Non-commutative Wick polynomials

Nikolas Tapia (WIAS/TU Berlin)
based on ongoing work with K. Ebrahimi-Fard (NTNU), F. Patras (U. Nice) and L. Zambotti (Sorbonne U.)

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Goals

1. Motivation
2. Moments and cumulants
3. Free Wick polynomials
4. If time permits:
   4.1 Modification of products
   4.2 Relation to power series

Classical Wick polynomials: A probabilist's approach

Definition

Let X be a r.v. with $\mathbb{E}X^n < \infty$ for all $n > 0$.
Recursive definition:

$$W_n'(x) = nW_{n-1}(x), \quad \mathbb{E}W_n(X) = 0.$$

For example: $W_1(x) = x - \mathbb{E}X, \quad W_2(x) = x^2 - 2x\mathbb{E}X + 2(\mathbb{E}X)^2 - \mathbb{E}X^2, \ldots$

Definition (Multivariate Wick polynomials)

$$\frac{\partial}{\partial x_i} W_n(x_1, \ldots, x_n) = W_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad \mathbb{E}W_n(X_1, \ldots, X_n) = 0.$$

Classical Wick polynomials: A naïve physics approach

Let $\mathcal{F}^\circ := \mathbb{C}\Omega \oplus H \oplus H^{\circ 2} \oplus \cdots$ be the symmetric Fock space over H.

For each $f \in H$ we have (bosonic) annihilation and creation operators $a(f)$, $a^\dagger(f)$ on $\mathcal{F}^\circ$ such that

$$a(f)\Omega = 0, \quad a^\dagger(f)\Omega = f$$

and

$$\begin{aligned} a(f)(f_1 \circ \cdots \circ f_n) &= \sum_{j=1}^{n} \langle f, f_j \rangle f_1 \circ \cdots \circ f_{j-1} \circ f_{j+1} \circ \cdots \circ f_n, \\ a^\dagger(f)(f_1 \circ \cdots \circ f_n) &= f \circ f_1 \circ \cdots \circ f_n. \end{aligned}$$

They satisfy the (canonical) commutation relation

$$a(f)a^\dagger(g) - a^\dagger(g)a(f) = \langle f, g \rangle 1.$$

Classical Wick polynomials: A naïve physics approach

The *normal order operator* N puts creation operators to the left of annihilation operators.

For example $\mathrm{N}(a^\dagger(f)a(g)) = a^\dagger(f)a(g)$ and $\mathrm{N}(a(f)a^\dagger(g)) = a^\dagger(g)a(f)$, etc.

The (unnormalized) position operators $p(f) := a(f) + a^\dagger(f)$ satisfy $\mathrm{N}(p(f)) = p(f)$ and

$$\begin{aligned} p(f)p(g) &= a(f)a(g) + a(f)a^\dagger(g) + a^\dagger(f)a(g) + a^\dagger(f)a^\dagger(g) \\ &= a(f)a(g) + a^\dagger(g)a(f) + a^\dagger(f)a(g) + a^\dagger(f)a^\dagger(g) + \langle f, g\rangle 1 \\ &= \mathrm{N}(p(f)p(g)) + \langle f, g\rangle 1, \end{aligned}$$

i.e. $\mathrm{N}(p(f)p(g)) = p(f)p(g) - \langle f, g\rangle 1$.

Denoting $X = p(f)$, $Y = p(g)$ and $\mathbb{E}(b) := \langle b\Omega, \Omega\rangle$ we see that $\mathrm{N}(XY) = XY - \mathbb{E}(XY) = W_2(X, Y)$.

By definition $\mathbb{E}[\mathrm{N}(p(f_1)\cdots p(f_n))] = 0$.

Classical Wick polynomials: Link to Dyson–Schwinger

Suppose we have a measure of the form $\mu(\mathrm{d}x) = \mathrm{e}^{-S(x)}\,\mathrm{d}x$ and set $I[f] = \int f\,\mathrm{d}\mu$.

Integrating by parts we get, for any nice function f, the Dyson-Schwinger equation

$$I[\partial_i f - (\partial_i S)f] = 0.$$

The measure μ is characterized by the values $I[f]$, for nice f, usually polynomials are enough.

For a 1-D Gaussian weight the equation is simply $I(f' - xf) = 0$ which entails $I(x^{2n}) = (n-1)!!$ and zero else.

This ultimately means that $Z[J] := I_x(\mathrm{e}^{Jx}) = \mathrm{e}^{\frac{1}{2}J^2}$, or $I[\mathrm{e}^{Jx-\frac{1}{2}J^2}] = 1$. Observe also that $(J - \frac{\mathrm{d}}{\mathrm{d}J})Z[J] = 0$.

Actually, the Dyson-Schwinger equaition implies that $I(H_{n+1}) = -I(H_n' - xH_n) = 0$, so that re-expanding x^n in the H_n basis:

$$I(x^n) = \sum_{k=0}^{n} \alpha_k I(H_k) = \alpha_0$$

Classical Wick polynomials: Link to Dyson–Schwinger

More generally, for a multidimensional Gaussian weight $S(x) = \frac{1}{2}g_{ij}x^i x^j$ a similar argument gives that

$$I(x^j x^{i_1} \cdots x^{i_n}) = \sum_{k=1}^{n} g^{ij_k} I(x^{i_1} \cdots \hat{x}^{i_k} \cdots x^{i_n}).$$

But, since we have that $I(H_n) = 0$ we can again re-expand any monomial in the H_n basis and recover the above formula from our knowledge of H_n.

In our annihilation–creator operator example:

$$\begin{aligned} \langle p(f_1)p(f_2)\Omega, \Omega\rangle &= \langle f_1, f_2\rangle \\ \langle p(f_1)p(f_2)p(f_3)p(f_4)\Omega, \Omega\rangle &= \langle f_1, f_2\rangle\langle f_3, f_4\rangle + \langle f_1, f_3\rangle\langle f_2, f_4\rangle + \langle f_1, f_4\rangle\langle f_2, f_3\rangle \end{aligned}$$

Classical Wick polynomials: Hopf-algebraic approach

Definition

A noncommutative probability space is a tuple (A, φ) where A is an associative algebra and $\varphi : A \longrightarrow k$ is unital, i.e. $\varphi(1_A) = 1$.

On $T(A) := \bigoplus_{n>0} A^{\otimes n}$ define $\Delta\colon T(A) \to T(A) \otimes T(A)$ by

$$\Delta^{\sqcup\!\sqcup}(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{[n] \setminus S}.$$

This induces a product on $T(A)^*$:

$$\mu \sqcup\!\sqcup \nu := (\mu \otimes \nu)\Delta^{\sqcup\!\sqcup}.$$

Classical Wick polynomials: Hopf-algebraic approach

Define $\phi : T(A) \to k$ by $\phi(a_1 \cdots a_n) := \varphi(a_1 \cdot_A \cdots \cdot_A a_n)$ and extend to $\overline{T}(A) := k1 \oplus T(A)$ by $\phi(1) = 1$.

There is $c : \overline{T}(A) \to k$ with $c(1) = 0$ such that $\phi = \exp^{\sqcup\!\sqcup}(c)$. In particular

$$\phi(a_1 \cdots a_n) = \sum_{\pi \in P(n)} \prod_{B \in \pi} c(a_B).$$

Definition

Since ϕ is invertible, we set $W := (\mathrm{id} \otimes \phi^{-1})\Delta^{\sqcup\!\sqcup}$.

Theorem

The map $W : \overline{T}(A) \to \overline{T}(A)$ is the unique linear map such that $\partial_a \circ W = W \circ \partial_a$ and $\phi \circ W = \varepsilon$. Its inverse is given by $W^{-1} = (\mathrm{id} \otimes \phi)\Delta^{\sqcup\!\sqcup}$.

Observe also that trivially $\varepsilon \circ W^{-1} = \phi = \exp^{\sqcup\!\sqcup}(c)$.

Moments and cumulants

We have other notions of independence: freeness, boolean idependence, monotone independence, etc. . .

Each is characterised by a set of cumulants: κ, β, ρ resp.

On the double tensor algebra $\overline{T}(T(A))$ consider

$$\Delta(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}.$$

This splits as

$$\Delta_{\prec}(a_1 \cdots a_n) := \sum_{1 \in S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S},$$

$$\Delta_{\succ}(a_1 \cdots a_n) := \sum_{1 \notin S \subseteq [n]} a_S \otimes a_{J_1^S} | \cdots | a_{J_k^S}.$$

Therefore, the convolution product $\mu * \nu := (\mu \otimes \nu)\Delta$ also splits:

$$\mu \prec \nu := (\mu \otimes \nu)\Delta_{\prec}, \quad \mu \succ \nu := (\mu \otimes \nu)\Delta_{\succ}.$$

Consider $\Phi \colon \overline{T}(T(A)) \longrightarrow k$ the unique character extension of ϕ.

Theorem (Ebrahimi-Fard,Patras; 2014, 2017)

The cumulants κ, β, ρ are the unique infinitesimal characters of $\overline{T}(T(A))$ such that

$$\begin{aligned}\Phi &= \varepsilon + \kappa \prec \Phi \\ &= \varepsilon + \Phi \succ \beta\end{aligned}$$

and $\Phi = \exp_*(\rho)$.

Moments and cumulants: Some known results

Theorem (Speicher; 1997)

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{\pi \in NC(n)} \prod_{B \in \pi} \kappa(a_B).$$

Theorem (Speicher, Woroudi; 1997)

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{\pi \in Int(n)} \prod_{B \in \pi} \beta(a_B).$$

Theorem (Hasebe, Saigo; 2011)

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{(\pi,\lambda) \in M(n)} \frac{1}{|\pi|!} \prod_{B \in \pi} \rho(a_B)$$

Moments and cumulants

We write

$$\Phi = \mathcal{E}_{\prec}(\kappa) = \mathcal{E}_{\succ}(\beta) = \exp_*(\rho).$$

Every character has an inverse for $*$. For Φ we have

$$\Phi^{-1} = \mathcal{E}_{\succ}(-\kappa) = \mathcal{E}_{\prec}(-\beta) = \exp_*(-\rho).$$

In fact, characters on $\overline{T}(T(A))$ form a group denoted by G.

Observe that $\Delta \colon T(A) \longrightarrow T(A) \otimes \overline{T}(T(A))$, i.e. we have a coaction.

Thus, the character group G acts on $\mathrm{End}(T(A))$.

Definition

By analogy, define $W : T(A) \longrightarrow T(A)$ by

$$W := (\mathrm{id} \otimes \Phi^{-1})\Delta.$$

Examples:

$$\begin{aligned}
W(a) &= a - \phi(a)1 \\
W(ab) &= ab - a\phi(b) - b\phi(a) + (2\phi(a)\phi(b) - \phi(a \cdot b))1 \\
W(abc) &= abc - \varphi(c)ab - \varphi(b)ac - \varphi(a)bc \\
&\quad - [\phi(b \cdot c) - 2\phi(b)\phi(c)]a + \phi(a)\phi(c)b - [\phi(a \cdot b) - 2\phi(a)\phi(b)]c \\
&\quad - [\phi(a \cdot b \cdot c) - 2\phi(a)\phi(b \cdot c) - 2\phi(c)\phi(a \cdot b) - \phi(b)\phi(a \cdot c) \\
&\quad + 5\phi(a)\phi(b)\phi(c)]1
\end{aligned}$$

By definition

$$\Phi \circ W = (\Phi \otimes \Phi^{-1})\Delta = \varepsilon$$

that is, $\Phi(W(a_1 \ldots a_n)) = 0$ for any $a_1, \ldots, a_n \in A$.

It's easy to check that W is invertible with $W^{-1} = (\mathrm{id} \otimes \Phi)\Delta$ and so $\Phi = \varepsilon \circ W^{-1}$.

In particular

$$a_1 \cdots a_n = \sum_{S \subseteq [n]} W(a_s)\Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S}).$$

Theorem (Anshelevich, 2004)

$$W(a_1 \cdots a_n) = \sum_{S \subseteq [n]} a_S \sum_{\substack{\pi \in \mathrm{Int}([n] \setminus S) \\ \pi \cup S \in NC(n)}} (-1)^{|\pi|} \prod_{B \in \pi} \kappa(a_B).$$

Theorem

The Wick polynomials satisfy the recursion

$$W(a_1 \cdots a_n) = a_1 W(a_2 \cdots a_n) - \sum_{j=0}^{n-1} W(a_{j+1} \cdots a_n)\kappa(a_1 \cdots a_j).$$

Proof.

$$\begin{aligned} W &= (\mathrm{id} \otimes \Phi^{-1})\Delta \\ &= \mathrm{id} \prec \Phi^{-1} + \mathrm{id} \succ \Phi^{-1} \\ &= \mathrm{id} \prec \Phi^{-1} - \mathrm{id} \succ (\Phi^{-1} \succ \kappa) \\ &= \mathrm{id} \prec \Phi^{-1} - W \succ \kappa. \end{aligned}$$

□

Wick polynomials

Now consider the *full Fock space* $\mathcal{F} = \mathbb{C}\Omega \oplus H \oplus H^{\otimes 2} \oplus \cdots$.

We have annihilation and creator operators

$$a(f)(f_1 \otimes \cdots \otimes f_n) = \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n, \quad a^*(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n.$$

This time they satisfy $a(f)a^*(g) = \langle f, g \rangle 1$.

Set as before $p(f) = (a(f) + a^*(f))$. We have

$$\langle p(f_1)p(f_2)\Omega, \Omega \rangle = \langle f_1, f_2 \rangle$$
$$\langle p(f_1)p(f_2)p(f_3)p(f_4)\Omega, \Omega \rangle = \langle f_1, f_2 \rangle \langle f_3, f_4 \rangle + \langle f_1, f_4 \rangle \langle f_2, f_3 \rangle.$$

We get a free version of Wick's theorem (Effros, Poppa; 2003).

Theorem

The Wick polynomials can be expressed in terms of boolean cumulants

$$W = (\mathrm{id} - \mathrm{id} \succ \beta) \prec \Phi^{-1}$$

Proof.

Previous theorem plus the fact that $\kappa = \Phi \succ \beta \prec \Phi^{-1}$. □

Definition

The Boolean Wick map is defined by

$$W' := \mathrm{id} - \mathrm{id} \succ \beta.$$

Therefore

$$W'(a_1 \cdots a_n) = a_1 \cdots a_n - \sum_{j=1}^{n} a_{j+1} \cdots a_n \beta(a_1 \cdots a_j).$$

Wick polynomials

Theorem

Boolean Wick polynomials are centered

Proof.

$\Phi \circ W' = \Phi - \Phi \succ \beta = \varepsilon$ $\square$

Theorem

We have

$$a_1 \cdots a_n = W'(a_1 \cdots a_n) + \sum_{j=1}^{n-1} \Phi(a_1 \cdots a_j) W'(a_{j+1} \cdots a_n).$$

From a previous computation $W' = W \prec \Phi$, that is

$$W'(a_1 \cdots a_n) = \sum_{1 \in S \subseteq [n]} W(a_S) \Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S}).$$

Assume we have a second state $\psi : A \to k$.

Definition

Two-state cumulants are defined implicitly by

$$\varphi(a_1 \cdot_A \cdots \cdot_A a_n) = \sum_{\pi \in NC(n)} \prod_{B \in \mathrm{Outer}(\pi)} R^{\varphi,\psi}(a_B) \prod_{B \in \mathrm{Inner}(\pi)} \kappa^{\psi}(a_B).$$

Theorem (Ebrahimi-Fard, Patras; 2018)

$R^{\varphi,\psi}$ *is the unique infinitesimal character of* $\overline{T}(T(A))$ *such that*

$$\Phi = \varepsilon + \Phi \succ (\Psi^{-1} \succ R^{\varphi,\psi} \prec \Psi).$$

Wick polynomials

Directly,

$$R^{\varphi,\psi} = \Psi \succ \beta^{\varphi} \prec \Psi^{-1}.$$

In particular,

$$\begin{aligned} R^{\varphi,\varphi} &= \Phi \succ \beta^{\varphi} \prec \Phi^{-1} = \kappa^{\varphi}, \\ R^{\varphi,\varepsilon} &= \beta^{\varphi}. \end{aligned}$$

Definition

The conditionally-free Wick polynomials are defined as

$$W^{c} := W \prec (\Phi * \Psi^{-1}).$$

This means

$$W^{c} = \Big(\mathrm{id} - \mathrm{id} \succ \Theta_{\Psi}(R^{\varphi,\psi})\Big) \prec \Psi^{-1}$$

where $\Theta_{\Psi}(\mu) := \Psi^{-1} \succ \mu \prec \Psi$.

The q-deformation

Back to the Fock space interpretation, we actually have a whole family parametrized by $q \in (-1, 1)$. The bosonic (i.e. symmetric) corresponds to $q = 1$, the free case corresponds to $q = 0$.

The next interesting case is the fermionic setting $q = -1$.

For any $q \in (-1, 1)$, the associated annihilation and creation operators are

$$a(f)(f_1 \otimes \cdots \otimes f_n) = \sum_{j=1}^{n} q^{j-1} \langle f, f_j \rangle f_1 \otimes \cdots \otimes \hat{f_j} \otimes \cdots \otimes f_n$$
$$a^*(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n.$$

satisfying the q-commutator relation

$$[a(f), a^*(g)]_q := a(f)a^*(g) - q a^*(g)a(f) = \langle f, g \rangle 1.$$

Thank you!

The q-deformation: Modification of products

Since W is invertible, one can induce a product on $T(A)$ by

$$x \bullet y = W(W^{-1}(x)W^{-1}(y))$$

Proposition

The $\bullet$ product admits the closed-form expression: for $x = a_1 \cdots a_n$, $y = a_{n+1} \cdots a_{n+m}$

$$x \bullet y = \sum_{S \subseteq [n+m]} a_S \Phi(a_{J_1^S}) \cdots \Phi(a_{J_k^S}).$$

Power series

The relations between moments and cumulants can also be encoded by power series.

In the classical case, one uses exponential generating functions:

$$\sum_{n\geqslant 0} m_n \frac{\lambda^n}{n!} = \exp\left(\sum_{k>0} c_k \frac{\lambda^k}{k!}\right).$$

In the noncommutative setting, these are replaced by ordinary generating functions.

Let

$$M(w) := 1 + \sum_{\alpha} \varphi(a_\alpha) w_\alpha, \quad R(w) := \sum_{\alpha} \kappa(a_\alpha) w_\alpha, \quad \eta(w) := \sum_{\alpha} \beta(a_\alpha) w_\alpha.$$

Considering a new set of variables $z_i = w_i M(w)$ we have

$$M(w) = 1 + R(z), \quad M(w) = 1 + \eta(w) M(w).$$

Power series

It turns out that the Hopf-algebraic language above describes two operations on power series.

Let G^p and G^c denote the group of invertible power series and formal diffeomorphisms, resp.

For $f, g \in G^p$ define

$$f^g(w) := g(w)f(z), \quad z_i = w_i g(w).$$

Also let

$$(f \curvearrowleft g)(w) := f(z), \quad z_i = w_i g(w).$$

Given $F : T(A) \longrightarrow k$ let $\Lambda(F) \in k[[w]]$ be given by

$$\Lambda(F)(w) = F(1) + \sum_{\alpha} F(a_\alpha) w_\alpha .$$

Power series

Theorem

$$\Lambda(F * G) = \Lambda(F)^{\Lambda(G)}$$

Theorem

$$\Lambda(F \prec G) = \Lambda(F) \curvearrowleft \Lambda(G).$$

Theorem

$$\Lambda(F \succ G) = \Lambda(F) \curvearrowright \Lambda(G).$$