The geometry of the space of branched Rough Paths

Nikolas Tapia[1], joint work w. Lorenzo Zambotti[2]

[1]NTNU Trondheim
[2]Sorbonne-Unversité

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Introduction

Rough paths were introduced by Terry Lyons near the end of the 90's to deal with stochastic integration (and SDEs) in a path-wise sense.

Some years later Massimiliano Gubinelli introduced controlled rough paths, and brached Rough Paths a decade after Lyons' work.

In 2014, Martin Hairer introduced Regularity Structures which generalize branched Rough Paths.

All of these objects consist of a mixture of algebraic and analytic properties.

Given $x \in C^1$ and $V \in C^\infty$, consider

$$\dot{y}_t = V(y_t)\dot{x}_t.$$

How can we get a local description of y? Note that, **setting** $\delta\psi_{st} := \psi_t - \psi_s$,

$$R^1_{st} := \delta y_{st} - V(y_s)\delta x_{st} = \int_s^t \left(V(y_u) - V(y_s)\right) \dot{x}_u \, \mathrm{d}u = o(|t-s|).$$

We can be more precise. Set $R^2_{st} := \delta y_{st} - V(y_s)\delta x_{st} - V'(y_s)V(y_s)\frac{(\delta x_{st})^2}{2}$.

$$\begin{aligned}
R^2_{st} &= \int_s^t (V(y_u) - V(y_s))\dot{x}_u \,\mathrm{d}u - V'(y_s)V(y_s)\int_s^t\int_s^u \dot{x}_r \,\mathrm{d}r\, \dot{x}_u \,\mathrm{d}u \\
&= V'(y_s)\int_s^t \delta y_{su}\dot{x}_u \,\mathrm{d}u - V'(y_s)V(y_s)\int_s^t\int_s^u \dot{x}_r \,\mathrm{d}r\, \dot{x}_u \,\mathrm{d}u + o(|t-s|^2) \\
&= V'(y_s)\int_s^t\int_s^u V(y_r)\dot{x}_r \,\mathrm{d}r\, \dot{x}_u \,\mathrm{d}u - V'(y_s)V(y_s)\int_s^t\int_s^u \dot{x}_r \,\mathrm{d}r\, \dot{x}_u \,\mathrm{d}u + o(|t-s|^2) \\
&= V'(y_s)\int_s^t\int_s^u (V(y_r) - V(y_s))\dot{x}_r \,\mathrm{d}r\, \dot{x}_u \,\mathrm{d}u + o(|t-s|^2) \\
&= o(|t-s|^2)
\end{aligned}$$

Geometric rough paths

Geometric rough paths (signatures) have recently found a number of applications in Data Analysis and Statistical Learning.

For a smooth path x, one defines its signature $S(x) : [0,1]^2 \to T(\mathbb{R}^d)^*$ as

$$\langle S(x)_{s,t}, e_{i_1\cdots i_n}\rangle = \int_s^t \int_s^{t_{n-1}} \cdots \int_s^{t_1} \mathrm{d}x^{i_1}_{u_1}\, \mathrm{d}x^{i_2}_{u_2} \cdots \mathrm{d}x^{i_n}_{u_n}$$

i.e. $S(x)$ is the collection of all iterated integrals of the components of x. Here, $e_{i_1\cdots i_n} := e_{i_1} \otimes \cdots \otimes e_{i_n}$ is a basis element of $T(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \cdots$

For example:

$$\langle S(x)_{s,t}, e_i\rangle = x^i_t - x^i_s$$

$$\langle S(x)_{s,t}, e_{ij}\rangle = \int_s^t (x^i_u - x^i_s)\,\mathrm{d}x^j_u, \quad \langle S(x)_{s,t}, e_{ii}\rangle = \frac{(x^i_t - x^i_s)^2}{2}$$

The vector space $T(\mathbb{R}^d)$ can be made into an algebra in two ways: the tensor (or concatenation) product, and the *shuffle product*.

Example:

$$e_i \sqcup\!\sqcup\, e_j = e_{ij} + e_{ji}, \quad e_{ij} \sqcup\!\sqcup\, e_{pq} = e_{ijpq} + e_{ipjq} + e_{pijq} + e_{ipqj} + e_{piqj} + e_{pqij}.$$

It also carries two coproducts: the deconcatenation coproduct Δ and the deshuffling coproduct $\Delta_{\sqcup\!\sqcup}$.

In fact, $(T(\mathbb{R}^d), \otimes, \Delta_{\sqcup\!\sqcup})$ and $(T(\mathbb{R}^d), \sqcup\!\sqcup, \Delta)$ are Hopf algebras, dual to one another.

The family of iterated integrals satisfies the so-called *shuffle relation*, implied by the integration-by-parts formula:

$$\langle S(x)_{s,t}, e_{i_1\cdots i_n} \sqcup\!\sqcup\, e_{i_{i+1}\cdots i_{n+m}}\rangle = \langle S(x)_{s,t}, e_{i_1\cdots i_n}\rangle\langle S(x)_{s,t}, e_{i_{n+1}\cdots i_{n+m}}\rangle.$$

For example, for $n = 1, m = 1$ we recover integration by parts:

$$\int_s^t \int_s^u \mathrm{d}x^i_{u_1}\,\mathrm{d}x^j_{u_2} + \int_s^t \int_s^u \mathrm{d}x^j_{u_1}\,\mathrm{d}x^i_{u_2} = \int_s^t \mathrm{d}x^i_u \int_s^t \mathrm{d}x^j_u.$$

It also satisfies the following identity, called *Chen's rule*, a generalization of $\int_s^u + \int_u^t = \int_s^t$:

$$\langle S(x)_{s,t}, e_{i_1\cdots i_n}\rangle = \langle S(x)_{s,u} \hat{\otimes} S(x)_{u,t}, \Delta e_{i_1\cdots i_n}\rangle$$

A classical theorem by Young tells us that the integration operator

$$I(f,g) := \int_0^1 f_s \, \mathrm{d}g_s$$

can be extended continuously from $C^0 \times C^1 \to C^1$ to $C^\alpha \times C^\beta \to C^\beta$ **if and only if** $\alpha + \beta > 1$.

Thus, finding the signature $S(x)$ as above is only possible for paths in C^α for $\alpha > \frac{1}{2}$.

Theorem (Lyons–Victoir (2007))

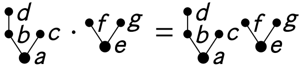
Given $\alpha < \frac{1}{2}$ with $\alpha^{-1} \notin \mathbb{N}$ and $x \in C^\alpha$, there exists a map $X : [0,1]^2 \to T(\!(\mathbb{R}^d)\!)$ such that $X_{s,t}$ is multiplicative, $X_{s,u} \otimes X_{u,t} = X_{s,t}$ and $|\langle X_{s,t}, e_{i_1 \cdots i_k} \rangle| \lesssim |t-s|^{k\gamma}$. It also satisfies $\langle X_{s,t}, e_i \rangle = \delta x^i_{st}$.

Branched rough paths

Let $(\mathscr{H}, \cdot, \Delta)$ be the Butcher–Connes–Kreimer Hopf algebra.

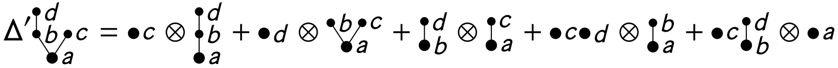
As an algebra, $\mathscr{H}$ is the commutative polynomial algebra over the set $\mathscr{T}$ of non-planar trees decorated by some alphabet A.

The product is simply the disjoint union of forests, e.g.



The empty forest 1 acts as the unit.

The coproduct Δ is described in terms of admissible cuts. For example



Consider again, for smooth x and V,

$$\dot{y}_t = V(y_t)\dot{x}_t.$$

Theorem (B-Series expansion (Gubinelli, 2010))

We have the expansion

$$\delta y_{st} = \sum_{\tau \in \mathscr{T}} \frac{1}{\sigma(\tau)} V_\tau(y_s)\langle X_{st}, \tau\rangle$$

Here V_τ is the *elementary differential*

$$V_{[\tau_1 \cdots \tau_k]}(y) = V^{(k)}(y)V_{\tau_1}(y) \cdots V_{\tau_k}(y).$$

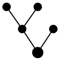
Example

$$V_{[\bullet]}(y) = V'(y)V(y), \quad V_{[[\bullet\bullet]\bullet]}(y) = V''(y)^2 V(y)^3.$$

The factor $\langle X_{st}, \tau \rangle$ is defined recursively:

$$\langle X_{st},\ [\tau_1 \cdots \tau_k] \rangle = \int_s^t \langle X_{su}, \tau_1 \rangle \cdots \langle X_{su}, \tau \rangle \, \dot{x}_u \mathrm{d}u$$

Example:

$\langle X_{st},$  $\rangle = \frac{1}{2}(x_t - x_s)^2, \quad \langle X_{st},$  $\rangle = \frac{1}{12}(x_t - x_s)^5$

Let G be the multiplicative functionals (characters) on $\mathscr{H}$.

Definition (Gubinelli (2010))

A branched Rough Path is a map $X : [0,1]^2 \to G$ *such that*

$$X_{su} \star X_{ut} = X_{st}, \quad |\langle X_{st}, \tau\rangle| \lesssim |t-s|^{\gamma|\tau|}.$$

Example: let $(B_t)_{t\geq 0}$ be a Brownian motion, set $\langle X_{st}, \bullet\rangle := B_t - B_s$ and

$$\langle X_{st}, [\tau_1 \cdots \tau_k]\rangle = \int_s^t \langle X_{su}, \tau_1\rangle \cdots \langle X_{su}, \tau_k\rangle \, \mathrm{d}B_u.$$

That is:

$$\langle X_{st}, [\bullet\,\bullet]\rangle = \int_s^t \left(\int_s^u \mathrm{d}B_r\right)\left(\int_s^u \mathrm{d}B_r\right) \mathrm{d}B_u = \int_s^t (B_u - B_s)^2 \, \mathrm{d}B_u.$$

Let $\mathscr{C}_k$ be the continuous functions in k variables vanishing when consecutive variables coincide.

Gubinelli (2003) defines an exact cochain complex

$$0 \to \mathbb{R} \to \mathscr{C}_1 \xrightarrow{\delta_1} \mathscr{C}_2 \xrightarrow{\delta_2} \mathscr{C}_3 \xrightarrow{\delta_3} \cdots$$

that is $\delta_{k+1} \circ \delta_k = 0$ and $\operatorname{im} \delta_k = \ker \delta_{k+1}$.

Remark

If $F \in \ker \delta_2$ then there exists $f \in \mathscr{C}_1$ such that $F_{st} = f_t - f_s$.
If $C \in \ker \delta_3$ then there exists $F \in \mathscr{C}_2$ such that $C_{sut} = F_{st} - F_{su} - F_{ut}$.

In general, none of these operators are injective: if $F = G + \delta_{k-1} H$ then $\delta_k F = \delta_k G$.

Can do more if we restrict to smaller spaces: let $\mathscr{C}_2^\mu$ be the $F \in \mathscr{C}_2$ such that

$$\|F\|_\mu := \sup_{s<t} \frac{|F_{st}|}{|t-s|^\mu} < \infty.$$

Similarly, $\mathscr{C}_3^\mu$ are the $C \in \mathscr{C}_3$ such that $\|C\|_\mu < \infty$ for some suitable norm.

Theorem (Gubinelli (2004))

There is a unique linear map $\Lambda : \mathscr{C}_3^{1+} \cap \ker \delta_3 \to \mathscr{C}_2^{1+}$ *such that* $\delta_2 \Lambda = \mathrm{id}$*. In each of* $\mathscr{C}_3^\mu$ *for* $\mu > 1$ *it satisfies*

$$\|\Lambda C\|_\mu \leq \frac{1}{2^\mu - 2} \|C\|_\mu.$$

Chen's rule reads

$$\langle X_{st}, \tau \rangle = \langle X_{su}, \tau \rangle + \langle X_{ut}, \tau \rangle + \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle.$$

or

$$\delta_2 F^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle$$

where $F^\tau_{st} := \langle X_{st}, \tau \rangle$.

The norm on $\mathscr{C}_3$ is such that the bound for X implies $\delta_2 F^\tau \in \mathscr{C}_3^{\gamma|\tau|}$.

The integer $N := \lfloor \gamma^{-1} \rfloor$ is special. Let G_N denote the multiplicative maps on the subcoalgebra

$$\mathscr{H}_N := \bigoplus_{n=0}^{N} \mathscr{H}_{(n)}.$$

Theorem (Gubinelli (2010))

Suppose $X : [0,1]^2 \to G_N$ satisfies $|\langle X_{st}, \tau\rangle| \lesssim |t-s|^{\gamma|\tau|}$. Then there exists a unique map $\hat{X} : [0,1]^2 \to G$ on $\mathscr{H}$ such that $\hat{X}\big|_{\mathscr{H}_N} = X$.

Proof.

Suppose $|\tau| = N+1$ is a tree and set $C^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta'\tau\rangle$.
First one shows that $C^\tau \in \ker \delta_3$ by using the coassociativity of Δ'.
The bound above implies that $C^\tau \in \mathscr{C}_3^{\gamma|\tau|}$.
Therefore C^τ lies in the domain of Λ and we can set

$$\langle X_{st}, \tau\rangle := (\Lambda C^\tau)_{st}.$$

Continue inductively. □

Results

The previous argument works only because $\gamma|\tau| > 1$ i.e. $|\tau| > N$.

If $\gamma|\tau| \le 1$, for any $g^\tau \in C^{\gamma|\tau|}$ (Hölder space) the function

$$G^\tau_{st} := F^\tau_{st} + \delta_1 g^\tau_{st}$$

also satisfies $\delta_2 G^\tau_{sut} = \langle X_{su} \otimes X_{ut}, \Delta'\tau\rangle$.

Let X and X' be two BRPs coinciding on $\mathscr{H}_{(1)}$.

Fix τ with $|\tau| = 2$ and let $F^\tau_{st} := \langle X_{st}, \tau\rangle$, $G^\tau_{st} := \langle X'_{st}, \tau\rangle$.

Then $\delta_2 F^\tau = \delta_2 G^\tau$ so there is $g^\tau \in \mathscr{C}_1$ such that

$$F^\tau_{st} = G^\tau_{st} + \delta_1 g^\tau_{st}.$$

Moreover $g^\tau \in C^{2\gamma}$.

This suggests that there might be an action of

$$\mathscr{D}^\gamma := \{(g^\tau)_{|\tau|\leq N} : g^\tau \in C^{\gamma|\tau|}, g_0^\tau = 0\}$$

on the space $\mathbf{BRP}^\gamma$ of branched Rough Paths.

Theorem (T.-Zambotti (2018))

Let $\gamma \in (0,1)$ such that $\gamma^{-1} \notin \mathbb{N}$. There is a regular action of $\mathscr{D}^\gamma$ on $\mathbf{BRP}^\gamma$.

This means we have a mapping

$$\mathscr{D}^\gamma \times \mathbf{BRP}^\gamma \ni (g, X) \to gX \in \mathbf{BRP}^\gamma$$

such that

- $g'(gX) = (g' + g)X$ for all $g, g' \in \mathscr{D}^\gamma$ and,
- for every pair $X, X' \in \mathbf{BRP}^\gamma$ there exists a *unique* $g \in \mathscr{D}^\gamma$ such that $X' = gX$.

$\mathbf{BRP}^\gamma$ is a *principal homogeneous space* for $\mathscr{D}^\gamma$.

Very rough sketch of proof

If $\gamma > \frac{1}{2}$ the result is easy: just set

$$\langle gX_{st}, \bullet_i \rangle = \langle X_{st}, \bullet_i \rangle + \delta g^{\bullet_i}_{st}$$

and $\langle gX, \tau \rangle$ for $|\tau| \geq 2$ is given by the Sewing Lemma.

If $\frac{1}{3} < \gamma < \frac{1}{2}$ the action is the same in degree 1. In degree 2 we must have

$$\delta_2 \langle gX, \overset{\bullet j}{\underset{\bullet i}{|}} \rangle_{sut} = (\delta x^j_{su} + \delta g^{\bullet_j}_{su})(\delta x^i_{ut} + \delta g^{\bullet_i}_{ut}).$$

The canonical choice (Young integral)

$$\int_s^t (\delta x^j_{su} + \delta g^{\bullet_j}_{su})\, \mathrm{d}(x^i_u + g^{\bullet_i}_u)$$

is not well defined since $2\gamma < 1$.

In higher degrees the expressions are more complicated.

We handle this by constructing an *anisotropic* geometric Rough Path $\bar{X}$ such that

$$\langle X_{st}, \tau \rangle = \langle \bar{X}_{st}, \psi(\tau) \rangle$$

where $\psi : (\mathscr{H}, \cdot, \Delta) \to (T(\mathscr{T}_n), \sqcup\!\sqcup, \bar{\Delta})$ is the Hairer–Kelly map.

Anisotropic means that letters (trees) are allowed to have different weights.

In addition to the standard grading by the number of letters we have a weight function, e.g.

$$\omega \left(\bullet_a \otimes \overset{\bullet_c}{\underset{\bullet_b}{|}} \right) = 3\gamma.$$

More concretely, $\bar{X}$ is a character over the shuffle algebra on the alphabet $\mathcal{T}_N$.

Single trees become letters in $T(\mathcal{T}_N)$, hence they are in degree one!

Set

$$\langle g\bar{X}, \tau\rangle := \langle \bar{X}, \tau\rangle + \delta g^{\tau}.$$

Then define

$$\langle gX, \tau\rangle = \langle g\bar{X}, \psi(\tau)\rangle.$$

1. Lifting of Chen's rule to the Lie algebra $\mathfrak{g}$. If $X_{st} = \exp_\star(\alpha_{st})$ then
$$\alpha_{st} = \mathrm{BCH}(\alpha_{su}, \alpha_{ut}) = \alpha_{su} + \alpha_{ut} + \mathrm{BCH}'(\alpha_{su}, \alpha_{ut}).$$
2. We use an explicit BCH formula due to Reutenauer.
3. We use the Lyons–Victoir (2007) method but in a constructive way, without invoking the axiom of choice.
4. However, the action is not unique nor canonical. The construction depends on a finite number of arbitrary choices.
5. We are able to construct γ-regular $\mathscr{H}$-rough paths over any $x \in C^\gamma(\mathbb{R}^d)$.

Next goals

1. Understand the algebraic picture. The action gX is not very easy to compute.
2. Relation with modification of products as explored in Ebrahimi-Fard, Patras, T. and Zambotti (2017).
3. Actions of an appropriate $\mathscr{D}^\gamma$ for the geometric case.
4. Clarify what the action means for controlled paths and RDEs.

Danke schön!