the functional $\Pi\mathcal{E}_\varepsilon^\kappa$ allows for $z:\Omega_{\mathrm{D}}\to[0,1]$. Therefore one has to use $\Pi^\varepsilon z$ in the second integral, since its integration domain is the untransformed domain $\Omega_{\mathrm{D}}^\varepsilon$. Only the integral containing the damage gradient is transformed from $\Omega_{\mathrm{D}}^\varepsilon$ to Ω_{D}. This requires to use $\nabla_\varepsilon z$ from (4.12) and involves a factor ε, which cancels out $1/\varepsilon$ in (4.1). Additionally we used that $\varepsilon\delta_{[\varepsilon^\gamma,1]}(z) = \delta_{[\varepsilon^\gamma,1]}(z)$ due to the properties of the characteristic function $\delta_{[\varepsilon^\gamma,1]}$. In view of the transformations (4.9), (4.11) we note that

$$\varepsilon^\gamma \le z \le 1 \text{ a.e. on } \Omega_{\mathrm{D}} \quad \text{is equivalent to} \quad \varepsilon^\gamma \le \Pi^\varepsilon z \le 1 \text{ a.e. on } \Omega_{\mathrm{D}}^\varepsilon\,. \tag{4.19}$$

Since we now use the common state space $\mathcal{Q}$ we also extend the dissipation potential. Transformation of the integral in (4.3) leads to the potential $\mathcal{R}:\mathcal{Z}\to[0,\infty]$,

$$\mathcal{R}(v) := \begin{cases} \int_{\Omega_{\mathrm{D}}} -\varrho v(y)\,\mathrm{d}y & \text{if } v\le 0 \text{ a.e. on } \Omega_{\mathrm{D}}, \\ \infty & \text{otherwise.} \end{cases} \tag{4.20}$$

For all $t\in[0,T]$ we define the stable sets of the transformed, approximating problems by

$$\mathcal{S}_\varepsilon^\kappa(t) := \{q=(u,z)\in\mathcal{Q}\,|\,\mathcal{E}_\varepsilon^\kappa(t,q)<\infty,\ \mathcal{E}_\varepsilon^\kappa(t,q)\le\mathcal{E}_\varepsilon^\kappa(t,\tilde q)+\mathcal{R}(\tilde z - z) \text{ for all } \tilde q=(\tilde u,\tilde z)\in\mathcal{Q}\}.$$

Using the common state space and the extended functionals we can rewrite the rate-independent systems $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}_\varepsilon^\kappa, \widetilde{\mathcal{R}}_\varepsilon)$ via the systems $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R})$. It remains to transfer the existence result stated in Proposition 4.1.1 for the systems $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}_\varepsilon^\kappa, \mathcal{R}_\varepsilon)$ to the systems $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R})$. For this, we first show that $\partial_t\mathcal{E}_\varepsilon^\kappa(t,q)$ is well-defined for all $q\in\mathcal{Q}$ if $\mathcal{E}_\varepsilon^\kappa(t_q,q)<\infty$ for some $t_q\in[0,T]$.

Lemma 4.1.2 (Well-posedness of $\partial_t\mathcal{E}_\varepsilon^\kappa$) *Let $\varepsilon\in(0,\varepsilon_0]$, $\kappa\in(0,\kappa_0]$ be fixed. Let the systems $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R})$ be given by (4.15), (4.18) and (4.20) such that (4.4) and (4.5) are valid. Then, for all $(t_q,q)\in[0,T]\times\mathcal{Q}$ with $\mathcal{E}_\varepsilon^\kappa(t_q,q)<\infty$ it holds $\mathcal{E}_\varepsilon^\kappa(\cdot,q)\in\mathrm{C}^1([0,T])$ with*

$$\partial_t\mathcal{E}_\varepsilon^\kappa(t,q) = \int_{\Omega_-^{\varepsilon_0}\cup\Omega_+^{\varepsilon_0}} \partial_e W(e(u+g(t))):\partial_t e(g(t))\,\mathrm{d}x\,. \tag{4.21}$$

Proof: Due to (4.1), (4.18) and (4.11) we have that $\mathcal{E}_\varepsilon^\kappa(t_q,u,z) = \widetilde{\mathcal{E}}_\varepsilon^\kappa(t_q,u,\Pi^\varepsilon z)<\infty$. Since $\int_{\Omega_{\mathrm{D}}}\frac{\kappa}{r}|\nabla_\varepsilon z|^r\,\mathrm{d}y$ with $z\in\mathcal{Z}_{\mathrm{D}}$ is not affected by $t\in[0,T]$ we may conclude that $\partial_t\mathcal{E}_\kappa^\varepsilon(t,u,z) = \partial_t\widetilde{\mathcal{E}}_\kappa^\varepsilon(t,u,\Pi^\varepsilon z)$, which is given by formula (4.21). ∎

This result is used to adapt Proposition 4.1.1 to the extended functionals.

Proposition 4.1.3 (Energetic solutions of $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$) *For all fixed $\varepsilon\in(0,\varepsilon_0]$ and $\kappa\in(0,\kappa_0]$ let the rate-independent system $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$ be defined via (4.15), (4.18) and (4.20) such that (4.4) and (4.5) hold true. Then for $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$ and for any initial state $q_0\in\mathcal{S}^\kappa_\varepsilon(0)$ there exists an energetic solution $q:[0,T]\to\mathcal{Q}$ of the initial value problem $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R},q_0)$.*

Proof: Consider $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$ with the initial state $q_0=(u_0,z_0)\in\mathcal{S}^\kappa_\varepsilon(0)$. By (4.18) and (4.20) we find that $(u_0,\Pi^\varepsilon z_0)\in\widetilde{\mathcal{S}}^\kappa_\varepsilon(0)$. Then Proposition 4.1.1 provides the existence of an energetic solution $q=(u,z):[0,T]\to\mathcal{Q}_\varepsilon$ of $(\mathcal{Q}_\varepsilon,\widetilde{\mathcal{E}}^\kappa_\varepsilon,\widetilde{\mathcal{R}}_\varepsilon)$ with $(u(0),z(0))=(u_0,\Pi^\varepsilon z_0)$. We want to show that $(u,\Pi_\varepsilon z)$ is an energetic solution of $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R},q_0)$. To verify that $(u(t),\Pi_\varepsilon z(t))\in\mathcal{S}^\kappa_\varepsilon(t)$ we use that $(u(t),z(t))\in\widetilde{\mathcal{S}}^\kappa_\varepsilon(t)$, implying for all $(\tilde u,\tilde z)\in\mathcal{Q}_\varepsilon$ that $\widetilde{\mathcal{E}}^\kappa_\varepsilon(t,u(t),z(t))\le\widetilde{\mathcal{E}}^\kappa_\varepsilon(t,\tilde u,\tilde z)+\widetilde{\mathcal{R}}_\varepsilon(\tilde z-z(t))$. From the bijectivity of $\Pi_\varepsilon:\mathcal{Z}_\varepsilon\to\mathcal{Z}_{\mathrm{D}}$ for all $\varepsilon\in(0,\varepsilon_0]$ and (4.19) we conclude that $\widetilde{\mathcal{E}}^\kappa_\varepsilon(t,\tilde u,\tilde z)<\infty$ is equivalent to $\mathcal{E}^\kappa_\varepsilon(t,\tilde u,\Pi_\varepsilon\tilde z)<\infty$. Applying Π_ε and transforming the integrals in stability condition (2.60(S)) yields the stability of $(u(t),\Pi_\varepsilon z(t))$, i.e. $\mathcal{E}^\kappa_\varepsilon(t,u(t),\Pi_\varepsilon z(t))\le\mathcal{E}^\kappa_\varepsilon(t,\tilde u,\Pi_\varepsilon\tilde z)+\mathcal{R}(\Pi_\varepsilon\tilde z-\Pi_\varepsilon z(t))$. The energy balance (2.60(E)) follows directly from $\mathrm{Diss}_{\mathcal{R}}(\Pi_\varepsilon z,[0,t])=\mathrm{Diss}_{\widetilde{\mathcal{R}}_\varepsilon}(z,[0,t])$ and Proposition 4.1.2, since $\partial_t\mathcal{E}^\kappa_\varepsilon(t,u(t),\Pi_\varepsilon z(t))=\partial_t\widetilde{\mathcal{E}}^\kappa_\varepsilon(t,u(t),z(t))$. ■

4.1.3 The Topologies $\mathcal{T}$, $\mathcal{T}_T$ and a Korn's Inequality

In the following we specify a suitable topology on the common state space $\mathcal{Q}$, which allows us to show that a subsequence of energetic solutions of the approximating systems $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$ converges to an energetic solution of the limit system as $\varepsilon\to0$ and as $\kappa\to0$ respectively.

For the analysis we will consider sequences of systems $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})_{\varepsilon\in(0,\varepsilon_0]}$ and sequences $(t_\varepsilon,q_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}\subset[0,T]\times\mathcal{Q}$ and thereby the notation $\varepsilon\in(0,\varepsilon_0]$ always stands for countably many indices $\varepsilon\in(0,\varepsilon_0]$ satisfying $\varepsilon\to0$. The indications $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})_{\kappa\in(0,\kappa_0]}$ and $(q_\kappa)_{\kappa\in(0,\kappa_0]}$ have to be understood similarly.

Since $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon,u_\varepsilon,z_\varepsilon)\le E$ for some $E\in[0,\infty)$ implies that $\|z_\varepsilon\|_{L^\infty(\Omega_{\mathrm{D}})}\le1$, a suitable topology on $\mathcal{Z}=L^\infty(\Omega_{\mathrm{D}})$ is the weak*-topology of $L^\infty(\Omega_{\mathrm{D}})$. In view of (4.18) and (4.5(H2)) we obtain that $\|e(u_\varepsilon+g(t_\varepsilon))\|_{L^p(\Omega^\varepsilon_-\cup\Omega^\varepsilon_+,\mathbb{R}^{d\times d})}\le E$. By the triangle inequality, assumption (4.5) and Korn's inequality on each of the domains $\Omega^\varepsilon_-\cup\Omega^\varepsilon_+$ we find a constant $\tilde E$ such that $\|u_\varepsilon\|_{W^{1,p}(\Omega^\varepsilon_-\cup\Omega^\varepsilon_+,\mathbb{R}^d)}\le\tilde E$, provided that the constants in Korn's inequality are uniformly bounded, which will be ensured below. Therefore the convergence of a sequence $(u_\varepsilon,z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ to a limit (u,z) has to be understood in the following sense

$$(u_\varepsilon,z_\varepsilon)\xrightarrow{\mathcal{T}}(u,z)\quad\Leftrightarrow\quad\begin{cases}u_\varepsilon\rightharpoonup u\text{ in }W^{1,p}(\Omega^\nu_-\cup\Omega^\nu_+,\mathbb{R}^d)\text{ for all }\nu\in(0,\varepsilon_0],\\ z_\varepsilon\overset{*}{\rightharpoonup}z\text{ in }L^\infty(\Omega_{\mathrm{D}}).\end{cases}\tag{4.22}$$

The following counterexample shows that $u_\varepsilon\rightharpoonup u$ in $W^{1,p}(\Omega^\nu_-\cup\Omega^\nu_+)$ for all $\nu\in(0,\varepsilon_0]$ does not imply $u_\varepsilon\rightharpoonup u$ in $W^{1,p}(\Omega_-\cup\Omega_+)$.

Example 4.1.4 *Let*

$$u_\varepsilon(x) := \begin{cases} -1 & \text{if } x_1 \in (-L,-\varepsilon), \\ \frac{1}{2\varepsilon}x_1 & \text{if } x_1 \in [-\varepsilon,\varepsilon], \\ 1 & \text{if } x_1 \in (\varepsilon, L) \end{cases} \qquad \text{and} \qquad u(x) := \begin{cases} -1 & \text{if } x_1 \in (-L,0), \\ 1 & \text{if } x_1 \in (0,L). \end{cases}$$

For all fixed $\nu \in (0,\varepsilon_0]$ we conclude that $u_\varepsilon \to u$ even strongly in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu)$. But $\|\partial_{x_1} u_\varepsilon\|^p_{L^p([-\varepsilon,0)\cup(0,\varepsilon])} = (2\varepsilon)^{1-p}$ is unbounded for all $p \in (1,\infty)$ and hence $u_\varepsilon \not\rightharpoonup u$ in $W^{1,p}(\Omega_- \cup \Omega_+)$.

To specify the convergence of sequences of pairs $(t_\varepsilon, q_\varepsilon) \in [0,T] \times \mathcal{Q}$ we define

$$(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t,q) \quad \Leftrightarrow \quad \begin{cases} t_\varepsilon \to t, \\ q_\varepsilon \xrightarrow{\mathcal{T}} q. \end{cases} \tag{4.23}$$

As already mentioned above it is required to prove a uniform Korn's inequality for the domains $\Omega_-^\varepsilon \cup \Omega_+^\varepsilon$. In particular it has to be shown that there is a constant $c_\mathcal{K}$ which is independent of $\varepsilon \in (0,\varepsilon_0]$. The proof is based on the transformation of the domains $\Omega_\pm^\varepsilon$ to the unit domains $\Omega_\pm$ and on the uniform boundedness of these transformations. Moreover the proof uses the classical ideas such as compactness arguments, which can be found in e.g. [KO88, Pom03].

Theorem 4.1.5 (Korn's inequality for a family of domains) *For all $0 < \varepsilon \le \varepsilon_0$ let $\Omega_\pm^\varepsilon \subset \Omega_\pm$ be the Lipschitz domains depicted in Fig.1a) and let $p \in (1,\infty)$. Then there is a constant $c_\mathcal{K} > 0$, such that for all $0 < \varepsilon \le \varepsilon_0$ and all $v \in W^{1,p}(\Omega_\pm^\varepsilon, \mathbb{R}^d)$ with $v = 0$ on Γ_{Dir} in trace sense it holds*

$$\|v\|_{W^{1,p}(\Omega_\pm^\varepsilon,\mathbb{R}^d)} \le c_\mathcal{K} \|e(v)\|_{L^p(\Omega_\pm^\varepsilon,\mathbb{R}^{d\times d})} . \tag{4.24}$$

Proof: It suffices to prove the result for Ω_+^ε and Ω_-^ε separately. We restrict ourselves to Ω_+^ε, the proof for Ω_-^ε is analogous.

We transform $\Omega_+^\varepsilon = (\varepsilon, L) \times \Gamma_\text{C}$ into $\Omega_+ = (0,L) \times \Gamma_\text{C}$ via the invertible mapping

$$\tau_\varepsilon : \Omega_+ \to \Omega_+^\varepsilon,\ (y_1, s) \mapsto (\varepsilon + \alpha(\varepsilon) y_1, s)\,, \text{ where } \alpha(\varepsilon) = (1 - \varepsilon/L)\,. \tag{4.25}$$

For $v_\varepsilon := v \circ \tau_\varepsilon \in W^{1,p}(\Omega_+, \mathbb{R}^d)$ we obtain that

$$\nabla_y v_\varepsilon = \nabla_x v \nabla_y \tau_\varepsilon \quad \text{and} \quad \nabla_x v = \nabla_y v_\varepsilon \nabla_x \tau_\varepsilon^{-1} \tag{4.26}$$

where $\nabla_y \tau_\varepsilon = \operatorname{diag}(\alpha(\varepsilon), 1, \ldots, 1)$, $y = (y_1, s) \in \Omega_+$ and $x = (x_1, s) \in \Omega_+^\varepsilon$ with $x_1 = \varepsilon + \alpha(\varepsilon) y_1$.

Using these relations and exploiting Korn's inequality on Ω_+ results in a uniform Korn's inequality for all $\varepsilon \in (0, \varepsilon_0]$:

$$\begin{aligned} \|v\|^p_{W^{1,p}(\Omega_+^\varepsilon)} &= \|v\|^p_{L^p(\Omega_+^\varepsilon)} + \|\nabla_x v\|^p_{L^p(\Omega_+^\varepsilon)} = \alpha(\varepsilon)\big(\|v_\varepsilon\|^p_{L^p(\Omega_+)} + \|\nabla_y v_\varepsilon \nabla_x \tau_\varepsilon^{-1}\|^p_{L^p(\Omega_+)}\big) \\ &\le \alpha(\varepsilon)^{-p+1}\big(\|v_\varepsilon\|^p_{L^p(\Omega_+)} + \|\nabla_y v_\varepsilon\|^p_{L^p(\Omega_+)}\big) \le \alpha(\varepsilon_0)^{-p+1} C^p_\mathcal{K} \|e(v_\varepsilon)\|^p_{L^p(\Omega_+)} \\ &\le \alpha(\varepsilon_0)^{-p} C^p_\mathcal{K} \|e(v)\|^p_{L^p(\Omega_+^\varepsilon)} . \end{aligned}$$

■

4.2 The First Γ-limit: Gradient Delamination

In this section we show that the damage models given by $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ approximate a model for gradient delamination as $\varepsilon \to 0$. In particular we prove that also their energetic solutions approximate an energetic solution of the limit system.

4.2.1 The Model for Gradient Delamination

Our aim for this section is to show that $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})_{\varepsilon\in(0,\varepsilon_0]}$ Γ-converges to the limit system $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ as $\varepsilon \to 0$, see Fig. 1b), where $\mathcal{E}^\kappa : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ is given by

$$\mathcal{E}^\kappa(t,q) := \begin{cases} \int\limits_{\Omega_-\cup\Omega_+} W(e(u+g(t)))\,\mathrm{d}x + \int\limits_{\Omega_\mathrm{D}} \Big(\frac{\kappa}{r}|\nabla z|^r + \delta_{[0,1]}(z)\Big)\,\mathrm{d}y & \text{if } q=(u,z)\in\mathcal{Q}_\mathrm{C}, \\ \infty & \text{if } q \in \mathcal{Q}\backslash\mathcal{Q}_\mathrm{C}\,, \end{cases} \tag{4.27}$$

$$\mathcal{Z}_\mathrm{C} := \{z \in W^{1,r}(\Omega_\mathrm{D}) \,|\, \partial_{y_1} z = 0\}\,, \tag{4.28}$$

$$\mathcal{Q}_\mathrm{C} := \big\{q = (u,z) \in \mathcal{U} \times \mathcal{Z}_\mathrm{C} \,\big|\, S_\mathrm{C} z[\![u]\!] = 0 \text{ and } [\![u\cdot \mathrm{n}_1]\!] \geq 0 \text{ a.e. on } \Gamma_\mathrm{C}\big\} \tag{4.29}$$

with $\mathcal{U}$ from (4.13). Here $S_\mathrm{C} z = z|_{\Gamma_\mathrm{C}}$ in the trace sense and $[\![\cdot]\!]$ denotes the jump of a function defined on $\Omega_-\cup\Omega_+$ across the interface Γ_C in the trace sense. Therefore the constraint $S_\mathrm{C} z[\![u]\!] = 0$ a.e. on Γ_C incorporates a transmission condition, namely $[\![u]\!] = 0$ whenever $S_\mathrm{C} z > 0$, which was already used in [Fré88]. Furthermore $\mathrm{n}_1 := (1,0,\ldots,0)$ stands for the unit normal vector to Γ_C. Thus the condition $[\![u\cdot\mathrm{n}_1]\!] \geq 0$ a.e. on Γ_C prevents the interpenetration of the material of Ω_- and Ω_+.

If $(u,z) \in \mathcal{Q}_\mathrm{C}$ and $v \in \mathcal{Z}_\mathrm{C}$ we find that $\mathcal{E}^\kappa(t,q)$ and $\mathcal{R}(v)$ equivalently read

$$\mathcal{E}^\kappa(t,u,z) = \int\limits_{\Omega_-\cup\Omega_+} W(e(u+g(t)))\,\mathrm{d}x + 2\int\limits_{\Gamma_\mathrm{C}} \Big(\frac{\kappa}{r}|\nabla_s S_\mathrm{C} z|^r + \delta_{[0,1]}(S_\mathrm{C} z)\Big)\,\mathrm{d}s \tag{4.30}$$

$$\mathcal{R}(v) = \begin{cases} 2\int_{\Gamma_\mathrm{C}} -\varrho S_\mathrm{C} v(s)\,\mathrm{d}s & \text{if } S_\mathrm{C} v \leq 0 \ \mathcal{L}^{d-1}\text{-a.e. on } \Gamma_\mathrm{C}\,, \\ \infty & \text{otherwise} \end{cases} \tag{4.31}$$

with $s := (x_2,\ldots,x_d)$ and $\nabla_s := (\partial_{x_2},\ldots,\partial_{x_d})$. This shows that the limit system indeed models delamination along the interface Γ_C. For all $t\in[0,T]$ we introduce the stable sets

$$\mathcal{S}^\kappa(t) := \{q=(u,z)\in\mathcal{Q} \,|\, \mathcal{E}^\kappa(t,q)<\infty,\ \mathcal{E}^\kappa(t,q)\leq\mathcal{E}^\kappa(t,\tilde{q})+\mathcal{R}(\tilde{z}-z) \text{ for all } \tilde{q}=(\tilde{u},\tilde{z})\in\mathcal{Q}\}\,.$$

The convergence result, which will be proven in the next subsection, is the following:

Theorem 4.2.1 (Γ-convergence of the damage problems) *Let the assumptions* (4.4) *and* (4.5) *be valid with* $p \in (1, \infty)$, $\gamma > (p-1)$, $r > d$, *and* $\kappa \in (0, \kappa_0]$ *fixed. For all* $\varepsilon \in (0, \varepsilon_0]$ *let* $q_\varepsilon : [0, T] \to \mathcal{Q}$ *be an energetic solution of* $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ *given by* (4.15), (4.18) *and* (4.20). *If the initial values satisfy* $q_0^\varepsilon \xrightarrow{\mathcal{T}} q_0$ *and* $\mathcal{E}^\kappa_\varepsilon(0, q_0^\varepsilon) \to \mathcal{E}^\kappa(0, q_0)$, *then the damage problems* $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})_{\varepsilon \in (0,\varepsilon_0]}$ Γ*-converge to the delamination problem* $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ *given by* (4.15), (4.27) *and* (4.20) *in the following sense:*
There is a subsequence $(q_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]}$, *such that for all* $t \in [0, T]$ *we have* $q_\varepsilon(t) \xrightarrow{\mathcal{T}} q(t)$ *and* $q = (u, z) : [0, T] \to \mathcal{Q}$ *is an energetic solution of* $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$, *i.e. in particular for all* $t \in [0, T]$ *it holds*

$$q(t) \in \mathcal{S}^\kappa(t) \quad \textit{and} \quad \mathcal{E}^\kappa(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(z, [0, t]) = \mathcal{E}^\kappa(0, q(0)) + \int_0^t \partial_\xi \mathcal{E}^\kappa(\xi, q(\xi))\, \mathrm{d}\xi\,.$$

Moreover, we have $\mathcal{E}^\kappa_\varepsilon(t, q_\varepsilon(t)) \to \mathcal{E}^\kappa(t, q(t))$ *and* $\mathrm{Diss}_{\mathcal{R}}(z_\varepsilon, [0, t]) \to \mathrm{Diss}_{\mathcal{R}}(z, [0, t])$ *for all* $t \in [0, T]$ *as well as* $\partial_t \mathcal{E}^\kappa_\varepsilon(t, q_\varepsilon(t)) \to \partial_t \mathcal{E}^\kappa(t, q(t))$ *a.e. in* $[0, T]$.

4.2.2 Proof of the Convergence Theorem

The proof of Theorem 4.2.1 consists of an application of Theorem 2.4.5. Thus, in the following we verify the conditions (2.69), (2.70) and (2.71). Additionally we have to prove the existence of a (sub-)sequence $(q_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]}$ of energetic solutions of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ which converges $q_\varepsilon(t) \xrightarrow{\mathcal{T}} q(t)$ for all $t \in [0, T]$, and q is an energetic solution of $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$.

Properties of $\mathcal{E}^\kappa_\varepsilon$ and $\mathcal{E}^\kappa$ and Verification of (2.69(E1)) and (2.71(C3))

In the following we show that the sublevels of the energy functionals $\mathcal{E}^\kappa_\varepsilon(t, \cdot)$ and $\mathcal{E}^\kappa(t, \cdot)$ are compact with respect to the topology $\mathcal{T}$ and that unions of sublevels of $\mathcal{E}^\kappa_\varepsilon(t, \cdot)$ are precompact. This complies with condition (2.69(E1)). Moreover we prove that the Γ-lim inf inequality (2.71(C3)) is satisfied.

As a direct consequence of stability (2.60(S)) one obtains that the energetic solutions of the approximating problems have an equibounded energy; to see this one may check (2.60(S)) for the energetic solutions and the states $(\hat{u}, \hat{z}_\varepsilon)$ with $\hat{u} = 0$ and $\hat{z}_\varepsilon = \varepsilon^\gamma$. Hence, we ensure by the next theorem that the equiboundedness of sequences enables us to extract weakly convergent subsequences and we verify that their limit indeed is an element of the set $\mathcal{Q}_{\mathrm{C}}$, given by (4.29). The Items 1. and 2.(a) below result from the coercivity inequality (4.5(H2)), which yields uniform boundedness of u_ε in $W^{1,p}(\Omega^\nu_- \cup \Omega^\nu_+, \mathbb{R}^d)$ for all fixed $\nu \in (0, \varepsilon_0]$ and hence, using Cantor's diagonal process, the convergence of a subsequence for all fixed ν. Moreover, 2.(b) results from the uniform boundedness of the gradient term for fixed $\kappa \in (0, \kappa_0]$. Additionally, the assumption $r > d$ implies the continuity of the sequence and its limit and leads to a subsequence converging uniformly on Ω_{D}. This enables us to obtain a lower bound $\delta_K \le z_\varepsilon$ on compact subsets $K \subset \Omega_{\rm D}$ away from the zero set of the limit z. This yields uniform coercivity on K and allows us to verify 2.(d). Moreover, 2.(c) is a consequence of the term $(e_{11}(u_\varepsilon))^-$ included to $W_{\rm D}$, see (4.2).

Theorem 4.2.2 (Properties of sequences with equibounded energies) *Let the energy functionals $\mathcal{E}_\varepsilon^\kappa$ be given by* (4.18) *such that the assumptions* (4.4) *and* (4.5) *hold. Let $\kappa \in (0, \kappa_0]$ fixed, $(t_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset [0,T]$, $p \in (1,\infty)$ and $r > d$. Assume that $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, u_\varepsilon, z_\varepsilon) \le E$ for all $\varepsilon \in (0, \varepsilon_0]$. Then*

1. *there is a subsequence $(u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}} (u, z)$ as $\varepsilon \to 0$,*
2. *for the limit holds $(u, z) \in \mathcal{Q}_{\rm C}$, i.e.*
   (a) *$u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, $u = 0$ on $\Gamma_{\rm Dir}$ in the trace sense,*
   (b) *$z \in W^{1,r}(\Omega_{\rm D})$, $0 \le z(y) \le 1$ and $\partial_{y_1} z(y) = 0$ for all $y \in \Omega_{\rm D}$,*
   (c) *$[\![u \cdot \mathrm{n}_1]\!] \ge 0$ a.e. on $\Gamma_{\rm C}$,*
   (d) *$S_{\rm C} z [\![u]\!] = 0$ a.e. on $\Gamma_{\rm C}$.*

Moreover, for $\gamma < p-1$ jumps are prevented at all, i.e. $[\![u]\!] = 0$ a.e. on $\Gamma_{\rm C}$.

Proof: Recall $\mathcal{Q}$ from (4.15), $\mathcal{E}_\varepsilon^\kappa$ from (4.18) and $\mathcal{Q}_{\rm C}$ from (4.29).

Ad 1. and 2.(a): From the equiboundedness of $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon)$ we infer that $\varepsilon^\gamma \le z_\varepsilon \le 1$ a.e. in $\Omega_{\rm D}$. Since the unit ball of $L^\infty(\Omega_{\rm D})$, which is the dual space of $L^1(\Omega_{\rm D})$, is weakly* sequentially compact by the theorem of Banach-Alaoglu we find a subsequence $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\rm D})$.

Keep $\nu \in (0, \varepsilon_0]$ fixed. Then $\Omega_-^\nu \cup \Omega_+^\nu \subseteq \Omega_-^\varepsilon \cup \Omega_+^\varepsilon$ for all $\varepsilon \le \nu$. From the equiboundedness, hypothesis (4.5(H2)), (4.4) and the uniform Korn's inequality (4.24), where we exploit the Dirichlet conditions on the Lipschitz domains $\Omega_\pm^\nu$, we infer that

$$
\begin{aligned}
E \ge \int_{\Omega_-^\nu \cup \Omega_+^\nu} W(e(u_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x &\ge c\|e(u_\varepsilon + g(t_\varepsilon))\|^p_{L^p(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^{d\times d})} \\
&\ge 2^{1-p} c\|e(u_\varepsilon)\|^p_{L^p(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^{d\times d})} - cc_g \ge \tfrac{2^{1-p}c}{c_{\mathcal{K}}}\|u_\varepsilon\|^p_{W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)} - cc_g\,,
\end{aligned}
\tag{4.32}
$$

which proves that $\|u_\varepsilon\|_{W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)}$ are uniformly bounded for all $\varepsilon \le \nu$. Thus there is a weakly converging subsequence, since $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ is a reflexive Banach space. For a countable set of indices ν with $\nu \to 0$ we conclude by Cantor's diagonal process that we have a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ as $\varepsilon \to 0$ for all ν. As $\nu \to 0$ we conclude that $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ with $u = 0$ on $\Gamma_{\rm Dir}$ in trace sense. This proves the existence of a subsequence $q_\varepsilon \xrightarrow{\mathcal{T}} q$.

Ad 2.(b): The equiboundedness of $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \le E$ together with $0 < \varepsilon \le \varepsilon_0 < 1$ yields that $\|z_\varepsilon\|^r_{W^{1,r}(\Omega_{\rm D})} \le \frac{r(E + \mathcal{L}^d(\Omega_{\rm D}))}{\kappa}$ and furthermore $\|\partial_{y_1} z_\varepsilon\|^r_{L^r(\Omega_{\rm D})} \le \frac{\varepsilon^r r E}{\kappa}$. Since $W^{1,r}(\Omega_{\rm D})$ is a reflexive Banach space there is a subsequence $z_\varepsilon \rightharpoonup z$ in $W^{1,r}(\Omega_{\rm D})$ with $\partial_{y_1} z = 0$ a.e. in $\Omega_{\rm D}$.

Due to the compact embedding of $W^{1,r}(\Omega_{\mathrm{D}}) \Subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$ for $r > d$ we have $z \in \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$ and thus $\partial_{y_1} z(y) = 0$ as well as $[\varepsilon^\gamma, 1] \ni z_\varepsilon(y) \to z(y) \in [0,1]$ for all $y \in \Omega_{\mathrm{D}}$.

Ad 2.(c): Let u_ε^i be the ith component of $u_\varepsilon \in \mathbb{R}^d$. For all $\nu > 0$ we define the linear and continuous trace operators $S_\nu^\pm : W^{1,p}(\Omega_\pm) \to L^1(\Gamma_{\mathrm{C}})$, $S_\nu^\pm v = (v(\pm\nu, s) - v_\pm)$, where $v_\pm$ is the trace of $v|_{\Omega_\pm} \in W^{1,p}(\Omega_\pm)$ onto Γ_{C}. Thereby $\|S_\nu^\pm\| = \nu^{\frac{p-1}{p}} (\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}))^{\frac{p-1}{p}}$, since for all v with $\|v\|_{W^{1,p}(\Omega_\pm)} = 1$ it is

$$\begin{aligned}\|S_\nu^\pm v\|_{L^1(\Gamma_{\mathrm{C}})} = \|v(\pm\nu, \cdot) - v_\pm\|_{L^1(\Gamma_{\mathrm{C}})} &\leq \nu^{\frac{p-1}{p}} (\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}))^{\frac{p-1}{p}} \int_{\Gamma_{\mathrm{C}}} \int_0^\nu |\partial_{x_1} v|^p \,\mathrm{d}x \\ &\leq \nu^{\frac{p-1}{p}} (\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}))^{\frac{p-1}{p}} \|v\|_{W^{1,p}(\Omega_\pm)} .\end{aligned}$$

In particular, this proves $\int_{\Gamma_{\mathrm{C}}} |u^1(\pm\nu, s) - u^1_\pm| \,\mathrm{d}s = \int_{\Gamma_{\mathrm{C}}} |S_\nu^\pm u^1| \,\mathrm{d}s \to 0$ as $\nu \to 0$ for the limit function $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$. Hence there is a subsequence $u^1(\pm\nu, \cdot) \to u^1_\pm$ pointwise a.e. on Γ_{C}, so that we may conclude by Fatou's lemma with $(f)^+ = \max\{0, f\}$

$$\int_{\Gamma_{\mathrm{C}}} \big(-[\![u\cdot \mathrm{n}_1]\!]\big)^+ \,\mathrm{d}s = \int_{\Gamma_{\mathrm{C}}} (u^1_- - u^1_+)^+ \,\mathrm{d}s \leq \liminf_{\nu \to 0} \int_{\Gamma_{\mathrm{C}}} (u^1(-\nu, s) - u^1(+\nu, s))^+ \,\mathrm{d}s . \tag{4.33}$$

Due to $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \leq E$ there is a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ for all fixed $\nu \in (0, \varepsilon_0]$. By the compactness of the trace operator $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu) \to L^p(\Gamma_{\mathrm{C}})$ and Fatou's lemma we conclude that

$$\begin{aligned}\int_{\Gamma_{\mathrm{C}}} (-[\![u^1]\!])^+ \,\mathrm{d}s &\leq \liminf_{\nu \to 0} \int_{\Gamma_{\mathrm{C}}} (u^1(-\nu, s) - u^1(\nu, s))^+ \,\mathrm{d}s \\ &\leq \liminf_{\nu \to 0} \liminf_{\varepsilon \to 0} \int_{\Gamma_{\mathrm{C}}} (u^1_\varepsilon(-\nu, s) - u^1_\varepsilon(\nu, s))^+ \,\mathrm{d}s.\end{aligned}$$

Since $(\cdot)^+$ is a convex function we may apply Jensen's inequality and hence we find

$$\int_{\Gamma_{\mathrm{C}}} (u^1_\varepsilon(-\nu) - u^1_\varepsilon(\nu))^+ \,\mathrm{d}s = \int_{\Gamma_{\mathrm{C}}} \Big(\int_{-\nu}^{\nu} -\partial_{x_1} u^1_\varepsilon \,\mathrm{d}x_1 \Big)^+ \,\mathrm{d}s \leq \int_{\Omega_{\mathrm{D}}^\nu} (-\partial_{x_1} u^1_\varepsilon)^+ \,\mathrm{d}x , \tag{4.34}$$

where $(-\partial_{x_1} u^1_\varepsilon)^+ = (-e_{11}(u_\varepsilon))^+$ occurs in $W_{\mathrm{D}}(e(u_\varepsilon))$ of (4.2). Due to $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, u_\varepsilon, z_\varepsilon) \leq E$ and the coercivity of W it holds $\|(-e_{11}(u_\varepsilon))^+\|^p_{L^p(\Omega)} \leq \max\{c^{-1}, 1\}E$. Hence there exists a limit $b \in L^p(\Omega)$, such that $(-e_{11}(u_\varepsilon))^+ \rightharpoonup b$ in $L^p(\Omega)$. Thus

$$\lim_{\nu \to 0} \lim_{\varepsilon \to 0} \int_{\Omega_{\mathrm{D}}^\nu} (-e_{11}(u_\varepsilon))^+ \,\mathrm{d}x = \lim_{\nu \to 0} \int_{\Omega_{\mathrm{D}}^\nu} b \,\mathrm{d}x = 0 .$$

This proves $\int_{\Gamma_{\mathrm{C}}} (u^1_- - u^1_+)^+ \,\mathrm{d}s = 0$, which implies that $[\![u\cdot \mathrm{n}_1]\!] \geq 0$ a.e. on Γ_{C}.

Ad 2.(d): In the following we verify $S_{\mathrm{C}} z[\![u]\!] = 0$ a.e. on Γ_{C} for the limit state (u, z). As it was proven in Ad 2.(b), there is a subsequence $z_\varepsilon \to z$ in $\mathrm{C}(\overline{\Omega_{\mathrm{D}}})$ by the compact embedding $W^{1,r}(\Omega_{\mathrm{D}}) \Subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$. I.e. for all $\alpha > 0$ there is an index $\varepsilon_\alpha > 0$ such that for all $\varepsilon < \varepsilon_\alpha$ it holds $\|z_\varepsilon - z\|_{C(\overline{\Omega_{\mathrm{D}}})} < \alpha$. Let $N_z = \{y \in \Omega_{\mathrm{D}} \,|\, z(y) = 0\}$. Choose a compact set $\hat{K} \Subset \Omega \backslash N_z$ with $\mathcal{L}^d(\hat{K} \cap \Omega^\varepsilon_{\mathrm{D}}) > 0$, such that $\hat{K} \cap \Omega^{\varepsilon_0}_\pm \neq \emptyset$ and so that $\Omega^{\varepsilon_0}_{\hat{K}} := \mathrm{int}(\overline{\Omega^{\varepsilon_0}_-} \cup \overline{\Omega^{\varepsilon_0}_+} \cup \hat{K})$ has a Lipschitz boundary. This choice is possible, since $\partial_{y_1} z = 0$ on Ω_{D}. Therefore $\Omega^{\varepsilon_0}_{\hat{K}}$ has the Dirichlet boundary Γ_{Dir} with $\mathcal{L}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0$ and thus Korn's inequality is applicable. Since $z \in \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$ there is $\delta_{\hat{K}} > 0$ such that $z(y) > \delta_{\hat{K}}$ for all $y \in \hat{K}$. Thus for $\alpha = \delta_{\hat{K}}/2$, for all $\varepsilon < \varepsilon_\alpha$ and for all $y \in \hat{K}$ we have $z_\varepsilon(y) > \delta_{\hat{K}}/2$. In other words: for all $x \in T_\varepsilon \hat{K} := \{(\varepsilon y_1, s) \,|\, (y_1, s) \in \hat{K}|_{\Omega_{\mathrm{D}}}\}$ it is $\Pi^\varepsilon z_\varepsilon(x) > \delta_{\hat{K}}/2$. Hence the equiboundedness $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, q_\varepsilon) \le E$ together with (4.5) yields

$$
\begin{aligned}
E &\ge \int_{\Omega^\varepsilon_- \cup \Omega^\varepsilon_+} W(e(u_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x + \int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi^\varepsilon z_\varepsilon \widetilde{W}(e(u_\varepsilon))\,\mathrm{d}x \\
&\ge \int_{\Omega^\varepsilon_- \cup \Omega^\varepsilon_+} W(e(u_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x + \int_{T_\varepsilon \hat{K}} \tfrac{\delta_{\hat{K}}}{2} \widetilde{W}(e(u_\varepsilon))\,\mathrm{d}x \ge \tfrac{\delta_{\hat{K}}}{2} \int_{\Omega^{\varepsilon_0}_{\hat{K}}} c|e(u_\varepsilon + g(t_\varepsilon))|^p\,\mathrm{d}x\,.
\end{aligned}
$$

Estimating similarly to (4.32) we find the bound $\|u_\varepsilon\|_{W^{1,p}(\Omega^{\varepsilon_0}_{\hat{K}})} \le \widehat{C}$ uniformly for all $\varepsilon \in (0, \varepsilon_0]$, where the constant $\widehat{C}$ now involves $c_{\mathcal{K}}(\Omega^{\varepsilon_0}_{\hat{K}})$ from Korn's inequality on the Lipschitz domain $\Omega^{\varepsilon_0}_{\hat{K}}$. Hence, there is a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega^{\varepsilon_0}_{\hat{K}}, \mathbb{R}^d)$. Since $\hat{K} \subset \Omega \backslash N_z$ was chosen arbitrarily and since $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ by Item 2.(a) we conclude that $u \in W^{1,p}_{\mathrm{loc}}(\Omega \backslash \{s \in \Gamma_{\mathrm{C}} \,|\, S_{\mathrm{C}} z(s) = 0\}, \mathbb{R}^d)$ with $S_{\mathrm{C}} z = z|_{\Gamma_{\mathrm{C}}}$. This proves that $[\![u]\!] = 0$ whenever $z > 0$ and $[\![u]\!] \neq 0$ is admissible whenever $z = 0$.

Ad $[\![u]\!]$: Using the ideas of Ad 2.(c) there is a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega^\nu_- \cup \Omega^\nu_+, \mathbb{R}^d)$ for all fixed $\nu \in (0, \varepsilon_0]$. Similarly to (4.34) we obtain for the ith component that

$$
\begin{aligned}
\int_{\Gamma_{\mathrm{C}}} |u^i_\varepsilon(\nu, s) - u^i_\varepsilon(-\nu, s)|\,\mathrm{d}s &= \int_{\Gamma_{\mathrm{C}}} \Big| \int_{-\nu}^{\nu} \partial_{x_1} u^i_\varepsilon(x_1, s)\,\mathrm{d}x_1 \Big|\,\mathrm{d}s \le \int_{\Omega^\nu_{\mathrm{D}}} |\partial_{x_1} u^i_\varepsilon|\,\mathrm{d}x \\
&\le \int_{\Omega^\varepsilon_{\mathrm{D}}} |\nabla u_\varepsilon|\,\mathrm{d}x + \int_{\Omega^\nu_{\mathrm{D}} \backslash \Omega^\varepsilon_{\mathrm{D}}} |\nabla u_\varepsilon|\,\mathrm{d}x\,.
\end{aligned} \tag{4.35}
$$

From $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, q_\varepsilon) \le E$ and the coercivity of $\widetilde{W}$ together with $\Pi^\varepsilon z_\varepsilon \ge \varepsilon^\gamma$, (4.19) and Korn's inequality in $W^{1,p}(\Omega, \mathbb{R}^d)$ we obtain for all $\varepsilon \in (0, \varepsilon_0]$ that

$$
\|\nabla u_\varepsilon\|_{L^p(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^{d\times d})} \le \|u_\varepsilon\|_{W^{1,p}(\Omega, \mathbb{R}^d)} \le c_{\mathcal{K}}(\Omega) E^{\frac{1}{p}} \varepsilon^{\frac{-\gamma}{p}} \tag{4.36}
$$

and by Hölder's inequality we find for the first term in (4.35) that

$$
\|\nabla u_\varepsilon\|_{L^1(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^{d\times d})} \le \varepsilon^{\frac{p-1}{p}} \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})^{\frac{p-1}{p}} c_{\mathcal{K}}(\Omega) E^{\frac{1}{p}} \varepsilon^{\frac{-\gamma}{p}}\,. \tag{4.37}
$$

If $\gamma < (p-1)$ we conclude from (4.37) that $\|\nabla u_\varepsilon\|_{L^1(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^{d\times d})} \to 0$.

Additionally the equiboundedness of the energies and the coercivity of W provide a constant $C > 0$ such that $\|\nabla u_\varepsilon\|_{L^p(\Omega^\varepsilon_- \cup \Omega^\varepsilon_+, \mathbb{R}^{d\times d})} \le C$. Thus, application of Hölder's inequality on the second term in (4.35) yields

$$\int_{\Omega_{\mathrm{D}}^{\nu}\setminus\Omega_{\mathrm{D}}^{\varepsilon}} |\nabla u_\varepsilon|\,\mathrm{d}x \le \big((\nu-\varepsilon)\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})\big)^{\frac{p-1}{p}} \|\nabla u_\varepsilon\|_{L^p((\Omega_-^\varepsilon\cup\Omega_+^\varepsilon)\setminus(\Omega_-^\nu\cup\Omega_+^\nu),\mathbb{R}^{d\times d})}$$
$$\le \big((\nu-\varepsilon)\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})\big)^{\frac{p-1}{p}} C \to 0 \quad \text{as } 0<\varepsilon\le\nu\to 0\,.$$

Using the ideas of Ad 2.(c) we obtain that $\int_{\Gamma_{\mathrm{C}}} |[\![u]\!]|\,\mathrm{d}s = 0$, if $\|\nabla u_\varepsilon\|_{L^1(\Omega_{\mathrm{D}}^\varepsilon,\mathbb{R}^{d\times d})} \to 0$. Hence jumps are prevented at all if $\gamma<(p-1)$. ■

The next lemma summarizes the properties of the limit energy $\mathcal{E}^\kappa$, which guarantee the existence of minimizers in the direct method of the calculus of variations, such as coercivity and lower semicontinuity. They yield the compactness of the sublevels of $\mathcal{E}^\kappa$.

Lemma 4.2.3 (Properties of the limit energy) *Let the assumptions* (4.4) *and* (4.5) *be satisfied. Then, for all $t\in[0,T]$ and all $\kappa\in(0,\kappa_0]$ the energy functional $\mathcal{E}^\kappa(t,\cdot):\mathcal{Q}_{\mathrm{C}}\to\mathbb{R}_\infty$ given by* (4.27) *and* (4.29) *is coercive and weakly sequentially lower semicontinuous on $\mathcal{Q}_{\mathrm{C}}$. In particular, with $C=cc_g+\frac{\kappa}{r}\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})$ it holds*

$$\mathcal{E}^\kappa(t,q) \ge \tfrac{2^{1-p}c}{C_K}\|u\|^p_{W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)} + \tfrac{\kappa}{r}\|z\|^r_{W^{1,r}(\Omega_{\mathrm{D}})} - C\,, \tag{4.38}$$

Moreover for all $E\in\mathbb{R}$ the sublevels $L_E^\kappa(t) := \{q\in\mathcal{Q}\,|\,\mathcal{E}^\kappa(t,q)\le E\}$ of the functional $\mathcal{E}^\kappa(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$ are weakly sequentially compact with respect to $\mathcal{T}$ from (4.22).

Proof: Keep $\kappa\in(0,\kappa_0]$ and $t\in[0,T]$ fixed. First assume that $(q_j)_{j\in\mathbb{N}}\subset\mathcal{Q}\backslash\mathcal{Q}_{\mathrm{C}}$. Then $\mathcal{E}^\kappa(t,q_j)=\infty$ for all $j\in\mathbb{N}$. Thus, for $\|u_j\|_{W^{1,p}(\Omega_-^\nu\cup\Omega_+^\nu,\mathbb{R}^d)}\to\infty$ for some $\nu\in(0,\varepsilon_0]$ the property $\mathcal{E}^\kappa(t,q_j)\to\infty$ is trivially satisfied. Similarly to estimate (4.32) coercivity inequality (4.38) follows from (4.5) for all $q\in\mathcal{Q}_{\mathrm{C}}$. Hence $\mathcal{E}^\kappa(t,\cdot)$ is coercive both on $\mathcal{Q}_{\mathrm{C}}$ and on $\mathcal{Q}$.

We now show the lower semicontinuity. Let $q_j \xrightarrow{\mathcal{T}} q$. If $q_j\in\mathcal{Q}\backslash\mathcal{Q}_{\mathrm{C}}$ for almost all $j\in\mathbb{N}$ then there is an index $j_0\in\mathbb{N}$ such that $q_j\in\mathcal{Q}\backslash\mathcal{Q}_{\mathrm{C}}$ for all $j\ge j_0$ and hence the lim inf-inequality trivially holds, i.e. $\liminf_{j\to\infty}\mathcal{E}^\kappa(t,q_j)=\infty\ge\mathcal{E}^\kappa(t,q)$. Assume that there is a subsequence (not relabelled) $(q_j)_{j\in\mathbb{N}}\subset\mathcal{Q}_{\mathrm{C}}$ with $u_j\rightharpoonup u$ in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ and $z_j\rightharpoonup z$ in $W^{1,r}(\Omega_{\mathrm{D}})$. Due to the compact embedding $W^{1,r}(\Omega_{\mathrm{D}})\subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$ it holds $z_j(x)\to z(x)$ for all $x\in\Omega_{\mathrm{D}}$. Let $u_j^\pm$, $u^\pm$ denote the traces of $u_j|_{\Omega_\pm}$ and $u|_{\Omega_\pm}$ on Γ_{C}. Then the compact trace operator $W^{1,p}(\Omega_\pm,\mathbb{R}^d)\to L^p(\Gamma_{\mathrm{C}},\mathbb{R}^d)$ implies that $u_j^\pm\to u^\pm$ in $L^p(\Gamma_{\mathrm{C}},\mathbb{R}^d)$ and that there is a subsequence $u_j^\pm\to u^\pm$ pointwise a.e. on Γ_{C}. Hence $[\![u\cdot \mathrm{n}_1]\!]\ge 0$ and $S_{\mathrm{C}}z[\![u]\!]=0$ a.e. on Γ_{C}, i.e. the limit $(u,z)\in\mathcal{Q}_{\mathrm{C}}$. Furthermore $\{z\in W^{1,r}(\Omega_{\mathrm{D}})\,|\,0\le z\le 1 \text{ a.e. on } \Omega_{\mathrm{D}}\}$ is a closed subset of $W^{1,r}(\Omega_{\mathrm{D}})$. Together with (4.5) one obtains lower semicontinuity of $\mathcal{E}^\kappa(t,\cdot)$ on $\mathcal{Q}_{\mathrm{C}}$.

Let now $(q_j)_{j\in\mathbb{N}}\subset L_E^\kappa(t)$. By coercivity inequality (4.38) there are constants $c_1(E), c_2(E)$ such that $\|u_j\|_{W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)}\le c_1(E)$ and $\|z_j\|_{W^{1,r}(\Omega_{\mathrm{D}})}\le c_2(E)$. Since $W^{1,p}(\Omega_-^+,\mathbb{R}^d)$ and $W^{1,r}(\Omega_{\mathrm{D}})$ are reflexive Banach spaces there are subsequences $u_j\rightharpoonup u$ in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ and $z_j\rightharpoonup z$ in $W^{1,r}(\Omega_{\mathrm{D}})$. From the lower semicontinuity of $\mathcal{E}^\kappa(t,\cdot)$ on $\mathcal{Q}_{\mathrm{C}}$ we now infer $E\ge\liminf\mathcal{E}^\kappa(t,q_j)\ge\mathcal{E}^\kappa(t,q)$, which proves that the sublevels of $\mathcal{E}^\kappa:\mathcal{Q}\to\mathbb{R}_\infty$ are compact in with respect to $\mathcal{T}$. ■

As a consequence of Theorem 4.2.2 and Lemma 4.2.3 the next corollary states that condition (2.69(E1)) is satisfied.

Corollary 4.2.4 *Keep $\kappa \in (0, \kappa_0]$ fixed and let the assumptions* (4.4) *and* (4.5) *hold true. Then, for all $\varepsilon \in (0, \varepsilon_0]$ the sublevels $L_E^{\varepsilon,\kappa}(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}_\varepsilon^\kappa(t,q) \leq E\}$ as well as the sublevels $L_E^\kappa(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}^\kappa(t,q) \leq E\}$ are compact and the unions $\cup_{\varepsilon\in(0,\varepsilon_0]} L_E^{\varepsilon,\kappa}(t)$ are precompact with respect to the topology $\mathcal{T}$, which is defined by* (4.22).

Proof: For all fixed $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in (0, \kappa_0]$ the weak sequential compactness of the sublevels $L_E^{\varepsilon,\kappa}(t)$ in $W^{1,p}(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega_\mathrm{D})$ can be obtained from Proposition 3.1.4 using that the composed density $\overline{W}$ defined in (4.6) satisfies hypotheses (4.5). Since $\mathcal{T}$ is coarser than the weak topology of $W^{1,p}(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega_\mathrm{D})$ we conclude the compactness of $L_E^{\varepsilon,\kappa}(t)$ with respect to $\mathcal{T}$. The precompactness of unions of sublevels with respect to $\mathcal{T}$ directly follows from Theorem 4.2.2 for $t_\varepsilon = t$ and the compactness of $L_E^\kappa(t)$ is due to Lemma 4.2.3. ■

In the following we prove the Γ-lim inf-inequality for $\mathcal{E}_\varepsilon^\kappa$, which corresponds to condition (2.71(C3)). The main idea in the proof is to exploit the lower semicontinuity of $\mathcal{E}_\varepsilon^\kappa(t, \cdot)$ on $L^p(\Omega_-^\nu \cup \Omega_-^\nu, \mathbb{R}^{d\times d}) \times L^r(\Omega_\mathrm{D}, \mathbb{R}^d)$ for all fixed $\nu \in (0, \varepsilon_0]$. The use of this space is admissible since the lower Γ-limit only has to be verified for stable sequences, so that their energies and therewith the damage gradients are uniformly bounded.

Lemma 4.2.5 (Lower Γ-limit of the energy functionals) *Keep $\kappa \in (0, \kappa_0]$ fixed. Let $(t_\varepsilon, u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, u, z)$ as $\varepsilon \to 0$ and $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}_\varepsilon^\kappa(t_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$. Then*

$$\mathcal{E}^\kappa(t, u, z) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^\kappa(t_\varepsilon, u_\varepsilon, z_\varepsilon)\,. \tag{4.39}$$

Proof: In view of (4.4) it holds $g(t_\varepsilon) \to g(t)$ in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$. Since $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}_\varepsilon^\kappa(t_\varepsilon)$ we find a constant $E > 0$ so that $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, u_\varepsilon, z_\varepsilon) \leq E$ for all $\varepsilon \in (0, \varepsilon_0]$. From Theorem 4.2.2 then follows that the limit $(u, z) \in \mathcal{Q}_\mathrm{C}$. Moreover we conclude that there is a subsequence $z_\varepsilon \rightharpoonup z$ in $W^{1,r}(\Omega_\mathrm{D})$, which is needed to show that $\liminf_{\varepsilon\to 0} \frac{\kappa}{r} \|\nabla_\varepsilon z_\varepsilon\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \geq \frac{\kappa}{r} \|\nabla z\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}$. Furthermore, we observe that $\int_{\Omega_-^\nu \cup \Omega_+^\nu} W(\cdot)\,\mathrm{d}x$ is weakly sequentially lower semicontinuous on $L^p(\Omega_-^\nu \cup \Omega_+^\nu; \mathbb{R}^{d\times d})$ by (4.5(H1)) and (4.5(C1)). In view of (4.5(H2)) and Theorem 4.2.2, Item 1 it holds for all $\nu > 0$

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^\kappa(t_\varepsilon, u_\varepsilon, z_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \int_{\Omega_-^\nu \cup \Omega_+^\nu} W(e(u_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x + \liminf_{\varepsilon\to 0} \int_{\Omega_\mathrm{D}} \tfrac{\kappa}{r} |\nabla_\varepsilon z_\varepsilon|^r\,\mathrm{d}y \\
&\geq \int_{\Omega_-^\nu \cup \Omega_+^\nu} W(e(u + g(t)))\,\mathrm{d}x + \liminf_{\varepsilon\to 0} \int_{\Omega_\mathrm{D}} \tfrac{\kappa}{r} |\nabla_s z_\varepsilon|^r\,\mathrm{d}y \\
&\geq \int_{\Omega_-^\nu \cup \Omega_+^\nu} W(e(u + g(t)))\,\mathrm{d}x + \int_{\Omega_\mathrm{D}} \tfrac{\kappa}{r} |\nabla_s z|^r\,\mathrm{d}y \\
&= \int_{\Omega_-^\nu \cup \Omega_+^\nu} W(e(u + g(t)))\,\mathrm{d}x + \int_{\Omega_\mathrm{D}} \tfrac{\kappa}{r} |\nabla z|^r\,\mathrm{d}y\,,
\end{aligned}$$

where the last equality is due to $\partial_{y_1} z = 0$. Since $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ by Theorem 4.2.2, Item 2 and hence $W(e(u+g(t))) \in L^1(\Omega_- \cup \Omega_+)$ by the upper growth estimate in (4.5(H2)) we obtain the desired lim inf-estimate as $\nu \to 0$. ■

Verification of (2.69(E2)), (2.69(E3)) and (2.71(C1))

Next, we verify the conditions concerning the time-derivatives of both the approximating and the limit energy functional. They correspond to the conditions (2.69(E2)), (2.69(E3)) and (2.71(C1)).

Lemma 4.2.6 (Properties of $\partial_t \mathcal{E}^\kappa_\varepsilon$, $\partial_t \mathcal{E}^\kappa$) *The energy functionals $\mathcal{E}^\kappa_\varepsilon : \mathcal{Q} \to \mathbb{R}_\infty$ and their limit $\mathcal{E}^\kappa : \mathcal{Q} \to \mathbb{R}_\infty$ satisfy:*

1. Uniform control of $\partial_t \mathcal{E}^\kappa_\varepsilon$, $\partial_t \mathcal{E}^\kappa$: *Let $\mathcal{F}$ be in place of both $\mathcal{E}^\kappa$ and $\mathcal{E}^\kappa_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$. There are constants $c_0 \in \mathbb{R}$, $c_1 > 0$ such that for all $q \in \mathcal{Q}$ it holds:*
   If $\mathcal{F}(t_\star, q) < \infty$ for any $t_\star \in [0,T]$, then $\mathcal{F}(\cdot, q) \in \mathrm{C}^1([0,T])$ and
   $$|\partial_t \mathcal{F}(t,q)| \le c_1(\mathcal{F}(t,q) + c_0) \quad \text{for all } t \in [0,T]\,. \tag{4.40}$$
   In particular $\partial_t \mathcal{E}^\kappa(t,q)$ takes the same form as $\partial_t \mathcal{E}^\kappa_\varepsilon(t,q)$ in (4.21).
2. Uniform time-continuity of $\partial_t \mathcal{E}^\kappa$: *For all $\nu>0$, for all $E\in\mathbb{R}$ there exists $\delta=\delta(\nu,E)>0$ so that for all $q \in \mathcal{Q}$ with $\mathcal{E}^\kappa(0,q) \le E$ and all t_1, t_2 with $|t_1 - t_2| < \delta$ it holds*
   $$|\partial_t \mathcal{E}^\kappa(t_1, q) - \partial_t \mathcal{E}^\kappa(t_2, q)| < \nu\,. \tag{4.41}$$
3. Conditioned continuous convergence: *For all $(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t,q)$ with $q_\varepsilon \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$ it holds:*
   $$\partial_t \mathcal{E}^\kappa_\varepsilon(t_\varepsilon, q_\varepsilon) \to \partial_t \mathcal{E}^\kappa(t,q)\,. \tag{4.42}$$

Proof: Recall that $\partial_t \mathcal{E}^\kappa_\varepsilon(t,q) = \int_{\Omega_-^{\varepsilon_0} \cup \Omega_+^{\varepsilon_0}} \partial_e W(e(u+g(t))) : \partial_t e(g(t))\,\mathrm{d}x$ for all $q \in \mathcal{Q}_\varepsilon$. If $\mathcal{E}^\kappa(t_q, u, z) < \infty$ we can prove that $\mathcal{E}^\kappa(\cdot, u, z) \in \mathrm{C}^1([0,T])$ by repeating the arguments of the proof of Theorem 3.1.7. Thus we find $\partial_t \mathcal{E}^\kappa(t,q) = \int_{\Omega_-^{\varepsilon_0} \cup \Omega_+^{\varepsilon_0}} \partial_e W(e(u+g(t))) : \partial_t e(g(t))\,\mathrm{d}x$ for all $q \in \mathcal{Q}_\mathrm{C}$ respectively.

Ad 1.: Property (4.40) is applied in the proof of Proposition 4.1.2 and for all fixed $\varepsilon \in (0, \varepsilon_0], \kappa \in (0, \kappa_0]$ it was verified in Theorem 3.1.7. The proof mainly uses the stress control (4.5(C2)) to derive a Gronwall estimate for the energy. Furthermore it relies on the assumptions (4.4) for g and on the coercivity inequalities (4.5(H2)). Since $\partial_t \mathcal{E}^\kappa_\varepsilon$ is independent of κ also the constants c_0, c_1 will not depend on κ. Due to the uniform Korn's inequality (4.24) these constants will also be independent of $\varepsilon \in (0, \varepsilon_0]$ and hence also apply to the limit energy.

Ad 2.: Since the limit energy satisfies (4.5), since it depends on $e(u)$ and ∇z but no more on z and since $\mathcal{Q}_\mathrm{C} \subset W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \times W^{1,r}(\Omega_\mathrm{D})$, Property *2.* may be deduced from Theorem 3.1.11.

Ad 3.: Based on (4.5(H3)) the proof can be adopted from Theorem 3.1.9. ■

Verification of (2.70) and (2.71(C4))

To make things complete we summarize the properties of the dissipation distances corresponding to conditions (2.70) and (2.71(C4)). The validity of conditions (2.70) can be concluded from Theorem 3.1.8.

Lemma 4.2.7 (Properties of the dissipation potentials) *For all $\varepsilon \in (0, \varepsilon_0]$ the dissipation potential $\mathcal{R} : \mathcal{Z} \to [0, \infty]$ from* (4.20) *is degree-1 homogeneous, convex, coercive and weakly sequentially lower semicontinuous, so that it satisfies the lower* Γ*-limit:*

$$\begin{aligned} &\textit{For all } (t_\varepsilon, u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, u, z) \textit{ and } (\tilde{t}_\varepsilon, \tilde{u}_\varepsilon, \tilde{z}_\varepsilon) \xrightarrow{\mathcal{T}_T} (\tilde{t}, \tilde{u}, \tilde{z}) \textit{ it holds:} \\ &\qquad \mathcal{R}(\tilde{z}-z) \leq \liminf_{\varepsilon \to 0} \mathcal{R}(\tilde{z}_\varepsilon - z_\varepsilon)\,. \end{aligned} \tag{4.43}$$

Moreover the following holds:

$$\begin{aligned} &\textit{for all compact } A \subset \mathcal{Z} \textit{ and all } (z_k)_{k\in\mathbb{N}} \subset A \\ &\textit{with } \min\{\mathcal{R}(z_k - z), \mathcal{R}(z - z_k)\} \to 0 : \quad z_k \to z \textit{ in } \mathcal{Z}\,. \end{aligned} \tag{4.44}$$

Proof: The convexity, coercivity and lower semicontinuity of $\mathcal{R}$ on $\mathcal{Z}$ are due to (4.20) and have already been verified in Theorem 3.1.8. The lower Γ-limit is a direct consequence of the lower semicontinuity.

To prove (4.44) we consider a compact set $A \subset \mathcal{Z}$ and a sequence $(z_k)_{k\in\mathbb{N}} \subset A$ satisfying $\min\{\mathcal{R}(z_k - z), \mathcal{R}(z - z_k)\} \to 0$. Due to the compactness of A, each subsequence of $(z_k)_{k\in\mathbb{N}}$ has a further subsequence $z_k \overset{*}{\rightharpoonup} \tilde{z}$ in $L^\infty(\Omega_{\mathrm{D}})$ for a limit $\tilde{z} \in \overline{A}$. We put $w_k = z_k - z$ if $z \geq z_k$ a.e. on Γ_{C} and $w_k = z - z_k$ otherwise. Hence $w_k \leq 0$ for all $k \in \mathbb{N}$ and thus $|\mathcal{R}(w_k)| = \int_{\Gamma_{\mathrm{C}}} \varrho |w_k| \,\mathrm{d}s \to 0$ implies that $z_k \to z$ in $L^1(\Gamma_{\mathrm{C}})$. But this implies that $\tilde{z} = z$, so that $z_k \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$, i.e. $z_k \to z$ in $\mathcal{Z}$ holds for the whole sequence. ■

Conditioned Upper Semicontinuity of Stable Sets (2.71(C2))

In the following we verify the property (2.71(C2)), saying that the limit of a stable sequence is stable. This will be done using a stronger formulation than the joint recovery condition stated in Lemma 2.4.6, namely for all stable sequences $(t_\varepsilon, u_\varepsilon, z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset [0, T] \times \mathcal{Q}$ with $(t_\varepsilon, u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, u, z)$ and for all $(\hat{u}, \hat{z}) \in \mathcal{Q}$ we construct a sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset \mathcal{Q}_{\mathrm{D}}$ satisfying $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \xrightarrow{\mathcal{T}} (\hat{u}, \hat{z})$ such that

$$\limsup_{\varepsilon \to 0} \left(\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, \hat{q}_\varepsilon) + \mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon)\right) \leq \left(\mathcal{E}^\kappa(t, \hat{q}) + \mathcal{R}(\hat{z} - z)\right)\,. \tag{4.45}$$

Clearly this implies the lim sup-estimate (2.73), since $\limsup \big(-\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, u_\varepsilon, z_\varepsilon)\big) \leq -\mathcal{E}^\kappa(t, u, z)$ holds true by the lim inf-estimate.

As for the joint recovery sequence in Theorem 3.1.14 we must ensure that $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) < \infty$ for all $\varepsilon \in (0, \varepsilon_0]$, so that $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) \to \mathcal{R}(\hat{z} - z)$ can be shown. Moreover $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ has to satisfy $\hat{u}_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^d)$ for all $\varepsilon \in (0, \varepsilon_0]$ to make sure that $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) < \infty$, whereas the limit $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ only. We will construct $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in such a way that $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) \to \mathcal{E}^\kappa(t, \hat{u}, \hat{z})$. This requires an interplay of $\hat{u}_\varepsilon$ and $\hat{z}_\varepsilon$.

The difficulty lies in the construction of $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$, which must allow it to prove that

$$\int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi^\varepsilon \hat{z}_\varepsilon W(e(\hat{u}_\varepsilon))\,\mathrm{d}x \to 0\,.$$

The construction will be based on reflecting both $\hat{u}_- = \hat{u}|_{\Omega_-}$ and $\hat{u}_+ = \hat{u}|_{\Omega_+}$ at the interface in the interval $(-\varepsilon, \varepsilon)$. With this method it is guaranteed that the new functions are in $W^{1,p}(\Omega_- \cup \Omega_+ \backslash N_{\hat{z}}, \mathbb{R}^d)$, where $N_{\hat{z}} := \{s \in \Gamma_{\mathrm{C}} \,|\, S_{\mathrm{C}}\hat{z}(s) = 0\}$. Moreover, to establish (4.45) it is crucial to exploit the relation between the limit condition $[\![\hat{u}\cdot \mathrm{n}_1]\!] = 0$ and the bulk term $\int_{\Omega^\varepsilon_{\mathrm{D}}} |(e_{11}(\hat{u}_\varepsilon))^-|^p \mathrm{d}x$. In particular, it is essential to construct the joint recovery sequence in such a way that the bulk term tends to 0 as $\varepsilon \to 0$.

Theorem 4.2.8 (Joint recovery sequences) *Keep $\kappa \in (0, \kappa_0]$ fixed. Let $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ be defined by* (4.15)-(4.29). *Assume that* (4.4) *and* (4.5) *hold true. Moreover, let $\gamma > (p-1)$, $p \in (1,\infty)$ and $r > d$. Then, for all $(t_\varepsilon, q_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset [0,T] \times \mathcal{Q}$ with $(t_\varepsilon, u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, u, z)$ as $\varepsilon \to 0$ and $q_\varepsilon = (u_\varepsilon, z_\varepsilon) \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$ and for every $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ there is a sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ such that* (4.45) *holds.*

Proof: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ and let $(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, q)$ as $\varepsilon \to 0$ with $q_\varepsilon \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$. Hence their energies are equibounded and Theorem 4.2.2 can be applied. Thus, $q \in \mathcal{Q}_{\mathrm{C}}$ with $0 \le z \le 1$ a.e. in Ω_{D}, so that $\mathcal{E}^\kappa(t, q)$ is at least finite. For an arbitrary $\hat{q} \in \mathcal{Q}$ we will now construct the joint recovery sequence $(\hat{q}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ with $\hat{q}_\varepsilon = (\hat{u}_\varepsilon, \hat{z}_\varepsilon)$.

If $\hat{q} \in \mathcal{Q}\backslash\mathcal{Q}_{\mathrm{C}}$, then $\mathcal{E}^\kappa(t_\varepsilon, \hat{q}) = \infty$ for all $\varepsilon \in (0, \varepsilon_0]$ so that (4.45) holds for $\hat{q}_\varepsilon = \hat{q}$. Let now $\hat{q} \in \mathcal{Q}_{\mathrm{C}}$. If $\hat{z} > z$ a.e. in Ω_{D}, then $\mathcal{R}(\hat{z}-z) = \infty$ and (4.45) trivially holds.

Hence, assume $\hat{z} \le z$ a.e. in Ω_{D} from now on. In order to keep $\mathcal{E}^\kappa_\varepsilon(t, \hat{u}_\varepsilon, \hat{z}_\varepsilon) + \mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon)$ finite, the sequence $(\hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ has to satisfy $\varepsilon^\gamma \le \hat{z}_\varepsilon \le z_\varepsilon$. Furthermore it is required that $\hat{u}_\varepsilon \in \mathcal{U}_{\mathrm{D}}$, i.e. $\hat{u}_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^d)$ with $\hat{u}_\varepsilon = 0$ on Γ_{Dir}, whereas $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ with $\hat{u} = 0$ on Γ_{Dir}, $S_{\mathrm{C}}\hat{z}[\![\hat{u}]\!] = 0$ and $[\![\hat{u}\cdot \mathrm{n}_1]\!] \ge 0$ a.e. on Γ_{C} only. We will first construct $(\hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ and prove the convergence of the energy terms which solely depend on the damage variable. Then we will construct $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in such a way that the interplay of $\hat{u}_\varepsilon$ with $\hat{z}_\varepsilon$ makes the remaining energy terms converge.

Construction of $\hat{z}_\varepsilon$: For every $\varepsilon \in (0, \varepsilon_0]$ we now construct $\hat{z}_\varepsilon$ in such a manner that $\hat{z}_\varepsilon \in \mathcal{Z}_{\mathrm{D}}$ and $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) < \infty$, i.e. the property $\varepsilon^\gamma \le \hat{z}_\varepsilon \le z_\varepsilon$ a.e. in Ω_{D} has to be ensured. Since $0 \le \hat{z} \le z$ a.e. in Ω_{D} this requirement is fulfilled by the sequence

$$\hat{z}_\varepsilon := \max\{\varepsilon^\gamma, \hat{z} - \|z_\varepsilon - z\|_{\mathrm{C}(\overline{\Omega_{\mathrm{D}}})}\}\,. \tag{4.46}$$

Due to $\|z_\varepsilon - z\|_{\mathrm{C}(\overline{\Omega_{\mathrm{D}}})} \to 0$ by the compact embedding $W^{1,r}(\Omega_{\mathrm{D}}) \Subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$ we find that $\hat{z}_\varepsilon \to \hat{z}$ in $W^{1,r}(\Omega_{\mathrm{D}})$ even strongly. Moreover this construction preserves that $\partial_{y_1}\hat{z}_\varepsilon = 0$ a.e.

in Ω_{D}. Therewith we obtain that

$$\int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r}|\nabla_\varepsilon \hat{z}_\varepsilon|^r \,\mathrm{d}y \to \int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r}|\nabla \hat{z}|^r \,\mathrm{d}y \quad \text{as well as} \quad \mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) \to \mathcal{R}(\hat{z}-z)\,. \tag{4.47}$$

Construction of $\hat{u}_\varepsilon$: For every $\varepsilon \in (0,\varepsilon_0]$ we now determine $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in such a way that $\hat{u}_\varepsilon \in \mathcal{U}_{\mathrm{D}}$, see (4.7). Since $(\hat{u},\hat{z}) \in \mathcal{Q}_{\mathrm{C}}$ we have $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, $\hat{u} = 0$ on Γ_{Dir}, $S_{\mathrm{C}}\hat{z}[\![\hat{u}]\!] = 0$ and $[\![\hat{u}\cdot \mathrm{n}_1]\!] \geq 0$ a.e. on Γ_{C}.

Let $\hat{u}^\pm := \hat{u}|_{\Omega_\pm}$. For our construction we reflect $\hat{u}^+|_{(0,\varepsilon)\times\Gamma_{\mathrm{C}}}$ and $\hat{u}^-|_{(-\varepsilon,0)\times\Gamma_{\mathrm{C}}}$ along the interface $\{0\}\times\Gamma_{\mathrm{C}}$. We move the reflection of $\hat{u}^+$ to the right by ε and the reflection of $\hat{u}^-$ to the left by $-\varepsilon$. Then we take the additive mean of these functions. Therewith we obtain an interpolated function $\hat{u}^\varepsilon_{\mathrm{inter}} \in W^{1,p}(\Omega^\varepsilon_{\mathrm{D}})$, which has the form

$$\hat{u}^\varepsilon_{\mathrm{inter}}(x_1,s) := \tfrac{x_1+\varepsilon}{2\varepsilon}\hat{u}_+(-x_1+\varepsilon,s) - \tfrac{x_1-\varepsilon}{2\varepsilon}\hat{u}_-(-x_1-\varepsilon,s)\,, \tag{4.48}$$

i.e. it holds

$$\hat{u}^\varepsilon_{\mathrm{inter}}(-\varepsilon,s) = \hat{u}_-(0,s)\,,\quad \hat{u}^\varepsilon_{\mathrm{inter}}(\varepsilon,s) = \hat{u}_+(0,s)\,,\quad \hat{u}^\varepsilon_{\mathrm{inter}}(0,s) = \tfrac12\hat{u}_+(\varepsilon,s) + \tfrac12\hat{u}_-(-\varepsilon,s).$$

Therewith we compose the functions $\hat{u}_\varepsilon$ as follows

$$\hat{u}_\varepsilon(x_1,s) := \begin{cases} \hat{u}_-(x_1{+}\varepsilon,s)\zeta_\varepsilon(x_1) & \text{if } (x_1,s)\in\Omega^\varepsilon_-,\\ \hat{u}^\varepsilon_{\mathrm{inter}}(x_1,s) & \text{if } (x_1,s)\in\Omega^\varepsilon_{\mathrm{D}},\\ \hat{u}_+(x_1{-}\varepsilon,s)\zeta_\varepsilon(x_1) & \text{if } (x_1,s)\in\Omega^\varepsilon_+, \end{cases} \tag{4.49}$$

where ζ_ε is a suitable cut-off function, which ensures that $\hat{u}_\varepsilon|_{\Gamma_{\mathrm{Dir}}} = 0$ in the trace sense, i.e.

$$\zeta_\varepsilon(x_1) := \begin{cases} \tfrac{1}{\varepsilon}x_1 + \tfrac{L}{\varepsilon} & \text{if } x_1 \in [-L,-L{+}\varepsilon),\\ 1 & \text{if } x_1\in(-L{+}\varepsilon, L{-}\varepsilon),\\ -\tfrac{1}{\varepsilon}x_1 + \tfrac{L}{\varepsilon} & \text{if } x_1 \in (L{-}\varepsilon, L]. \end{cases} \tag{4.50}$$

With this method we have ensured that $\hat{u}_\varepsilon \in W^{1,p}(\Omega,\mathbb{R}^d)$ and now we show that

$$\limsup_{\varepsilon\to0} \int_{\Omega^\varepsilon_-\cup\Omega^\varepsilon_+} W(e(\hat{u}_\varepsilon{+}g(t_\varepsilon)))\,\mathrm{d}x = \int_{\Omega_-\cup\Omega_+} W(e(\hat{u}{+}g(t)))\,\mathrm{d}x. \tag{4.51}$$

We prove this explicitly on Ω_-. The proof on Ω_+ uses exactly the same ideas. To shorten the presentation we write L^p for $L^p(\Omega^\varepsilon_-,\mathbb{R}^{d\times d})$ and if misunderstanding is excluded we omit indicating the dependence of a function $f:\Omega_-\to\mathbb{R}^d$ on s. Moreover we will often apply the estimate below, relying on the fact that $f:\Omega_-\to\mathbb{R}^d$ satisfies $f(-L,s)=0$ for a.a. $s\in\Gamma_{\mathrm{C}}$ and on Hölder's inequality. Using $m:=\max\{a,b\}$ we find:

$$\begin{aligned} &\int_{-L+a\varepsilon}^{-L+b\varepsilon} |f(y_1)|^p\,\mathrm{d}y_1 \leq \int_{-L+a\varepsilon}^{-L+b\varepsilon}\left|\int_{-L}^{y_1}\partial_\xi f(\xi)\,\mathrm{d}\xi\right|^p \mathrm{d}y_1 \leq \int_{-L+a\varepsilon}^{-L+b\varepsilon}\left(\int_{-L}^{y_1}|\partial_\xi f(\xi)|\,\mathrm{d}\xi\right)^p \mathrm{d}y_1\\ &\leq \int_{-L+a\varepsilon}^{-L+b\varepsilon}\int_{-L}^{y_1}|\partial_\xi f(\xi)|^p\,\mathrm{d}\xi (y_1+L)^{p-1}\,\mathrm{d}y_1 \leq \int_{-L}^{-L+m\varepsilon}|\partial_\xi f(\xi)|^p\,\mathrm{d}\xi\int_{-L+a\varepsilon}^{-L+b\varepsilon}(y_1+L)^{p-1}\,\mathrm{d}y_1\\ &\leq (b^p-a^p)\varepsilon^p\int_{-L}^{-L+m\varepsilon}|\partial_\xi f(\xi)|^p\,\mathrm{d}\xi\,. \end{aligned} \tag{4.52}$$

By (4.5(H2)) and the mean value theorem of differentiability we obtain for (4.51) that

$$
\begin{aligned}
&\int_{\Omega_-^\varepsilon} |W(e(\hat{u}_\varepsilon{+}g(t_\varepsilon))) - W(e(\hat{u}{+}g(t)))| \,\mathrm{d}x \\
&\leq \int_{\Omega_-^\varepsilon} 2^{p-2}\hat{c}\big(|e(\hat{u}{+}g(t))|^{p-1} + |e(\hat{u}_\varepsilon{+}g(t_\varepsilon))|^{p-1} + \hat{C}\big)|e(\hat{u}_\varepsilon{+}g(t_\varepsilon)) - e(\hat{u}{+}g(t))| \,\mathrm{d}x \\
&\leq 2^{p-2}\hat{c}\Big(\|e(\hat{u}{+}g(t))\|_{L^p}^{p-1} + \|e(\hat{u}_\varepsilon{+}g(t_\varepsilon))\|_{L^p}^{p-1} + \hat{C}\Big)\|e(\hat{u}_\varepsilon{+}g(t_\varepsilon)){-}e(\hat{u}{+}g(t))\|_{L^p}\,,
\end{aligned}
$$

where $L^p = L^p(\Omega_-^\varepsilon, \mathbb{R}^{d\times d})$. Due to (4.5) it holds that $g(t_\varepsilon) \to g(t)$ in $W^{1,p}(\Omega_-, \mathbb{R}^d)$. Moreover, in view of $\big|\|e(\hat{u}_\varepsilon)\|_{L^p(\Omega_-^\varepsilon)} - \|e(\hat{u})\|_{L^p(\Omega_-^\varepsilon)}\big| \leq \|\nabla\hat{u}_\varepsilon - \nabla\hat{u}\|_{L^p(\Omega_-^\varepsilon)}$ it suffices to show that $\|\nabla\hat{u}_\varepsilon{-}\nabla\hat{u}\|^p_{L^p(\Omega_-^\varepsilon)} \to 0$. From (4.49) and (4.50) we infer by the chain rule that $\partial_{x_1}\hat{u}_\varepsilon(x_1,s){=}\big(\zeta_\varepsilon(x_1)\partial_{x_1}\hat{u}(x_1{+}\varepsilon,s){+}\hat{u}(x_1{+}\varepsilon,s)\partial_{x_1}\zeta_\varepsilon(x_1)\big)$ as well as $\nabla_s\hat{u}_\varepsilon(x_1,s){=}\zeta_\varepsilon(x_1)\nabla_s\hat{u}(x_1{+}\varepsilon,s)$. Using that $\mathrm{C}^\infty(\Omega_-,\mathbb{R}^d)\cap W^{1,p}(\Omega_-,\mathbb{R}^d)$ is dense in $W^{1,p}(\Omega_-,\mathbb{R}^d)$, we find a sequence $(\phi_j)_{j\in\mathbb{N}}$ such that for all $\varepsilon \in (0,\varepsilon_0]$ there is an index $j_0 \in \mathbb{N}$ so that it holds $\|\hat{u}-\phi_j\|_{W^{1,p}(\Omega_-,\mathbb{R}^d)} < \varepsilon$ for all $j \geq j_0$. Moreover, this sequence satisfies $\phi_j{=}\hat{u}$ on $\partial\Omega_-$ by [Bur98, p. 57, Cor. 3]. Hence, for $\phi_j^\varepsilon(x_1,s){=}\phi_j(x_1{+}\varepsilon,s)\zeta_\varepsilon(x_1)$ it holds

$$
\|\nabla\hat{u}_\varepsilon{-}\nabla\hat{u}\|^p_{L^p} \leq 2^{p-1}\Big(\Big(\|\nabla(\hat{u}_\varepsilon{-}\phi_j^\varepsilon)\|_{L^p}{+}\|\nabla(\hat{u}{-}\phi_j)\|_{L^p}\Big)^p{+}\|\nabla(\phi_j^\varepsilon{-}\phi_j)\|^p_{L^p}\Big). \tag{4.53}
$$

Thereby we have $\|\nabla(\hat{u}-\phi_j)\|_{L^p(\Omega_-^\varepsilon)} < \varepsilon$ for all $j \geq j_0$. To estimate the first term in (4.53) we introduce $x_1^\varepsilon := x_1 + \varepsilon$ and we apply (4.52) both on $\hat{u}$ and on ϕ_j:

$$
\begin{aligned}
&\|\nabla(\hat{u}_\varepsilon - \phi_j^\varepsilon)\|^p_{L^p} \\
&\leq 2^{p-1}\!\int_{\Omega_-^\varepsilon}\big(\zeta_\varepsilon(x_1)^p|\nabla(\hat{u}(x_1^\varepsilon) - \phi_j(x_1^\varepsilon))|^p{+}|(\hat{u}(x_1^\varepsilon){-}\phi_j(x_1^\varepsilon))\partial_{x_1}\zeta_\varepsilon(x_1)|^p\big)\,\mathrm{d}x \\
&\leq 2^{p-1}\int_{\Gamma_\mathrm{C}}\int_{-L+\varepsilon}^0|\nabla(\hat{u}(y_1){-}\phi_j(y_1))|^p\mathrm{d}y_1\,\mathrm{d}s + \tfrac{2^{2p-2}}{\varepsilon^p}\int_{\Gamma_\mathrm{C}}\int_{-L+\varepsilon}^{-L+2\varepsilon}\big(|\hat{u}(y_1)|^p{+}|\phi_j(y_1)|^p\big)\,\mathrm{d}y_1\mathrm{d}s \\
&\leq \tfrac{(2\varepsilon)^p}{2}{+}\tfrac{2^{2p-2}}{\varepsilon^p}\int_{\Gamma_\mathrm{C}}\int_{-L}^{-L+2\varepsilon}\big(|\partial_\xi\hat{u}(\xi)|^p + |\partial_\xi\phi_j(\xi)|^p\big)\,\mathrm{d}\xi(2^p{-}1)\varepsilon^p\,\mathrm{d}s \quad \to 0 \text{ as } \varepsilon \to 0\,.
\end{aligned}
$$

Exploiting (4.50) and applying the triangle inequality on the third term in (4.53) yields

$$
\begin{aligned}
\|\nabla(\phi_j^\varepsilon{-}\phi_j)\|^p_{L^p(\Omega_-^\varepsilon)} \leq{}& \int_{\Gamma_\mathrm{C}}\Big(\int_{-L}^{-L+\varepsilon} + 2^{2p-2}\int_{-L+\varepsilon}^{-L+2\varepsilon}\Big)|\nabla\phi_j(x_1)|^p\,\mathrm{d}x_1\,\mathrm{d}s \\
&+ 2^{p-1}\int_{\Gamma_\mathrm{C}}\int_{-L+\varepsilon}^{-L+2\varepsilon}\tfrac{1}{\varepsilon^p}|\phi_j(x_1^\varepsilon)|^p\,\mathrm{d}x_1\,\mathrm{d}s \\
&+ \int_{\Gamma_\mathrm{C}}\Big(\int_{-L}^{-L+\varepsilon} + 2^{2p-2}\int_{-L+\varepsilon}^{-L+2\varepsilon}\Big)|\nabla(\phi_j(x_1^\varepsilon) - \phi_j(x_1))|^p\,\mathrm{d}x_1\,\mathrm{d}s\,.
\end{aligned}
$$

Clearly the first integral tends to 0 as $\varepsilon \to 0$. Using the triangle inequality and the transformation $y_1 = x_1 + \varepsilon$ we verify that also the third integral vanishes as $\varepsilon \to 0$. Again by

(4.52) we obtain for the second integral that

$$\tfrac{1}{\varepsilon^p}\int_{\Gamma_\mathrm{C}}\int_{-L}^{-L+\varepsilon}|\phi_j(x_1{+}\varepsilon,s)|^p\,\mathrm{d}x_1\,\mathrm{d}s \le \tfrac{(2^p-1)\varepsilon^p}{\varepsilon^p}\int_{\Gamma_\mathrm{C}}\int_{-L}^{-L+2\varepsilon}|\partial_\xi\phi_j(\xi,s)|^p\,\mathrm{d}\xi\,\mathrm{d}s \;\to 0 \text{ as } \varepsilon\to 0\,.$$

Thus, we have verified that $\|\nabla\hat{u}_\varepsilon-\nabla\hat{u}\|_{L^p(\Omega_-^\varepsilon,\mathbb{R}^{d\times d})}\to 0$ as $\varepsilon\to 0$. Exactly the same ansatz can be applied to prove that also $\|\nabla\hat{u}_\varepsilon-\nabla\hat{u}\|_{L^p(\Omega_+^\varepsilon,\mathbb{R}^{d\times d})}\to 0$. This proves (4.51).

Hence it remains to show that $\int_{\Omega_\mathrm{D}^\varepsilon}W_\mathrm{D}(e(\hat{u}_\varepsilon),\Pi_\varepsilon^{-1}\hat{z}_\varepsilon)\,\mathrm{d}x\to 0$. From the construction (4.46) we infer that $\partial_{x_1}\Pi_\varepsilon^{-1}\hat{z}_\varepsilon=0$ and that $\Pi_\varepsilon^{-1}\hat{z}_\varepsilon(x)=\varepsilon^\gamma$ if $\hat{z}(x)=0$ for all $\varepsilon\in(0,\varepsilon_0]$. With $N_{\hat{z}}:=\{s\in\Gamma_\mathrm{C}\,|\,S_\mathrm{C}\hat{z}(s)=0\}$ and (4.5(H2)) we infer that

$$\int_{\Omega_\mathrm{D}^\varepsilon}\Pi_\varepsilon^{-1}\hat{z}_\varepsilon|\nabla\hat{u}_\mathrm{int}(x_1,s)|^p\,\mathrm{d}x \le \int_{N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}\varepsilon^\gamma|e(\hat{u}_\mathrm{int}^\varepsilon)|^p\,\mathrm{d}x_1\,\mathrm{d}s+\int_{\Gamma_\mathrm{C}\backslash N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}|e(\hat{u}_\mathrm{int}^\varepsilon)|^p\,\mathrm{d}x_1\,\mathrm{d}s\,,$$

where $|e(\hat{u}_\mathrm{int}^\varepsilon)|^p\le 2^{p-1}\big(|\partial_{x_1}\hat{u}_\mathrm{int}^\varepsilon|^p+|\nabla_s\hat{u}_\mathrm{int}^\varepsilon|^p\big)$. Using (4.48) and $x_1^\varepsilon=x_1{+}\varepsilon$ we get

$$\begin{aligned}\partial_{x_1}\hat{u}_\mathrm{int}^\varepsilon &= \tfrac{1}{2\varepsilon}\big(\hat{u}_+(-x_1{+}\varepsilon)-\hat{u}_-(-x_1^\varepsilon)\big)+\tfrac{x_1+\varepsilon}{2\varepsilon}\partial_{x_1}\hat{u}_+(-x_1{+}\varepsilon)-\tfrac{x_1-\varepsilon}{2\varepsilon}\partial_{x_1}\hat{u}_-(-x_1^\varepsilon)\,,\\ \nabla_s\hat{u}_\mathrm{int}^\varepsilon &= \tfrac{x_1+\varepsilon}{2\varepsilon}\nabla_s\hat{u}_+(-x_1{+}\varepsilon,s)-\tfrac{x_1-\varepsilon}{2\varepsilon}\nabla_s\hat{u}_-(-x_1{-}\varepsilon,s)\,.\end{aligned}$$

For the first term of $\partial_{x_1}\hat{u}_\mathrm{int}^\varepsilon$ we use the estimate below with $\varphi=\mathrm{id}$

$$\begin{aligned}&|\varphi\big(\hat{u}_+(-x_1{+}\varepsilon)-\hat{u}_-(-x_1{-}\varepsilon)\big)|^p\\ &\le 3^{p-1}\big(|\varphi\big(\hat{u}_+(-x_1{+}\varepsilon)-\hat{u}_+(0)\big)|^p+|\varphi\big(\hat{u}_-(0)-\hat{u}_-(-x_1{-}\varepsilon)\big)|^p+|\varphi\big([\![\hat{u}]\!]\big)|^p\big)\,.\end{aligned}\tag{4.54}$$

Applying (4.52) on the first two terms and using that $\gamma>(p{-}1)$ for the last term yields

$$\begin{aligned}&\tfrac{\varepsilon^{\gamma-p}}{2^p}\int_{N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}|\hat{u}_\pm(-x_1{\pm}\varepsilon,s)-\hat{u}_\pm(0,s)|^p\,\mathrm{d}x_1\,\mathrm{d}s\le\tfrac{\varepsilon^\gamma}{2^p}\int_{N_{\hat{z}}}\int_0^{2\varepsilon}|\partial_\xi\hat{u}_\pm(\xi,s)|^p\,\mathrm{d}\xi\,\mathrm{d}s\;\to 0,\\ &\varepsilon^\gamma\int_{N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}\tfrac{1}{(2\varepsilon)^p}|\hat{u}_+(0,s)-\hat{u}_-(0,s)|^p\,\mathrm{d}x_1\,\mathrm{d}s=\tfrac{\varepsilon^{\gamma+1-p}}{2^p}\|[\![\hat{u}]\!]\|_{L^p(\Gamma_\mathrm{C},\mathbb{R}^d)}^p\to 0.\end{aligned}$$

Due to $|\frac{x_1\pm\varepsilon}{2\varepsilon}|\le 1$ on $(-\varepsilon,\varepsilon)$ the remaining terms of $\partial_{x_1}\hat{u}_\mathrm{int}^\varepsilon$ can be estimated as follows

$$\begin{aligned}&\varepsilon^\gamma\int_{N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}\big(|\tfrac{x_1+\varepsilon}{2\varepsilon}|^p|\partial_{x_1}\hat{u}_+(-x_1{+}\varepsilon,s)|^p+|\tfrac{x_1-\varepsilon}{2\varepsilon}|^p|\partial_{x_1}\hat{u}_-(-x_1{-}\varepsilon,s)|^p\big)\,\mathrm{d}x\\ &\le\varepsilon^\gamma\int_{N_{\hat{z}}}\int_0^{2\varepsilon}|\partial_{y_1}\hat{u}_+(y_1,s)|^p\,\mathrm{d}y_1\,\mathrm{d}s-\varepsilon^\gamma\int_{N_{\hat{z}}}\int_0^{-2\varepsilon}|\partial_{y_1}\hat{u}_-(y_1,s)|^p\,\mathrm{d}y_1\,\mathrm{d}s\,,\end{aligned}\tag{4.55}$$

tending to 0 as $\varepsilon\to 0$ also without the prefactor ε^γ, which will be used below.

On $\Gamma_\mathrm{C}\backslash N_{\hat{z}}$ it holds that $\hat{u}\in W^{1,p}(\Omega\backslash N_{\hat{z}})$ and hence we find that

$$\begin{aligned}\int_{\Gamma_\mathrm{C}\backslash N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}|\partial_{x_1}\hat{u}_\mathrm{int}|^p\,\mathrm{d}x_1\,\mathrm{d}s&\le 3^{p-1}\int_{\Gamma_\mathrm{C}\backslash N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}\Big|\int_{-x_1-\varepsilon}^{-x_1+\varepsilon}\partial_\xi\hat{u}(\xi,s)\,\mathrm{d}\xi\Big|^p\,\mathrm{d}x_1\,\mathrm{d}s\\ &+3^{p-1}\int_{\Gamma_\mathrm{C}\backslash N_{\hat{z}}}\int_{-\varepsilon}^{\varepsilon}\big(|\tfrac{x_1+\varepsilon}{2\varepsilon}|^p|\partial_{x_1}\hat{u}(-x_1{+}\varepsilon,s)|^p+|\tfrac{x_1-\varepsilon}{2\varepsilon}|^p|\partial_{x_1}\hat{u}(-x_1{-}\varepsilon,s)|^p\big)\,\mathrm{d}x_1\,\mathrm{d}s\,.\end{aligned}$$

Calculations similar to estimate (4.55) show that the second term on the right-hand side tends to 0 as $\varepsilon \to 0$. Applying Hölder's inequality on the first term yields

$$\int_{\Gamma_\mathrm{C}\setminus N_{\hat z}} \int_{-\varepsilon}^{\varepsilon} \left| \int_{-x_1-\varepsilon}^{-x_1+\varepsilon} \partial_\xi \hat u(\xi, s)\,\mathrm{d}\xi \right|^p \mathrm{d}x_1\,\mathrm{d}s \le (2\varepsilon)^p \int_{\Gamma_\mathrm{C}\setminus N_{\hat z}} \int_{-2\varepsilon}^{2\varepsilon} |\partial_\xi \hat u(\xi, s)|^p\,\mathrm{d}\xi\,\mathrm{d}s\,.$$

For the term $\nabla_s \hat u^\varepsilon_\mathrm{int}$ we calculate that

$$\int_{-\varepsilon}^{\varepsilon} |\nabla_s \hat u_\mathrm{int}(x_1, s)|^p\,\mathrm{d}x_1 \le 2^{p-1}\Big(\int_0^{2\varepsilon} |\nabla_s \hat u_+(y_1, s)|^p\,\mathrm{d}y_1 - \int_0^{-2\varepsilon} |\nabla_s \hat u_-(y_1, s)|^p\,\mathrm{d}y_1 \Big). \tag{4.56}$$

After integration over Γ_C we see that this term tends to 0 as $\varepsilon \to 0$. This proves that $\int_{\Omega^\varepsilon_\mathrm{D}} \Pi_\varepsilon^{-1} \hat z_\varepsilon \widetilde{W}(e(\hat u^\varepsilon_\mathrm{int}))\,\mathrm{d}x \to 0$.

It remains to show that also $\int_{\Omega^\varepsilon_\mathrm{D}} |(-e_{11}(\hat u^\varepsilon_\mathrm{inter}))^+|^p\,\mathrm{d}x \to 0$. For this we use that

$$\begin{aligned}(e_{11}(\hat u^\varepsilon_\mathrm{inter}))^- \le{}& \tfrac{1}{2\varepsilon}\big(\hat u^1_+(-x_1{+}\varepsilon, s) - \hat u^1_-(-x_1{-}\varepsilon, s)\big)^- \\ &+ |\tfrac{x_1+\varepsilon}{2\varepsilon}|(\partial_{x_1}\hat u^1_+(-x_1{+}\varepsilon, s))^- + |\tfrac{x_1-\varepsilon}{2\varepsilon}|(\partial_{x_1}\hat u^1_-(-x_1{-}\varepsilon, s))^+\,,\end{aligned}$$

where both $|\frac{x_1+\varepsilon}{2\varepsilon}| \le 1$ and $|\frac{x_1-\varepsilon}{2\varepsilon}| \le 1$ on $(-\varepsilon, \varepsilon)$. By calculations similar to (4.55) one obtains that the terms in the second line lead to integrals that vanish as $\varepsilon \to 0$. Moreover, (4.54) can be applied to the first term with $\varphi(\hat u) = (\hat u^1)^-$. Here $\big([\![\hat u \cdot \mathrm{n}_1]\!]\big)^- = 0$ since $\mathcal{E}^\kappa(t, \hat u, \hat z) < \infty$. By integration by parts, Jensen's and Hölder's inequality we find

$$\big(\pm(\hat u^1_\pm(-x_1{\pm}\varepsilon) {-} \hat u^1_\pm(0))\big)^- \le \int_0^{-x_1\pm\varepsilon} |\partial_\xi \hat u_\pm(\xi)|\,\mathrm{d}\xi \le \varepsilon^{\frac{p-1}{p}} C\,.$$

In view of these informations, integration of (4.54) over $\Omega^\varepsilon_\mathrm{D}$ then finishes the proof of $\int_{\Omega^\varepsilon_\mathrm{D}} (e_{11}(\hat u_\varepsilon))^-\,\mathrm{d}x \to 0$. ∎

The sequence $(\hat u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ given by (4.49) together with $\hat z_\varepsilon := \max\{\varepsilon^\gamma, \hat z\}$ forms a recovery sequence for the functionals $\mathcal{E}^\kappa_\varepsilon(t,\cdot,\cdot)$ and $\mathcal{E}^\kappa(t,\cdot,\cdot)$. Hence also the Γ-convergence of $\mathcal{E}^\kappa_\varepsilon$ to $\mathcal{E}^\kappa$ is proven.

Corollary 4.2.9 (Γ-convergence of $\mathcal{E}^\kappa_\varepsilon$) *Keep $t \in [0, T]$ fixed. Let $\mathcal{E}^\kappa_\varepsilon$ and $\mathcal{E}^\kappa$ be defined by* (4.18) *and* (4.27) *such that the assumptions* (4.4) *and* (4.5) *are satisfied. Then, for all $\hat q \in \mathcal{Q}$ there exists a recovery sequence $\hat q_\varepsilon \xrightarrow{\mathcal{T}} \hat q$ such that*

$$\limsup_{\varepsilon\to 0} \mathcal{E}^\kappa_\varepsilon(t, \hat q_\varepsilon) \le \mathcal{E}^\kappa(t, \hat q)\,.$$

Hence, together with the lower Γ-limit stated in Theorem 4.2.5, it holds $\mathcal{E}^\kappa_\varepsilon(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}^\kappa(t,\cdot)$.

Uniform Bounds and Convergence of a Subsequence of Solutions for all $t \in [0,T]$

Up to now we have verified the conditions (2.69), (2.70) and (2.71). They are formulated for any fixed time $t \in [0,T]$ and ensure that for all $t \in [0,T]$ a (t-dependent) subsequence of the energetic solutions of the approximating systems converges to a (t-dependent) limit, which satisfies the properties (2.60(S)) & (2.60(E)) for the limit system at time $t \in [0,T]$. In order to make sure that the energetic solutions indeed approximate an *energetic* solution of the limit system, it remains to show the existence of a subsequence of energetic solutions converging for all $t \in [0,T]$ with respect to $\mathcal{T}$.

Lemma 4.2.10 (Uniform bounds & convergence of a subsequence of solutions) *Let the assumptions (4.4), (4.5) be valid and $\kappa \in (0,\kappa_0]$ fixed. For all $\varepsilon \in (0,\varepsilon_0]$ let $q_\varepsilon : [0,T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R}, q_0^\varepsilon)$. Additionally assume that the initial values satisfy $q_0^\varepsilon \xrightarrow{\mathcal{T}} q_0$ and $\mathcal{E}^\kappa_\varepsilon(0,q_0^\varepsilon) \to \mathcal{E}^\kappa(0,q_0)$. Then there are constant $\tilde{E}, C$ which are independent of κ and ε, such that for all $t \in [0,T]$ and for all fixed $\nu \in (0,\varepsilon_0]$ the following uniform bounds are valid*

$$\|u_\varepsilon(t)\|_{W^{1,p}(\Omega^\nu_\pm,\mathbb{R}^d)} \le \tilde{E}\,, \qquad \|z_\varepsilon(t)\|_{L^\infty(\Omega_{\mathrm{D}})} \le 1\,, \tag{4.57a}$$

$$\mathrm{Diss}_{\mathcal{R}}(z_\varepsilon,[0,t]) \le C\,. \tag{4.57b}$$

Moreover there is a subsequence $(q_\varepsilon)_{\varepsilon\in(0,\varepsilon]}$ with $q_\varepsilon(t) \xrightarrow{\mathcal{T}} q(t)$ in $\mathcal{Q}$ for all $t \in [0,T]$.

Proof: For all $\varepsilon \in (0,\varepsilon_0]$ the functions $q_\varepsilon : [0,T] \to \mathcal{Q}$ supply an energetic solution of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$. Hence, for all $t \in [0,T]$ they satisfy $\mathcal{E}^\kappa_\varepsilon(t, q_\varepsilon(t)) < \infty$, which implies that $\varepsilon^\gamma \le z_\varepsilon(t,x) \le 1$ for a.e. $x \in \Omega_{\mathrm{D}}$, for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$. Exploiting stability inequality (2.60(S)) for $q_\varepsilon(t)$ and $\tilde{q} = (0,\varepsilon^\gamma)$ yields $\mathcal{E}^\kappa_\varepsilon(t,q_\varepsilon(t)) \le \mathcal{E}^\kappa_\varepsilon(t,\tilde{q}) + \mathcal{R}(\tilde{z} - z_\varepsilon(t)) \le E$ for all $t \in [0,T]$ by (4.4), so that $(t, q_\varepsilon(t))_{\varepsilon\in(0,T]}$ is a stable sequence and their energies are equibounded for all $t \in [0,T]$. Using estimate (4.32) finishes the proof of (4.57a).

Estimate (4.57a) together with Lemma 4.2.8 implies that $\mathcal{S}^\kappa(t) \neq \emptyset$ for all $t \in [0,T]$.

We now prove the existence of a subsequence that converges pointwise for all $t \in [0,T]$. This can be done using ideas similar to the proof of [MM05, Th. 3.2]. It has to be used that $\mathcal{E}^\kappa_\varepsilon(0, q_\varepsilon(0)) \le C$ and that $\int_0^t \partial_\xi \mathcal{E}^\kappa_\varepsilon(\xi, q_\varepsilon(\xi))\,\mathrm{d}\xi \le c_g T(\hat{c}E + \hat{C}\mathcal{L}^d(\Omega))$ for all $t \in [0,T]$ by stress control (4.5(C2)). Hence energy balance (2.60(E)) yields $\mathrm{Diss}_{\mathcal{R}}(z_\varepsilon,[0,T]) < C$ for a fixed constant C.

We now define $\varphi_\varepsilon : [0,T] \to [0,C]$, $t \mapsto \mathrm{Diss}_{\mathcal{R}}(z_\varepsilon,[0,t])$. These functions are nondecreasing and hence the classical scalar Helly's selection principle guarantees the existence of a subsequence and a limit function $\varphi : [0,T] \to [0,C]$ with $\varphi_\varepsilon(t) \to \varphi(t)$ for all $t \in [0,T]$. Since φ is monotone and bounded the set J of all discontinuity points is at most countable. We choose a countable set $M \subset [0,T]$, which is dense in $[0,T]$ and which satisfies $J \subset M$ and $0 \in M$. Due to the uniform bound on $(z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$, the arguments of the proof of Theorem 4.2.2, Item 2.(b) and Cantor's diagonal process we find a further subsequence and a limit z with $z(\tau) \in W^{1,r}(\Omega_{\mathrm{D}})$ and $\partial_{y_1} z(\tau) = 0$, such that $z_\varepsilon(\tau) \overset{*}{\rightharpoonup} z(\tau)$ in $L^\infty(\Omega_{\mathrm{D}})$ for all $\tau \in M$, i.e. $z : M \to W^{1,r}(\Omega_{\mathrm{D}})$ is well defined. Now we show that this subsequence also converges for $t \in [0,T]\backslash M$. Again by the uniform bounds we find a further, t-dependent subsequence $(z_{\tilde\varepsilon})_{\tilde\varepsilon\in(0,\varepsilon_0]} \subset (z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ so that $z_{\tilde\varepsilon}(t) \overset{*}{\rightharpoonup} z(t) \in W^{1,r}(\Omega_{\mathrm{D}})$ in $L^\infty(\Omega_{\mathrm{D}})$ and we must prove the uniqueness of this accumulation point. Thus, we choose a sequence $(\tau_k)_{k\in\mathbb{N}} \subset M$ with $\tau_k \to t$ as $k \to \infty$. The lower semicontinuity of $\mathcal{R}$ provides that

$$\begin{aligned}\min\{\mathcal{R}(z(\tau_k)-\tilde z_t), \mathcal{R}(\tilde z_t - z(\tau_k)\} &\le \liminf_{\tilde\varepsilon\to 0} \max\{\varphi_{\tilde\varepsilon}(t)-\varphi_{\tilde\varepsilon}(\tau_k), \varphi_{\tilde\varepsilon}(\tau_k)-\varphi_{\tilde\varepsilon}(t)\}\\ &= |\varphi(t)-\varphi(\tau_k)| \to 0\end{aligned}$$

as $k\to\infty$ due to the continuity of φ at t. Since $0 \le z(\tau_k) \le 1$ on Γ_{C} for all $k \in \mathbb{N}$ the set $(z(\tau_k))_{k\in\mathbb{N}}$ is compact in $L^\infty(\Gamma_{\mathrm{C}})$. Hence, (4.44) implies that $z(\tau_k) \overset{*}{\rightharpoonup} z(t)$ in $L^\infty(\Omega_{\mathrm{D}})$. This limit is unique. Therefore $(z_\varepsilon(t))_{\varepsilon\in(0,T]}$ has the unique accumulation point $z(t)$. Thus, we have found a subsequence $z_\varepsilon(t) \to z(t)$ in $\mathcal{Z}$ for all $t \in [0,T]$.

For the corresponding subsequence $(u_\varepsilon)_{\varepsilon\in(0,T]}$ the uniform bound (4.57a) provides a further subsequence $u_{\tilde\varepsilon}(t) \rightharpoonup u(t)$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ uniformly for a countable choice of indices $\nu \to 0$ and Lemma 4.2.8 implies that $(u(t), z(t)) \in \mathcal{S}^\kappa(t)$ for all $t \in [0,T]$, since q_ε are energetic solutions. That $u_\varepsilon(t) \rightharpoonup u(t)$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ for all $\nu \in (0,\varepsilon_0]$ and all $t\in[0,T]$ for the whole subsequence can be concluded from the strict convexity of W, see (4.5(H1)). Therewith $\mathcal{E}^\kappa(t,\cdot,z(t))$ has a unique minimizer, so that $u(t)$ is the only accumulation point of this subsequence. ■

4.3 The Second Γ-limit: Griffith-Type Delamination

In this section we prove that the gradient delamination models obtained in Section 4.2 approximate a model for Griffith-type delamination as $\kappa \to 0$. In particular we show that a subsequence of energetic solutions of the systems for gradient delamination converges for all $t\in[0,T]$ in $\mathcal{T}$ to an energetic solution of the system for Griffith-type delamination.

The model for Griffith-type delamination is discussed in Section 4.3.1 and the convergence proof is elaborated in Section 4.3.2. Due to the vanishing delamination gradient the main difficulty of the proof lies in the construction of a joint recovery sequence, which has to be sufficiently smooth and which must respect the transmission condition.

4.3.1 The Model for Griffith-type Delamination

Our aim in this section is to show that the first limit problems $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})_{\kappa\in(0,\kappa_0]}$ converge to the limit system $(\mathcal{Q},\mathcal{E},\mathcal{R})$, where $\mathcal{R}:\mathcal{Z}\to[0,\infty]$ from (4.20) and

$$\mathcal{E}(t,q) := \begin{cases} \int_{\Omega_-\cup\Omega_+} W(e(u+g(t)))\,\mathrm{d}x & \text{if } q=(u,z)\in\mathcal{Q}_{\mathrm{G}},\\ \infty & \text{if } q\in\mathcal{Q}\backslash\mathcal{Q}_{\mathrm{G}},\end{cases} \tag{4.58}$$

$$\mathcal{Z}_{\mathrm{G}} := \{z \in L^\infty(\Omega_{\mathrm{D}}) \,|\, 0\le z\le 1 \text{ and } \partial_{y_1} z = 0 \text{ a.e. in } \Omega_{\mathrm{D}}\}, \tag{4.59}$$

$$\mathcal{Q}_{\mathrm{G}} := \big\{(u,z)\in\mathcal{U}\times\mathcal{Z}_{\mathrm{G}} \,|\, [\![u\cdot\mathrm{n}_1]\!] \ge 0 \text{ and } S_{\mathrm{G}} z [\![u]\!] = 0 \text{ a.e. on } \Gamma_{\mathrm{C}}\big\}, \tag{4.60}$$

with $\mathcal{U}$ as in (4.13) and with S_{G} defined in (4.61). The study of sequences $(u_\kappa, z_\kappa)_{\kappa\in(0,\kappa_0]}$ with equibounded energies yields that there is $z \in L^\infty(\Omega_{\mathrm{D}})$ such that $z_\kappa \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$ for a subsequence and due to $\partial_{y_1} z_\kappa = 0$ in Ω_{D} for all $\kappa\in(0,\kappa_0]$ we find that $z \in L^\infty(\Omega_{\mathrm{D}})$ is constant a.e. with respect to y_1-direction. Using the definition of the weak derivative we can verify that $z \in L^\infty(\Omega_{\mathrm{D}})$ has a weak derivative in y_1-direction, namely $\partial_{y_1} z = 0$ a.e. in Ω_{D} : By definition $\partial_{y_1} z$ is the weak y_1-derivative of $z \in L^\infty(\Omega_{\mathrm{D}})$ iff

$$\int_{\Omega_{\mathrm{D}}} \partial_{y_1} z\phi \,\mathrm{d}y = -\int_{\Omega_{\mathrm{D}}} z\partial_{y_1}\phi \,\mathrm{d}y \quad \text{for all } \phi \in \mathrm{C}^1_0(\Omega_{\mathrm{D}})\,.$$

For z being constant a.e. in y_1-direction we obtain for the right-hand side that indeed $-\int_{\Omega_{\mathrm{D}}} z\partial_{y_1}\phi\,\mathrm{d}y = -\int_{\Gamma_{\mathrm{C}}} z(s)\int_{-1}^1 \partial_{y_1}\phi(y_1,s)\,\mathrm{y}_1\,\mathrm{d}s = -\int_{\Gamma_{\mathrm{C}}} z(s)\big(\phi(1,s)-\phi(-1,s)\big)\mathrm{d}s = 0$.

But although the weak y_1-derivative of $z\in L^\infty(\Omega_{\mathrm{D}})$ exists, the trace of z on Γ_{C} may not be well-defined. To replace the trace operator S_{C} we introduce

$$S_{\mathrm{G}} z(s) = \tfrac{1}{2}\int_{-1}^1 z(y_1,s)\,\mathrm{d}y_1\,. \tag{4.61}$$

Then, for all $z\in\mathcal{Z}_{\mathrm{C}}$ from (4.28) it holds $S_{\mathrm{G}}z = S_{\mathrm{C}}z = z|_{\Gamma_{\mathrm{C}}}$ and for all $v \in \mathcal{Z}_{\mathrm{G}}$ it is

$$\mathcal{R}(v) = \begin{cases} 2\int_{\Gamma_{\mathrm{C}}} -\varrho S_{\mathrm{G}} v\,\mathrm{d}s & \text{if } S_{\mathrm{G}} v \le 0 \text{ a.e. on } \Gamma_{\mathrm{C}},\\ \infty & \text{otherwise,}\end{cases} \tag{4.62}$$

so that $(\mathcal{Q}_{\mathrm{G}}, \mathcal{E}, \mathcal{R})$ indeed models Griffith-type delamination along the interface Γ_{C}.

For all $t\in[0,T]$ the stable sets of the approximating and the limit problem are given by

$$\begin{aligned}\mathcal{S}^\kappa(t) &:= \{q=(u,z)\in\mathcal{Q}\,|\,\mathcal{E}^\kappa(t,q)<\infty,\ \mathcal{E}^\kappa(t,q)\le\mathcal{E}^\kappa(t,\tilde q)+\mathcal{R}(\tilde z - z) \text{ for all } \tilde q=(\tilde u,\tilde z)\in\mathcal{Q}\}\,,\\ \mathcal{S}(t) &:= \{q=(u,z)\in\mathcal{Q}\,|\,\mathcal{E}(t,q)<\infty,\ \mathcal{E}(t,q)\le\mathcal{E}(t,\tilde q)+\mathcal{R}(\tilde z - z) \text{ for all } \tilde q=(\tilde u,\tilde z)\in\mathcal{Q}\}.\end{aligned}$$

A function $S_{\mathrm{G}}z \in L^\infty(\Gamma_{\mathrm{C}})$ is only defined $\mathcal{L}^{d-1}$-a.e. on Γ_{C}. In order to define its support $\operatorname{supp} S_{\mathrm{G}} z$ and its zero set N_z one can understand $S_{\mathrm{G}}z$ as the Radon-Nikodym density with respect to the Lebesgue-measure $\mathcal{L}^{d-1}$ of a measure μ_z. Due to [Fed69, p. 60] the support of the measure μ_z is given by $\operatorname{supp}\mu_z := \Gamma_{\mathrm{C}}\backslash \cup\{\mathcal{O}\subset\Gamma_{\mathrm{C}}\,|\,\mathcal{O} \text{ open}, \mu_z(\mathcal{O}) = 0\}$. Since it is more convenient for later application we use an equivalent definition for $\operatorname{supp}\mu_z$, which can be obtained with De Morgan's laws:

$$\begin{aligned}\operatorname{supp} S_{\mathrm{G}} z &:= \cap\{A\subset\Gamma_{\mathrm{C}}\,|\,A \text{ closed}, \mathcal{L}^{d-1}\big(\{s\in\Gamma_{\mathrm{C}}\,|\,S_{\mathrm{G}}z(s)\neq 0\}\backslash A\big) = 0\}\,,\\ N_z &:= \Gamma_{\mathrm{C}}\backslash \operatorname{supp} S_{\mathrm{G}} z = \cup\{\mathcal{O}\subset\Gamma_{\mathrm{C}}\,|\,\mathcal{O} \text{ open}, \mathcal{L}^{d-1}\big(\mathcal{O}\cap\{s\in\Gamma_{\mathrm{C}}\,|\,S_{\mathrm{G}}z(s)\neq 0\}\big) = 0\}\,.\end{aligned} \tag{4.63}$$

Clearly, $\operatorname{supp} S_{\mathrm{G}} z$ is a closed and N_z is an open set. Moreover, for all $S_{\mathrm{G}}z_1, S_{\mathrm{G}}z_2 \in L^\infty(\Gamma_{\mathrm{C}})$ with $z_1 = z_2$ a.e. on Γ_{C} we have $\operatorname{supp} S_{\mathrm{G}}z_1 = \operatorname{supp} S_{\mathrm{G}} z_2$ and $N_{z_1} = N_{z_2}$.

Lemma 4.3.1 *Let $f \in L^\infty(\Gamma_{\mathrm{C}})$ and $g \in \mathrm{C}^0(\overline{\Gamma_{\mathrm{C}}})$. Then*

$$f(s)g(s) = 0 \textit{ for a.e. } s\in\Gamma_{\mathrm{C}} \quad \textit{is equivalent to} \quad \operatorname{supp} f\cap \operatorname{OS} g = \emptyset\,, \tag{4.64}$$

where $\operatorname{OS} g := \{s\in\Gamma_{\mathrm{C}}\,|\,g(s)\neq 0\}$ is the open support, which is an open set since $g\in\mathrm{C}^0(\overline{\Gamma_{\mathrm{C}}})$.

Proof: First, let $f(s)g(s) = 0$ for a.e. $s \in \Gamma_{\mathrm{C}}$. Assume that there is a set $B \neq \emptyset$ with $B \subset \operatorname{supp} f \cap \operatorname{OS} g$. Then $f(s)g(s) \neq 0$ for all $s \in B$. Since $\operatorname{supp} f$ is closed and $\operatorname{OS} g$ is open, there exists a point $s \in B$ and $\varepsilon > 0$ such that the open ball $B_\varepsilon(s) \subset \Gamma_{\mathrm{C}}$ around s is contained in B. This implies that $\mathcal{L}^{d-1}(B) > 0$, which is in contradiction to the requirement $f(s)g(s) = 0$ a.e. on Γ_{C}. Hence, the assumption $B \neq \emptyset$ is wrong and we conclude that $\operatorname{supp} f \cap \operatorname{OS} g = \emptyset$.

Let now $\operatorname{supp} f \cap \operatorname{OS} g = \emptyset$ hold true and we have to show that $f(s)g(s) = 0$ for a.e. $s \in \Gamma_{\mathrm{C}}$. Since g is continuous we infer that $g(s) = 0$ for all $s \in \Gamma_{\mathrm{C}} \backslash \operatorname{OS} g$. Moreover, $\operatorname{supp} f \subset \Gamma_{\mathrm{C}} \backslash \operatorname{OS} g$, which is a closed set. By the definition of the support we conclude that $f = 0$ a.e. on $\operatorname{OS} g$, which proves that $f(s)g(s) = 0$ for a.e. $s \in \Gamma_{\mathrm{C}}$. ■

The following example emphasizes the interaction of u and z for $(u, z) \in \mathcal{Q}_{\mathrm{G}}$.

Example 4.3.2 *Let $M \subset \Gamma_{\mathrm{C}}$ be closed and nowhere dense, i.e. M has an empty interior. Let $0 < \mathcal{L}^{d-1}(M) < \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})$. Such a set can be constructed similarly to Cantor's middle third set, see e.g.* [Els02, p. 70 & Exercise 8.9].

Consider $z = 1 - I_M \in L^\infty(\Gamma_{\mathrm{C}})$, i.e. $z = 0$ on M and $z = 1$ on $\Gamma_{\mathrm{C}} \backslash M$. Then $N_z = \emptyset \neq M$. Let $(u, z) \in \mathcal{Q}_{\mathrm{G}}$. Thus, it holds $[\![u]\!] = 0$ on $\Gamma_c \backslash M$ and $[\![u]\!] \geq 0$ on M. Due to $p > d$ we have that $[\![u]\!] \in \mathrm{C}^0(\Gamma_{\mathrm{C}})$. Hence, since $(0, \infty)$ is open we find that $\{s \in \Gamma_{\mathrm{C}} \,|\, [\![u]\!] > 0\}$ is open as well. By $\operatorname{int} M = \emptyset$ we conclude that $\{s \in \Gamma_{\mathrm{C}} \,|\, [\![u]\!] > 0\} = \emptyset$, i.e. $[\![u]\!] = 0$ on Γ_{C}. This means, if $z = 0$ holds only on a nowhere dense subset of Γ_{C}, then u cannot jump on Γ_{C} at all, although possibly $\mathcal{L}^{d-1}(M) > 0$.

As can be seen from (4.58), the values of $\mathcal{E}(t, u, z)$ are independent of the particular values of z. Moreover Example 4.3.2 shows that, for $p > d$ only the set N_z is of importance. The proposition below states that the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ for Griffith-type delamination favours energetic solutions (u, z) with either $z(t, y) = 0$ or $z(t, y) = z_0(t, y)$ rather than $0 < z(t, y) < z_0(y)$ for all $(t, y) \in [0, T] \times \Omega_{\mathrm{D}}$ with z_0 as a given initial condition.

Proposition 4.3.3 *Let $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ be given by* (4.15), (4.58) *and* (4.20) *such that assumptions* (4.4) *and* (4.5) *hold true. Assume that $p > d$. Let $(u_0, z_0) \in \mathcal{Q}$ be a given initial value such that $(u_0, z_0) \in \mathcal{S}(0)$. Then for all $t \in [0, T]$ and a.a. $y \in \Omega_{\mathrm{D}}$ an energetic solution $(u, z) : [0, T] \to \mathcal{Q}$ satisfies $z(t, y) \in \{0, z_0(y)\}$.*

Consequently, if $z_0 = 1$ a.e. in Ω_{D}, then $z(t, y) \in \{0, 1\}$ for a.a. $(t, y) \in [0, T] \times \Omega_{\mathrm{D}}$.

Proof: Let (u, z) be an energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{R}, q_0)$. Consider $\tilde{z}(t) \in L^\infty(\Omega_{\mathrm{D}})$ with $\tilde{z}(t, y) = z_0(y)$ if $z(t, y) > 0$ and $z(t, y) = 0$ if $z(t, y) = 0$.

We show that $(u, \tilde{z})$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{R}, q_0)$ as well. First, we check the stability condition (2.60(S)) for an arbitrary state $(\hat{u}, \hat{z})$. If $\hat{z} > \tilde{z}$ on a set of positive measure, then $\mathcal{R}(\hat{z} - \tilde{z}) = \infty$ and (2.60(S)) is trivially satisfied. Hence it remains to investigate the case $\hat{z} \leq \tilde{z}$ a.e. on Ω_{D}.

If $z \leq \hat{z} \leq \tilde{z}$ a.e., then we have already $\mathcal{E}(t, \hat{u}, \hat{z}) \geq \mathcal{E}(t, \tilde{u}, \tilde{z})$, so that (2.60(S)) holds for this choice of $(\hat{u}, \hat{z})$. Assume now that $\hat{z} \leq z \leq \tilde{z}$. The stability of (u, z) and the fact that

$\hat{z} \geq z$ then yield $\mathcal{E}(t,\hat{u},\hat{z}) = \mathcal{E}(t,u,z) \leq \mathcal{E}(t,\hat{u},\hat{z}) + \mathcal{R}(\hat{z}-z) \leq \mathcal{E}(t,\hat{u},\hat{z}) + \mathcal{R}(\hat{z}-\tilde{z})$. Finally consider $\hat{z}$ such that $\hat{z} \leq z \leq \tilde{z}$ on $A \subset \Omega_{\mathrm{D}}$ and $\tilde{z} > \hat{z} > z$ on $\Omega_{\mathrm{D}} \backslash A$ for a set $A \subset \Omega_{\mathrm{D}}$ with $\mathcal{L}^d(A) > 0$. we introduce a function $\bar{z}$ such that $\bar{z} := \hat{z}$ in A and $\bar{z} := z$ in $\Omega_{\mathrm{D}} \backslash A$. We obtain from the stability of (u,z)

$$\mathcal{E}(t,\tilde{u},\tilde{z}) = \mathcal{E}(t,u,z) \leq \mathcal{E}(t,\bar{u},\bar{z}) + \mathcal{R}(\bar{z}-z) \leq \mathcal{E}(t,\hat{u},\hat{z}) + \mathcal{R}(\hat{z}-\tilde{z}),$$

due to $\mathcal{R}(\bar{z}-z) = \int_A (z-\hat{z})\,\mathrm{d}y \leq \int_A (\tilde{z}-\hat{z})\,\mathrm{d}y \leq \mathcal{R}(\hat{z}-\tilde{z})$.

It remains to verify the energy balance (2.60(E)). We have $\mathcal{E}(t,u(t),\tilde{z}(t)) = \mathcal{E}(t,u(t),z(t))$ and $\partial_t\mathcal{E}(t,u(t),\tilde{z}(t)) = \partial_t\mathcal{E}(t,u(t),z(t))$. Moreover, due to the monotonicity of $\tilde{z}$ and z with $\tilde{z} \geq z$ it holds that

$$\mathrm{Diss}_{\mathcal{R}}(\tilde{z},[0,t]) = \mathcal{R}(\tilde{z}(t)-z_0) \leq \mathcal{R}(z(t)-z_0) = \mathrm{Diss}_{\mathcal{R}}(z,[0,t])\,. \tag{4.65}$$

This implies the upper energy estimate for $(u,\tilde{z}) : [0,T] \to \mathcal{Q}$. The lower energy estimate, which is a direct consequence of stability (see e.g. [FM06, p. 70] for a proof) then yields equality in (2.60(E)). This implies equality in (4.65) and we conclude that $\tilde{z}(t,y) = z(t,y)$ for all $t \in [0,T]$ and a.e. $y \in \Omega_{\mathrm{D}}$. ■

We now state the Γ-convergence result from gradient delamination to Griffith-type delamination. The proof will be carried out in the next section. Up to now, the construction of a joint recovery sequence requires the assumption $p > d$.

Theorem 4.3.4 (Γ-convergence of the delamination problems) *Let all the assumptions* (4.4) *and* (4.5) *hold with $p > d$ and $r > d$. For all $\kappa \in (0,\kappa_0]$, let $q_\kappa : [0,T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$. If the initial values satisfy $q_0^\kappa \xrightarrow{\mathcal{T}} q_0$ and $\mathcal{E}^\kappa(0,q_0^\kappa) \to \mathcal{E}(0,q_0)$, then the delamination problems $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})_{\kappa\in(0,\kappa_0]}$ Γ-converge to the limit delamination problem $(\mathcal{Q},\mathcal{E},\mathcal{R})$ in the following sense: There is a subsequence $(q_\kappa)_{\kappa\in(0,\kappa_0]}$, such that for all $t \in [0,T]$ we have $q_\kappa(t) \xrightarrow{\mathcal{T}} q(t)$ and $q = (u,z) : [0,T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{R})$. In particular, for all $t \in [0,T]$ it holds*

$$q(t) \in \mathcal{S}(t) \quad \textit{and} \quad \mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{R}}(z,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_\xi\mathcal{E}(\xi,q(\xi))\,\mathrm{d}\xi\,.$$

Moreover we have $\mathcal{E}^\kappa(t,q_\kappa(t)) \to \mathcal{E}(t,q(t))$, $\mathrm{Diss}_{\mathcal{R}}(z_\kappa,[0,t]) \to \mathrm{Diss}_{\mathcal{R}}(z,[0,t])$ as well as $\partial_t\mathcal{E}^\kappa(t,q_\kappa(t)) \to \partial_t\mathcal{E}(t,q(t))$ for all $t \in [0,T]$.

4.3.2 Proof of the Convergence Theorem

The procedure of the proof is the same as in Section 4.2, i.e. the assumptions of Theorem 2.4.5 have to be verified. First of all we note that $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ is independent of κ. Hence the results of Lemma 4.2.7 also cover $\kappa \to 0$, so that the conditions (2.70) and (2.71(C4)) hold true. Furthermore, for all q with finite energy it holds $\partial_t\mathcal{E}(t,q) = \partial_t\mathcal{E}^\kappa(t,q)$, which is given by formula (4.21). Therefore also Lemma 4.2.6 may directly be adopted, which proves the conditions (2.69(E2)), (2.69(E3)) and (2.71(C1)). The existence of a subsequence $(q_\kappa)_{\kappa\in(0,\kappa_0]}$ of energetic solutions to $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R},q_0^\kappa)$, which converges with respect to $\mathcal{T}$ for all $t\in[0,T]$ can be established by repeating the arguments of the proof of Lemma 4.2.10. Thus it remains to prove the lower Γ-limit (2.71(C3)) and the existence of joint recovery sequences (2.71(C2)) as well as the compactness of the energy sublevels (2.69(E1)).

Properties of $\mathcal{E}$ and Verification of (2.69(E1)) and (2.71(C3))

In Lemma 4.2.3 it has been verified that the sublevels of the energy functionals $\mathcal{E}^\kappa(t,\cdot)$ are compact in the topology $\mathcal{T}$. In order to complete the proof of (2.69(E1)) it remains to verify that unions of sublevels with respect to κ are precompact in $\mathcal{T}$ and that also the sublevels of the limit energy $\mathcal{E}$ are compact in $\mathcal{T}$. Moreover, we will show that the sublevels of $\mathcal{E}$ are even sequentially compact in the weak topology of $\mathcal{Q}$, i.e. particularly in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ for the displacements, which is important for the proof of the Γ-lim inf-inequality (2.71(C3)).

Theorem 4.3.5 (Properties of sequences with equibounded energies) *For all $\kappa\in(0,\kappa_0]$ let the energy functionals $\mathcal{E}^\kappa$ be given by* (4.27) *such that the assumptions* (4.4) *and* (4.5) *hold. Moreover, let $E\in\mathbb{R}$ and $(t_\kappa)_{\kappa\in(0,\kappa_0]}\subset[0,T]$. Assume that $\mathcal{E}^\kappa(t_\kappa,u_\kappa,z_\kappa)\le E$ for all $\kappa\in(0,\kappa_0]$. Then*

1. *there is a subsequence $(u_\kappa,z_\kappa)\rightharpoonup(u,z)$ in $\mathcal{Q}$ as $\kappa\to0$, hence also $(u_\kappa,z_\kappa)\xrightarrow{\mathcal{T}}(u,z)$,*
2. *for the limit holds $(u,z)\in\mathcal{Q}_{\mathrm{G}}$, see* (4.60)*, and $0\le S_{\mathrm{G}}z\le1$ a.e. on Γ_{C}.*

Proof: **Ad 1.:** From the equiboundedness of the energies (4.27) together with coercivity estimate (4.38) we find a uniform bound $\tilde{E}$ on u_κ, i.e. $\|u_\kappa\|_{W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)}\le\tilde{E}$. Since $\mathcal{U}\subset W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ is a real, reflexive Banach space there is a subsequence $u_\kappa\rightharpoonup u$ in $\mathcal{U}$ and hence also in $W^{1,p}(\Omega_-^\nu\cup\Omega_+^\nu,\mathbb{R}^d)$ for all $\nu\in(0,\varepsilon_0]$. Furthermore the equiboundedness of the energies $\mathcal{E}^\kappa(t_\kappa,u_\kappa,z_\kappa)$ from (4.27) implies that $\|z_\kappa\|_{L^\infty(\Omega_{\mathrm{D}})}\le1$ for all $\kappa\in(0,\kappa_0]$. Since the unit ball of $L^\infty(\Omega_{\mathrm{D}})$ as the dual of the Banach space $L^1(\Omega_{\mathrm{D}})$ is weakly sequentially compact there is a subsequence $z_\kappa\overset{*}{\rightharpoonup}z$ in $L^\infty(\Omega_{\mathrm{D}})$. This proves that the subsequence $(u_\kappa,z_\kappa)_{\kappa\in(0,\kappa_0]}$ converges to (u,z) both in the weak topology of $\mathcal{Q}$ and in the topology $\mathcal{T}$.

Ad 2.: For the limit (u,z) of the subsequence $(u_\kappa,z_\kappa)_{\kappa\in(0,\kappa_0]}\subset\mathcal{U}\times\mathcal{Z}_{\mathrm{C}}$ from above we now show that $(u,z)\in\mathcal{Q}_{\mathrm{G}}$. Since $\mathcal{U}$ is a Banach space it clearly holds $u\in\mathcal{U}$. For $z_\kappa\overset{*}{\rightharpoonup}z$ in $L^\infty(\Omega_{\mathrm{D}})$ with $z_\kappa\in W^{1,r}(\Omega_{\mathrm{D}})$, $\partial_{y_1}z_\kappa=0$ and $0\le z_\kappa\le1$ a.e. in Ω_{D} it remains to prove that $z\in\mathcal{Z}_{\mathrm{G}}$, see (4.59). We first verify that $0\le z\le1$ a.e. in Ω_{D}. Testing the weak*-convergence with $L^1_+(\Omega_{\mathrm{D}})=\{\varphi\in L^1(\Omega_{\mathrm{D}})\,|\,\varphi\ge0\text{ a.e. in }\Omega_{\mathrm{D}}\}$ yields $0\le\lim_{\kappa\to0}\int_{\Omega_{\mathrm{D}}}\varphi z_\kappa\,\mathrm{d}y=\int_{\Omega_{\mathrm{D}}}\varphi z\,\mathrm{d}y$ for all $\varphi\in L^1_+(\Omega_{\mathrm{D}})$. We want to conclude that $z\ge0$ a.e. on Ω_{D}. For this, we assume that $z<0$ on $A\subset\Omega_{\mathrm{D}}$ with $\mathcal{L}^d(A)>0$. For the indicator function $I_A:\Omega_{\mathrm{D}}\to\{0,1\}$ of the set A holds $I_A\in L^1_+(\Omega_{\mathrm{D}})$, but $\int_A z\,\mathrm{d}y<0$, which is a contradiction to $\int_{\Omega_{\mathrm{D}}}\varphi z\,\mathrm{d}y\ge0$ for all $\varphi\in L^1_+(\Omega_{\mathrm{D}})$. Hence it indeed holds that $z\ge0$ a.e. in Ω_{D}. Using the same arguments we obtain that $0 \leq \lim_{\kappa\to 0} \int_{\Omega_\mathrm{D}} \varphi(1-z_\kappa)\,\mathrm{d}y = \int_{\Omega_\mathrm{D}} \varphi(1-z)\,\mathrm{d}y$ for all $\varphi \in L^1_+(\Omega_\mathrm{D})$, which yields that $z \leq 1$ a.e. in Ω_D.

Now we verify that z is constant a.e. with respect to the y_1-direction. For all $\kappa \in (0,\kappa_0]$ we find $0 = -\int_{\Omega_\mathrm{D}} \partial_{y_1} z_\kappa \varphi\,\mathrm{d}y = \int_{\Omega_\mathrm{D}} z_\kappa \partial_{y_1}\varphi\,\mathrm{d}y$ for all $\varphi \in \mathrm{C}^\infty_0(\Omega_\mathrm{D})$. Hence by the weak*-convergence it holds $0 = \lim_{\kappa\to 0}\int_{\Omega_\mathrm{D}} z_\kappa \partial_{y_1}\varphi\,\mathrm{d}y = \int_{\Omega_\mathrm{D}} z\partial_{y_1}\varphi\,\mathrm{d}y$ for all $\varphi \in \mathrm{C}^\infty_0(\Omega_\mathrm{D})$. The fundamental lemma of the calculus of variations then implies that z is constant a.e. in y_1-direction.

Moreover, since $0 \leq z \leq 1$ a.e. in Ω_D and since $S_\mathrm{G} z = \frac{1}{2}\int_{-1}^{1} z(y_1,s)\,\mathrm{d}y_1$ we obtain that $0 = S_\mathrm{G} 0 \leq S_\mathrm{G} z \leq S_\mathrm{G} 1 = 1$.

It remains to verify the transmission and noninterpenetration conditions. Due to the weak*-convergence on Ω_D we may use testfunctions $f \in L^1(\Omega_\mathrm{D})$, which are constant a.e. in y_1-direction, i.e. $f(y_1,s) = f(s)$. Then we find

$$
\begin{aligned}
2\int_{\Gamma_\mathrm{C}} f(s) S_\mathrm{G} z_\kappa(s)\,\mathrm{d}s &= \int_{\Gamma_\mathrm{C}}\int_{-1}^{1} f(y_1,s) z_\kappa(y_1,s)\,\mathrm{d}y_1\mathrm{d}s \\
\to \int_{\Gamma_\mathrm{C}}\int_{-1}^{1} f(y_1,s) z(y_1,s)\,\mathrm{d}y_1\mathrm{d}s &= 2\int_{\Gamma_\mathrm{C}} f(s) S_\mathrm{G} z(s)\,\mathrm{d}s.
\end{aligned}
$$

This proves in particular that $0 = \int_{\Gamma_\mathrm{C}} S_\mathrm{G} z_\kappa \big|[\![u_\kappa]\!]\big|\,\mathrm{d}s \to \int_{\Gamma_\mathrm{C}} S_\mathrm{G} z \big|[\![u]\!]\big|\,\mathrm{d}s$, since the compactness of the trace operator $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \to L^p(\Gamma_\mathrm{C},\mathbb{R}^d)$ yields $[\![u_\kappa]\!] \to [\![u]\!]$ strongly in $L^p(\Gamma_\mathrm{C},\mathbb{R}^d)$. Therefore we find a subsequence which converges pointwise a.e. on Γ_C and hence $0 \leq \lim_{\kappa\to 0}[\![u_\kappa\cdot \mathrm{n}_1]\!] = [\![u\cdot\mathrm{n}_1]\!]$ a.e. on Γ_C. ∎

For $t_\kappa = t$ fixed the above theorem states the precompactness of unions of sublevels both in the weak topology of $\mathcal{Q}$ and in $\mathcal{T}$. It remains to verify the compactness of the sublevels of the limit energy functional $\mathcal{E}(t,\cdot)$.

Lemma 4.3.6 (Properties of the limit energy) *Let the energy functional $\mathcal{E}$ be given by* (4.58) *such that the assumptions* (4.4) *and* (4.5) *hold true. Then for all $t \in [0,T]$ the functional $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ is coercive and weakly sequentially lower semicontinuous on $\mathcal{Q}$. In particular*

$$
\mathcal{E}(t,q) \geq \tfrac{2^{1-p}c}{C_K}\|u\|_{W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)} - cc_g\,. \tag{4.66}
$$

Moreover for all $E \in \mathbb{R}$ the sublevels $L_E(t) := \{q \in \mathcal{Q}\,|\,\mathcal{E}(t) \leq E\}$ of the functional $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ are sequentially compact in the weak topology of $\mathcal{Q}$ and hence in $\mathcal{T}$.

Proof: Estimate (4.66) is a direct consequence of (4.5(H2)), (4.4) and Korn's inequality (4.24). This estimate together with the fact that $\mathcal{E}(t,u,z) = \infty$ if $\|z\|_{L^\infty(\Gamma_\mathrm{C})} > 1$ proves the coercivity of $\mathcal{E}(t,\cdot)$ on $\mathcal{Q}$. Lower semicontinuity follows from convexity (4.5(H1)) and the closedness of $\mathcal{Q}_\mathrm{G} \cap \{(u,z) \in W^{1,p}(\Omega_- \cup \Omega_+,\mathbb{R}^d) \times L^\infty(\Omega_\mathrm{D})\,|\,0 \leq z \leq 1 \text{ a.e. in } \Omega_\mathrm{D}\}$ in $W^{1,p}(\Omega_- \cup \Omega_+,\mathbb{R}^d) \times L^\infty(\Omega_\mathrm{D})$, which can be shown as in the proof of Lemma 4.2.3 using the ideas of the proof of Theorem 4.3.5, Item 2. Then the compactness of the sublevels in the weak topology of $\mathcal{Q}$ directly follows from the lower semicontinuity and coercivity as in the proof of Lemma 4.2.3. Since $\mathcal{T}$ is coarser than the the weak topology of $\mathcal{Q}$ the compactness of the sublevels in $\mathcal{T}$ follows. ■

In the following we establish the Γ-lim inf-estimate for $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ as $\kappa \to 0$. For this, we use that stable sequences have equibounded energies, so that there is a subsequence which weakly converges even in $\mathcal{Q}$.

Theorem 4.3.7 (Lower Γ-limit of the energy functionals) *Let $\mathcal{E}^\kappa$ and $\mathcal{E}$ be given by* (4.27) *and* (4.58) *such that the assumptions* (4.4) *and* (4.5) *hold. Let $(t_\kappa, q_\kappa) \xrightarrow{\mathcal{T}_T} (t, q)$ as $\kappa \to 0$ with $q_\kappa \in \mathcal{S}^\kappa(t_\kappa)$ for all $\kappa \in (0, \kappa_0]$. Then*

$$\mathcal{E}(t, q) \leq \liminf_{\kappa \to 0} \mathcal{E}^\kappa(t_\kappa, q_\kappa)\,. \tag{4.67}$$

Proof: Since $q_\kappa = (u_\kappa, z_\kappa) \in \mathcal{S}^\kappa(t_\kappa)$ for all $\kappa \in (0, \kappa_0]$ there is a constant $E > 0$ such that $\mathcal{E}^\kappa(t_\kappa, u_\kappa, z_\kappa) \leq E$. Thus Theorem 4.3.5 can be applied and yields the existence of a subsequence $(u_\kappa, z_\kappa) \rightharpoonup (u, z)$ in $\mathcal{Q}$ with $(u, z) \in \mathcal{Q}_{\mathrm{G}}$.

Due to assumptions (4.5) we obtain that the functional $\int_{\Omega_- \cup \Omega_+} W(\cdot)\,\mathrm{d}x$ is weakly sequentially lower semicontinuous on $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$. Together with (4.4) we deduce $\liminf_{\kappa \to 0} \int_{\Omega_- \cup \Omega_+} W(e(u_\kappa + g(t_\kappa)))\,\mathrm{d}x \geq \int_{\Omega_- \cup \Omega_+} W(e(u + g(t)))\,\mathrm{d}x$. Furthermore it clearly holds $\liminf_{\kappa \to 0} \frac{\kappa}{r} \int_{\Gamma_{\mathrm{C}}} |\nabla_s z_\kappa|^r\,\mathrm{d}s \geq 0$, so that (4.67) is established. ■

Conditioned Upper Semicontinuity of the Stable Sets (2.71(C2))

We show condition (2.71(C2)) by proving the existence of a joint recovery sequence. Hence, for any sequence $(t_\kappa, q_\kappa) \xrightarrow{\mathcal{T}_T} (t, q)$ with $q = (u, z)$ and with $q_\kappa = (u_\kappa, z_\kappa) \in \mathcal{S}^\kappa(t_\kappa)$ for all $\kappa \in (0, \kappa_0]$ and for all $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ our task is to construct a joint recovery sequence $(\hat{q}_\kappa)_{\kappa \in (0, \kappa_0]}$ with $\hat{q}_\kappa = (\hat{u}_\kappa, \hat{z}_\kappa)$ such that

$$\limsup_{\kappa \to 0} \big(\mathcal{E}^\kappa(t_\kappa, \hat{q}_\kappa) + \mathcal{R}(\hat{z}_\kappa - z_\kappa) - \mathcal{E}^\kappa(t_\kappa, q_\kappa)\big) \leq \mathcal{E}(t, \hat{q}) + \mathcal{R}(\hat{z} - z) - \mathcal{E}(t, q)\,. \tag{4.68}$$

In order to constitute $(\hat{z}_\kappa)_{\kappa \in (0, \kappa_0]} \subset W^{1,r}(\Omega_{\mathrm{D}})$ for a given function $\hat{z} \in L^\infty(\Omega_{\mathrm{D}})$ we have to mollify $\hat{z}$ by a sequence of suitable mollifiers $(\eta_\kappa)_{\kappa(0,\kappa_0]} \subset \mathrm{C}_0^\infty(\mathbb{R}^d)$ in such a way that $\int_{\Omega_{\mathrm{D}}} \frac{\kappa}{r}\big(|\nabla \hat{z}_\kappa|^r - |\nabla z_\kappa|^r\big)\,\mathrm{d}y$ vanishes. For this, we use mollifiers of the form

$$\tilde{\eta}_1(y) := \begin{cases} c\exp(-1/(1-|y|^2)) & \text{if } |y| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{\eta}_\rho(y) := \tfrac{1}{\rho^d}\tilde{\eta}_1(y/\rho)\,, \quad \eta_\kappa = \tilde{\eta}_{\rho(\kappa)}\,, \tag{4.69}$$

where c is defined in such a way that $\|\tilde{\eta}_1\|_{L^1(\mathbb{R}^d)} = 1$ and $\rho(\kappa) \to 0$ as $\kappa \to 0$ suitably.

Due to $\hat{z} \in L^\infty(\Omega_{\mathrm{D}})$ the mollification guarantees that $\hat{z}_\kappa \to \hat{z}$ in $L^q(\Omega_{\mathrm{D}})$ for all $q \in [1, \infty)$, see [Ada75, p. 29, L. 2.18]. Moreover, by [Jan71, p. 33 Th. 39.1] it holds that

$$\operatorname{supp}(\hat{z} * \tilde{\eta}_\rho) \subset \operatorname{supp}\hat{z} + B_\rho(0) = \{s + \tilde{s} \,|\, s \in \operatorname{supp}\hat{z}, \tilde{s} \in B_\rho(0)\}\,, \tag{4.70}$$

where $\hat{z}*\tilde{\eta}_\rho(y) := \int_{\Omega_{\rm D}} \hat{z}(\tilde{y})\eta_\rho(\tilde{y}-y)\,{\rm d}\tilde{y}$ and where $B_\rho(0)$ is the closed ball of radius ρ around 0. Hence, using $\hat{z} \geq 0$, for all $\rho > 0$ we have

$$\operatorname{supp} S_{\rm G}\hat{z} \subset \operatorname{supp} S_{\rm G}(\hat{z} * \tilde{\eta}_\rho) \quad \text{and} \quad N_{\hat{z}*\tilde{\eta}_\rho} \subset N_{\hat{z}}\,, \tag{4.71}$$

where the support and the zero set are defined by (4.63). Moreover, by (4.70) we conclude that $\mathcal{L}^{d-1}(N_{\hat{z}} \backslash N_{\hat{z}*\tilde{\eta}_\rho}) \to 0$ as $\rho \to 0$, which is not generally true for arbitrary sequences $\hat{z}_\rho \to \hat{z}$ in $L^q(\Omega_{\rm D})$.

Example 4.3.8 *Let $\Gamma_{\rm C} = [-1, 1]$ and $\hat{z}_\rho := \rho$ for all $\rho > 0$. Clearly $\hat{z}_\rho \to 0 = \hat{z}$ uniformly on $\Gamma_{\rm C}$ as well as in $L^q(\Gamma_{\rm C})$ for all $q \in [1, \infty]$. But for all $\rho > 0$ it is $N_{\hat{z}_\rho} = \emptyset$ and hence $\mathcal{L}(N_{\hat{z}_\rho}) = 0$, whereas $N_{\hat{z}} = \Gamma_{\rm C}$ with $\mathcal{L}(\Gamma_{\rm C}) = 2$.*

Since in general $N_{\hat{z}_\kappa} \subsetneq N_{\hat{z}}$, where $\hat{z}_\kappa = \hat{z} * \eta_\kappa$ with η_κ defined by (4.69), it is necessary to modify $\hat{u}$ so that the modified functions $\hat{u}_\kappa$ satisfy $[[\![\hat{u}_\kappa]\!] > 0] \subset N_{\hat{z}_\kappa}$. In fact, the next example demonstrates that it is in general not possible to set $\hat{u}_\kappa := \hat{u}$ and to exchange $\hat{z}$ by ζ_κ, which is 0 in a sufficiently large neighborhood U_κ around $N_{\hat{z}}$, so that $S_{\rm G}\zeta_\kappa * \eta_\kappa = 0$ on $[[\![\hat{u}]\!] > 0] = \{s \in \Gamma_{\rm C} \,|\, [\![\hat{u}(s)]\!] > 0\}$.

Example 4.3.9 *Let $\Gamma_{\rm C} = [-1, 1]$ and consider a closed set $M \subset [-1, 1]$ which is nowhere dense with $0 < \mathcal{L}(M) < 2$. For all $s \in M$ put $\hat{u}(s) := (\inf_{\tilde{s}\in M}(s, \tilde{s}))^2$. Then $\hat{u}(s) = 0$ for all $s \in M$ and $\hat{u}(s) > 0$ otherwise. Moreover, $\hat{u}$ is continuous. There is a function $\hat{z} \in L^\infty([-1, 1])$ such that $\hat{z} > 0$ on M and $\hat{z} = 0$ on $\Gamma_{\rm C}\backslash M$. Since M contains no open ball of radius $r > 0$ we conclude that*

$$M \subset U_r := \bigcup_{s\in\Gamma_{\rm C}\backslash M} B_r(s) \quad \text{for all } r > 0\,,$$

where $B_r(s)$ is the closed ball of radius r around s. This implies that $U_r \cap \Gamma_{\rm C} = \Gamma_{\rm C}$ for all $r > 0$ and hence $U_r \not\to \Gamma_{\rm C}\backslash M$ as $r \to 0$.

This shows that we cannot avoid to modify $\hat{u}$. In order to verify (4.68) it is helpful if $\mathcal{E}^\kappa(t_\kappa, \hat{u}_\kappa, \hat{z}_\kappa) \to \mathcal{E}(t, \hat{u}, \hat{z})$. This can be guaranteed if $\hat{u}_\kappa \to \hat{u}$ strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$. In other words, the following conjecture must hold.

Conjecture 4.3.10 *Let $(\hat{u}, \hat{z}) \in \mathcal{Q}_{\rm G}$. Choose a sequence of mollifiers $(\eta_\kappa)_{\kappa\in(0,\kappa_0]} \subset {\rm C}_0^\infty(\mathbb{R}^d)$ as in (4.69) and set $\hat{z}_\kappa := \hat{z} * \eta_\kappa$. Then there is a sequence $(\hat{u}_\kappa)_{\kappa\in(0,\kappa_0]} \subset W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ with $\hat{u}_\kappa \in W^{1,p}(\Omega\backslash N_{\hat{z}_\kappa}, \mathbb{R}^d)$, $\hat{u}_\kappa = 0$ a.e. on $\Gamma_{\rm D}$ and $[\![\hat{u}_\kappa \cdot {\rm n}_1]\!] \geq 0$ a.e on $\Gamma_{\rm C}$ in trace sense, which satisfies*

$$\|\hat{u}_\kappa - \hat{u}\|_{W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)} \to 0 \quad \text{as } \kappa \to 0\,.$$

In the following we prove the Conjecture 4.3.10 for the case $p > d$, since the continuity of $[\![\hat{u}]\!]$ on $\overline{\Gamma_{\rm C}}$ then allows us to conclude from Lemma 4.3.1 that $(\hat{u}, \hat{z}) \in \mathcal{Q}_{\rm G}$ is equivalent to $\operatorname{supp} S_{\rm G}\hat{z} \cap \operatorname{OS}[\![\hat{u}]\!] = \emptyset$. The basic idea of our construction is to split $\hat{u} = \hat{u}_{\rm sym} + \hat{u}_{\rm anti}$

in its symmetric part $\hat{u}_{\mathrm{sym}}(x_1,s)=\frac{1}{2}\big(\hat{u}(x_1,s)+\hat{u}(-x_1,s)\big)$ and its antisymmetric part $\hat{u}_{\mathrm{anti}}(x_1,s)=\frac{1}{2}\big(\hat{u}(x_1,s)-\hat{u}(-x_1,s)\big)$. It is easy to see that, $[\![\hat{u}_{\mathrm{sym}}]\!]=0$ on Γ_{C}, whereas $[\![\hat{u}_{\mathrm{anti}}]\!]=[\![\hat{u}]\!]$, i.e. $\hat{u}_{\mathrm{sym}}\in W^{1,p}(\Omega,\mathbb{R}^d)$ and $\hat{u}^{\pm}_{\mathrm{anti}}=\hat{u}_{\mathrm{anti}}|_{\Omega_\pm}\in W^{1,p}(\Omega_\pm,\mathbb{R}^d)$ with $\hat{u}^{\pm}_{\mathrm{anti}}=0$ on $\operatorname{supp}S_{\mathrm{G}}\hat{z}$. Then, we multiply $\hat{u}_{\mathrm{anti}}$ by cut-off functions ξ_κ, which push $\hat{u}^{\kappa}_{\mathrm{anti}}:=\xi_\kappa\hat{u}_{\mathrm{anti}}$ to 0 in a suitable neighborhood of

$$\hat{M}:=\operatorname{supp}S_{\mathrm{G}}\hat{z}\,. \tag{4.72}$$

In order to prove that $\xi_\kappa\hat{u}_{\mathrm{anti}}\to\hat{u}_{\mathrm{anti}}$ strongly in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ we will apply the following generalized Hardy inequality:

Proposition 4.3.11 ([Lew88, p. 190]) *Let $\hat{M}\subset\overline{\Omega_0}$ be closed, let $\Omega_0\subset\mathbb{R}^d$ be bounded and $p>d$. Let $d_{\widetilde{M}}(x):=\min_{\hat{x}\in\hat{M}\cup\partial\Omega_0}|x-\hat{x}|$ for all $x\in\overline{\Omega_0}$. Then there is a constant $C_0>0$ such that for all $u\in W^{1,p}_0(\Omega_0\backslash\hat{M},\mathbb{R}^d):=\{\tilde{u}\in W^{1,p}(\Omega_0\backslash\hat{M},\mathbb{R}^d)\,|\,\tilde{u}=0$ on $\hat{M}\cup\partial\Omega_0\}$ it holds*

$$\big\|u/d_{\widetilde{M}}\big\|_{L^p(\Omega_0\backslash\hat{M},\mathbb{R}^d)}\leq C_0\,\|\nabla u\|_{L^p(\Omega_0\backslash\hat{M},\mathbb{R}^{d\times d})}\,. \tag{4.73}$$

Regarding Ω_0 as a suitable extension of $\Omega_\pm$ one can prove that $(u/d_{\hat{M}})\in L^p(\Omega_\pm,\mathbb{R}^d)$ for all $u\in W^{1,p}_{\hat{M}}(\Omega_\pm,\mathbb{R}^d)$.

Corollary 4.3.12 *Let $\hat{M}\subset\overline{\Gamma_{\mathrm{C}}}$ be closed and let $\Omega_\pm\subset\mathbb{R}^d$ be given as in Fig. 4.1. Assume that $p>d$. Let $d_{\hat{M}}(x):=\min_{\hat{x}\in\hat{M}}|x-\hat{x}|$ for all $x\in\overline{\Omega_\pm}$. For all $u\in W^{1,p}_{\hat{M}}(\Omega_\pm,\mathbb{R}^d)$ with $W^{1,p}_{\hat{M}}(\Omega_\pm,\mathbb{R}^d):=\{\tilde{u}\in W^{1,p}(\Omega_\pm,\mathbb{R}^d)\,|\,\tilde{u}=0$ on $\hat{M}\}$ it holds that $(u/d_{\hat{M}})\in L^p(\Omega_\pm,\mathbb{R}^d)$.*

Proof: We carry out the proof for $u\in W^{1,p}_{\hat{M}}(\Omega_+,\mathbb{R}^d)$, where $\Omega_+:=(0,+L)\times(-H,+H)^{d-1}$. With the same arguments one can prove a similar relation for $u\in W^{1,p}_{\hat{M}}(\Omega_-,\mathbb{R}^d)$ with $\Omega_-:=(-L,0)\times(-H,+H)^{d-1}$.

For Ω_+ from above one can choose the bounded domain Ω_0 from Lemma 4.3.11 sufficiently large, so that $d_{\widetilde{M}}(x)=d_{\hat{M}}(x)$ for all $x\in\Omega_+$, i.e. $\min_{\hat{x}\in\hat{M}}|x-\hat{x}|\leq\min_{\hat{x}\in\partial\Omega_0}|x-\hat{x}|$ must hold for all $x\in\Omega_+$. Clearly this requirement can be satisfied by a bounded domain Ω_0 due to the geometry of Ω_+. Moreover, the functions of $W^{1,p}_{\hat{M}}(\Omega_+,\mathbb{R}^d)$ have to be extended to functions of $W^{1,p}_0(\Omega_0\backslash\hat{M},\mathbb{R}^d)$. Such an extension exists, since one can extend the functions of $W^{1,p}(\Omega_+,\mathbb{R}^d)$ as $W^{1,p}$-functions onto the entire $\mathbb{R}^d$ by [Ada75, p. 91, Th. 4.32], i.e. for all $u\in W^{1,p}(\Omega_+,\mathbb{R}^d)$ there is $Eu\in W^{1,p}(\mathbb{R}^d,\mathbb{R}^d)$ such that $Eu|_{\Omega_+}=u$ and such that $\|Eu\|_{W^{1,p}(\mathbb{R}^d,\mathbb{R}^d)}\leq K\|u\|_{W^{1,p}(\Omega_+,\mathbb{R}^d)}$ with K fixed for all $u\in W^{1,p}(\Omega_+,\mathbb{R}^d)$. Using a cut-off function $\zeta\in\mathrm{C}^\infty_0(\Omega_0)$, which satisfies $\zeta=1$ on Ω_+, $\zeta=0$ on $\Omega_0\backslash(-L,+2L)\times(-2H,+2H)$ and $\zeta\in(0,1)$ on $(-L,+2L)\times(-2H,+2H)\backslash\Omega_+$ yields that $(\zeta Eu)\in W^{1,p}_0(\Omega_0\backslash M,\mathbb{R}^d)$.

By Hardy's inequality (4.73) we obtain that

$$\begin{aligned}\|u/d_{\hat{M}}\|_{L^p(\Omega_+,\mathbb{R}^d)}&\leq\|(\zeta Eu)/d_{\widetilde{M}}\|_{L^p(\Omega_0\backslash\hat{M},\mathbb{R}^d)}\leq C_0\|\nabla(\zeta Eu)\|_{L^p(\Omega_0\backslash\hat{M},\mathbb{R}^{d\times d})}\\&\leq C_0\big(\|\nabla\zeta\|_{\mathrm{C}(\overline{\Omega_0})}\|Eu\|_{L^p(\Omega_0\backslash\hat{M},\mathbb{R}^d)}+\|\nabla Eu\|_{L^p(\Omega_0\backslash\hat{M},\mathbb{R}^{d\times d})}\big)\\&\leq 2^{\frac{p-1}{p}}KC_0\max\{1,\|\nabla\zeta\|_{\mathrm{C}(\overline{\Omega_0})}\}\|u\|_{W^{1,p}(\Omega_+,\mathbb{R}^d)}\,.\end{aligned}$$

■

Thus, we can prove Conjecture 4.3.10 for $p > d$.

Corollary 4.3.13 *Let $p > d$ and $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+ \cup \hat{M}, \mathbb{R}^d)$ with $\hat{u} = 0$ on Γ_{Dir} and $[\![\hat{u}]\!] = 0$ on $\hat{M}$. Put*

$$\hat{u}_{\mathrm{sym}}(x_1, s) = \tfrac{1}{2}\big(\hat{u}(x_1, s) + \hat{u}(-x_1, s)\big) \ \text{ and } \ \hat{u}_{\mathrm{anti}}(x_1, s) = \tfrac{1}{2}\big(\hat{u}(x_1, s) - \hat{u}(-x_1, s)\big)\,.$$

Let $\xi_\rho(x) := \min\{\frac{1}{\rho}\big(d_{\hat{M}}(x) - \rho\big)^+, 1\}$. Set $\hat{u}^\rho_{\mathrm{anti}} := \xi_\rho \hat{u}_{\mathrm{anti}}$ and $\hat{u}^\rho := \hat{u}_{\mathrm{sym}} + \hat{u}^\rho_{\mathrm{anti}}$.

Then $\hat{u}^\rho \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ with $[\![\hat{u}^\rho \cdot \mathrm{n}_1]\!] \geq 0$ for all $\rho > 0$ and $\hat{u}^\rho \to \hat{u}_{\mathrm{anti}}$ strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$.

Proof: Since ξ_ρ is positive we infer that $[\![\hat{u}^\rho_{\mathrm{anti}} \cdot \mathrm{n}_1]\!] \geq 0$. Moreover, from the definition of ξ_ρ we see that $\xi_\rho \to \xi$ pointwise in Ω, where

$$\xi_\rho(x) := \begin{cases} 0 & \text{if } d_{\hat{M}}(x) \leq \rho, \\ \in (0,1) & \text{if } \rho < d_{\hat{M}}(x) \leq 2\rho, \\ 1 & \text{if } 2\rho < d_{\hat{M}}(x), \end{cases} \qquad \text{and} \qquad \xi(x) := \begin{cases} 0 & \text{if } x \in \hat{M}, \\ 1 & \text{otherwise.} \end{cases}$$

By the dominated convergence theorem we obtain that

$$\|\hat{u}^\rho_{\mathrm{anti}} - \hat{u}_{\mathrm{anti}}\|^p_{L^p(\Omega,\mathbb{R}^d)} = \int\limits_{[d_{\hat{M}}(x) \leq \rho]} |\hat{u}_{\mathrm{anti}}|^p \,\mathrm{d}x + \int\limits_{[\rho < d_{\hat{M}}(x) \leq 2\rho]} |(\xi_\rho - \xi)\hat{u}_{\mathrm{anti}}|^p \,\mathrm{d}x + \int\limits_{[2\rho < d_{\hat{M}}(x)]} |0|^p \,\mathrm{d}x$$
$$\to 0$$

due to $\mathcal{L}^d\big([d_{\hat{M}}(x) \leq \rho]\big) \to 0$, $\mathcal{L}^d\big([\rho < d_{\hat{M}}(x) \leq 2\rho]\big) \to 0$ and $|\xi_\rho - \xi| \leq 1$ for all $\rho > 0$.

By the chain rule we calculate that $\nabla \hat{u}^\rho_{\mathrm{anti}} = \xi_\rho \nabla \hat{u}_{\mathrm{anti}} + \hat{u}_{\mathrm{anti}} \otimes \nabla \xi_\rho$. Thus,

$$\|\nabla(\hat{u}^\rho_{\mathrm{anti}} - \hat{u}_{\mathrm{anti}})\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^{d\times d})} \leq \|(1-\xi_\rho)\nabla \hat{u}_{\mathrm{anti}}\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^{d\times d})} + \|\hat{u}_{\mathrm{anti}} \otimes \nabla \xi_\rho\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^{d\times d})}\,,$$

where $\|(1-\xi_\rho)\nabla \hat{u}_{\mathrm{anti}}\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^{d\times d})} \to 0$ again by the dominated convergence theorem.

It remains to show that $\|\hat{u}_{\mathrm{anti}} \otimes \nabla \xi_\rho\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^{d\times d})} \to 0$. We obtain that

$$|\nabla \xi_\rho| = \begin{cases} 0 & \text{if } 0 \leq d_{\hat{M}}(x) \leq \rho, \\ \frac{1}{\rho} & \text{if } \rho < d_{\hat{M}}(x) \leq 2\rho, \\ 0 & \text{if } 2\rho < d_{\hat{M}}(x)\,, \end{cases}$$

i.e. $|\nabla \xi_\rho| \leq \frac{1}{\rho}$. Since $d_{\hat{M}}(x) \in [\rho, 2\rho]$ it holds that $\frac{1}{\rho} \leq \frac{2}{d_{\hat{M}}(x)}$ for all $x \in \Omega$. Hence we conclude that

$$\|\hat{u}_{\mathrm{anti}} \otimes \nabla \xi_\rho\|^p_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^{d\times d})} \leq 2^p \int_{B_{2\rho}(\hat{M}) \setminus B_\rho(\hat{M})} \left| \frac{\hat{u}_{\mathrm{anti}}(x)}{d_{\hat{M}}(x)} \right|^p \mathrm{d}x \to 0,$$

since $\|\hat{u}_{\mathrm{anti}}/d_{\hat{M}}\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^d)}$ is bounded by Corollary 4.3.12 and since

$$\mathcal{L}^d\big(B_{2\rho}(\hat{M}) \setminus B_\rho(\hat{M})\big) \to 0 \text{ for } B_{2\rho}(\hat{M}) \setminus B_\rho(\hat{M}) = \{x \in \Omega \,|\, \rho < d_{\hat{M}}(x) \leq 2\rho\}.$$

■

With these tools at hand we now prove the existence of a joint recovery sequence under the assumption that $r > d$. In particular we have to determine the mollifiers η_κ in such a way that their slopes grow sufficiently slow, so that $\int_{\Omega_\mathrm{D}} \frac{\kappa}{r}\big(|\nabla \hat{z}_\kappa|^r - |\nabla z_\kappa|^r\big)\,\mathrm{d}y$ vanishes. In order to show that this holds true, we will exploit the Lipschitz continuity of $|\cdot|^r$.

Theorem 4.3.14 (Joint recovery sequences) *Let the systems* $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$ *and* $(\mathcal{Q},\mathcal{E},\mathcal{R})$ *be given by* (4.15), (4.27), (4.20) *and* (4.58), *such that the assumptions* (4.4) *and* (4.5) *hold true with* $p>d$ *and* $r>d$. *Then, for all* $(t_\kappa,q_\kappa) \xrightarrow{\mathcal{T}_T} (t,q)$ *with* $q_\kappa=(u_\kappa,z_\kappa)\in\mathcal{S}^\kappa(t_\kappa)$ *for all* $\kappa\in(0,\kappa_0]$ *and for every* $\hat{q}=(\hat{u},\hat{z})\in\mathcal{Q}$ *there is a sequence* $(q_\kappa)_{\kappa\in(0,\kappa_0]}$ *with* $\hat{q}_\kappa=(\hat{u}_\kappa,\hat{z}_\kappa)$ *such that*

$$\limsup_{\kappa\to 0}\left(\mathcal{E}^\kappa(t_\kappa,\hat{q}_\kappa)+\mathcal{R}(\hat{z}_\kappa-z_\kappa)-\mathcal{E}^\kappa(t_\kappa,q_\kappa)\right)\le\left(\mathcal{E}(t,\hat{q})+\mathcal{R}(\hat{z}-z)-\mathcal{E}(t,q)\right). \tag{4.74}$$

Proof: Let $(t_\kappa,u_\kappa,z_\kappa)\xrightarrow{\mathcal{T}_T}(t,u,z)$ with $q_\kappa=(u_\kappa,z_\kappa)\in\mathcal{S}^\kappa(t_\kappa)$ for every $\kappa\in(0,\kappa_0]$. Consider $\hat{q}=(\hat{u},\hat{z})\in\mathcal{Q}$. If $\hat{q}\in\mathcal{Q}\backslash\mathcal{Q}_\mathrm{G}$, then $\mathcal{E}(t_\kappa,\hat{q})=\infty$ for all $\kappa\in(0,\kappa_0]$ and (4.74) trivially holds. Hence, assume that $\hat{q}\in\mathcal{Q}_\mathrm{G}$. Additionally let $0\le\hat{z}\le z$ a.e. in Ω_D, otherwise $\mathcal{R}(\hat{z}-z)=\infty$ and (4.74) would again be trivially satisfied. For every $\kappa\in(0,\kappa_0]$ we now have to construct the joint recovery sequence $(\hat{u}_\kappa,\hat{z}_\kappa)_{\kappa\in(0,\kappa_0]}\subset\mathcal{Q}$ in such a way that $\hat{q}_\kappa=(\hat{u}_\kappa,\hat{z}_\kappa)\in\mathcal{Q}_\mathrm{C}$ and $\mathcal{R}(\hat{z}_\kappa-z_\kappa)<\infty$ for all $\kappa\in(0,\kappa_0]$. This means in particular that $\hat{z}_\kappa\in W^{1,r}(\Omega_\mathrm{D})$, whereas $\hat{z}$ lies only in $L^\infty(\Omega_\mathrm{D})$, and additionally is required that $\hat{z}_\kappa\le z_\kappa$ a.e. in Ω_D. The construction of $(\hat{z}_\kappa)_{\kappa\in(0,\kappa_0]}$ will be done in Step 1. In Step 2 we verify that $\int_{\Omega_\mathrm{D}}\frac{\kappa}{r}\big(|\nabla\hat{z}_\kappa|^r-|\nabla z_\kappa|^r\big)\,\mathrm{d}s\to 0$. Finally, in Step 3, we specify $\hat{u}_\kappa$ using Corollary 4.3.13.

Step 1: For all $\kappa\in(0,\kappa_0]$ we now construct $\hat{z}_\kappa$. We have $\hat{z}\in L^\infty(\Omega_\mathrm{D})$ with $0\le\hat{z}\le 1$ being constant a.e. in y_1-direction, whereas the recovery sequence has to fulfill $\hat{z}_\kappa\in W^{1,r}(\Omega_\mathrm{D})$ with $\partial_{y_1}\hat{z}_\kappa=0$ and $0\le\hat{z}_\kappa\le 1$. First, we put

$$\zeta:=\begin{cases}\frac{\hat{z}}{z} & \text{if } z>0\,,\\ 0 & \text{if } z=0.\end{cases} \tag{4.75}$$

Due to the assumption $0\le\hat{z}\le z$ it clearly holds that $0\le\zeta\le 1$ a.e. in Ω_D. We mollify ζ by convolution with a sequence $(\eta_\kappa)_{\kappa\in(0,\kappa_0]}\subset\mathrm{C}_0^\infty(\mathbb{R}^d)$ similar to (4.69), where the dependence of ρ on κ will be specified below. For all $\kappa\in(0,\kappa_0]$ the convolution leads to functions $\zeta_\kappa=\zeta*\eta_\kappa$ which satisfy $\zeta_\kappa\to\zeta$ strongly in $L^q(\Omega_\mathrm{D})$ for all $q\in[1,\infty)$ by [Ada75, Lemma 2.18] since $\hat{z}/z\in L^q(\Omega_\mathrm{D})$. As the final recovery sequence we introduce

$$\hat{z}_\kappa:=z_\kappa\zeta_\kappa\quad\text{for all }\kappa\in(0,\kappa_0]\,, \tag{4.76}$$

which satisfies $0\le\hat{z}_\kappa\le z_\kappa$ due to $\|\zeta_\kappa\|_{L^\infty(\Omega_\mathrm{D})}=\|\zeta\|_{L^\infty(\Omega_\mathrm{D})}\|\eta_\kappa\|_{L^1(\Omega_\mathrm{D})}\le 1$ by the properties of a standard mollifier, see [Ada75, p. 29]. Since $z_\kappa\overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_\mathrm{D})$ by assumption and $\zeta_\kappa\to\zeta$ in $L^1(\Omega_\mathrm{D})$ it holds that $\hat{z}_\kappa\rightharpoonup\hat{z}$ in $L^1(\Omega_\mathrm{D})$ and hence

$$\lim_{\kappa\to 0}\mathcal{R}(\hat{z}_\kappa-z_\kappa)=\lim_{\kappa\to 0}\varrho\int_{\Omega_\mathrm{D}}(z_\kappa-\hat{z}_\kappa)\,\mathrm{d}y=\mathcal{R}(\hat{z}-z)\,. \tag{4.77}$$

Application of the chain rule yields that $\nabla\hat{z}_\kappa = \nabla(\zeta_\kappa z_\kappa) = \zeta_\kappa\nabla z_\kappa + z_\kappa\nabla\zeta_\kappa \in L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)$ as well as $\partial_{y_1}\hat{z}_\kappa = 0$ due to $\partial_{y_1}z_\kappa = 0$ and $\partial_{y_1}\zeta_\kappa = \partial_{y_1}(\zeta * \eta_\kappa) = 0$, since both z and $\hat{z}$ are constant a.e. with respect to y_1-direction.

In order to ensure that $\frac{\kappa}{r}\|\nabla\hat{z}_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \to 0$ as $\kappa \to 0$ we now determine the mollifiers η_κ suitably. For this, we consider for all $\rho > 0$ the mollifier $\tilde{\eta}_\rho$ from (4.69). Then it holds

$$\|\nabla(\zeta * \tilde{\eta}_\rho)\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \le \|\zeta\|_{L^\infty(\Omega_{\mathrm{D}})}\|\nabla\tilde{\eta}_\rho\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \le \|\nabla\tilde{\eta}_1\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)}\rho^{-rd}\,. \tag{4.78}$$

In order to guarantee that $\kappa\rho^{-rd} \to 0$ we choose $\rho(\kappa) = \kappa^{\frac{1}{2rd}}$ and define $\eta_\kappa = \tilde{\eta}_{\rho(\kappa)}$.

Step 2: Up to now our construction makes sure that $\frac{\kappa}{r}\|\nabla\zeta_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \to 0$ as $\kappa \to 0$. Since $\frac{\kappa}{r}\|\nabla z_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)}$ is only uniformly bounded by the properties of stable sequences, we conclude that $\frac{\kappa}{r}\|\nabla\hat{z}_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)}$ may not vanish completely. However in the lim sup-estimate (4.74) we can compensate the remaining terms by the term $-\frac{\kappa}{r}\|\nabla z_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)}$ that occurs in $\mathcal{E}^\kappa(t_\kappa, u_\kappa, z_\kappa)$. In order to show that these terms indeed cancel out we use the following Lipschitz estimate for $w(x) = x^r$ with $r \in (1,\infty)$ and $x \ge 0$, which can be obtained by a Taylor expansion:

$$|w(a) - w(b)| = \Big|\int_0^1 w'(b + \alpha(a-b))(a-b)\,\mathrm{d}\alpha\Big| \le 2^{r-1}(a^{r-1} + b^{r-1})|a-b| \tag{4.79}$$

for all $a, b \ge 0$. Using the fact that both $0 \le \zeta_\kappa \le 1$ and $0 \le z_\kappa \le 1$ a.e. in Ω_{D}, and with (4.79) together with Hölder's inequality we now conclude that

$$\begin{aligned}
&\int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r}\big(|\nabla\hat{z}_\kappa|^r - |\nabla z_\kappa|^r\big)\,\mathrm{d}y \le \int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r}\big((|\nabla\zeta_\kappa| + |\nabla z_\kappa|)^r - |\nabla z_\kappa|^r\big)\,\mathrm{d}y\\
&\le 2^{r-1}\int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r}\big(2^{r-1}|\nabla\zeta_\kappa|^{r-1} + 2^r|\nabla z_\kappa|^{r-1}\big)|\nabla\zeta_\kappa|\,\mathrm{d}y\\
&\le \tfrac{2^{2r-2}}{r}\kappa\|\nabla\zeta_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} + \tfrac{2^{2r-1}}{r}\kappa^{\frac{r-1}{r}}\|\nabla z_\kappa\|^{r-1}_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)}\kappa^{\frac{1}{r}}\|\nabla\zeta_\kappa\|_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \;\to 0\,,
\end{aligned}$$

since $\kappa\|\nabla\zeta_\kappa\|^r_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \to 0$ by construction and $\kappa^{\frac{r-1}{r}}\|\nabla z_\kappa\|^{r-1}_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \le C$ due to the properties of stable sequences.

Step 3: Due to $\hat{z}_\kappa = z_\kappa\zeta_\kappa$ we find that $\operatorname{supp} S_{\mathrm{G}}\hat{z}_\kappa = \operatorname{supp} S_{\mathrm{G}}z_\kappa \cap \operatorname{supp} S_{\mathrm{G}}\zeta_\kappa$. Hence it suffices that $[\![\hat{u}_\kappa]\!] = 0$ on $\operatorname{supp} S_{\mathrm{G}}\zeta_\kappa$. Since $p > d$ we can apply Corollary 4.3.13 and set

$$\hat{u}_\kappa := \hat{u}_{\mathrm{sym}} + \hat{u}^{\rho(\kappa)}_{\mathrm{anti}}\,,$$

where $\rho(\kappa) = \kappa^{\frac{1}{2rd}}$ was obtained in (4.78). From (4.70), the definition of $\hat{u}^{\rho(\kappa)}_{\mathrm{anti}}$ in Corollary 4.3.13 and Lemma 4.3.1 we infer that $S_{\mathrm{G}}\hat{z}_\kappa[\![\hat{u}_\kappa]\!] = 0$ a.e. on Γ_{C}. By Corollary 4.3.13 it holds that $\hat{u}_\kappa \to \hat{u}$ strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ and $(\hat{u}_\kappa + g(t_\kappa)) \to (\hat{u} + g(t))$ strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ follows from assumption (4.4). Due to assumption (4.5(H2)) we can prove by Taylor's expansion that $\int_{\Omega_-\cup\Omega_+} W(e(\hat{u}_\kappa + g(t_\kappa)))\,\mathrm{d}x \to \int_{\Omega_-\cup\Omega_+} W(e(\hat{u} + g(t)))\,\mathrm{d}x$.

This finishes the proof of the lim sup-estimate (4.74). ■

The above construction of the joint recovery sequence gives an idea for a suitable recovery sequence to show that the energy functionals $\mathcal{E}^\kappa(t,\cdot)$ Γ-converge to $\mathcal{E}(t,\cdot)$: Defining $\hat{u}_\kappa$ by Corollary 4.3.13 and $\hat{z}_\kappa := \hat{z} * \eta_\kappa$ with η_κ defined by (4.69) and (4.78) yields the Γ-lim sup-inequality, since $\frac{\kappa}{r}\|\nabla\hat{z}_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \to 0$.

Corollary 4.3.15 (Γ-convergence of $\mathcal{E}^\kappa(t,\cdot)$) *Keep $t \in [0,T]$ fixed. Let $\mathcal{E}^\kappa$ and $\mathcal{E}$ be defined by (4.27) and (4.58) such that the assumptions (4.4) and (4.5) hold true. Then, for all $\hat{q} \in \mathcal{Q}$ there exists a recovery sequence $\hat{q}_\kappa \xrightarrow{\mathcal{T}} \hat{q}$ such that*

$$\limsup_{\kappa\to 0} \mathcal{E}^\kappa(t,\hat{q}_\kappa) \leq \mathcal{E}(t,\hat{q})\,. \tag{4.80}$$

Hence, together with the lower Γ-limit stated in Theorem 4.3.7, we have $\mathcal{E}^\kappa(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}(t,\cdot)$.

Remark 4.3.16 *The product ansatz used in the proof of Theorem 4.3.14 for the construction of $\hat{z}_\kappa$ cannot be applied in the settings of partial damage or gradient delamination (see $(\mathcal{Q},\mathcal{E},\mathcal{D})$ defined by (3.8), (3.9), (3.5), (3.2) and $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$ defined by (4.15), (4.18), (4.20)). In order to make sure that $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) \to \mathcal{R}(\hat{z}-z)$ it is required to put*

$$\hat{z}_\varepsilon := z_\varepsilon \tfrac{\hat{z}}{z}\,.$$

But $\frac{\hat{z}}{z}$ not necessarily is a $W^{1,r}$-function, so that mollifiers have to be used. Their gradients will blow up as $\varepsilon \to 0$ and since both models contain the damage/delamination gradient this blow up cannot be compensated in the way it was exploited in the passage from gradient delamination to Griffith-type delamination.

Remark 4.3.17 *Although the product ansatz from the proof of Theorem 4.3.14 supplies a method to suppress the delamination gradient, one cannot prove with this technique that the partial damage models given by (3.8), (3.9), (3.5), (3.2), which include the damage gradient, converge to a damage model without the damage gradient. In contrast to the delamination models $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$ and $(\mathcal{Q},\mathcal{E},\mathcal{R})$ the model for partial damage includes an energy term which usually consists of the product of the damage variable and the strain tensor, i.e. the energy density may take the form $W(e,z) = z|e|^p$. Thus, in order to guarantee the weak sequential lower semicontinuity of the energy functional, the weak*-convergence of the damage variables in $L^\infty(\Omega)$ is not sufficient. Since the model only provides weak convergence of the strain tensors in $L^p(\Omega,\mathbb{R}^{d\times d})$ it is required that the damage variables converge strongly in $L^{p'}(\Omega)$. For this, the regularizing gradient term is needed.*

4.4 Simultaneous Convergence

One can merge the results from Sections 4.2 and 4.3 to a simultaneous convergence of solutions of $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$, whose existence was claimed in Proposition 4.1.3, directly to solutions of the Griffith-type delamination problem $(\mathcal{Q},\mathcal{E},\mathcal{R})$.

Theorem 4.4.1 (Simultaneous convergence) *Let the assumptions of Theorems 4.2.1 and 4.3.4 hold. For all $\varepsilon \in (0,\varepsilon_0]$, $\kappa \in (0,\kappa_0]$ let $q_\varepsilon^\kappa = (u_\varepsilon^\kappa, z_\varepsilon^\kappa)$ denote energetic solutions of $(\mathcal{Q},\mathcal{E}_\varepsilon^\kappa,\mathcal{R},q_0^{\varepsilon\kappa})$. There is a function $G:\mathbb{R}^+\to\mathbb{R}^+$ so that every subsequence $(u_\varepsilon^\kappa(t), z_\varepsilon^\kappa(t))_{\varepsilon\in(0,\varepsilon_0],\kappa\in(0,\kappa_0],\varepsilon\le G(\kappa)}$ of energetic solutions, which converges for all $t\in[0,T]$ with respect to the topology $\mathcal{T}$, has an energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{R},q_0)$ as its limit.*

Proof: In [MRS08, Th. 3.3] it is stated that there exist measurable energetic solutions for rate independent systems, which satisfy the properties proven in Theorems 4.2.2, 4.3.5 and Lemmata 4.2.3, 4.3.6. Hence the uniform bounds (4.57a) provide temporal L^∞-bounds. Furthermore, the uniform bound (4.57b) implies that $\|z_\varepsilon^\kappa\|_{BV([0,T],L^1(\Omega_\mathrm{D}))} \le C$ uniformly for all $\varepsilon\in(0,\varepsilon_0]$, $\kappa\in(0,\kappa_0]$. Additionally the uniform boundedness of the energies provides that $\|z_\varepsilon^\kappa\|_{L^\infty([0,T],W^{1,r}(\Omega_\mathrm{D}))} \le \big(\frac{r}{\kappa}(E+\mathcal{L}^d(\Omega_\mathrm{D}))\big)^{\frac{1}{r}}$ for all $\varepsilon\in(0,\varepsilon_0]$ and all $\kappa\in(0,\kappa_0]$, which is a κ-dependent bound. In contrast to the Sobolev-spaces involved in (4.57a) the L^1-space occurring in the BV-estimate above has no separable predual. Thus, we enlarge $\mathcal{M}([0,T];L^1(\Omega_\mathrm{D}))$ to $\mathcal{M}([0,T]\times\overline{\Omega_\mathrm{D}})$ so that the rate of damage and delamination $\partial_t z_\varepsilon^\kappa$ and $\partial_t z^\kappa$ are a-priori bounded in $\mathrm{C}([0,T]\times\overline{\Omega_\mathrm{D}})^*$. Moreover, from now on we denote $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ equipped with the topology defined in (4.22) by $W^{1,p}_\mathcal{T}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$. Hence, for the preduals of the spaces $L^\infty([0,T],W^{1,p}_\mathcal{T}(\Omega_-\cup\Omega_+,\mathbb{R}^d))$, $L^\infty([0,T]\times\Omega_\mathrm{D})$, $L^\infty([0,T],W^{1,r}(\Omega_\mathrm{D}))$ and $\mathcal{M}([0,T]\times\overline{\Omega_\mathrm{D}})$ we find a countable dense set. In this way, we ensure that all the occurring weak*-topologies are compact and metrizable if restricted on any closed ball centered at 0 of finite radius referring to the norms in (4.57a) and (4.57b). In view of (4.57a), (4.57b) and the above estimates, we can set the radii equal to $\tilde{E}$ for $\|\cdot\|_{L^\infty([0,T],W^{1,p}(\Omega_\pm^\nu,\mathbb{R}^d))}$, equal to 1 for $\|\cdot\|_{L^\infty([0,T]\times\Omega_\mathrm{D})}$, equal to C for $\|\cdot\|_\mathcal{M}$ and equal to $\big(\frac{r}{\kappa}(E+\mathcal{L}^d(\Omega_\mathrm{D}))\big)^{\frac{1}{r}}$ for $\|\cdot\|_{L^\infty([0,T],W^{1,r}(\Omega_\mathrm{D}))}$. These closed balls are used to construct a weak metric on the corresponding spaces, i.e. D_p, D_∞, D_r^κ and $D_\mathcal{M}$ are the weak* metrics on $L^\infty([0,T],W^{1,p}_\mathcal{T}(\Omega_-\cup\Omega_+,\mathbb{R}^d))$, $L^\infty([0,T]\times\Omega_\mathrm{D})$ or $L^\infty([0,T],W^{1,r}(\Omega_\mathrm{D}))$ (restricted to the κ-dependent ball) and $\mathcal{M}([0,T]\times\overline{\Omega_\mathrm{D}})$.

We now introduce $\mathrm{S}=\{q:[0,T]\to\mathcal{Q}_\mathrm{G}\,|\,q$ is an energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{R},q_0)\}$. By Proposition 4.1.3 and Theorems 4.2.1, 4.3.4 it is ensured that $\mathrm{S}\neq\emptyset$. For $\rho>0$ we define

$$\mathrm{N}^1_\rho(\mathrm{S}) := \{\hat{q}=(\hat{u},\hat{z}):[0,T]\to\mathcal{Q}\,|\,\exists\, q=(u,z)\in\mathrm{S}: D(q,\hat{q})<\rho\},$$

where $D(q,\hat{q}) = D_p(u,\hat{u})+D_\infty(z,\hat{z})+D_\mathcal{M}(\partial_t z,\partial_t\hat{z})$.

Similarly we put $\mathrm{S}^\kappa:=\{q:[0,T]\to\mathcal{Q}_\mathrm{C}\,|\,q$ is an energetic solution of $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R},q_0^\kappa)\}$, which are nonempty by Proposition 4.1.3 and Theorem 4.2.1, and we prove that there is an index κ_ρ so that $\mathrm{S}^\kappa\subset\mathrm{N}^1_\rho(\mathrm{S})$ for all $\kappa\le\kappa_\rho$. Assume the contrary, i.e. for all $\kappa>0$ there is a particular $q^\kappa\in\mathrm{S}^\kappa$, but $q^\kappa\notin\mathrm{N}^1_\rho(\mathrm{S})$. Due to the estimates (4.57a), (4.57b) the sequence $(q^\kappa)_{\kappa\in(0,\kappa_0]}$ contains a subsequence which converges in the topology $\mathcal{T}$ to some $q\in\mathrm{S}$ by the same arguments as in the proof of Lemma 4.2.10. In particular this implies for this subsequence that there is an index κ_ρ such that all elements with $\kappa\le\kappa_\rho$ satisfy $q^\kappa\in\mathrm{N}^1_\rho(\mathrm{S})$, in contradiction to the assumption.

For all $\rho>0$ and all $\kappa\in(0,\kappa_0]$ we now introduce the neighborhoods

$$\mathrm{N}^2_\rho(\mathrm{S}^\kappa) := \{\hat{q}=(\hat{u},\hat{z}):[0,T]\to\mathcal{Q}\,|\,\exists\, q=(u,z)\in\mathrm{S}^\kappa: D_\kappa(q,\hat{q})<\rho\},$$

where $D_\kappa(q,\hat{q}) = D_p(u,\hat{u}) + D_r^\kappa(z,\hat{z}) + D_\infty(z,\hat{z}) + D_{\mathcal{M}}(\partial_t z, \partial_t \hat{z})$. Due to $\mathrm{S}^\kappa \subset \mathrm{N}^1_{\frac{\varrho}{2}}(\mathrm{S})$ for all $\kappa \le \kappa_{\frac{\varrho}{2}}$, it holds that $\mathrm{N}^2_{\frac{\varrho}{2}}(\mathrm{S}^\kappa) \subset \mathrm{N}^1_\rho(\mathrm{S})$ for all $\kappa \le \kappa_{\frac{\varrho}{2}}$, since $\mathrm{N}^1_{\frac{\varrho}{2}}(q_\kappa) \subset \mathrm{N}^1_\rho(q)$ for all $q_\kappa \in \mathrm{N}^1_{\frac{\varrho}{2}}(q)$ and since D is coarser than D_κ for all $\kappa > 0$.

Now we define $\mathrm{S}^\kappa_\varepsilon := \{q : [0,T] \to \mathcal{Q}_\varepsilon \,|\, q \text{ is an energetic solution of } (\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R}, q_0^{\kappa\varepsilon})\}$. With the same contradiction argument we find that there is an index $\varepsilon^\kappa_{\frac{\varrho}{2}}$ such that $(u_\varepsilon, z_\varepsilon) \in \mathrm{N}^2_{\frac{\varrho}{2}}(\mathrm{S}^\kappa)$ if $\varepsilon \le \varepsilon^\kappa_{\frac{\varrho}{2}}$.

For fixed $\kappa \in (0, \kappa_0]$ we have to show that there is $G(\kappa) > 0$ such that it holds even $\mathrm{S}^\kappa_\varepsilon \subset \mathrm{N}^2_{\frac{\kappa}{2}}(\mathrm{S}^\kappa)$ for all $\varepsilon \le G(\kappa)$. For this, assume the contrary, i.e. for every $G > 0$ there is some solution q_{ε_G} with $(u_{\varepsilon_G}, z_{\varepsilon_G}) \notin \mathrm{N}^2_{\frac{\kappa}{2}}(\mathrm{S}^\kappa)$. But q_{ε_G} are energetic solutions of $(\mathcal{Q}, \mathcal{E}^\kappa_{\varepsilon_G}, \mathcal{R}, q_0^{\kappa\varepsilon_G})$ for all $G > 0$ and thus satisfy the uniform bounds from above, so that by Lemma 4.2.10 and Theorem 4.2.1 there is a further subsequence that converges in the topology $\mathcal{T}$ for all $t \in [0,T]$ to an energetic solution q of $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R}, q_0^\kappa)$. This states a contradiction to the assumption, since it means that all the elements of this further subsequence are contained in $\mathrm{N}^2_{\frac{\kappa}{2}}(q)$ from a particular index on.

Now the considerations from above imply $\mathrm{S}^\kappa_\varepsilon \subset \mathrm{N}^2_{\frac{\kappa}{2}}(\mathrm{S}^\kappa) \subset \mathrm{N}^2_{\frac{\varrho}{2}} \subset \mathrm{N}^1_\rho(\mathrm{S})$ for all $\varepsilon < G(\kappa)$ and all $\kappa < \kappa_{\frac{\varrho}{2}}$ which proves the existence both of $G : \mathbb{R}_+ \to \mathbb{R}_+$ and of a subsequence of energetic solutions $(q^\kappa_\varepsilon)_{\varepsilon \in (0,\varepsilon_0], \kappa \in (0,\kappa_0], \varepsilon \le G(\kappa)}$, which converges in the topology $\mathcal{T}$ to an energetic solution q of the limit system. ■

Chapter 5

On the Temporal Regularity of Energetic Solutions

The properties (2.60(S)) and (2.60(E)) provide a general, but only very weak result on the temporal regularity of an energetic solution $q = (u, z) : [0, T] \to \mathcal{Q}$, namely:

$$z \in \mathrm{BV}([0, T], L^1(\Omega)) \cap L^\infty([0, T], W^{1,r}(\Omega)) \quad \text{and} \quad u \in L^\infty([0, T], W^{1,p}(\Omega, \mathbb{R}^d)) \,.$$

The BV-estimate is due to $\mathrm{Var}_{L^1(\Omega)}(z, [r, s]) \leq \frac{1}{\varrho_0}\mathrm{Diss}_{\mathcal{R}}(z, [r, s]) < \infty$, which is a consequence of the energy balance. In fact, the monotonicity $z(t_1, x) \geq z(t_2, x)$ for $t_1 < t_2$ implies $\mathrm{Var}_{L^1(\Omega)}(z, [r, s]) = \int_\Omega \big(z(r, x) - z(s, x)\big)\,\mathrm{d}x \leq \mathcal{L}^d(\Omega)$. The L^∞-bound in $W^{1,p}(\Omega, \mathbb{R}^d) \times W^{1,r}(\Omega)$ is due to the energy bound $\mathcal{E}(t, q(t)) \leq E_*$, which results from stability.

It was first obtained in [MT04] that the temporal regularity of an energetic solution can be improved, if $\mathcal{E}$ has additional convexity properties. If for all $t \in [0, T]$ the energy functional $\mathcal{E}(t, \cdot)$ is strictly convex, one obtains that all energetic solutions are continuous in time. This is due to the fact that strict convexity implies the uniqueness of minimizers, which is already sufficient for temporal continuity. We develop this result in Section 5.1.

Furthermore, it was proven in [MT04] that even Lipschitz continuity can be achieved for energy functionals that are uniformly convex, i.e. for all $\theta \in [0, 1]$, $q_1, q_2 \in \mathcal{Q}$ it holds

$$\mathcal{E}(t, \theta q_1 + (1-\theta) q_2) \leq \theta \mathcal{E}(t, q_1) + (1-\theta)\mathcal{E}(t, q_2) - c\theta(1-\theta)\|q_1 - q_2\|_{\mathcal{Q}}^\alpha \tag{5.1}$$

with constants $c > 0$ and $\alpha = 2$. In Section 5.2 we will see that (5.1) depends on the choice of $\|\cdot\|_{\mathcal{Q}}$ and that uniform convexity is not restricted to the exponent $\alpha = 2$. We provide properties of energy densities W that lead to uniform convexity on sublevels with an exponent $\alpha \geq 2$. In such a situation we prove Hölder continuity in time. Moreover we demonstrate in Section 5.3 that the temporal regularity of energetic solutions can be improved if estimate (5.1) is derived with respect to the norm of a bigger Banach space $\mathcal{V} \supset \mathcal{Q}$.

Most of the results and examples presented in this chapter will appear in [TM10]. Moreover, parts of the examples in Section 5.3.3 were carried out for [GKNT09].

Before we go into the analysis we provide an example of an energy density W that satisfies all the assumptions from (3.6) and additionally the uniform convexity conditions that will be used later. The fact that joint convexity is compatible with damage models was first exploited in [Rou08].

Example 5.0.2 *A simple example for a suitable W generating a uniformly convex energy functional is given by*

$$W(x,e,z) = \frac{1}{2\big(1+\eta(1-z)\big)^{\gamma}}\, e{:}\mathbb{B}{:}e + \frac{a}{2}\, z^2,$$

where $\eta, a > 0$, $\gamma \in (0,1)$, and $\mathbb{B}$ is a symmetric and positive definite linear operator on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. Such densities are discussed in detail in Section 5.3.1.

5.1 Temporal Continuity

The first result provides continuity in time, which means that energetic solutions cannot have jumps. The idea is to use that under the assumption of strict convexity energetic solutions $q : [0,T] \to \mathcal{Q}$ have weak left and right limits $q_+(t)$ and $q_-(t)$ for all $t \in [0,T]$. Moreover, it can be shown that $q_-(t)$, $q(t)$ and $q_+(t)$ have to be minimizers of the functional $q \mapsto \mathcal{E}(t,q) + \mathcal{D}(q_-(t),q)$. By strict convexity one then concludes that all three values must coincide and weak continuity follows. Strong continuity is concluded by an argument of Visintin (cf. [Vis84]), which allows us to convert weak convergence and energy convergence into strong convergence by exploiting the strict convexity once again.

We now develop the details. We first provide a result that does not explicitly use the strict convexity of $\mathcal{E}(t,\cdot)$; for stable states $q = (u,z) \in \mathcal{S}(t)$ it only requires the uniqueness of the minimizer of $\mathcal{E}(t,\cdot,z)$, which then is u.

Lemma 5.1.1 (Jump relations) *Assume that $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfies (2.62), (2.65) as well as (2.66). Moreover,*

$$\forall\, t \in [0,T]\ \forall\, q = (u,z) \in \mathcal{S}(t): \quad \{u\} = \operatorname*{Argmin}_{\widetilde{u}\in\mathcal{U}} \mathcal{E}(t,\widetilde{u},z). \tag{5.2}$$

Then, for all $t \in [0,T]$ the weak limits $q_-(t) = \text{w-lim}_{\tau\to t^-} q(\tau)$ and $q_+(t) = \text{w-lim}_{\tau\to t^+} q(\tau)$ (where $q_-(0) := q(0)$ and $q_+(T) := q(T)$) exist and satisfy

$$\mathcal{E}(t,q_-(t)) = \mathcal{E}(t,q(t)) + \mathcal{D}(q_-(t),q(t)), \tag{5.3a}$$
$$\mathcal{E}(t,q(t)) = \mathcal{E}(t,q_+(t)) + \mathcal{D}(q(t),q_+(t)), \tag{5.3b}$$
$$\mathcal{D}(q_-(t),q_+(t)) = \mathcal{D}(q_-(t),q(t)) + \mathcal{D}(q(t),q_+(t)). \tag{5.3c}$$

Proof: From $\mathrm{Diss}_{\mathcal{D}}(z,[0,T]) < \infty$ we conclude that the limits $z_-(t) = \text{w-lim}_{\tau\to t^-} z(\tau)$ and $z_+(t) = \text{w-lim}_{\tau\to t^+} z(\tau)$ exist, cf. [MM05]. Now, fix t, choose $v_\pm \in \mathcal{U}$ and subsequences

$(t_k^{\pm})_{k\in\mathbb{N}}$ such that $u(t_k^{\pm}) \rightharpoonup v_{\pm}$, where $t_k^{\pm} \to t$ with $\pm(t_k^{\pm}-t) > 0$. Then, (2.66(C2)) guarantees $(v_{\pm}, z_{\pm}(t)) \in \mathcal{S}(t)$. Exploiting the assumption (5.2) we find that $v^{\pm}$ are uniquely determined and cannot depend on the subsequence. Hence, the function $u : [0,T] \to \mathcal{U}$ has the desired left-hand and right-hand limits $u_{\pm}(t)$ in the weak sense.

To obtain the desired energy identities (5.3) we exploit the energy balance

$$\mathcal{E}(s, q(s)) + \mathrm{Diss}_{\mathcal{D}}(z, [r, s]) = \mathcal{E}(r, q(r)) + \int_r^s \partial_\tau \mathcal{E}(\tau, q(\tau))\,\mathrm{d}\tau, \quad 0 \le r < s \le T.$$

For relation (5.3a) we let $s = t$ and consider $r \to t^-$. Using the obvious relation $\mathrm{Diss}_{\mathcal{D}}(z, [r, t]) \to \mathcal{D}(z_-(t), z(t))$ we find

$$\mathcal{E}(t, q(t))+\mathcal{D}(z_-(t), z(t)) \le \limsup_{r\to t^-} \mathcal{E}(r, q(r)) \le \mathcal{E}(t, q_-(t)) \le \mathcal{E}(t, q(t))+\mathcal{D}(z_-(t), z(t)),$$

where the second estimate follows from the stability of $q(r)$ at time r as $r \to t^-$, while the third estimate is due to the stability of $q_-(t)$. This establishes (5.3a).

The second relation (5.3b) follows by setting $r = t$ and taking the limit $s \to t^+$:

$$\begin{aligned}\mathcal{E}(t, q_+(t))+\mathcal{D}(z(t), z_+(t)) &\le \liminf_{s\to t^+} \mathcal{E}(s, q(s))+\mathcal{D}(z(t), z(s)) = \mathcal{E}(t, q(t)) + 0\\ &\le \mathcal{E}(t, q_+(t))+\mathcal{D}(z(t), z_+(t)),\end{aligned}$$

where we first used lower semicontinuity (2.62(E1)), then the energy balance, and finally the stability of $q(t)$. Thus, relation (5.3b) holds.

The third relation (5.3c) follows from (2.65(D1)) and the first two relations:

$$\mathcal{D}(z_-(t), z_+(t)) \le \mathcal{D}(z_-(t), z(t))+\mathcal{D}(z(t), z_+(t)) = \mathcal{E}(t, q_-(t))-\mathcal{E}(s, q_+(s)) \le \mathcal{D}(z_-(t), z_+(t)),$$

where the last estimate uses the stability of $q_-(t)$. ■

The next result provides the continuity of the energetic solutions if $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ is strictly convex. In fact, the proof only uses the weaker property that for stable states $q=(u, z)\in\mathcal{S}(t)$ the functional $\widetilde{q} \mapsto \mathcal{E}(t, \widetilde{q}) + \mathcal{D}(z, \widetilde{z})$ has a unique minimizer, see [MR08].

Theorem 5.1.2 (Continuity by strict convexity) *Let the assumptions of the existence theorem 3.1.1 hold. Moreover, assume that $W(x,\cdot,\cdot) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [z_\star, 1] \to \mathbb{R}$ is strictly convex for a.a. $x \in \Omega$. Then, any energetic solution $q : [0,T] \to \mathcal{Q}$ is (norm-) continuous with respect to time, i.e. $q \in \mathrm{C}^0([0,T], \mathcal{Q})$.*

Proof: We first observe that for each $t \in [0,T]$ the functional $\mathcal{E}(t,\cdot)$ is strictly convex, since it results from the strictly convex density $(e, z, A) \mapsto W(x, e+e_D(t, x), z)+\frac{\kappa}{r}|A|^r$ and the linear term $l(t)$ with arguments $(e, z, A)=(e(u), z, \nabla z)$ depending linearly on $(u, z)\in\mathcal{Q}$. Moreover, for each $z\in\mathcal{Z}$ the mapping $\widetilde{z} \mapsto \mathcal{D}(z, \widetilde{z})$ is convex. Thus, for each $t\in[0,T]$ the functional $\mathcal{Q} \ni \widetilde{q} = (\widetilde{u}, \widetilde{z}) \mapsto \mathcal{E}(t, \widetilde{q}) + \mathcal{D}(z_-(t), \widetilde{z})$ has a unique minimizer.

Exploiting the jump relations (5.3) we easily find that $q_-(t)$, $q(t)$, and $q_+(t)$ all provide the same value $\mathcal{E}(t, q_-(t))$, which must be the global minimum by the stability of $q_-(t)$. Hence, the three values must coincide, and Lemma 5.1.1 allows us to conclude weak continuity of $q : [0, T] \to \mathcal{Q}$, namely $q(\tau) \rightharpoonup q(t)$ for $\tau \to t$.

Applying the jump relations (5.3) once again we have $\mathcal{E}(\tau, q(\tau)) \to \mathcal{E}(t, q(t))$ for $\tau \to t$. Fixing t and employing Lipschitz estimate (2.63) we also get $\mathcal{E}(t, q(\tau)) \to \mathcal{E}(t, q(t))$. Thus, we may apply Proposition 5.1.3 below to the family $V(\tau) = (e(u(\tau)) + e_D(t), z(\tau), A(\tau))$, which provides that $(e(u(\tau)) + e_D(t), z(\tau), \nabla z(\tau)) \to (e(u(\tau)) + e_D(t), z(\tau), \nabla z(t))$ strongly in $L^p(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times L^r(\Omega) \times L^r(\Omega, \mathbb{R}^d)$. Using Korn's inequality (3.12) the desired strong convergence $q(\tau) \to q(t)$ in $\mathcal{Q}$ follows. ∎

The following result was used in the proof above. It is a variant of [Vis84, §2 & Th. 8].

Proposition 5.1.3 *Let Ω satisfy* (3.6(A1)) *and let $\mathbf{C}$ be a nonempty, closed, convex subset of $\mathcal{V} := L^p(\Omega, \mathbb{R}^N)$, where $1 \le p < \infty$ and $N \ge 1$. Assume that $\phi : \Omega \times \mathbb{R}^N \to [0, \infty]$ is a Carathéodory function such that $\phi(x, \cdot)$ is strictly convex on $\mathbb{R}^N$ for a.e. $x \in \Omega$. For $V \in \mathbf{C}$ set $\Phi(V) := \int_\Omega \phi(x, V(x))\,\mathrm{d}x$. Then, the following holds:*

$$\left.\begin{array}{l} V_k \rightharpoonup V \ \textit{in}\ \mathcal{V}, \\ \Phi(V_k) \to \Phi(V), \end{array}\right\} \implies \left\{\begin{array}{l} V_k \to V \ \textit{in}\ \mathcal{V}, \\ \phi(\cdot, V_k(\cdot)) \to \phi(\cdot, V(\cdot)) \ \textit{in}\ L^1(\Omega). \end{array}\right.$$

5.2 Temporal Hölder and Lipschitz Continuity

In this section we generalize the ideas developed in [MT04, MR07], where Lipschitz continuity with respect to time was derived. Our generalization has two aspects. First we emphasize that the convexity properties can be formulated with respect to a norm $\|\cdot\|_{\mathcal{V}}$ that may differ significantly from that underlying the state space $\mathcal{Q}$. In particular, if $\mathcal{Q}$ is a closed, convex subset of a Banach space $\mathcal{X}$, which specifies the topology for the existence analysis, and if $\mathcal{X}$ is chosen as small as possible under preservation of the coercivity of $\mathcal{E}$, see (2.62(E1)), it may be advantageous to investigate the temporal regularity of energetic solutions with respect to the norm of an even bigger Banach space $\mathcal{V} \supset \mathcal{X} \supset \mathcal{Q}$, since temporal regularity may improve. Second we generalize the notion of uniform convexity by allowing for a weaker lower bound in (5.4). Previous work asked $\alpha = 2$ and $\beta = 1$ and enforced the condition on the entire space $\mathcal{Q}$, while we only pose it on sublevels.

After establishing the main abstract result in Theorem 5.2.1, we will show how the main assumptions can be satisfied for integral functionals in Lemma 5.2.2. The effective use of the spaces $\mathcal{V}$ and $\mathcal{Q}$ will be demonstrated in Section 5.3.2, pages 112–115.

Theorem 5.2.1 (Temporal Hölder continuity) *Let $(\mathcal{Q},\mathcal{E},\mathcal{D})$ be a rate-independent system, where $\mathcal{Q}$ is a closed, convex subset of a Banach space $\mathcal{X}$. Let the energy sublevels be denoted with $L_E(t)=\{q\in\mathcal{Q}\,|\,\mathcal{E}(t,q)\le E\}$. Assume that there is a Banach space $\mathcal{V}$ and that there are constants $\alpha\ge 2$, $\beta\le 1$ such that for all E_* there exist constants $C_*,\ c_*>0$ so that for all $t\in[0,T]$, $q_0,q_1\in L_{E_\star}(t)$ and all $\theta\in[0,1]$ the following holds:*

$$\begin{aligned}\mathcal{E}(t,q_\theta)+&\mathcal{D}(z_0,z_\theta)+c_*\theta(1-\theta)\|q_1-q_0\|_{\mathcal{V}}^{\alpha}\\ &\le(1-\theta)\big(\mathcal{E}(t,q_0)+\mathcal{D}(z_0,z_0)\big)+\theta\big(\mathcal{E}(t,q_1)+\mathcal{D}(z_0,z_1)\big)\end{aligned}\tag{5.4a}$$

$$|\partial_t\mathcal{E}(t,q_1)-\partial_t\mathcal{E}(t,q_0)|\le C_*\|q_1-q_0\|_{\mathcal{V}}^{\beta},\tag{5.4b}$$

where $(u_\theta,z_\theta)=q_\theta=(1-\theta)q_0+\theta q_1$.

Then, any energetic solution $q:[0,T]\to\mathcal{Q}$ of $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is Hölder continuous from $[0,T]$ to $\mathcal{V}$ with the exponent $1/(\alpha-\beta)$, i.e. there is a constant $C_{\mathrm{H}}>0$ such that

$$\|q(s)-q(t)\|_{\mathcal{V}}\le C_{\mathrm{H}}|t-s|^{1/(\alpha-\beta)}\quad\textit{for all } s,t\in[0,T]\,.\tag{5.5}$$

Proof: We proceed in three steps. First we derive an improved stability condition (2.60(S)), where an additional term of the form $c_*\theta(1-\theta)\|q_1-q_0\|_{\mathcal{V}}^{\alpha}$ appears on the left-hand side. Second, following [MT04, MR07], we derive an estimate for $\|q(s)-q(t)\|_{\mathcal{V}}$ and finally we use a differential inequality to obtain (5.5).

Step 1: *Improved stability estimate*
Choose E_* such that $\mathcal{E}(t,q(t))\le E_*$ for all t. For fixed $s,t\in[0,T]$ we apply (5.4a) with $q_0=q(t)$ and $q_1=q(s)$. By the stability of $q(t)$ we find

$$\mathcal{E}(t,q_0)\le\mathcal{E}(t,q_\theta)+\mathcal{D}(z_,z_\theta)\le(1-\theta)\mathcal{E}(t,q_0)+\theta\big(\mathcal{E}(t,q_1)+\mathcal{D}(z_0,z_1)\big)-c_*\theta(1-\theta)\|q_1-q_0\|_{\mathcal{V}}^{\alpha}\,.$$

After subtracting $\mathcal{E}(t,q_0)$ from both sides we may divide by θ and pass to the limit $\theta\to 0^+$. Recalling $q_0=q(t)$ and $q_1=q(s)$ this leads to

$$\mathcal{E}(t,q(t))+c_*\|q(t)-q(s)\|_{\mathcal{V}}^{\alpha}\le\mathcal{E}(t,q(s))+\mathcal{D}(z(t),z(s))\,,\tag{5.6}$$

which is the desired improved stability estimate. (In fact, in place of $q(s)$ we could have taken any $\widetilde{q}$ with $\mathcal{E}(t,\widetilde{q})\le E_*$; or vice versa, we could have weakened condition (5.4) by assuming it only for stable states.)

Step 2: *Estimate for $\|q(t)-q(s)\|_{\mathcal{V}}$*
Now we assume $0\le s\le t\le T$ and interchange the role of s and t in (5.6). Employing $\mathcal{D}(z(s),z(t))\le\mathrm{Diss}_{\mathcal{D}}(z;[s,t])$ and the energy balance we find

$$\begin{aligned}c_*\|q(t)-q(s)\|_{\mathcal{V}}^{\alpha}&\le\mathcal{E}(s,q(t))+\mathcal{D}(z(s),z(t))-\mathcal{E}(s,q(s))\\ &\le\mathcal{E}(s,q(t))-\mathcal{E}(t,q(t))+\mathcal{E}(t,q(t))+\mathrm{Diss}_{\mathcal{D}}(z;[s,t])-\mathcal{E}(s,q(s))\\ &=\int_s^t\big(\partial_\xi\mathcal{E}(\xi,q(t))-\partial_\xi\mathcal{E}(\xi,q(\xi))\big)\,\mathrm{d}\xi\le\int_s^t C_*\|q(t)-q(\xi)\|_{\mathcal{V}}^{\beta}\,\mathrm{d}\xi,\end{aligned}$$

where we used (5.4b) in the last estimate.

Step 3: *Hölder estimate*
Putting $h(\tau) := \int_{t-\tau}^{t} \|q(\xi) - q(t)\|_{\mathcal{V}}^{\beta}\,\mathrm{d}\xi$ for $\tau \in [0, t-s]$ yields $h'(\tau) \leq \big(\frac{C_*}{c_*}h(\tau)\big)^{\beta/\alpha}$. Using $h(0) = 0$ leads to $h(\tau) \leq C_1\tau^{\alpha/(\alpha-\beta)}$ with a constant C_1 depending only on C_*, c_*, α and β. Hence we conclude

$$\|q(s) - q(t)\|_{\mathcal{V}} = h'(t-s)^{1/\beta} \leq \Big(\frac{C_*}{c_*}h(t-s)\Big)^{1/\alpha} \leq \Big(\frac{C_*C_1}{c_*}h(t-s)\Big)^{1/\alpha}(t-s)^{1/(\alpha-\beta)},$$

which is the desired result. ∎

The lemma below is useful to establish the assumptions in (5.4) for integral functionals.

Lemma 5.2.2 (On the convexity assumptions)
(A) *Assume that $\mathcal{D}(z_0,\cdot) : \mathcal{Z} \to [0,\infty]$ and $\mathcal{C} : \mathcal{Q} \to \mathbb{R}_\infty$ are convex and that $\mathcal{W} : \mathcal{Q} \to \mathbb{R}_\infty$ satisfies the following:*

$$\begin{aligned}&\forall\, E_*\ \exists\, C_W, c_w > 0\ \forall\, q_0, q_1 \text{ with } \mathcal{W}(q_0), \mathcal{W}(q_1) \leq w_*\ \forall\, \theta \in [0,1] :\\ &\mathcal{W}((1-\theta)q_0+\theta q_1) + c_w\theta(1-\theta)\|q_1-q_0\|_{\mathcal{V}}^{\alpha} \leq (1-\theta)\mathcal{W}(q_0) + \theta\mathcal{W}(q_1).\end{aligned} \tag{5.7}$$

Then, with $\mathcal{E}(t,\cdot) = \mathcal{W} + \mathcal{C}$ condition (5.4a) *holds.*

(B) *For $j \in \{1,\ldots,m\}$ let $V_j \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}\}$ and let $\mathbb{V} := \times_{j=1}^{m} V_j$. Assume that $\mathbb{W} : \Omega \times \mathbb{V} \to [0,\infty]$ is a Carathéodory function and that there exist $k \in \{0,1,...,m\}$, $C_1, c_1, c_0 > 0$ and $p_j > 1$ with $p_j \geq 2$ for $j \leq k$ and $p_j < 2$ for $j > k$ such that for a.a. $x \in \Omega$ and all b, b^0, $b^1 \in \mathbb{V}$ the following estimates hold:*

$$\mathbb{W}(x,b) \geq c_0\sum_{j=1}^{m}|b_j|^{p_j} - C_1, \tag{5.8a}$$

$$\begin{aligned}&\mathbb{W}(x,(1-\theta)b^0+\theta b^1) + c_1\theta(1-\theta)\Big(\textstyle\sum_{j=1}^{k}|b_j^1-b_j^0|^{p_j} + \sum_{j=k+1}^{m}\frac{|b_j^1-b_j^0|^2}{(1+\mathbb{W}(x,b^0)+\mathbb{W}(x,b^1))^{\gamma_j}}\Big)\\ &\quad\leq (1-\theta)\mathbb{W}(x,b^0) + \theta\mathbb{W}(x,b^1),\end{aligned} \tag{5.8b}$$

where $\gamma_j = (2-p_j)/p_j \in (0,1)$. Then, with $\mathcal{V} = \times_{j=1}^{m} L^{p_j}(\Omega)$ and $\mathcal{W}(v) = \int_\Omega \mathbb{W}(x,v(x))\,\mathrm{d}x$ the condition (5.7) *holds with $\alpha = \max\{p_1,...,p_k,2\}$.*

(C) *Assume that for a.a. $x \in \Omega$ we have $\mathbb{W}(x,\cdot) \in \mathrm{C}^1(\mathbb{V})$ and that there is a constant $c_* > 0$ such that the following holds for all $b^0, b^1 \in \mathbb{V}$:*

$$\begin{aligned}&\mathbb{W}(x,b^1) - \mathbb{W}(x,b^0) - \partial_b\mathbb{W}(b^0)\cdot(b^1-b^0)\\ &\geq c_*\sum_{j=1}^{k}|b_j^1-b_j^0|^{p_j} + c_*\sum_{j=k+1}^{m}\frac{|b_j^1-b_j^0|^2}{(1+\mathbb{W}(x,b^0)+\mathbb{W}(x,b^1))^{\gamma_j}}\end{aligned} \tag{5.9}$$

for p_j, γ_j as in part (B). *Then $\mathbb{W}$ satisfies* (5.8b).

(D) *Let $\mathbb{P} : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function satisfying*

$$|\mathbb{P}(x,b)| \leq C_2\mathbb{W}(x,b) + C_3, \tag{5.10a}$$

$$|\mathbb{P}(b^1)-\mathbb{P}(b^0)| \leq C_4\sum_{j=1}^{m}(1+\mathbb{W}(x,b^0)+\mathbb{W}(x,b^1))^{\delta_j}|b_j^1-b_j^0|, \tag{5.10b}$$

where $\delta_j = (p_j - 1)/p_j \in (0,1)$ *and* $\mathbb{W}$ *fulfills* (5.8). *For* $\mathcal{W}(v) < \infty$ *set* $\mathcal{P}(v) = \int_\Omega \mathbb{P}(x, v(x))\,\mathrm{d}x$. *Then, for each* E_* *there exists* $C_*^{\mathcal{P}}$ *such that for all* $v_0, v_1 \in \mathcal{V}$ *with* $\mathcal{W}(v_0)$, $\mathcal{W}(v_1) \le E_*$ *we have* $|\mathcal{P}(v_1) - \mathcal{P}(v_0)| \le C_*^{\mathbb{P}} \|v_1 - v_0\|_{\mathcal{V}}$.

Proof: Part (A) follows simply by using the convexity of $\mathcal{D}(z,\cdot)$ and $\mathcal{C}$ and adding it to the estimate provided by (5.7).

For Part (B) be first note that $\mathcal{W}(v^0)$, $\mathcal{W}(v^1) \le E_*$ together with (5.8a) implies that there is a constant Λ_* such that

$$\|v_j^n\|_{L^{p_j}(\Omega)} \le \Lambda_* \quad \text{for } n \in \{0,1\} \text{ and } j = 1, ..., m. \tag{5.11}$$

Setting $b^n = v^n(x)$ and integrating both sides of (5.8b) over the domain Ω it remains to estimate the left-hand side from below. For $j > k$ we derive a so-called reverse Hölder's inequality for the the quotient $u^2/N^{-\gamma}$ via

$$\int_\Omega u^{2/(1+\gamma)}\,\mathrm{d}x \le \Big(\int_\Omega u^2/N^\gamma\,\mathrm{d}x\Big)^{1/(1+\gamma)} \Big(\int_\Omega N\,\mathrm{d}x\Big)^{\gamma/(1+\gamma)}$$

where $u = |v_j^1 - v_j^0|$ and $N = 1 + \mathbb{W}(v^0) + \mathbb{W}(v^1)$. This provides the lower bound

$$\begin{aligned}&(1-\theta)\mathcal{W}(v^0) + \theta\mathcal{W}(v^1) - \mathcal{W}((1-\theta)v^0 + \theta v^1)\\ &\ge c_1\theta(1-\theta)\Big(\textstyle\sum_{j=1}^k \|v_j^1 - v_j^0\|_{L^{p_j}}^{p_j} + \sum_{j=k+1}^m \frac{\|v_j^1 - v_j^0\|_{L^{p_j}}^2}{(\mathcal{L}^d(\Omega) + 2E_*)^{\gamma_j}}\Big).\end{aligned}$$

Since $\alpha = \max\{p_1, ..., p_k, 2\}$ condition (5.7) follows from $\|v_j^1 - v_j^0\|_{L^{p_j}}^\rho \ge \|v_j^1 - v_j^0\|_{L^{p_j}}^\alpha/(2\Lambda_*)^{\alpha-\rho}$ for all $\rho \in \{p_1, ..., p_k, 2\}$ and for Λ_* from (5.11).

To establish Part (C) we let $b^\theta = (1-\theta)b^0 + \theta b^1$ and apply (5.9) with b^0 replaced by b^θ. Dropping x for notational simplicity and using $b^1 - b^\theta = (1-\theta)(b^1 - b^0)$ we find

$$\begin{aligned}&\mathbb{W}(b^1) - \mathbb{W}(b^\theta) - (1-\theta)\partial_b\mathbb{W}(b^\theta)\cdot(b^1 - b^0)\\ &\ge c_*\sum_{j=1}^k (1-\theta)^{p_j}|b_j^1 - b_j^0|^{p_j} + c_*(1-\theta)^2 \sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(b^1) + \mathbb{W}(b^\theta))^{\gamma_j}}.\end{aligned} \tag{5.12}$$

Now we replace b^1 by b^0 in (5.9) and b^0 by b^θ, respectively; with $b^0 - b^\theta = -\theta(b^1 - b^0)$ it is

$$\begin{aligned}&\mathbb{W}(b^0) - \mathbb{W}(b^\theta) + \theta\partial_b\mathbb{W}(b^\theta)\cdot(b^1 - b^0)\\ &\ge c_*\sum_{j=1}^k \theta^{p_j}|b_j^1 - b_j^0|^{p_j} + c_*\theta^2 \sum_{j=k+1}^m \frac{|b_j^1 - b_j^0|^2}{(1 + \mathbb{W}(b^0) + \mathbb{W}(b^\theta))^{\gamma_j}}.\end{aligned} \tag{5.13}$$

Multiplying (5.12) by θ, (5.13) by $1-\theta$ and adding the results, the term with the partial derivative cancels and we obtain

$$\begin{aligned}&(1-\theta)\mathbb{W}(b^0) + \theta\mathbb{W}(b^1) - \mathbb{W}(b^\theta)\\ &\ge c_*\textstyle\sum_{j=1}^k \big(\theta(1-\theta)^{p_j} + (1-\theta)\theta^{p_j}\big)|b_j^1 - b_j^0|^{p_j} + c_*\theta(1-\theta)\sum_{j=k+1}^m A_j(\theta, b^1, b_0)|b_j^1 - b_j^0|^2\end{aligned}$$

where $A_j(\theta, b^1, b_0) = \dfrac{1-\theta}{(1 + \mathbb{W}(b^1) + \mathbb{W}(b^\theta))^{\gamma_j}} + \dfrac{\theta}{(1 + \mathbb{W}(b^0) + \mathbb{W}(b^\theta))^{\gamma_j}}$.

Since $\theta(1-\theta)^{p_j} + (1-\theta)\theta^{p_j} \geq \theta(1-\theta)/2^{p_j}$ it suffices to estimate the terms A_j from below. Letting $w_n = \mathbb{W}(b^n)$ convexity gives $\mathbb{W}(b^\theta) \leq (1-\theta)w_0 + \theta w_1$. Using $\theta \in [0,1]$ we find

$$\begin{aligned} A_j(\theta, b^1, b_0) &\geq \frac{1-\theta}{(1+(1+\theta)w_1+(1-\theta)w_0)^{\gamma_j}} + \frac{\theta}{(1+(2-\theta)w_1+\theta w_0)^{\gamma_j}} \\ &\geq \left(\frac{1-\theta}{(1+\theta)^{\gamma_j}} + \frac{\theta}{(2-\theta)^{\gamma_j}}\right)\frac{1}{(1+w_1+w_0)^{\gamma_j}} \geq \frac{(2/3)^{\gamma_j}}{(1+w_1+w_0)^{\gamma_j}}. \end{aligned}$$

Thus, (5.8) is established and Part (C) is proven.

Part (D) follows from Hölder's inequality with $C_*^{\mathcal{P}} = \max\{(\mathcal{L}^d(\Omega)+2E_*)^{\delta_j} \mid j=1,...,m\}$

$$|\mathcal{P}(v^1)-\mathcal{P}(v^0)| \leq C_4 \sum\nolimits_{j=1}^{m} \left(\mathcal{L}^d(\Omega)+2E_*\right)^{\delta_j} \|v_j^1-v_j^0\|_{L^{p_j}} \leq C_*^{\mathcal{P}} \|v^1-v^0\|_{\mathcal{V}}$$

■

Part (D) will be applied to $\mathcal{P}(q)=\partial_t\mathcal{E}(t,q)$, see (3.17). Clearly the linear terms involving $l(t)$ and $\dot{l}(t)$ can be estimated directly. Thus, for fixed $t \in [0,T]$ the density $\mathbb{P}$ has the form $\mathbb{P}(x,e,z) = \partial_e W(x, e+e_D(t,x)){:}\dot{e}_D(t,x)$ with $e_D = e(g)$ for g as in (3.6(A2)), see also Corollary 5.3.4 for more details.

5.3 Examples

In this section we give examples fitting to the setup of Lemma 5.2.2. To simplify notations we drop the explicit dependence on the material coordinates $x \in \Omega$. Of course, the results can be generalized to heterogeneous materials, if all the estimates are uniform as assumed in the previous sections. Sections 5.3.1 and 5.3.2 deal with examples on the different types of convexity. They all use Part (C) of Lemma 5.2.2. In Section 5.3.3 Theorem 5.2.1 is applied to models for plasticity and phase transformations.

5.3.1 Examples on Joint, Strict and Uniform Convexity

In the modeling of damage the inner variable often influences the stored energy density in form of a product. The density analyzed in the following was first introduced in [Rou08]. There, it was shown that such product can be jointly convex in the two variables e and z. With regard to Lemma 5.2.2 we summarize several properties of W in the next lemma.

Lemma 5.3.1 *For $h \in \mathrm{C}^2([0,1],(0,1])$, $a \geq 0$ and $\mathbb{B} \in \mathbb{R}^{(d\times d)\times(d\times d)}$ symmetric and positive definite let*

$$W(e,z) := \frac{1}{2h(z)}\, e{:}\mathbb{B}{:}e + \frac{a}{2}\, z^2,$$

where we further assume $1 = h(0) > h(1) > 0$, $h'(z) \leq 0$ and $h''(z) \leq -\gamma \leq 0$ for $z \in [0,1]$. Then, $W : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times [0,1] \to \mathbb{R}$ is convex and there exists a constant $C > 0$ such that for all

e, $\widehat{e}$, z, and $\widehat{z}$ we have

$$|\partial_e W(e,z)| \leq C\,(W(e,z){+}1)\,, \tag{5.14}$$
$$|\partial_e W(e,z){-}\partial_e W(\widehat{e},\widehat{z})| \leq C|e{-}\widehat{e}| + C\,\big(1{+}W(e,z){+}W(\widehat{e},\widehat{z})\big)^{1/2}|z{-}\widehat{z}|\,. \tag{5.15}$$

If additionally $a>0$ and $\gamma>0$, then there exists $c_*>0$ such that

$$W(\widehat{e},\widehat{z}) - W(e,z) - \partial_e W(e,z){:}(\widehat{e}{-}e) - \partial_z W(e,z)(\widehat{z}{-}z) \geq \frac{c_*}{2}\big(|\widehat{e}{-}e|^2 + |\widehat{z}{-}z|^2\big). \tag{5.16}$$

Proof: Estimates (5.14) and (5.15) are due to the linear structure of $\partial_e W(e,z) = \frac{1}{h(z)}\mathbb{B}{:}e$ and the positive definiteness of $\mathbb{B}$, namely $W(e,z) \geq c_1|e|^2$ for all e and z.

To establish the convexity properties we calculate the Hessian D^2W explicitly. Omitting the argument z in h and its derivatives we obtain

$$D^2W(e,z)\left[\binom{E}{Z},\binom{E}{Z}\right] = \frac{1}{h^3}(hE{-}h'Ze){:}\mathbb{B}{:}(hE{-}h'Ze) + \frac{-h''}{2h^2}e{:}\mathbb{B}{:}eZ^2 + aZ^2, \tag{5.17}$$

which provides convexity since all terms on the right-hand side are nonnegative.

To derive strict convexity we let $\delta(z) = h'(z)/h(z) \in [-\delta_0,\delta_0]$ and use $h''(z) \leq -\gamma < 0$ to find c_2, $c_3 > 0$ such that

$$\begin{aligned} D^2W(e,z)\left[\binom{E}{Z},\binom{E}{Z}\right] &\geq c_2|E{-}\delta Ze|^2 + c_3|e|^2Z^2 + aZ^2 \\ &\geq \frac{c_2\varepsilon}{1+\varepsilon}|E|^2 + (c_3{-}\varepsilon\delta_0^2c_2)|e|^2Z^2 + aZ^2. \end{aligned}$$

Choosing $\varepsilon = c_3/(\delta_0^2c_2)$ we obtain (5.16) with $c_* = \min\{a, c_2c_3/(c_3{+}\delta_0^2)\}$ by classical convexity arguments. ■

The above lemma states that the stored energy density $W(e,z) = \frac{1}{\eta-z}\,e{:}\mathbb{B}{:}e + az^2/2$ with $\eta>1$, $a\geq 0$, and $\mathbb{B}$ symmetric and positive definite is convex. For $a=0$, it is not strictly convex, since $W(0,z)=0$ for $z\in[0,1]$. For $a>0$ we gain strict convexity but still do not have uniform convexity for W on $\mathbb{R}^{d\times d}_{\mathrm{sym}}\times[0,1]$, since $h''\equiv 0$, i.e., $\gamma=0$. For C^2 functions uniform convexity is equivalent to $D^2W(e,z)\left[\binom{E}{Z},\binom{E}{Z}\right]\geq c_*(|E|^2{+}Z^2)$ for some fixed $c_*>0$. However, inserting $(E,Z)=(\delta e,1)$ into the formula (5.17) gives $D^2W(e,z)\left[\binom{\delta e}{1},\binom{\delta e}{1}\right]=a$, while $|\delta e|^2{+}1$ may be arbitrarily big, since $\delta(z)=h'(z)/h(z) = -1/(\eta{-}z)<0$.

5.3.2 More Examples on Uniform Convexity

Here we construct an example for uniform convex stored elastic energy densities that have variables being parts of the strain tensor, like its deviator or its trace. The following lemma provides a rich set of uniformly convex functions.

Lemma 5.3.2 *Let $V \in \{\mathbb{R}, \mathbb{R}^d, \mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}\}$ have the scalar product $A_1 \cdot A_2 \in \mathbb{R}$ for all $A_1, A_2 \in V$. For $\kappa, \varepsilon > 0$, and $p \in (1,\infty)$ let $Z_{p\kappa\varepsilon}(A) := \frac{\kappa}{p}(\varepsilon + |A|^2)^{\frac{p}{2}}$ for $A \in V$. Then there are constants $c_{p\kappa\varepsilon}$, C_p, $k_{p\kappa\varepsilon} > 0$ and $\lambda_p \in \{0, \varepsilon\}$ such that for all $A_1, A_2, A \in V$ we have*

$$Z_{p\kappa\varepsilon}(A_1) - Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2)\cdot(A_1 - A_2) + c_{p\kappa\varepsilon}(\lambda_p + |A_1| + |A_2|)^{p-2}|A_1 - A_2|^2, \quad (5.18)$$

$$|\partial_A Z_{p\kappa\varepsilon}(A)| \leq C_p(Z_{p\kappa\varepsilon}(A) + 1), \quad (5.19)$$

$$|\partial_A Z_{p\kappa\varepsilon}(A_1) - \partial_A Z_{p\kappa\varepsilon}(A_2)| \leq \begin{cases} k_{p\kappa\varepsilon}|A_1 - A_2| & \text{if } 1 < p < 2, \\ k_{p\kappa\varepsilon}(\sqrt{\varepsilon} + |A_1| + |A_2|)^{p-2}|A_1 - A_2| & \text{if } p \geq 2. \end{cases} \quad (5.20)$$

Proof: In the proof we omit the subscripts p, κ, and ε. Direct computations give

$$\partial_A Z(A_2)\cdot A_1 = \kappa(\varepsilon + |A_2|^2)^{\frac{p-2}{2}} A_2 \cdot A_1,$$
$$\partial_A^2 Z(A_2)[A_1, A_3] = (p-2)\kappa(\varepsilon + |A_2|^2)^{\frac{p-4}{2}}(A_2\cdot A_1)(A_2 \cdot A_3) + \kappa(\varepsilon + |A_2|^2)^{\frac{p-2}{2}} A_1 \cdot A_3 \,.$$

Estimate (5.18) can be verified by a Taylor expansion of $\xi \mapsto Z(A_2 + \xi(A_1 - A_2))$ in the point $\xi = 0$ with a remainder term of order 2 using the ideas of [Kne04]. Estimate (5.19) is obtained, with $C_p = p^{(p-1)/p}$, via

$$|\partial_A Z(A)| \leq \kappa(\varepsilon + |A|^2)^{\frac{p-2}{2}}(\varepsilon + |A|^2)^{\frac{1}{2}} = (pZ(A))^{(p-1)/p} \leq C_p(Z(A) + 1).$$

In the following we carry out the proof of estimate (5.20) using a Taylor expansion of $f(\xi) := \partial_A Z(A_2 + \xi(A_1 - A_2))$ in the point $\xi = 0$ with a remainder term of order 1:

$$|\partial_A Z(A_1) - \partial_A Z(A_2)| = |f(1) - f(0)| \leq \int_0^1 \left|\tfrac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right| \mathrm{d}\xi \,.$$

We let $A^\xi := A_2 + \xi(A_1 - A_2)$. For $1 < p \leq 2$ we have

$$\left|\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right| = \left|\partial_A^2 Z(A^\xi)[A_1 - A_2, \cdot]\right| \leq \Big((2-p)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-4}{2}}|A^\xi|^2 + \kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}}\Big)|A_1 - A_2|$$
$$\leq (3-p)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}}|A_1 - A_2| \leq (3-p)\kappa\varepsilon^{\frac{p-2}{2}}|A_1 - A_2|.$$

This provides the upper estimate in (5.20). Similarly, for $p \geq 2$ we have

$$\left|\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi}\right| = \left|\partial_A^2 Z_{p\kappa\varepsilon}(A^\xi)[A_1 - A_2, \cdot]\right|$$
$$\leq (p-1)\kappa(\varepsilon + |A^\xi|^2)^{\frac{p-2}{2}}|A_1 - A_2| \leq (p-1)\kappa(\sqrt{\varepsilon} + |A_1| + |A_2|)^{p-2}|A_1 - A_2|,$$

which is the lower estimate in (5.20). ■

We introduce linear operators $g_i, g_j, g_{kl} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ in the form

deviator:	$g_i(e) = e^D := e - \dfrac{\operatorname{tr} e}{d}\mathrm{Id}$	(5.21a)
volumetric strain:	$g_j(e) := \dfrac{\operatorname{tr} e}{d}\mathrm{Id}$	(5.21b)
kl-th component of e :	$g_{kl}(e) := e_{kl}\mathrm{M}_{kl}$ for $k, l \in 1, \ldots, d$, where M_{kl} has the entry 1 at positions kl and lk and 0 else.	(5.21c)

The operators in (5.21) are orthogonal projections with respect to the scalar product $e:\hat{e}=\sum_{k,l} e_{kl}\hat{e}_{kl}$ and form building blocks for our examples. The following lemma provides convexity properties by combining these g_i with Lemma 5.3.2. Here, the index i indicates that the operator is defined by one of the expressions in (5.21).

Lemma 5.3.3 *For $1 < q, p_i, r, \tilde{r} < \infty$, $\varepsilon_q, \varepsilon_i, \kappa_q, \kappa_i, \kappa, \tilde{\kappa} > 0$, $\tilde{\varepsilon} \geq 0$ and $N \in \mathbb{N}_0$ let*

$$\begin{aligned}\overline{W}(t,x,e,z,A) &:= \widehat{W}(e{+}e_D(t,x),z){+}Z_{q\kappa_q\varepsilon_q}(e{+}e_D(t,x))\\ &\quad+\sum_{i=1}^{N} Z_{p_i\kappa_i\varepsilon_i}(g_i(e{+}e_D(t,x))){+}Z_{\tilde{r}\tilde{\kappa}\tilde{\varepsilon}}(z){+}Z_{r\kappa 0}(A),\end{aligned} \tag{5.22}$$

where $\widehat{W}$ is as in Lemma 5.3.1 with $\gamma, a > 0$ and the linear operators $g_i : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ are as in (5.21)*. Then, $\overline{W}$ and $\partial_t\overline{W}$ satisfy the counterparts to* (5.8) *and* (5.10)*, respectively, namely there exist constants $c_*, C_* > 0$ such that*

$$\begin{aligned}&c_1\theta(1{-}\theta)\Big(T_2(E,w^{1,0}){+}T_2(Z,w^{1,0}){+}T_q(E,w^{1,0}){+}\sum_{i=1}^{N} T_{p_i}(G_i,w^{1,0}){+}T_{\tilde{r}}(Z,w^{1,0}){+}T_r(A,w^{1,0})\Big)\\ &\quad\leq \theta\overline{W}(t,x,e^1,z^1,A^1){+}(1{-}\theta)\overline{W}(t,x,e^0,z^0,A^0){-}\overline{W}(t,x,\theta(e^1,z^1,A^1){+}(1{-}\theta)(e^0,z^0,A^0)),\\ &|\partial_e\overline{W}(t,x,e^1,z^1,A^1) - \partial_e\overline{W}(t,x,e^0,z^0,A^0)|\\ &\quad\leq C_*\Big(S_2(E,w^{1,0}){+}S_2(Z,w^{1,0}){+}S_q(E,w^{1,0}){+}\sum_{i=1}^{N} S_{p_i}(G_i,w^{1,0}){+}S_{\tilde{r}}(Z,w^{1,0}){+}S_r(A,w^{1,0})\Big)\end{aligned}$$

with $w^{1,0}{=}\overline{W}(t,x,e^1,z^1,A^1){+}\overline{W}(t,x,e^0,z^0,A^0)$, $E{=}|e^1{-}e^0|$, $Z{=}|z^1{-}z^0|$, $G_i{=}|g_i(e^1{-}e^0)|$, $A{=}|A^1{-}A^0|$, and where T_p, S_p are defined via

$$T_p(\xi,w) = \begin{cases} |\xi|^p & \text{if } p\geq 2,\\ |\xi|^2/(1{+}w)^{(2-p)/p} & \text{if } p\in[1,2],\end{cases}\qquad S_p(\xi,w) = \begin{cases} (1{+}w)^{(p-1)/p}|\xi| & \text{if } p\geq 2,\\ |\xi| & \text{if } p\in[1,2].\end{cases}$$

Proof: Set $\mathbb{W}(e,\tilde{e},g_1,\ldots,g_N,z,A){:=}\widehat{W}(e,z){+}Z_{q\kappa_q\varepsilon_q}(\tilde{e}){+}\sum_1^N Z_{p_i\kappa_i\varepsilon_i}(g_i){+}Z_{\tilde{r}\tilde{\kappa}\tilde{\varepsilon}}(z){+}Z_{r\kappa 0}(A)$ and $\hat{b}(e,z,A) := (e,e,g_1(e),\ldots,g_N(e),z,A)$. Then $\overline{W}(t,x,e,z,A) = \mathbb{W}(\hat{b}(e{+}e_D(t,x),z,A))$ and by the chain rule we find that $\partial_t\overline{W}(t,x,e,z,A) = \mathbb{P}(\hat{b}(e{+}e_D(t,x)),z,A);\dot{e}_D(t,x))$ with $\mathbb{P}(b;\dot{e}) = \partial_e\mathbb{W}(b){:}\dot{e} + \partial_{\tilde{e}}\mathbb{W}(b){:}\dot{e} + \sum_1^N \partial_{g_j}\mathbb{W}(b){:}g_j(\dot{e})$. We also used the fact that each g_i is linear, self-adjoint and idempotent.

As a consequence it suffices to show the desired estimates for each term $Z_{p_j\kappa_j\varepsilon_j}$ separately. For $i = 1,2$ let A_i be the corresponding jth component of the vectors b^i. For simplicity, we suppress the subscript j in the sequel. Inequality (5.8a) is obvious, so that we only prove (5.8b) in detail by showing (5.9). From (5.18) in Lemma 5.3.2 we derive for $p\geq 2$ that $Z_{p\kappa\varepsilon}(A_1){-}Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}c_{p\kappa\varepsilon}|A_1{-}A_2|^p$ and for $1<p<2$:

$$\begin{aligned}&Z_{p\kappa\varepsilon}(A_1){-}Z_{p\kappa\varepsilon}(A_2) \geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}c_{p\kappa\varepsilon}(\lambda_p{+}|A_1|{+}|A_2|)^{p-2}|A_1{-}A_2|^2\\ &\geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}(\kappa/p)^{\frac{2-p+(p-2)}{p}}3^{\frac{(p-1)(p-2)}{p}}(\lambda_p^p{+}(\varepsilon{+}|A_1|^2)^{\frac{p}{2}}{+}(\varepsilon{+}|A_2|^2)^{\frac{p}{2}})^{\frac{p-2}{p}}|A_1{-}A_2|^2\\ &\geq \partial_A Z_{p\kappa\varepsilon}(A_2){\cdot}(A_1{-}A_2){+}(\kappa/p)^{\gamma}3^{-\gamma(p-1)}\max\{1,\lambda_p^p\}^{-\gamma}(1{+}\mathbb{W}(b^1){+}\mathbb{W}(b^2))^{-\gamma}|A_1{-}A_2|^2\end{aligned}$$

with $\gamma = (2-p)/p$. In view of (5.16) this proves (5.8b).

Estimate (5.10a) holds, since $|\partial_t \widehat{W}(e{+}e_D, z)| {=} |\partial_e \widehat{W}(e{+}e_D, z){:}\dot{e}_D| {\leq} c_g \tilde{c}(\overline{W}(t,e,z,A){+}1)$ by (5.14) and since

$$|\partial_A Z_{p\kappa\varepsilon}(A)| = |\kappa(\varepsilon{+}|A|^2)^{\frac{p-2}{2}} A| \leq C_{p\kappa\varepsilon}(1{+}Z_{p\kappa\varepsilon}(A))\,.$$

Inequality (5.10b) follows from (5.15) together with (5.20), since in the case $p \geq 2$ we have

$$\begin{aligned}
&|\partial_A Z_{p\kappa\varepsilon}(A_1){-}\partial_A Z_{p\kappa\varepsilon}(A_2)| \leq k_{p\kappa\varepsilon}(\varepsilon^{\frac{p}{2p}}{+}(\varepsilon{+}|A_1|^2)^{\frac{p}{2p}}{+}(\varepsilon{+}|A_2|^2)^{\frac{p}{2p}})^{p-2}|A_1{-}A_2|\\
&\leq k_{p\kappa\varepsilon} 3^{\frac{(p-1)(p-2)}{p}}(\varepsilon^{\frac{p}{2}}{+}(\varepsilon{+}|A_1|^2)^{\frac{p}{2}}{+}(\varepsilon{+}|A_2|^2)^{\frac{p}{2}})^{\frac{p-2}{p}}|A_1{-}A_2|\\
&\leq k_{p\kappa\varepsilon} 3^{\frac{(p-1)(p-2)}{p}} \max\{1, \varepsilon^{\frac{p-2}{2}}\}(1{+}\mathbb{W}(b^1){+}\mathbb{W}(b^2))^{\frac{p-1}{p}}|A_1{-}A_2|\,.
\end{aligned}$$

■

As a first simple consequence we obtain Lipschitz continuity of energetic solutions for a reasonably wide class of stored energy densities $\widehat{W}$.

Corollary 5.3.4 *If the assumptions of Theorem 3.1.1 hold with $1{<}r{\leq}2$ and if W is given as in Lemma 5.3.1, then all energetic solutions $q : [0,T] \to \mathcal{Q}$ satisfy $q \in \mathrm{C}^{Lip}([0,T],\mathcal{V})$ with $\mathcal{V} = H^1(\Omega,\mathbb{R}^d) \times W^{1,r}(\Omega)$.*

With the results of Lemmata 5.3.1 and 5.3.2 at hand we now discuss the effective use of the spaces $\mathcal{Q}$, $\mathcal{X}$ and $\mathcal{V}$ for particular examples of the type introduced in Lemma 5.3.3. We emphasize that different choices for $\mathcal{V}$ lead to different results that cannot be obtained by just using one space $\mathcal{V}$. We start with the simpler case of time-independent Dirichlet conditions.

The Space $\mathcal{V}$ in Case of Time-independent Dirichlet Conditions

We consider the energy density

$$W(x,e,z,A) := \frac{|e+e(g(x))|^2}{2(1+2(1-z))^{\frac{1}{2}}} + |\mathrm{tr}(e{+}e(g(x))|^p + |A|^3 + |z|^2 \quad \text{with } p > 2\,. \tag{5.23}$$

The process is driven by time-dependent volume forces $l \in \mathrm{C}^1([0,T], H^{-1}(\Omega,\mathbb{R}^d))$ so that the energy is given by $\mathcal{E}(t,u,z) {:=} \int_\Omega W(x,e(u),z,\nabla z)\,\mathrm{d}x {-} \langle l(t), u{+}g\rangle$, where $g {\in} H^1(\Omega,\mathbb{R}^d)$ models the time-independent Dirichlet conditions. The existence of an energetic solution to $(\mathcal{Q},\mathcal{E},\mathcal{R})$ with $\mathcal{R}$ as in (3.1) can be proven using $\mathcal{Q} := \{(u,z) {\in} \mathcal{X} \,|\, z_\star {\leq} z {\leq} 1 \text{ a.e. in } \Omega\}$ with $\mathcal{X} := \{u \in H^1(\Omega,\mathbb{R}^d) \,|\, \mathrm{tr}\, e(u) \in L^p(\Omega),\ u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\} \times W^{1,3}(\Omega)$.

Let $q {=} (u,z)$, $\hat{q} {=} (\hat{u},\hat{z}) {\in} L_{E_*}(t)$ with $e {=} e(u)$ and $\hat{e} {=} e(\hat{u})$. Due to the time-independent Dirichlet conditions we have $\partial_t \mathcal{E}(t,u,z) {=} \langle \dot{l}(t), u{+}g\rangle$ and (5.4b) takes the form

$$|\partial_t \mathcal{E}(t,q) - \partial_t \mathcal{E}(t,\hat{q})| \leq c_l \|u - \hat{u}\|_{H^1} \leq c_l \|q - \hat{q}\|_{\mathcal{V}} \tag{5.24}$$

for any norm $\|\cdot\|_{\mathcal{V}}$ satisfying $\|u\|_{H^1} \leq \|(u,z)\|_{\mathcal{V}}$. Using the uniform convexity inequalities (5.16), (5.18) and Lemma 5.2.2 Parts (B), (C) we find for $\theta \in (0,1)$ and $(u_\theta, z_\theta) = (\theta\hat{u}+(1-\theta)u, \theta\hat{z}+(1-\theta)z)$

$$\begin{aligned}&\theta\mathcal{E}(t,\hat{u},\hat{z}) + (1-\theta)\mathcal{E}(t,u,z) - \mathcal{E}(t,u_\theta,z_\theta)\\ &\quad\geq \frac{\theta(1-\theta)c_*}{1+c_K^2}\big(\|e-\hat{e}\|_{L^2}^2 + \|\mathrm{tr}(e-\hat{e})\|_{L^p}^p + \|\nabla z - \nabla\hat{z}\|_{L^3}^3 + \|z-\hat{z}\|_{L^2}^2\big)\,.\end{aligned} \tag{5.25}$$

We introduce four different norms, namely for $j,k \in \{0,1\}$ we let

$$\begin{aligned}&\|(u,z)\|_{\mathcal{V}_{jk}} := \|e(u)\|_{L^2} + \|\operatorname{tr} e(u)\|_{\mathcal{U}_j} + \|z\|_{\mathcal{Z}_k}\,,\\ &\text{where } \|\tau\|_{\mathcal{U}_0} = 0,\ \|\tau\|_{\mathcal{U}_1} = \|\tau\|_{L^p},\ \|z\|_{\mathcal{Z}_0} = \|z\|_{L^2},\ \|\tau\|_{\mathcal{Z}_1} = \|z\|_{W^{1,3}}.\end{aligned}$$

We may define the associated Banach space $\mathcal{V}_{jk}$ and find the embeddings $\mathcal{V}_{11} \subset \mathcal{V}_\alpha \subset \mathcal{V}_{00}$ for $\alpha =$'01' or '10'.

In light of (5.24) we are free to drop the second or third term in the right-hand side of (5.25), which leads to the lower estimate

$$\theta\mathcal{E}(t,\hat{q}) + (1-\theta)\mathcal{E}(t,q) - \mathcal{E}(t,q_\theta) \geq \theta(1-\theta)c_{jk}\|\hat{q}-q\|_{\mathcal{V}_{jk}}^{\alpha_{jk}} \tag{5.26}$$

with $\alpha_{00} = 2$, $\alpha_{10} = p$, $\alpha_{01} = 3$, and $\alpha_{11} = \max\{3,p\}$. Applying Theorem 5.2.1 gives the following temporal regularities:

$$q \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathcal{V}_{00}) \cap \mathrm{C}^{1/2}([0,T],\mathcal{V}_{01}) \cap \mathrm{C}^{1/(p-1)}([0,T],\mathcal{V}_{10}) \cap \mathrm{C}^{1/(\max\{p,3\}-1)}([0,T],\mathcal{V}_{11})\,.$$

The Space $\mathcal{V}$ in Case of Time-dependent Dirichlet Conditions

Next we show the importance of the effective use of the space $\mathcal{V}$ in the case of time-dependent Dirichlet conditions. We consider $\overline{W}(t,x,e,z,A) := W(x,e+e(g(t,x)),z,A)$ with W from (5.23). Now, $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}$ reads $\mathcal{E}(t,u,z) := \int_\Omega \overline{W}(t,x,e(u),z,\nabla z)\,\mathrm{d}x$ with $\mathcal{Q}$ as above. For $\mathcal{V}$ we again use the spaces $\mathcal{V}_{jk}$ which provide the same convexity exponents α_{jk} for (5.4a) as in (5.26).

It remains to study the exponents β_{jk} for the Hölder estimate of $\partial_t\mathcal{E}(t,q)$ in (5.4b). Thereto we consider states $q=(u,z)$, $\hat{q}=(\hat{u},\hat{z}) \in L_{E_*}(t)$ with $e=e(u)$, $\hat{e}=e(\hat{u})$ and we put $e_D(t,x) := e(g(t,x))$. Lemma 5.3.1 applies to $\widehat{W}(t,x,e,z) = \frac{|e+e_D(t,x)|^2}{2(1+2(1-z))^{1/2}} + |z|^2$. Hence estimate (5.15) and Hölder's inequality with $\tilde{p}=2$ yield

$$\begin{aligned}&\Big|\partial_t \int_\Omega \widehat{W}(t,x,e,z) - \widehat{W}(t,x,\hat{e},\hat{z})\,\mathrm{d}x\Big|\\ &\leq Cc_g\mathcal{L}^d(\Omega)^{\frac12}\|e-\hat{e}\|_{L^2} + C(\mathcal{L}^d(\Omega)+\mathcal{E}(t,q)+\mathcal{E}(t,\hat{q}))^{\frac12}\|z-\hat{z}\|_{L^2}\\ &\leq C(1+c_g)(\mathcal{L}^d(\Omega)+2E_*)^{\frac12}\big(\|e-\hat{e}\|_{L^2} + \|z-\hat{z}\|_{L^2}\big)\,,\end{aligned}$$

where ∇z and $\nabla\hat{z}$ vanish because of ∂_t. Hölder's inequality for $\tilde{p}=p$ and (5.10b) lead to

$$\left|\partial_t\int_\Omega |\operatorname{tr}(e{+}e_D(t,x))|^p{-}|\operatorname{tr}(\hat{e}{+}e_D(t,x))|^p\,\mathrm{d}x\right| \le C_4c_g(\mathcal{L}^d(\Omega){+}2E_*)^{\frac{p-1}{p}}\,\|\operatorname{tr}(e{-}\hat{e})\|_{L^p}\,. \quad (5.27)$$

Adding these two estimates and putting $\tilde{C}:=(C{+}C_4)(1{+}c_g)(\mathcal{L}^d(\Omega){+}2E_*)^{\frac{p-1}{p}}$ results in

$|\partial_t\mathcal{E}(t,q){-}\partial_t\mathcal{E}(t,\hat{q})|\le C\big(\|e{-}\hat{e}\|_{L^2}+\|\operatorname{tr}(e{-}\hat{e})\|_{L^p}+\|z{-}\hat{z}\|_{L^2}\big)\le\tilde{C}\|q{-}\widehat{q}\|_{\mathcal{V}_{1k}}$ for $k=0$ or 1.

The uniform convexity inequality (5.26) (for $j=1$) together with Theorem 5.2.1 leads to the following temporal regularity results:

$$q\in \mathrm{C}^{1/(p-1)}([0,T],\mathcal{V}_{10})\cap\mathrm{C}^{1/(\max\{p,3\}-1)}([0,T],\mathcal{V}_{11})$$

for all energetic solutions of $(\mathcal{Q},\mathcal{E},\mathcal{D})$, i.e. the largest Hölder exponent we can reach is $1/2$.

The result can still be improved by strengthening (5.27). For example, for $2\le p\le 4$ we have

$$\begin{aligned}\left|\partial_t\int_\Omega |\operatorname{tr}(e{+}e_D)|^p{-}|\operatorname{tr}(\hat{e}{+}e_D)|^p\,\mathrm{d}x\right| &\le \int_\Omega C\big(1{+}|\operatorname{tr}(e{+}e_D)|{+}|\operatorname{tr}(\hat{e}{+}e_D)|\big)^{p-2}|\operatorname{tr}(e{-}\hat{e})|\,\mathrm{d}x\\ \le C\big\|1{+}|\operatorname{tr}(e{+}e_D)|{+}|\operatorname{tr}(\hat{e}{+}e_D)|^{p-2}\big\|_{L^2}\|e{-}\hat{e}\|_{L^2} &\le C_4c_g(\mathcal{L}^d(\Omega){+}2E_*)^{\frac{p-1}{p}}\|q{-}\hat{q}\|_{\mathcal{V}_{00}}\,,\end{aligned}$$

where we first applied (5.20), then the Cauchy-Schwarz inequality, and finally the estimate $\||\operatorname{tr}(\hat{e}{+}e_D)|^{p-2}\|_{L^2}\le C\|\operatorname{tr}(\hat{e}{+}e_D)\|_{L^p}$, which holds for $2<p\le 4$. Moreover, $p-2=0$ for $p=2$, so that the above estimate also holds for $p=2$. Thus, for $p\in[2,4]$ we obtain Lipschitz continuity of $q=(u,z):[0,T]\to\mathcal{V}_{00}=H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega)$.

Note that in the above examples it is not possible to reduce the powers 3 or p by estimating $\|\nabla z{-}\nabla\hat{z}\|^3_{L^3(\Omega)}$ and $\|\operatorname{tr}(e{-}\hat{e})\|^p_{L^p}$ by a lower norm, since the application of Hölder's inequality only changes the Lebesgue norm, but not its power: $\|Z\|^3_{L^3}\ge\mathcal{L}^d(\Omega)^{-1/6}\|Z\|^3_{L^2}$ Moreover, an interpolation $\|Z\|_{L^q}\le\|Z\|^\theta_{L^r}\|Z\|^{1-\theta}_{L^3}$ with $1<r<q<3$ and $\frac{1}{q}=\frac{\theta}{p}+\frac{(1-\theta)}{3}$ even leads to larger powers. Furthermore, for $p>4$ the term $\|\operatorname{tr}(e{-}\hat{e})\|^p_{L^p}$ in estimate (5.25) cannot be dropped, as we did it with $\|\nabla z{-}\nabla\hat{z}\|^3_{L^3(\Omega)}$, since it appears in the estimates (5.27), (5.3.2) because of the time-dependent Dirichlet conditions.

Improvement of Temporal Regularity by Interpolation

In the following we demonstrate how additional regularity results can be obtained by using two different spaces for $\mathcal{V}$ and by subsequent interpolation. We consider

$$W(x,e,z,A):=\frac{|e{+}e_D(x)|^2}{2(1{+}2(1{-}z))^{\frac12}}+|e{+}e_D(x)|^p+|e+e_D(x)|^4+|A|^3+|z|^2 \quad \text{with } 2{<}p{<}4$$

and $\mathcal{E}(t,u,z):=\int_\Omega W(t,x,e(u),z,\nabla z)\,\mathrm{d}x-\langle l(t),u{+}g\rangle$ with $l\in\mathrm{C}^1([0,T],H^{-1}(\Omega))$ and with $g\in H^1(\Omega,\mathbb{R}^d)$, $e_D=e(g)$ modeling the time-independent Dirichlet conditions. The existence of energetic solutions can be proven in $\mathcal{Q} := \{(u,z) \in \mathcal{X} \mid z_\star \le z \le 1 \text{ in } \Omega\}$ with $\mathcal{X} := \{u \in H^1(\Omega,\mathbb{R}^d) \mid e(u) \in L^4(\Omega,\mathbb{R}^{d\times d}),\ u=0 \text{ on } \Gamma_{\text{Dir}}\} \times W^{1,3}(\Omega)$.

The previous examples show that energetic solutions $q=(u,z) : [0,T] \to \mathcal{Q}$ have the temporal regularity

$$
\begin{aligned}
&u \in \mathrm{C}^{\mathrm{Lip}}([0,T],H^1(\Omega,\mathbb{R}^d)) \cap \mathrm{C}^{1/(p-1)}([0,T],W^{1,p}(\Omega,\mathbb{R}^d)) \cap \mathrm{C}^{1/3}([0,T],W^{1,4}(\Omega,\mathbb{R}^d)),\\
&z \in \mathrm{C}^{\mathrm{Lip}}([0,T],L^2(\Omega)) \cap \mathrm{C}^{1/2}([0,T],W^{1,3}(\Omega)).
\end{aligned}
$$

We have $u(t) \in H^1(\Omega,\mathbb{R}^d) \cap W^{1,4}(\Omega,\mathbb{R}^d)$ for all $t \in [0,T]$ and hence $u(t) \in W^{1,q}(\Omega,\mathbb{R}^d)$ for all $2<q<4$. Applying the interpolation theorem with $\frac{1}{q} = \frac{\theta}{2} + \frac{(1-\theta)}{4}$ we obtain that $\|u(t)\|_{W^{1,q}(\Omega,\mathbb{R}^d)} \le \|u(t)\|^{\theta}_{H^1(\Omega,\mathbb{R}^d)} \|u(t)\|^{(1-\theta)}_{W^{1,4}(\Omega,\mathbb{R}^d)}$ and hence

$$
\|u(s)-u(t)\|_{W^{1,q}(\Omega,\mathbb{R}^d)} \le \|u(s)-u(t)\|^{\theta}_{H^1(\Omega,\mathbb{R}^d)} \|u(s)-u(t)\|^{(1-\theta)}_{W^{1,4}(\Omega,\mathbb{R}^d)} \le C|s-t|^{\theta+\frac{1-\theta}{3}}.
$$

In particular, if $q=p$, we have $\theta = \frac{4-p}{p}$ and the new Hölder exponent $h = \theta + \frac{1-\theta}{3} = \frac{8-p}{3p}$, which satisfies $h > \frac{1}{p-1}$ for all $p \in (2,4)$. Thus we have obtained by interpolation that $u \in \mathrm{C}^{(8-p)/(3p)}([0,T],W^{1,p}(\Omega,\mathbb{R}^d))$.

5.3.3 Examples from Other Applications

In the following we apply the results of Section 5.2 to rate-independent models for plasticity and phase transformations. All these models use quadratic energy functionals. For all of the applications we may consider the dissipation potential

$$
\mathcal{R}(v) = \int_\Omega R(v)\,\mathrm{d}x \quad \text{with } R : \mathbb{R}^{d\times d}_{\text{sym,dev}} \to [0,\infty),\ R(v) = |v|, \tag{5.28}
$$

where $\mathbb{R}^{d\times d}_{\text{sym,dev}} := \{A \in \mathbb{R}^{d\times d} \mid A = A^\top,\ \mathrm{tr}\, A = 0\}$.

We will obtain that $\mathcal{V} = \mathcal{Q}$ in these settings, that $\alpha = 2$ and $\beta = 1$, so that energetic solutions are Lipschitz continuous with respect to time. This regularity is in good accordance with the results proven in [MT04].

Linear Elasticity Coupled with Kinematic Hardening

In the framework of plasticity the state $q=(u,z)$ is given by the displacement field $u : \Omega \to \mathbb{R}^d$ and the plastic strain $z : \Omega \to \mathbb{R}^{d\times d}_{\text{sym,dev}}$. We consider a density of the form

$$
W(e(u),z) := \frac{\lambda}{2}(\mathrm{tr}(e(u)-z))^2 + \mu|e(u)-z|^2 + \frac{K}{2}|z|^2, \tag{5.29}
$$

where $\lambda, \mu > 0$ are the Lamé constants and $K > 0$ denotes the hardening parameter. Using a binomic formula one obtains that the three components of W satisfy the following convexity inequalities

$$\big(\operatorname{tr} w_\theta\big)^2 = \theta(\operatorname{tr} w_1)^2 + (1-\theta)(\operatorname{tr} w_0)^2 - \theta(1-\theta)\big(\operatorname{tr}(w_1-w_0)\big)^2 < \theta(\operatorname{tr} w_1)^2 + (1-\theta)(\operatorname{tr} w_0)^2 \quad (5.30)$$

$$|w_\theta|^2 = \theta|w_1|^2 + (1-\theta)|w_0|^2 - \theta(1-\theta)|w_1-w_0|^2 < \theta|w_1|^2 + (1-\theta)|w_0|^2 \quad (5.31)$$

$$|z_\theta|^2 = \theta|z_1|^2 + (1-\theta)|z_0|^2 - \theta(1-\theta)|z_1-z_0|^2 < \theta|z_1|^2 + (1-\theta)|z_0|^2 \quad (5.32)$$

for all $(u_0, z_0), (u_1, z_1) \in \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}$, all $\theta \in (0,1)$, $w_i := e(u_i)-z_i$ and $w_\theta := \theta w_1+(1-\theta)w_0$. Since the term comprising the work of the external loadings is linear, the energy functional $\mathcal{E}(t,u,z) = \int_\Omega W(e(u+g(t),z))\,\mathrm{d}x - \langle l(t), u+g(t)\rangle$ is strictly convex with respect to $(u,z) \in \mathcal{Q} := \{\tilde u \in H^1(\Omega,\mathbb{R}^d) \,|\, \tilde u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\} \times L^2(\Omega,\mathbb{R}^{d\times d})$. Moreover we verify that the energy functional satisfies the assumptions (5.4a) and (5.4b) of Theorem 5.2.1 for $l \in \mathrm{C}^1([0,T], H^{-1}(\Omega,\mathbb{R}^d))$ and $g \in \mathrm{C}^1([0,T], H^1(\Omega,\mathbb{R}^d))$ with $c_l := \|l\|_{\mathrm{C}^1([0,T],H^{-1}(\Omega,\mathbb{R}^d))}$ and $c_g := \|g\|_{\mathrm{C}^1([0,T],H^1(\Omega,\mathbb{R}^d))}$. We calculate that

$$\partial_t\mathcal{E}(t,q) = \int_\Omega \lambda\big(\operatorname{tr}(e(u+g(t))-z)\operatorname{tr} e(\dot g(t)) + 2\mu(e(u+g(t))-z) : e(\dot g(t))\big)\,\mathrm{d}x \\ -\langle \dot l(t), u+g(t)\rangle - \langle l(t), \dot g(t)\rangle.$$

With Hölder's inequality and $q_i = (u_i, z_i) \in \mathcal{Q}$, $i = 0,1$, we get Lipschitz estimate (5.4b), i.e.

$$\big|\partial_t\mathcal{E}(t,q_1) - \partial_t\mathcal{E}(t,q_0)\big| \le \big((\lambda+2\mu)c_g + c_l\big)\big(\|u_1-u_0\|_{H^1(\Omega,\mathbb{R}^d)} + \|z_1-z_0\|_{L^2(\Omega,\mathbb{R}^{d\times d})}\big)\,.$$

To prove the uniform convexity inequality (5.4a) we apply estimates (5.30)-(5.32) on W to gain a convexity estimate on $\mathcal{E}$ by integration. Moreover the triangle inequality and Korn's inequality yield

$$\|w_1-w_0\|_{L^2}^2 \ge \frac{1}{2}\|e(u_1)-e(u_0)\|_{L^2}^2 - \|z_1-z_0\|_{L^2}^2 \ge \frac{1}{2C_K}\|u_1-u_0\|_{H^1}^2 - \|z_1-z_0\|_{L^2}^2\,. \quad (5.33)$$

Thus, if $\mu < K$ (see equation (5.29)) the following uniform convexity inequality holds

$$\mathcal{E}(t,q_\theta) \le \theta\mathcal{E}(t,q_1) + (1-\theta)\mathcal{E}(t,q_0) + \theta(1-\theta)c_*\big(\|u_1-u_0\|_{H^1(\Omega,\mathbb{R}^d)} + \|z_1-z_0\|_{L^2(\Omega,\mathbb{R}^{d\times d})}\big)$$

for $c_* := \min\{\mu/(2C_K), (K-\mu)\}$. The uniform convexity inequality for $\mathcal{E}(t,\cdot) + \mathcal{D}(z_0,\cdot)$ holds true due to the 1-homogeneity of $\mathcal{D}$.

Thus we have $\mathcal{V} = \mathcal{Q} = \{\tilde u \in H^1(\Omega,\mathbb{R}^d) \,|\, \tilde u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\} \times L^2(\Omega,\mathbb{R}^{d\times d})$, $\alpha = 2$ and $\beta = 1$, so that an energetic solution satisfies $q \in \mathrm{C}^{Lip}([0,T],\mathcal{Q})$.

Elasto-plasticity with Cosserat micropolar effects

We consider the energy density

$$W(u,Q,z) = \mu|e(u)-z|^2 + \mu_c|\operatorname{skew}(\nabla u - Q)|^2 + \tfrac{\lambda}{2}|\operatorname{tr}\nabla u|^2 + \gamma|\nabla Q|^2\,,$$

where the plastic strain $z\in\mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$ is the inner variable and where $Q\in\mathbb{R}^{d\times d}_{\mathrm{skew}}$ denote the micro-rotations. Here, $\mathbb{R}^{d\times d}_{\mathrm{skew}}:=\{A\in\mathbb{R}^{d\times d}\,|\,A^\top=-A\}$. Similar to estimates (5.30)-(5.32) we find that

$$\begin{aligned}|\operatorname{skew} v_\theta|^2 &= \theta|\operatorname{skew} v_1| + (1-\theta)|\operatorname{skew} v_0| - \theta(1-\theta)|\operatorname{skew}(v_1-v_0)|^2\\ &< \theta|\operatorname{skew} v_1| + (1-\theta)|\operatorname{skew} v_0|\,,\end{aligned}\tag{5.34}$$

where $v_i := \nabla u_i - Q_i$ and $v_\theta := \theta v_1 + (1-\theta)v_0$ for all $(u_i,Q_i)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{skew}}$, $i{=}0,1$, and all $\theta\in(0,1)$. Again we assume that the plastic deformation is driven by the external loadings $l\in \mathrm{C}^1([0,T],H^{-1}(\Omega,\mathbb{R}^d))$ with $c_l:=\|l\|_{\mathrm{C}^1([0,T],H^{-1}(\Omega,\mathbb{R}^d))}$. Moreover we consider time-dependent Dirichlet data $g\in\mathrm{C}^1([0,T],H^2(\Omega,\mathbb{R}^d))$ with $c_g:=\|g\|_{\mathrm{C}^1([0,T],H^2(\Omega,\mathbb{R}^d))}$, since we also impose Dirichlet conditions on the micro-rotations by $Q_g(t):=\operatorname{skew}\nabla g(t)$ such as e.g. in [NK08]. Thus we have $\mathcal{E}(t,q):=\int_\Omega W((u{+}g(t),Q{+}Q_g(t)),z)\,\mathrm{d}x-\langle l(t),u{+}g(t)\rangle$ for the state $q=(u,Q,z)$ and the partial time derivative is given by

$$\begin{aligned}\partial_t\mathcal{E}(t,q) = &\int_\Omega \big(2\mu(e(u{+}g(t)){-}z) : e(\dot g(t)){+}2\gamma\nabla(Q{+}Q_g(t)) : \nabla\dot Q_g(t)\big)\,\mathrm{d}x\\ &+\int_\Omega 2\mu_c(\operatorname{skew}(\nabla(u{+}g(t)){-}(Q{+}Q_g(t)))) : (\operatorname{skew}(\nabla\dot g(t){-}\dot Q_g(t))\,\mathrm{d}x\\ &+\int_\Omega \big(\lambda(\operatorname{tr}\nabla(u{+}g(t)))\operatorname{tr}\nabla\dot g(t)\big)\,\mathrm{d}x-\langle\dot l(t),u{+}g(t)\rangle-\langle l(t),\dot g(t)\rangle\end{aligned}$$

As in the example of kinematic hardening we find with the aid of Hölder's inequality that

$$\begin{aligned}&|\partial_t\mathcal{E}(t,q_1)-\partial_t\mathcal{E}(t,q_0)|\\ &\le(2\mu c_g{+}2\gamma c_g{+}4\mu_c c_g{+}\lambda c_g{+}c_l)\big(\|u_1{-}u_0\|_{H^1}{+}\|z_1{-}z_0\|_{L^2}{+}\|Q_1{-}Q_0\|_{H^1}\big)\,,\end{aligned}$$

where we simply included the missing terms $\|u_1{-}u_0\|_{L^2(\Omega,\mathbb{R}^d)}$, $\|\nabla Q_1-\nabla Q_0\|_{L^2(\Omega,\mathbb{R}^{d\times d})}$. Hence $\beta=1$ in (5.4b).

Using estimates (5.30)-(5.34), applying Poincaré-Friedrich's inequality on $\|Q\|_{H^1}$ and assuming that $\mu_c<\gamma/(C_K^2)$ allows us to verify convexity inequality (5.4a) for the constants $\alpha{=}2$, $c_*:=\min\{\mu/(2C_K^2),\mu,(\gamma/C_P^2){-}\mu_c\}$ in the space $\mathcal{V}=\mathcal{Q}$ given by

$$\mathcal{V}=\{\tilde u\in H^1(\Omega,\mathbb{R}^d)\,|\,\tilde u{=}0\text{ on }\Gamma_{\mathrm{Dir}}\}\times\{\tilde Q\in H^1(\Omega,\mathbb{R}^{d\times d}_{\mathrm{skew}})\,|\,\tilde Q{=}0\text{ on }\Gamma_{\mathrm{Dir}}\}\times\{z\in L^2(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym,dev}})\}.$$

Therefore any energetic solution satisfies $q=(u,Q,z)\in\mathrm{C}^{Lip}([0,T],\mathcal{Q})$.

The Souza-Auricchio model for phase transformations in shape memory alloys

In the context of phase transformations in shape memory alloys the internal variable $z:\Omega\to\mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$ is the mesoscopic transformation strain reflecting the phase distribution. The dissipation distance, which measures the energy dissipated due to phase transformation, is assumed to take the form $\mathcal{D}(z,\tilde z)=\varrho\|z{-}\tilde z\|_{L^1(\Omega)}$ with $\varrho>0$.

The phase transformations are considered to be thermally induced. For a body that is small in at least one direction, it is reasonable to assume that the temperature $\vartheta \in \mathrm{C}^1([0,T], H^1(\Omega))$ with $C_\vartheta := \|\vartheta\|_{\mathrm{C}^1([0,T],H^1(\Omega))}$ is a priori given, since it influences the transformation process like an applied load, see [MP07, Aur01]. With $F(u,z) := (e(u), z, \nabla z)$, the energy density therefore takes the form

$$W(F(u,z),\vartheta) = \frac{1}{2}\big(e(u){-}z\big) : \mathbb{B}(\vartheta) : \big(e(u){-}z\big) + h(z,\vartheta) + \frac{\sigma}{2}|\nabla z|^2$$

with the constant $\sigma > 0$ and the elasticity tensor $\mathbb{B} \in \mathrm{C}^1([\vartheta_{\min}, \vartheta_{\max}], \mathbb{R}^{(d\times d)\times(d\times d)})$ being symmetric and positive definite for all ϑ. This means that there are constants $c_1^{\mathbb{B}}, c_2^{\mathbb{B}} > 0$ so that $c_1^{\mathbb{B}}|A|^2 \leq A : \mathbb{B} : A \leq c_2^{\mathbb{B}}|A|^2$ for all $A \in \mathbb{R}^{d\times d}$ and $c_\vartheta^{\mathbb{B}} := \|\mathbb{B}\|_{\mathrm{C}^1([\vartheta_{\min},\vartheta_{\max}],\mathbb{R}^{(d\times d)\times(d\times d)})}$. The function $h : \mathbb{R}^{d\times d}_{\mathrm{sym,dev}} \times \mathbb{R} \to \mathbb{R}$ is given by

$$h(z,\vartheta) := c_1(\vartheta)\sqrt{\delta^2 + |z|^2} + c_2(\vartheta)|z|^2 + \frac{1}{\delta}(|z|{-}c_3(\vartheta))_+^3\,,$$

where $\delta > 0$ is constant and $c_i \in \mathrm{C}^1([\vartheta_{\min}, \vartheta_{\max}])$ with $0 < c_i^1 \leq c_i(\vartheta)$ for all $\vartheta \in [\vartheta_{\min}, \vartheta_{\max}]$ and $c_i^\vartheta := \|c_i\|_{\mathrm{C}^1([\vartheta_{\min},\vartheta_{\max}])}$, $i = 1,2,3$. Here, $c_1(\vartheta)$ is an activation threshold for the initiation of martensitic phase transformations, $c_2(\vartheta)$ measures the occurrence of an hardening phenomenon with respect to the internal variable z and $c_3(\vartheta)$ represents the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants. Furthermore $(f)_+ := \max\{0, f\}$. For given data $l \in \mathrm{C}^1([0,T], H^{-1}(\Omega,\mathbb{R}^d))$ as well as $g \in \mathrm{C}^1([0,T], H^1(\Omega,\mathbb{R}^d))$ the energy functional is defined by

$$\mathcal{E}(t,q) = \int_\Omega W(F(u{+}g(t), z),\vartheta)\,\mathrm{d}x - \langle l(t), u{+}g(t)\rangle.$$

Hence we have

$$\partial_t\mathcal{E}(t,q) = \int_\Omega \big(\partial_u W(F(u{+}g,z),\vartheta) : e(\dot g) {+} \dot\vartheta\,\partial_\vartheta W(F(u{+}g,z),\vartheta)\big)\,\mathrm{d}x - \langle \dot l, u{+}g\rangle - \langle l, \dot g\rangle \text{ with}$$

$$\partial_u W(F(u{+}g,z),\vartheta) : \dot g = (e(u{+}g){-}z){:}\mathbb{B}(\vartheta){:}e(\dot g)\,,$$

$$\dot\vartheta\,\partial_\vartheta W(F(u{+}g,z),\vartheta) = \dot\vartheta\,\big((e(u{+}g){-}z){:}\partial_\vartheta\mathbb{B}(\vartheta){:}(e(u{+}g){-}z) + \partial_\vartheta h(\vartheta,z)\big)\,.$$

To gain a Lipschitz estimate for $\partial_t\mathcal{E}(t,\cdot)$ for the present model it is important that Theorem 5.2.1 is formulated for energy-sublevels $L_{E_\star}(t) = \{q \in \mathcal{Q} \,|\, \mathcal{E}(t,q) \leq E_\star\}$, since this provides the bound $\|u_i\|_{H^1}{+}\|z_i\|_{H^1} \leq C_{E_\star}$. Thus for all $(u_0,z_0),(u_1,z_1) \in L_{E_\star}(t)$ it holds

$$\begin{aligned}
&\int_\Omega |\dot\vartheta\big(e(u_1{-}u_0){-}(z_1{-}z_0)\big){:}\partial_\vartheta\mathbb{B}(\vartheta){:}\big(e(u_1{-}u_0){-}(z_1{-}z_0)\big)|\,\mathrm{d}x\\
\leq& C_\vartheta c_\vartheta^{\mathbb{B}}\big(\|e(u_1{-}u_0)\|_{L^2}{+}\|z_1{-}z_0\|_{L^2}\big)^2\\
\leq& C_\vartheta c_\vartheta^{\mathbb{B}}\Big(\sum_{i=0}^{1}\|e(u_i)\|_{L^2}{+}\|z_i\|_{L^2}\Big)\big(\|e(u_1{-}u_0)\|_{L^2}{+}\|z_1{-}z_0\|_{L^2}\big)\\
\leq& 2C_{E_\star}C_\vartheta c_\vartheta^{\mathbb{B}}\big(\|u_1{-}u_0\|_{H^1}{+}\|z_1{-}z_0\|_{L^2}\big)\,.
\end{aligned}$$

Furthermore the application of the main theorem on differentiable functions yields

$$\begin{aligned}\big|\sqrt{\delta^2+|z_1|^2}-\sqrt{\delta^2+|z_0|^2}\big| &\le |z_1-z_0|,\\ \big||z_1|^2-|z_0|^2\big| &\le 2(|z_1|+|z_0|)|z_1-z_0|,\\ |(|z_1|-c_3(\vartheta))_+^3-(|z_0|-c_3(\vartheta))_+^3| &\le 2(|z_1|+|z_0|)^2|z_1-z_0|,\end{aligned}$$

so that

$$\int_\Omega |\partial_\vartheta h(\vartheta,z_1)-\partial_\vartheta h(\vartheta,z_0)| \le \|z_1-z_0\|_{L^1}\big(c_1^\vartheta+2(\mathcal{L}^d(\Omega)C_{E_\star})^{\frac12}c_2^\vartheta+\frac{6}{\delta}c_3^\vartheta C_{E_\star}\big) \le \tilde{C}_\star\|z_1-z_0\|_{L^2}$$

with $\tilde{C}_\star := \mathcal{L}^d(\Omega)^{\frac12}\big(c_1^\vartheta+2(\mathcal{L}^d(\Omega)C_{E_\star})^{\frac12}c_2^\vartheta+\frac{6}{\delta}c_3^\vartheta C_{E_\star}\big)$, where $\mathcal{L}^d(\Omega)$ denotes the d-dimensional Lebesgue-measure of Ω. Therefore Lipschitz estimate (5.4b) holds true with $\beta = 1$ and $C_\star = (\tilde{C}_\star + 2C_{E_\star}C_\vartheta c_\vartheta^{\mathbb{B}} + c_\vartheta^{\mathbb{B}}c_g + c_l)$.

To verify that W is uniformly convex with respect to $F(u,z)$ we proceed as in estimate (5.31). Hence we calculate that

$$w_\theta{:}\mathbb{B}(\vartheta){:}w_\theta \le \theta w_1{:}\mathbb{B}(\vartheta){:}w_1+(1-\theta)w_0{:}\mathbb{B}(\vartheta){:}w_0-\theta(1-\theta)c_1^{\mathbb{B}}|w_1-w_0|^2$$

for $w_i{=}e_i{-}z_i$ with $(e_i,z_i)\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{sym,dev}}$, $i=0,1$, $w_\theta=\theta w_1+(1{-}\theta)w_0$ with $\theta\in(0,1)$. Here, a binomic formula and the positive definiteness of $\mathbb{B}(\vartheta)$ for all ϑ were applied. The uniform convexity of the gradient term is already stated in (5.32). We now show that h is uniformly convex. We immediately see that $\tilde{h}_1(z) := (\delta^2+|z|^2)^{\frac12}$ is convex in z. Furthermore, since $\tilde{h}_3(z) := (|z|-c_3(\vartheta))_+^3$ is the composition of the monotone function x^3 and the convex function $(\cdot)_+$, we conclude that also $\tilde{h}_3(z)$ is convex in z. Additionally we obtain with similar calculations as applied for the other quadratic terms that $\tilde{h}_2(z) := |z|^2$ is uniformly convex. Since $c_i(\vartheta)\ge c_i^1>0$ for all $\vartheta\in[\vartheta_{\min},\vartheta_{\max}]$ and $i=1,2,3$ we have proven that h is uniformly convex in z with $h(z_\theta,\vartheta)\le\theta h(z_1,\vartheta)+(1{-}\theta)h(z_0,\vartheta)-\theta(1{-}\theta)c_2^1|z_1{-}z_0|^2$. Summing up all terms and taking into account all prefactors yields a uniform convexity estimate for W, which leads to

$$\mathcal{E}(t,q_\theta)\le\theta\mathcal{E}(t,q_1)+(1{-}\theta)\mathcal{E}(t,q_0)-\theta(1{-}\theta)\left(\frac{c_1^{\mathbb{B}}}{2}\|w_1-w_0\|_{L^2}^2+\frac{\sigma}{2}\|\nabla(z_1-z_0)\|_{L^2}^2+c_2^1\|z_1-z_0\|_{L^2}^2\right).$$

For this we have used that the term describing the work of the external loadings is linear in u. By estimate (5.33) and the assumption $(c_2^1-(c_1^{\mathbb{B}}/2))>0$ we conclude that (5.4a) holds for $\alpha=2$, $c_* := \min\big\{c_1^{\mathbb{B}}/(2C_K^2),\sigma/2,(c_2^1{-}(c_1^{\mathbb{B}}/2))\big\}$ and the space

$$\mathcal{V}=\mathcal{Q}=\{\tilde{u}\in H^1(\Omega,\mathbb{R}^d)\,|\,\tilde{u}=0 \text{ on } \Gamma_{\mathrm{Dir}}\}\times\{\tilde{z}\in H^1(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym,dev}})\}.$$

Thus any energetic solution $q{:}[0,T]\to\mathcal{Q}$ is Lipschitz continuous with respect to time, i.e. $q\in \mathrm{C}^{Lip}([0,T],\mathcal{Q})$.

Chapter 6

Conclusions and Outlook

This thesis is concerned with the analytical study of rate-independent damage and delamination processes in their energetic formulation. The main results are deduced in the Chapters 3–5.

In Chapter 3 the existence result of previous work [MR06] is extended to a larger class of models describing partial, isotropic damage processes in physically and geometrically nonlinearly elastic materials. While the existence proof in [MR06] requires damage variables $z \in W^{1,r}(\Omega)$ with $r > d$ (for $\Omega \subset \mathbb{R}^d$) the existence theorems 3.1.1 and 3.2.7 cover all exponents $r \in (1, \infty)$ at small and finite strains respectively. This extended existence result is obtained by a new technique for the construction of a joint recovery sequence, see Theorems 3.1.14 and 3.2.13 respectively.

The existence result for the small strain setting is used in Chapter 4 to deduce a delamination model as the Γ-limit of models describing the partial damage of the middle constituent of a three-component-sandwich-structure as the thickness of the middle constituent is flattened to 0. As an important result it is obtained in Theorem 4.2.2 that transmission conditions and unilateral contact conditions, which define the interface and the crack in the limit model, result from the nature of the partial damage models. With the convergence theorems 4.2.1 and 4.3.4 it is proven that the delamination model obtained in the limit is indeed the one discussed in [RSZ09], which is based on the contact-with-adhesion-model proposed in [Fré88].

In Chapter 5 the temporal regularity of energetic solutions is investigated. Theorem 5.1.2 provides the temporal continuity of energetic solutions in the case of jointly strictly convex energy functionals $\mathcal{E}(t, \cdot)$, which is similar to the results of previous work [MT04], whereas Theorem 5.2.1 on the temporal Lipschitz and Hölder continuity due to the uniform convexity of the energy functional $\mathcal{E}(t, \cdot)$ establishes an important extension of the respective results in [MT04]. While [MT04] requires the uniform convexity of $\mathcal{E}(t, \cdot)$ on the entire state space $\mathcal{Q}$ and restricts this kind of convexity more or less to quadratic energy functionals leading to the temporal Lipschitz continuity of energetic solutions, it is shown in Chapter 5 that non-quadratic energy functionals can also satisfy uniform convexity inequalities on energy sublevels with respect to a bigger Banach space $\mathcal{V} \supset \mathcal{Q}$, which lead to the temporal Hölder continuity of energetic solutions. Moreover it is demonstrated in Section 5.3.2 that the temporal regularity essentially depends on the space $\mathcal{V}$ and that the regularity of the separate state components can be improved by considering different spaces $\mathcal{V}$.

Since Chapter 3 only ensures the existence of energetic solutions $q = (u, z) : [0, T] \to \mathcal{Q}$ with $z \in W^{1,r}(\Omega)$, where $r \in (1, \infty)$, cf. formula (3.3), it would be of interest to treat the case $r = 1$ as well. The Banach space $W^{1,1}(\Omega)$ is not reflexive. This requires the use of a larger Banach space, namely the space of functions with bounded variation $BV(\Omega)$.

The convergence which has to be considered in this setting, is the weak*-convergence in $BV(\Omega)$, i.e. by [AFP05, Definition 3.11] this means

$$z_k \overset{*}{\rightharpoonup} z \text{ in } BV(\Omega) \;\Leftrightarrow\; \left\{ \begin{array}{cl} z_k \to z & \text{in } L^1(\Omega), \\ \int_\Omega \varphi \, \mathrm{d}\mathrm{D}z_k \to \int_\Omega \varphi \, \mathrm{d}\mathrm{D}z & \text{for all } \varphi \in \mathrm{C}(\Omega), \end{array} \right.$$

where $\mathrm{D}z_k$ and $\mathrm{D}z$ are the distributional derivatives of z_k and z respectively, which are represented by finite Radon measures. By [AFP05, Proposition 3.13] the convergence $z_k \overset{*}{\rightharpoonup} z$ in $BV(\Omega)$ is equivalent to $(z_k)_{k\in\mathbb{N}}$ being bounded in $BV(\Omega)$ and converging to z strongly in $L^1(\Omega)$.

The main difficulty to prove the existence of energetic solutions for damage variables in $BV(\Omega)$ then lies in the construction of a joint recovery sequence $(\hat{u}_k, \hat{z}_k)_{k\in\mathbb{N}}$, see Definition 3.1.12, where $\hat{z}_k \in BV(\Omega)$ and $\hat{z}_k \overset{*}{\rightharpoonup} \hat{z}$ in $BV(\Omega)$. Using the definition of BV-functions

$$v \in BV(\Omega) \quad \Leftrightarrow \quad \sup\left\{ \int_\Omega v \,\mathrm{div}\, \phi \, \mathrm{d}x \mid \phi \in \mathrm{C}_c^1(\Omega, \mathbb{R}^d),\ |\phi| \le 1 \right\} < \infty \,,$$

where $\mathrm{C}_c^1(\Omega, \mathbb{R}^d)$ denotes the $\mathrm{C}^1(\Omega, \mathbb{R}^d)$-functions with compact support, it is easy to see that $\hat{z}_k := \min\{\hat{z} - \delta_k, z_k\} \in BV(\Omega)$ for all $\hat{z}, z_k \in BV(\Omega)$ and $\delta_k \in \mathbb{R}$. Moreover, for $\delta_k := \|z - z_k\|_{L^1(\Omega)}^{\frac{1}{2}}$ we can verify that $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$. But it has to be checked whether $\mathrm{D}\hat{z}_\kappa$ can be expressed by $\mathrm{D}\hat{z}$ and $\mathrm{D}z_\kappa$, i.e. whether the composition lemma 3.1.13 can be adapted to BV-functions. If not so, the methods of Step 2 in the proof of Theorem 3.1.14 are not applicable and a completely different construction of $\hat{z}_k$ must be developed.

The results of Chapter 4 allow to approximate a delamination model by models describing partial isotropic damage. This means in particular that the two local conditions $[\![u \cdot \mathrm{n}_1]\!] \ge 0$ and $z[\![u]\!] = 0$ a.e. on the interface Γ_C (see (4.60)) can be approximated by the nonlocal terms $\int_{\Omega_\mathrm{D}^\varepsilon} W_\mathrm{D}(e(u_\varepsilon), \Pi^\varepsilon z_\varepsilon) \, \mathrm{d}x + \int_{\Omega_\mathrm{D}} \frac{\kappa}{r} |\nabla_\varepsilon z_\varepsilon|^r + \delta_{[\varepsilon^\gamma, 1]} \, \mathrm{d}y$, see formula (4.1) and (4.2). Since the local transmission and noninterpenetration conditions are difficult to implement numerically, it might be advantageous to use the approximating damage problems for numerical simulations instead. Therefore it is of importance to propose an algorithm for the numerical computation of the approximating problems and to perform the numerical analysis for this procedure.

Up to now the simultaneous convergence of the damage models to the Griffith-type delamination model could only be verified if $p > d$, i.e. if $[\![u]\!]$ is continuous. This was required for the construction of the joint recovery sequence when passing from gradient delamination to Griffith-type delamination, see Section 4.3, in particular Lemma 4.3.1 and Corollaries 4.3.12, 4.3.13. To extend the convergence result to $p \in (1, \infty)$ would be challenging, since it requires a new ansatz to prove the Conjecture 4.3.10.

Moreover, Chapter 4 only treated the convergence of the partial damage models to the delamination model in the small strain setting. Since Section 3.2 also provides the existence of energetic solutions for the partial damage model at finite strains it is of interest to investigate the convergence also in this setting. In particular it has to be studied whether the noninterpenetration condition $[\![u \cdot \mathrm{n}_1]\!] \geq 0$ can be gained with a polyconvex energy density, since making W_{D} in (4.2) polyconvex requires to replace $(-e_{11})^+$ by $(-\operatorname{tr} \nabla\varphi)^+$. In view of Ad 2.(c) in the proof of Theorem 4.2.2 it is unclear whether this replacement enables us to prove the noninterpenetration condition.

In delamination problems it seems to be physically reasonable to allow for friction on the crack surfaces. This would require to include a friction law to the delamination model (4.58)–(4.62).

Delamination admits movements of originally bonded material points $x \in \Gamma_{\mathrm{C}}$ relatively to each other. But the points forming the opposite crack surfaces may hinder each other in moving, which can be described by Coulomb sliding friction. This type of friction occurs if the relative velocity of the opposite crack surfaces is nonzero. If the relative velocity becomes zero again, then Coulomb static friction takes place, which can be understood as another kind of adhesion. Therefore it might be reasonable to change the dissipation potential capturing the energy dissipated due to delamination $\mathcal{R}(\dot{z})$ from (4.62) e.g. to

$$\mathcal{R}(z, \dot{z}) := \begin{cases} \int_\Omega -\dot{z}(2 - z)\,\mathrm{d}x & \text{if } \dot{z} \leq 0 \text{ or } z = 0, \\ \infty & \text{otherwise,} \end{cases}$$

which would allow for an increase of adhesion if $z = 0$, so that static friction is included.

This extended delamination model including sliding and static friction at finite strains might even be used to model the contact of tectonic plates in order to predict earthquakes, since the movement of tectonic plates can be interpreted as quasistatic. With this extended model an earthquake would be most likely in those regions of two tectonic plates in contact, where $z = 0$ and where the relative velocity of points in this contact zone is nonzero.

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