On Large Deviation Efficiency in Statistical Inference

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Abstract

We present a general approach to statistical problems with criteria based on probabilities of large deviations. Our main idea, which originates from similarity in the definitions of the large deviation principle and weak convergence, is to develop a large deviation analogue of asymptotic decision theory. We introduce the concept of the large deviation principle (LDP) for sequences of statistical experiments, which parallels the concept of weak convergence of experiments, and prove that, in analogy with Le Cam's minimax theorem, the LDP provides an asymptotic lower bound for the sequence of appropriately defined minimax risks. We also show that the bound is tight and give a method of constructing decisions whose asymptotic risk is arbitrarily close to the bound. The construction is further specified for hypothesis testing and estimation problems.

We apply the results to a number of standard statistical models: an independent-identically-distributed sample, regression, the change-point model and others. For each model, we check the LDP; after that, considering first a hypothesis testing problem and then an estimation problem, we calculate the asymptotic minimax risks and indicate associated decisions.

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Contents

1 Introduction

The approach to statistical problems that bases its conclusions on the study of probabilities of large deviations has been in use in statistical inference since the papers by Chernoff, 1952 and Bahadur, 1960.

Chernoff, 1952 considering the problem of discriminating between two simple hypotheses showed that, if the hypotheses are fixed, the error probabilities decrease exponentially fast as the sample size tends to infinity; the corresponding optimal exponent is specified by what is now known as Chernoff's function.

Basu, 1956 and Bahadur, 1960 proposed a criterion for comparing statistical estimators based on the view that the quality of an estimator is characterised by the probability that the true value of the parameter is covered by the confidence interval of a given width $2c$ with centre at the estimate. If the width $2c$ is held fixed as the sample size grows then the probabilities that the true value of the parameter is not covered are typically exponentially small. The estimator giving the fastest decay is called now Bahadur efficient. Later, Bahadur et al., 1980, for the model of independent identically distributed observations showed that in the class of consistent estimators the optimal rate instead of Chernoff's function is specified by the Kullback–Leibler information.

The ideas of Chernoff and Bahadur have been developed in various directions. Ibragimov and Radavichius, 1981, Kallenberg, 1981, Ibragimov and Khasminskii, 1981, Radavichius, 1983 and Radavichius, 1991 studied the properties of maximum likelihood estimators from the point of view of Bahadur's criterion. Fu, 1982, Borovkov and Mogulskii, 1992b and Borovkov and Mogulskii, 1992a analysed the terms of the second and higher order of asymptotic expansions of Bahadur risks. Kallenberg, 1983, Rao, 1963, Wieand, 1976, and Ermakov, 1993 considered intermediate criteria for statistical estimators when the width of the confidence interval goes to zero at certain rate. Sievers, 1978 and Rubin and Rukhin, 1983 evaluated Bahadur risks for particular statistical models.

Lately this direction in mathematical statistics has received a new impetus, mostly in papers by Korostelev, 1993, Korostelev, 1995, see also Korostelev and Spokoiny, 1995, Korostelev and Leonov, 1995, where the classical large deviation set-up is considered in the minimax nonparametric framework.

Our aim here is to give a unified treatment of the statistical problems that use large deviation considerations. The idea is to capitalise on analogies between large deviation theory and weak convergence theory (see Lynch and Sethuraman, 1987; Vervaat, 1988; Puhalskii, 1991) and develop a large deviation analogue of asymptotic decision theory, Strasser, 1985. The approach of invoking the methods of weak convergence theory to obtain results about large deviations has proved its worth in various set-ups, Puhalskii, 1991, 1993, 1994a, 1994b, 1995a, 1995b, 1996. We show that it can successfully be applied to statistical problems too.

We begin by defining in Section 2 the concept of the large deviation principle (LDP) for a sequence of statistical experiments. Analogously to the concept of weak convergence of statistical experiments, it is a short cut for saying that the distributions of suitably defined likelihood processes satisfy the large deviation principle, Varadhan, 1966; Varadhan, 1984. We illustrate the general definition by considering a number of standard statistical models (the Gaussian shift model, the model of independent identically distributed observations, the "signal + white noise" model, the regression model with Gaussian and non–Gaussian errors, with deterministic and random design, and the change–point model). We next study properties of the LDP for statistical experiments and give a sufficient condition for it which is analogous to the local asymptotic normality condition by LeCam, 1960.

The classical minimax theorem by Le Cam states that if statistical experiments weakly converge then the minimax risks are asymptotically bounded from below by the corresponding risk for the limit model, see LeCam, 1972, LeCam, 1986, Strasser, 1985. In Section 3, we show that, similarly, if a sequence of statistical experiments obeys the LDP, then there is an asymptotic lower bound for appropriately defined minimax risks. The problem of evaluating the bound is a minimax optimization problem. Further in Section 3, we study the question of sharpness of the lower bound. We show that it is sharp under a strengthened version of the LDP. This allows us to define large deviation (LD) efficient decisions as the ones that attain the lower bound. We give a method of obtaining nearly LD efficient decisions, i.e., those whose LD asymptotic risk is arbitrarily close to the lower bound.

Sections 4 and 5 deal with applications. Section 4 adapts the results of Section 3 to the cases of hypothesis testing and estimation problems and presents explicit constructions of nearly LD efficient decisions. In Section 5, we apply the machinery to the models introduced in Section 2: we check the LDP, give conditions when the lower bounds are attained, calculate them for hypothesis testing and estimation problems, and indicate nearly LD efficient decisions. An appendix contains extensions and auxiliary results.

The results of the first four sections are new. The results that we obtain for the models are partly new and partly cover or extend earlier results.

2 The Large Deviation Principle for Statistical Experiments

Let $\{\mathcal{E}_n, n \geq 1\}$ be a sequence of statistical experiments $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta}, \theta \in \Theta)$ with a parameter set Θ, Strasser, 1985. In this section, we give the definition of the large deviation principle for $\{\mathcal{E}_n, n \geq 1\}$ and study its properties. We start with the case of dominated experiments.

2.1 The Dominated Case

Let us assume that each experiment $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta}, \theta \in \Theta)$ is dominated by a probability measure P_n, i.e., $P_{n,\theta} \ll P_n$ for all $\theta \in \Theta$. We abbreviate this by writing $\{\mathcal{E}_n, P_n, n \geq 1\}$. Denote

$$Z_{n,\theta} = \left(\frac{dP_{n,\theta}}{dP_n} \right)^{1/n}, \qquad \theta \in \Theta, \tag{2.1}$$

and let $Z_{n,\Theta} = (Z_{n,\theta}, \theta \in \Theta)$. We endow R_+^Θ with the Tihonov (product) topology and Borel σ–field so that $Z_{n,\Theta}$ is a random element of R_+^Θ; $\mathcal{L}(Z_{n,\Theta}|P_n)$ denotes the distribution of $Z_{n,\Theta}$ on R_+^Θ under P_n. Roughly speaking, the large deviation principle for $\{\mathcal{E}_n, P_n, n \geq 1\}$ means that the sequence $\{\mathcal{L}(Z_{n,\Theta}|P_n), n \geq 1\}$ of distributions on R_+^Θ obeys the large deviation principle, so we recall some basic notions of large deviation theory.

We use Varadhan's original definitions of the rate function and the large deviation principle (Varadhan, 1966; Varadhan, 1984). Let S be a Hausdorff topological space. We say that a function $I : S \to [0, \infty]$ is a rate function on S if the sets $I^{-1}([0,a])$ are compact in S for all $a \geq 0$. A sequence $\{Q_n, n \geq 1\}$ of probability measures on the Borel σ–field of S is said to obey the large deviation principle (LDP) with the rate function I if

$$\varliminf_{n\to\infty} \frac{1}{n} \ln Q_n(G) \geq -\inf_{x \in G} I(x)$$

for all open $G \subset S$ and

$$\varlimsup_{n\to\infty} \frac{1}{n} \ln Q_n(F) \leq -\inf_{x \in F} I(x)$$

for all closed $F \subset S$.

We also say that I is a probability rate function if $\inf_{x\in S} I(x) = 0$. Obviously, if I appears in the LDP, it is a probability rate function.

Recall that the contraction principle states that continuous mappings preserve the LDP, Varadhan, 1966; Varadhan, 1984.

Next, we say that the sequence $\{\mathcal{E}_n, P_n, n \geq 1\}$ satisfies condition (U) if

$$(U) \qquad \lim_{H\to\infty} \overline{\lim_{n\to\infty}} E_n^{1/n} Z_{n,\theta}^n 1(Z_{n,\theta} > H) = 0, \ \theta \in \Theta .$$

Here and below E_n denotes an expectation with respect to P_n and, by definition, $E_n^{1/n}\xi = (E_n\xi)^{1/n}$, $P_n^{1/n}(A) = (P_n(A))^{1/n}$.

Definition 2.1 *We say that a sequence $\{\mathcal{E}_n, P_n, n \geq 1\}$ of dominated statistical experiments obeys* the dominated large deviation principle (LDP) *if*

1. *the sequence $\{\mathcal{L}(Z_{n,\Theta}|P_n), n \geq 1\}$ obeys the LDP with some (probability) rate function I,*
2. *condition (U) holds.*

A critical part of the definition is condition 1. Condition (U) plays a subordinate though essential role. If we disregard condition (U), the definition is analogous to the definition of weak convergence of dominated statistical experiments (Strasser, 1985) which states that the likelihood ratios weakly converge. The role of condition (U) will become clear shortly: it ensures the compatibility of this definition with a more general one which does not depend on a choice of dominating measures and incorporates the nondominated case too. In particular, condition (U) implies that the lower bound that we obtain in Section 3 for the sequence of so called large deviation risks does not depend on dominating measures either (see Remark 3.2 below). Note that an analogue of condition (U) in the theory of weak convergence of statistical experiments is a consequence of weak convergence of the likelihood ratios and does not have to be singled out.

In applications, rather than considering $Z_{n,\theta}$, it is more convenient to deal with log-likelihood ratios $\Xi_{n,\theta}$ defined as $\Xi_{n,\theta} = \frac{1}{n}\ln\frac{dP_{n,\theta}}{dP_n}$. Let us introduce $\Xi_{n,\Theta} = (\Xi_{n,\theta}, \theta \in \Theta)$ and denote by $\mathcal{L}(\Xi_{n,\Theta}|P_n)$ the distribution of $\Xi_{n,\Theta}$ on R^Θ under P_n, where R^Θ is supplied with the Tihonov topology and Borel σ–field. If the $\Xi_{n,\theta}$ are well defined then, by the contraction principle, the LDP for the sequence $\{\mathcal{L}(\Xi_{n,\Theta}|P_n), n \geq 1\}$ implies the LDP for the sequence $\{\mathcal{L}(Z_{n,\Theta}|P_n), n \geq 1\}$.

Now we consider a number of statistical models which, on the one hand, show that the LDP for the log-likelihood ratios arises quite naturally and, on the other hand, motivate and illustrate theoretical developments below. We stop short of giving rigourous proofs of the LDP for the models deferring this until Section 5.

Example 2.1 *Gaussian Observations*

Let us observe a sample of n independent real-valued random variables $\mathbf{X}_n = (X_{1,n}, \ldots, X_{n,n})$ normally distributed with $\mathcal{N}(\theta, 1)$, $\theta \in \Theta \subset R$. For this model, $\Omega_n = R^n$

and $P_{n,\theta} = (\mathcal{N}(\theta,1))^n$, $\theta \in \Theta$. We take $P_{n,0}$ as a dominating measure P_n. Then the corresponding log-likelihood ratios are of the form

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n} (\mathbf{X}_n) = \frac{1}{n} \sum_{k=1}^{n} (\theta X_{k,n} - \frac{1}{2}\theta^2) = \theta Y_n - \frac{1}{2}\theta^2,$$

where

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_{k,n}, \ n \geq 1.$$

The sequence $\{\mathcal{L}(Y_n|P_n), n \geq 1\}$ obeys the LDP in R with rate function $I^N(y) = y^2/2, y \in R$ (see, e.g., Freidlin and Wentzell, 1979). This yields by the contraction principle the LDP for the log-likelihood ratios $\Xi_{n,\theta}$.

Example 2.2 *An Independent-Identically-Distributed Sample*

Let $\mathbf{X}_n = (X_{1,n}, \ldots, X_{n,n})$ be an independent–identically–distributed sample from a distribution $P_\theta, \theta \in \Theta$, on the real line. We do not specify the nature of the parameter set Θ. For example, it can be a subset of a finite–dimensional space, a set of distributions on R (or their probability density functions), etc. We assume that the family $\mathcal{P}$ is dominated by a probability measure P, i.e., $P_\theta \ll P$, $\theta \in \Theta$. This model is described by dominated experiments $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta}, \theta \in \Theta)$ with $\Omega_n = R^n$, $\mathcal{F}_n = \mathcal{B}(R^n)$, $P_{n,\theta} = P_\theta^n$, $\theta \in \Theta$ and $P_n = P^n$.

We have

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(\mathbf{X}_n) = \sum_{k=1}^{n} \frac{1}{n} \ln \frac{dP_\theta}{dP}(X_{k,n}) = \int_R \ln \frac{dP_\theta}{dP}(x) F_n(dx),$$

where

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} 1(X_{k,n} \leq x), \qquad x \in R,$$

is an empirical distribution function.

Let $\mathcal{Y}$ be the space of cumulative distribution functions on R with the topology of weak convergence of associated probability measures. By Sanov's theorem (Sanov, 1957, Deuschel and Stroock, 1989, 3.2.17), the sequence $\{\mathcal{L}(F_n|P_n), n \geq 1\}$ obeys the LDP in $\mathcal{Y}$ with rate function $I^S(F) = K(F, P)$, $F \in \mathcal{Y}$, where $K(F,P)$ denotes the Kullback–Leibler information:

$$K(F,P) = \begin{cases} \displaystyle\int_R \frac{dF}{dP}(x) \ln \frac{dF}{dP}(x)\, P(dx), & \text{if } F \ll P, \\ \infty, & \text{otherwise.} \end{cases}$$

Let us also denote, for $\theta \in \Theta$ and $F \in \mathcal{Y}$,

$$\zeta_\theta(F) = \int_R \ln \frac{dP_\theta}{dP}(x) F(dx).$$

If the density functions $\frac{dP_\theta}{dP}(x)$ are bounded from above, bounded away from zero and continuous in x for all $\theta \in \Theta$, then the $\zeta_\theta(F)$ are continuous functions on $\mathcal{Y}$ and, since $\Xi_{n,\theta} = \zeta_\theta(F_n)$, the contraction principle yields the LDP for the sequence $\{\Xi_{n,\Theta}, n \geq 1\}$.

Example 2.3 *"Signal + White Noise"*

We observe a real-valued stochastic process $X_n = (X_n(t), t \in [0,1])$ obeying the stochastic differential equation

$$dX_n(t) = \theta(t)dt + \frac{1}{\sqrt{n}} dW(t), \ 0 \leq t \leq 1,$$

where $W = (W(t), t \in [0,1])$ is a standard Wiener process and $\theta(\cdot)$ is an unknown function assumed to belong to some set Θ of real-valued continuous functions on $[0,1]$.

This model is described by statistical experiments $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta}, \theta \in \Theta)$, where Ω_n is $C[0,1]$, the space of continuous functions on $[0,1]$ with the uniform metric and Borel σ–field, and $P_{n,\theta}$ is the distribution of X_n on $C[0,1]$ for θ. We take $P_n = P_{n,0}$, where $P_{n,0}$ corresponds to the zero function $\theta(\cdot) \equiv 0$. Then $P_{n,\theta} \ll P_n$ and, moreover, by Girsanov's formula, P_n–almost surely,

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) = \int_0^1 \theta(t) dX_n(t) - \frac{1}{2}\int_0^1 \theta^2(t)dt. \tag{2.2}$$

Let $C_0[0,1]$ be the subset of $C[0,1]$ of the functions $x(\cdot)$ that are absolutely continuous with respect to Lebesgue measure and equal 0 at 0. Then the sequence $\{\mathcal{L}(X_n | P_n), n \geq 1\}$ obeys the LDP in $C[0,1]$ with rate function

$$I^W(x(\cdot)) = \begin{cases} \frac{1}{2}\int_0^1 (\dot{x}(t))^2 \, dt, & \text{if } x(\cdot) \in C_0[0,1], \\ \infty, & \text{otherwise}, \end{cases}$$

$\dot{x}(t)$ denoting the derivative of $x(\cdot)$ at t (see, e.g., Freidlin and Wentzell, 1979).

Let us denote, for functions $\theta(\cdot) \in \Theta$ and $x(\cdot) \in C_0[0,1]$,

$$\zeta_\theta(x) = \int_0^1 \theta(t)dx(t) - \frac{1}{2}\int_0^1 \theta^2(t)dt,$$

where the integral is understood as a Lebesgue–Stiltjes integral.

Again the log-likelihood ratio $\Xi_{n,\theta}$ can formally be represented as $\Xi_{n,\theta} = \zeta_\theta(X_n)$. Note however that the first integral in (2.2) is an Ito integral, so the latter equality as well as the continuity property for ζ_θ actually hold for functions $\theta(\cdot)$ of special sort (e.g., piecewise constant or differentiable). For these functions, the contraction principle again implies the LDP for $\{\Xi_{n,\Theta}, n \geq 1\}$. A general case is studied in Section 5.

Example 2.4 *Gaussian Regression*

We consider the regression model

$$X_{k,n} = \theta(t_{k,n}) + \xi_{k,n}, \qquad t_{k,n} = \frac{k}{n}, \quad k = 1, \dots, n, \tag{2.3}$$

where errors $\xi_{k,n}$ are independent standard normal and $\theta(\cdot)$ is an unknown real–valued continuous function.

In this model, $\Omega_n = R^n$, $\Theta \subset C[0,1]$ and $P_{n,\theta}$ is the distribution of $\mathbf{X}_n = (X_{1,n}, \dots, X_{n,n})$ for $\theta(\cdot)$. As above, we take $P_n = P_{n,0}$. Then

$$\begin{aligned}
\Xi_{n,\theta} &= \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) \\
&= \frac{1}{n} \sum_{k=1}^{n} \theta(t_{k,n}) X_{k,n} - \frac{1}{2n} \sum_{k=1}^{n} \theta^2(t_{k,n}) \\
&= \int_0^1 \theta(t)\, dX_n(t) - \frac{1}{2n} \sum_{k=1}^{n} \theta^2\left(t_{k,n}\right),
\end{aligned}$$

where

$$X_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_{k,n}, \qquad 0 \le t \le 1.$$

Let $\mathcal{Y}$ be the space of right-continuous with left-hand limits functions on $[0,1]$ with the uniform metric (for measurability of X_n, see Billingsley, 1968, §8).

Since the $X_{k,n}$ are $\mathcal{N}(0,1)$–distributed under P_n, the sequence $\{\mathcal{L}\left(X_n | P_n\right), n \ge 1\}$ obeys the LDP in $\mathcal{Y}$ with rate function I^W (Mogulskii, 1976, Theorem 2).

Since the function $\theta(\cdot)$ is continuous, we have, for large n, the approximate equality

$$\frac{1}{n} \sum_{k=1}^{n} \theta^2\left(t_{k,n}\right) \approx \int_0^1 \theta^2(t)\, dt$$

and hence $\Xi_{n,\theta} \approx \zeta_\theta(X_n)$ with the same function ζ_θ as in the preceding example. If the θ are differentiable, integration by parts shows that the $\Xi_{n,\theta}$ are continuous functions of the X_n, and the LDP for $\{\Xi_{n,\Theta}, n \ge 1\}$ follows by the contraction principle. Again, a general case is deferred until Section 5.

Example 2.5 *Non–Gaussian Regression*

We consider the same regression model (2.3) but now assume that independent identically distributed errors $\xi_{k,n}$ have a distribution P with a positive probability density function $p(x)$ with respect to Lebesgue measure on the real line. An unknown regression function $\theta(\cdot)$ is assumed to be continuous, so $\Theta \subset C[0,1]$.

As above, for a regression function $\theta(\cdot)$, we denote by $P_{n,\theta}$ the distribution of $X_n = (X_{1,n}, \ldots, X_{n,n})$. We have, with $P_n = P_{n,0}$,

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) = \frac{1}{n} \sum_{k=1}^{n} \ln \frac{p(X_{k,n} - \theta(t_{k,n}))}{p(X_{k,n})}.$$

Introducing an empirical process $F_n = F_n(x,t)$, $x \in R$, $t \in [0,1]$, by

$$F_n(x,t) = \frac{1}{n} \sum_{k=1}^{[nt]} 1(X_{k,n} \le x),$$

we have that

$$\Xi_{n,\theta} = \int_0^1 \int_R \ln \frac{p(x - \theta(t))}{p(x)} F_n(dx, dt). \tag{2.4}$$

Let us define $\mathcal{Y}$ as the space of cumulative distribution functions $F = F(x,t)$, $x \in R$, $t \in [0,1]$, on $R \times [0,1]$ with the weak topology. Let $\mathcal{Y}_0$ be the subset of $\mathcal{Y}$ of absolutely continuous with respect to Lebesgue measure on $R \times [0,1]$ functions $F(x,t)$ with densities $p_t(x)$ such that $\int_R p_t(x)\, dx = 1$ for $t \in [0,1]$.

It is shown in Dembo and Zajic, 1995 (see also Theorem 1 in Puhalskii, 1996) that the sequence $\{\mathcal{L}(F_n | P_n), n \ge 1\}$ obeys the LDP in $\mathcal{Y}$ with rate function $I^{SK}(F)$ given by

$$I^{SK}(F) = \begin{cases} \displaystyle\int_0^1 \int_R \ln \frac{p_t(x)}{p(x)}\, p_t(x)\, dx\, dt, & \text{if } F \in \mathcal{Y}_0, \\ \infty, & \text{otherwise.} \end{cases}$$

Denote, for $F \in \mathcal{Y}_0$ and $\theta \in \Theta$,

$$\zeta_\theta(F) = \int_0^1 \int_R \ln \frac{p(x - \theta(t))}{p(x)} F(dx, dt).$$

Then by (2.4), $\Xi_{n,\theta} = \zeta_\theta(F_n)$ and if the log's in the integrals in the definition of the ζ_θ are bounded and continuous, we have the LDP for $\{\Xi_{n,\Theta}, n \ge 1\}$.

Example 2.6 *The Change–Point Model*

Let us observe a sample $X_n = (X_{1,n}, \ldots, X_{n,n})$ of real–valued random variables, where, for some $k_n \ge 1$, the observations $X_{1,n}, \ldots, X_{k_n,n}$ are independent identically distributed with a distribution P_0 and the observations $X_{k_n+1,n}, \ldots, X_{n,n}$ are independent identically distributed with a distribution P_1. We assume that P_0 and P_1 are known and k_n is unknown. Let us also assume that $k_n = [n\theta]$, where $\theta \in \Theta = [0,1]$. For this model, $\Omega_n = R^n$ and $P_{n,\theta}$ stands for the distribution of X_n for θ.

Let a probability measure P dominate P_0 and P_1 and

$$f_0(x) = \frac{dP_0}{dP}(x), \;\; f_1(x) = \frac{dP_1}{dP}(x), \;\; x \in R,$$

be respective densities. Assume that $f_0(x)$ and $f_1(x)$ are positive and continuous. Denoting $P_n = P^n$, we have

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) = \frac{1}{n} \sum_{i=1}^{[n\theta]} \ln f_0(X_{i,n}) + \frac{1}{n} \sum_{i=[n\theta]+1}^{n} \ln f_1(X_{i,n})$$

so that defining an empirical process again by

$$F_n(x,t) = \frac{1}{n} \sum_{i=1}^{[nt]} 1(X_{i,n} \le x), \; x \in R, t \in [0,1],$$

we obtain the representation

$$\Xi_{n,\theta} = \int_0^\theta \int_R \ln f_0(x) \; F_n(dx,dt) + \int_\theta^1 \int_R \ln f_1(x) \; F_n(dx,dt).$$

Let a space $\mathcal{Y}$ be defined as for the preceding model and $\mathcal{Y}_P$ be the set of those $F \in \mathcal{Y}$ that are absolutely continuous with respect to the measure $P(dx) \times dt$ and admit densities $p_t(x)$ such that $\int_R p_t(x)P(dx) = 1\,, t \in [0,1]$. As above, the F_n obey the LDP with rate function I_P^{SK} of the form

$$I_P^{SK}(F) = \begin{cases} \displaystyle\int_0^1 \int_R p_t(x) \ln p_t(x) P(dx) \; dt, & \text{if } F \in \mathcal{Y}_P, \\ \infty, & \text{otherwise.} \end{cases}$$

Define next for $F \in \mathcal{Y}_P$

$$\zeta_\theta(F) = \int_0^\theta \int_R \ln f_0(x) \; F(dx,dt) + \int_\theta^1 \int_R \log f_1(x) \; F(dx,dt)\,.$$

Then again $\Xi_{n,\theta} = \zeta_\theta(F_n)$, and the LDP for $\{\Xi_{n,\Theta}, n \ge 1\}$ holds, e.g., when $\log f_0(x)$ and $\log f_1(x)$ are bounded and continuous.

Example 2.7 *Regression with Random Design*

We consider the model

$$X_{k,n} = \theta(t_{k,n}) + \xi_{k,n}, \qquad k = 1, \ldots, n,$$

where real-valued errors $\xi_{k,n}$ and design points $t_{k,n}$ are independent with respective distributions P and Π dominated by Lebesgue measure. We denote the respective densities by $p(x)$ and $\pi(t)$. We also assume that the prior measure Π has a compact support D, $\pi(t)$ is continuous and positive on the support, $p(x)$ is continuous and positive on R, and an unknown regression function $\theta(\cdot)$ is continuous.

In this model, $P_{n,\theta}$ is the joint distribution of $X_n = (X_{1,n},\ldots,X_{n,n})$ and $t_n = (t_{1,n},\ldots,t_{n,n})$ for θ. Let F_n be the joint empirical distribution function of X_n and t_n:

$$F_n(A,B) = \frac{1}{n}\sum_{k=1}^{n} 1(X_{k,n}\in A, t_{k,n}\in B)$$

for Borel sets $A\subset R$, $B\subset D$, and let $\mathcal{Y}$ be the space of distributions on $R\times D$ with the weak topology. Set also $P_n = P_{n,0} = (P\times\Pi)^n$.

With these definitions,

$$\begin{aligned}\Xi_{n,\theta} &= \frac{1}{n}\ln\frac{dP_{n,\theta}}{dP_n}(X_n,t_n)\\ &= \frac{1}{n}\sum_{k=1}^{n}\ln\frac{p(X_{k,n}-\theta(t_{k,n}))}{p(X_{k,n})}\\ &= \int_D\int_R \ln\frac{p(x-\theta(t))}{p(x)}F_n(dx,dt).\end{aligned}$$

Let $\mathcal{Y}_1$ be the set of the cumulative distribution functions on R^2 that are absolutely continuous with respect to Lebesgue measure on R^2 and have support in $R\times D$. Under P_n, the random pairs $(X_{k,n},t_{k,n})$ are independent identically distributed with the distribution $P\times\Pi$, and hence, by Sanov's theorem, the LDP holds for the F_n with rate function $I^{SS}(F)$ given by

$$I^{SS}(F) = \begin{cases}\displaystyle\int_D\int_R \ln\frac{p(x,t)}{p(x)\pi(t)}\,p(x,t)\,dx\,dt, & \text{if } F\in\mathcal{Y}_1,\\ \infty, & \text{otherwise.}\end{cases}$$

Here $F(dx,dt)=p(x,t)\,dx\,dt$. The LDP for this model follows now in a manner similar to the case of an independent-identically-distributed sample.

We end the subsection with a simple but useful remark. It is noticeable that the definition of the LDP given above uses the same letter n both to subscript probability measures and associated random elements, and denote a scaling parameter. One might wonder whether this is not a loss of generality and how n should be chosen when considering particular models. The answer to the first question is in the negative and making n play the two roles is a matter of convenience for economy of notation. Indeed, if we have a sequence of probability measures $\{Q_n, n\ge 1\}$ with $\log Q_n$ having the right rate $b_n\to\infty$ as $n\to\infty$, we can always reduce this case to the above "standard" set–up by "relabeling" the measures, i.e., by introducing measures Q'_n such that $Q'_{b_n}=Q_n$; taking b_n as a new n then gives $\log Q'_n$ the rate n as required. This argument originating from Varadhan, 1984 also answers the second question: n in our formalism has the meaning of the right scale rather than "the natural parameter of the model". Of course, the two can coincide as in most of the examples we considered where n is a sample size, but not always as Example 2.3 shows. On the other hand, it is clear from the above that if we want n to be "the natural parameter", we can do this by introducing some $b_n \to \infty$ as a scale.

2.2 Sufficient Conditions for the Dominated LDP

We now study properties of the LDP for statistical experiments and begin with a sufficient condition for the LDP. The condition serves two purposes further: first, in particular statistical models it can easier be checked than the definition of the LDP, second, this condition comes in handy when constructing asymptotically optimal decisions, see Section 4. The idea behind the condition is similar to the one used in the condition of local asymptotic normality by LeCam, 1960 for studying weak convergence of experiments, or, more generally, in the condition of λ–convergence by Shiryaev and Spokoiny, 1995.

Given a sequence of dominated statistical experiments $\{\mathcal{E}_n, P_n, n \geq 1\}$, assume that there exist statistics Y_n on $(\Omega_n, \mathcal{F}_n)$ with values in a Hausdorff space $\mathcal{Y}$ such that the sequence $\{\mathcal{L}(Y_n|P_n), n \geq 1\}$ obeys the LDP and the Y_n are asymptotically sufficient in the sense that $Z_{n,\theta} \approx \mathfrak{z}_\theta(Y_n)$ for some nonrandom functions $\mathfrak{z}_\theta$ on $\mathcal{Y}$. In the above examples the statistic Y_n is easily identified: it is the empirical mean $(X_{1,n}+\ldots+X_{n,n})/n$ in the case of a sample from normal distribution in Example 2.1, the empirical distribution function F_n in the case of an independent-identically-distributed sample in Example 2.2, the observation process X_n for the "signal + white noise" model, the empirical process F_n for the regression model with non-Gaussian errors and the change-point model, etc.

If the functions $\mathfrak{z}_\theta$ are continuous then, by the contraction principle, the LDP for the sequence $\{\mathcal{L}(Y_n|P_n), n \geq 1\}$ implies the LDP for the sequence $\{\mathcal{L}(\mathfrak{z}_\theta(Y_n)|P_n), n \geq 1\}$ and hence for $\{\mathcal{L}\left(Z_{n,\theta}|P_n\right), n \geq 1\}$. Unfortunately, by contrast with the theory of weak convergence of experiments, in applications the functions $\mathfrak{z}_\theta$ typically are not continuous. For instance, the functions $\zeta_\theta(y) = \ln \mathfrak{z}_\theta(y)$ generally are not continuous in the above examples for an independent-identically-distributed sample, the "signal + white noise" model, the regression models and the change-point model. To overcome this difficulty, we need to introduce "regularisations" $\mathfrak{z}_{\theta,\delta}(y)$ of $\mathfrak{z}_\theta(y)$ that, on the one hand, are continuous functions and, on the other hand, converge to $\mathfrak{z}_\theta(y)$ as $\delta \to 0$.

Before stating the condition, let us review some more facts about large deviations used in the sequel. Recall (Varadhan, 1966; Varadhan, 1984; Deuschel and Stroock, 1989; Bryc, 1990) that if a sequence of probability measures $\{Q_n, n \geq 1\}$ on the Borel σ–field of a Hausdorff space S obeys the LDP with rate function I then, for all non-negative bounded continuous functions f on S,

$$\lim_{n\to\infty}\left[\int_S (f(x))^n Q_n(dx)\right]^{1/n} = \sup_{x\in S} f(x)V(x), \tag{2.5}$$

where $V(x)=\exp(-I(x))$. If S is a metric, or, more generally, a Tihonov (i.e., completely regular) space (Engelking, 1977; Kelley, 1957) then (2.5) also is sufficient for the LDP (Puhalskii, 1993).

Moreover, the LDP implies (2.5) for unbounded continuous non-negative functions f too under "the uniform exponential integrability condition" (Varadhan, 1984; Deuschel and Stroock, 1989)

$$\lim_{H\to\infty}\ \overline{\lim_{n\to\infty}}\ \left[\int_S (f(x))^n\, 1(f(x)>H) Q_n(dx)\right]^{1/n} = 0. \tag{2.6}$$

Also, if f is a lower semicontinuous non-negative function then

$$\underline{\lim}_{n\to\infty}\ \left[\int_S (f(x))^n Q_n(dx)\right]^{1/n} \geq \sup_{x\in S} f(x)V(x). \tag{2.7}$$

The function $V(x)$ is further referred to as a deviability. Equivalently, a deviability is defined as a function $V: S\to[0,1]$ such that $\sup_{x\in S}V(x)=1$ and the inverse images $V^{-1}([a,1])$ are compact sets for all $a>0$. Obviously, there is one-to-one correspondence between probability rate functions and deviabilities. We say that $\{Q_n, n\geq 1\}$ large deviation (LD) converges to V and write $Q_n \overset{l.d.}{\to} V$ $(n\to\infty)$ if (2.5) holds for all bounded continuous non-negative functions f (Puhalskii, 1994a). Below we use the fact that, if S is metric, then one can only require that the functions f be uniformly continuous (analogously to weak convergence theory, Billingsley, 1968, Theorem 2.1). By the above, if S is a Tihonov space then $Q_n \overset{l.d.}{\to} V(n\to\infty)$ if and only if $\{Q_n\}$ obeys the LDP with $I=-\ln V$. All the spaces we consider below are Tihonov and we mostly use the formulation of the LDP as LD convergence as more convenient in theoretical considerations.

Next, let S and S' be Hausdorff spaces and V a deviability on S. Denote

$$\Phi_V(a)=\{x\in S:\ V(x)\geq a\},\quad a>0. \tag{2.8}$$

As in Puhalskii, 1995b (cf. Schwartz, 1973), we say that a map $\varphi: S\to S'$ is V–Luzin measurable if it is continuous in restriction to each set $\Phi_V(a), a>0$. [2] Deviabilities

[2] The name is motivated by the following analogy with Luzin's theorem in measure theory. Let us extend V to a set function on S by defining $V(\Gamma)=\sup_{x\in\Gamma}V(x)$, $\Gamma\subset S$. Then V as a set function is an analogue of probability (see Puhalskii, 1991, 1994, 1995a for a discussion), and, equivalently, a function φ is V–Luzin measurable if, for every $\varepsilon>0$, there exists a set $A\subset S$ with $V(S\backslash A)<\varepsilon$ such that φ is continuous in restriction to A. It is also interesting to note that one can prove an analogue of Egorov's theorem for sequences of Luzin measurable functions Puhalskii, 1991, 1995b.

are preserved under Luzin measurable maps: for any V–Luzin measurable map φ, the function $V\circ\varphi^{-1}$ on S' defined by $V\circ\varphi^{-1}(x') = \sup_{x\in\varphi^{-1}(x')} V(x)$, $x'\in S'$, is a deviability on S' (see Deuschel and Stroock, 1989, 2.1.4, also the argument of Puhalskii, 1991, Lemma 2.1 applies).

Also, we say that a function $\varphi: S\to S'$ is V–almost everywhere (V–a.e.) continuous if it is continuous at every $x\in S$ with $V(x)>0$. Obviously, each V–a.e. continuous function is V–Luzin measurable.

Some more notational conventions are in order. We denote by $\mathcal{A}(\Theta)$ the family of all finite subsets of Θ. Elements of R_+^Θ are denoted by $z_\Theta = (z_\theta, \theta\in\Theta)$, and elements of R_+^Λ, where $\Lambda\in\mathcal{A}(\Theta)$, by $z_\Lambda = (z_\theta,\theta\in\Lambda)$. Maps π_Λ and $\pi_{\Lambda'\Lambda}$, where $\Lambda\in\mathcal{A}(\Theta)$, $\Lambda'\in\mathcal{A}(\Theta)$ and $\Lambda\subset\Lambda'$, are the natural projections of R_+^Θ onto R_+^Λ and of $R_+^{\Lambda'}$ onto R_+^Λ, respectively: $\pi_\Lambda(z_\theta,\theta\in\Theta) = (z_\theta,\theta\in\Lambda)$ and $\pi_{\Lambda'\Lambda}(z_\theta,\theta\in\Lambda') = (z_\theta,\theta\in\Lambda)$. Since R_+^Θ and R_+^Λ, $\Lambda\in\mathcal{A}(\Theta)$, are supplied with the Tihonov topology, the projections are continuous.

We now state and prove the sufficient condition for the LDP. We assume in it that the statistics Y_n take values in a metric space which is enough for applications though this restriction can be relaxed.

Lemma 2.1 *Let $\{\mathcal{E}_n, P_n, n\ge 1\}$ be a sequence of dominated experiments and $Z_{n,\theta}$, $\theta\in\Theta$, be defined by (2.1).*

Assume that the following condition holds:

(Y) there exist statistics $Y_n:\Omega_n\to\mathcal{Y}$ with values in a metric space $\mathcal{Y}$ with the Borel σ–field, functions $\mathfrak{z}_\theta:\mathcal{Y}\to R_+$, $\theta\in\Theta$, and $\mathfrak{z}_{\theta,\delta}:\mathcal{Y}\to R_+$, $\theta\in\Theta$, $\delta>0$, such that

(Y.1) *the sequence $\{\mathcal{L}(Y_n|P_n), n\ge 1\}$ of distributions on $\mathcal{Y}$ LD converges to a deviability $V(y), y\in\mathcal{Y}$;*

(Y.2) *for all $\delta>0$, the functions $\mathfrak{z}_{\theta,\delta}:\mathcal{Y}\to R_+$, $\theta\in\Theta$, are Borel measurable and V–a.e. continuous;*

(Y.3) $$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ P_n^{1/n}(|Z_{n,\theta}-\mathfrak{z}_{\theta,\delta}(Y_n)|>\varepsilon) = 0 \text{ for all } \varepsilon>0 \text{ and } \theta\in\Theta;$$

(Y.4) $$\lim_{\delta\to 0}\ \sup_{y\in\Phi_V(a)} |\mathfrak{z}_{\theta,\delta}(y)-\mathfrak{z}_\theta(y)| = 0 \text{ for all } a>0 \text{ and } \theta\in\Theta.$$

Then $\mathcal{L}(Z_{n,\Theta}|P_n)\overset{l.d.}{\to} V_\Theta$ $(n\to\infty)$, where $V_\Theta = V\circ\mathfrak{z}_\Theta^{-1}$, $\mathfrak{z}_\Theta = (\mathfrak{z}_\theta, \theta\in\Theta)$.

Proof. Conditions (Y.2) and (Y.4) obviously imply that $\mathfrak{z}_\Theta:\mathcal{Y}\to R_+^\Theta$ is V–Luzin measurable, hence V_Θ is a deviability on R_+^Θ.

Let $\Lambda\in\mathcal{A}(\Theta)$. We first prove that

$$\mathcal{L}(Z_{n,\Lambda}|P_n)\overset{l.d.}{\to} V_\Lambda\ \ (n\to\infty), \tag{2.9}$$

where $Z_{n,\Lambda} = (Z_{n,\theta}, \theta \in \Lambda)$, $V_\Lambda = V \circ \mathfrak{z}_\Lambda^{-1}$ and $\mathfrak{z}_\Lambda = (\mathfrak{z}_\theta, \theta \in \Lambda)$. Let $f : R_+^\Lambda \to R_+$ be bounded and uniformly continuous. Since, by the definition of V_Λ, $\sup_{z_\Lambda \in R_+^\Lambda} f(z_\Lambda) V_\Lambda(z_\Lambda) = \sup_{y \in \mathcal{Y}} f(\mathfrak{z}_\Lambda(y)) V(y)$, we need to prove that

$$\lim_{n\to\infty} E_n^{1/n} f^n(Z_{n,\Lambda}) = \sup_{y\in\mathcal{Y}} f(\mathfrak{z}_\Lambda(y)) V(y). \tag{2.10}$$

Let $\mathfrak{z}_{\Lambda,\delta} = (\mathfrak{z}_{\theta,\delta}, \theta \in \Lambda)$. Condition $(Y.3)$ implies in view of the boundedness and uniform continuity of f that

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \, |E_n^{1/n} f^n(Z_{n,\Lambda}) - E_n^{1/n} f^n(\mathfrak{z}_{\Lambda,\delta}(Y_n))| = 0, \tag{2.11}$$

Since the sequence $\{\mathcal{L}(Y_n|P_n), n \ge 1\}$ LD converges to V and the map $\mathfrak{z}_{\Lambda,\delta} : \mathcal{Y} \to R_+^\Lambda$ is V–a.e. continuous, the sequence $\{\mathcal{L}(\mathfrak{z}_{\Lambda,\delta}(Y_n)|P_n), n \ge 1\}$ LD converges to $V \circ (\mathfrak{z}_{\Lambda,\delta})^{-1}$, Puhalskii, 1991, Theorem 2.2. Thus, since f is non-negative, bounded and continuous,

$$\lim_{n\to\infty} E_n^{1/n} f^n(\mathfrak{z}_{\Lambda,\delta}(Y_n)) = \sup_{y\in\mathcal{Y}} f(\mathfrak{z}_{\Lambda,\delta}(y)) V(y). \tag{2.12}$$

By (2.11) and (2.12), for (2.10) it remains to show that

$$\lim_{\delta\to 0} \sup_{y\in\mathcal{Y}} f(\mathfrak{z}_{\Lambda,\delta}(y)) V(y) = \sup_{y\in\mathcal{Y}} f(\mathfrak{z}_\Lambda(y)) V(y) \tag{2.13}$$

which is an easy consequence of condition $(Y.4)$. Convergence (2.9) is proved. The assertion of the lemma now follows by the Dawson–Gärtner theorem on the projective limits of large deviation systems (Dawson and Gärtner, 1987, Theorem 3.3) if we note that $\mathcal{L}(Z_{n,\Theta}|P_n)$ is the projective limit of $\{\mathcal{L}(Z_{n,\Lambda}|P_n), \Lambda \in \mathcal{A}(\Theta)\}$ and $V_\Lambda = V_\Theta \circ \pi_\Lambda^{-1}$, $\Lambda \in \mathcal{A}(\Theta)$. □

Remark 2.1 *Since R_+^Θ is a Tihonov space, the lemma implies that, under conditions (Y) and (U), the sequence $\{\mathcal{E}_n, P_n, n \ge 1\}$ obeys the dominated LDP.*

Remark 2.2 *As we have seen, in applications it is more convenient to manipulate rate functions and log-likelihood ratios given by*

$$\Xi_{n,\theta} = \ln Z_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}, \quad \theta \in \Theta.$$

Accordingly, it is useful to state condition (Y) in these terms. Assume that the $\Xi_{n,\theta}$ are well defined. It is easy to see that condition (Y) is implied by the following condition

(Y') *there exist statistics $Y_n : \Omega_n \to \mathcal{Y}$ with values in a metric space $\mathcal{Y}$ with the Borel σ–field, functions $\zeta_\theta : \mathcal{Y} \to R$, $\theta \in \Theta$, and $\zeta_{\theta,\delta} : \mathcal{Y} \to R$, $\theta \in \Theta$, $\delta > 0$, such that*

$(Y'.1)$ *the sequence $\{\mathcal{L}(Y_n|P_n),\ n \ge 1\}$ of distributions on $\mathcal{Y}$ obeys the LDP with rate function $I(y)$, $y \in \mathcal{Y}$;*

$(Y'.2)$ *for all* $\delta > 0$, *the functions* $\zeta_{\theta,\delta} : \mathcal{Y} \to R$, $\theta \in \Theta$, *are Borel measurable and continuous at each point* y *such that* $I(y) < \infty$;

$(Y'.3)$ $$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(Y_n)| > \varepsilon) = 0 \text{ for all } \varepsilon > 0 \text{ and } \theta \in \Theta;$$

$(Y'.4)$ $$\lim_{\delta\to 0} \sup_{y\in\Phi'_I(a)} |\zeta_{\theta,\delta}(y) - \zeta_\theta(y)| = 0 \text{ for all } a \geq 0 \text{ and } \theta \in \Theta,$$

where $\Phi'_I(a) = \{y \in \mathcal{Y} : I(y) \leq a\}$.

Condition (U) *takes the form*

(U') $$\lim_{H\to\infty} \overline{\lim_{n\to\infty}} E_n^{1/n} \exp(n\Xi_{n,\theta}) 1(\Xi_{n,\theta} > H) = 0,\ \theta \in \Theta .$$

By Lemma 2.1, conditions (Y') *and* (U') *imply the dominated LDP.*

2.3 The General Case

The above definition of the large deviation principle for statistical experiments covers only the dominated case and depends on a choice of dominating measures. We present now another definition which is free of these defects. It is motivated by Le Cam's definition of weak convergence of experiments, see, e.g., Strasser, 1985.

Let $|\Lambda|$ denote the number of elements in $\Lambda \in \mathcal{A}(\Theta)$. For $z_\Lambda = (z_\theta, \theta \in \Lambda) \in R_+^\Lambda$ and $z_\Theta = (z_\theta, \theta \in \Theta) \in R_+^\Theta$, we set $\|z_\Lambda\|_\Lambda = \max_{\theta\in\Lambda} z_\theta$ and $\|z_\Theta\|_\Theta = \max_{\theta\in\Theta} z_\theta$, respectively, and define $S_\Lambda = \{z_\Lambda \in R_+^\Lambda : \|z_\Lambda\|_\Lambda = 1\}$ and $S_\Theta = \{z_\Theta \in R_+^\Theta : \|z_\Theta\|_\Theta = 1\}$. Not to overburden notation, we sometimes omit the subscript Λ in $\|\cdot\|_\Lambda$ if there is no risk of confusion.

Next, given a sequence of statistical experiments $\{\mathcal{E}_n , n \geq 1\}$, where $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n, P_{n,\theta}, \theta \in \Theta)$, set, for $\Lambda \in \mathcal{A}(\Theta)$,

$$\begin{aligned} P_{n,\Lambda} &= \frac{1}{|\Lambda|} \sum_{\theta\in\Lambda} P_{n,\theta}, \\ \mathbf{Z}_{n,\theta;\Lambda} &= \left(\frac{dP_{n,\theta}}{dP_{n,\Lambda}}\right)^{1/n}, \qquad \theta \in \Lambda, \\ \mathbf{Z}_{n,\Lambda} &= (\mathbf{Z}_{n,\theta;\Lambda},\ \theta \in \Lambda). \end{aligned} \tag{2.14}$$

The definitions immediately imply that, $P_{n,\Lambda}$–almost surely,

$$\sum_{\theta\in\Lambda} \mathbf{Z}_{n,\theta;\Lambda}^n = |\Lambda| \tag{2.15}$$

and

$$1 \leq \|\mathbf{Z}_{n,\Lambda}\| \leq |\Lambda|^{1/n}. \tag{2.16}$$

Definition 2.2 *A sequence* $\{\mathcal{E}_n\,, n \geq 1\}$ *of statistical experiments obeys* the large deviation principle (LDP) *if, for each* $\Lambda \in \mathcal{A}(\Theta)$, *the sequence* $\{\mathcal{L}(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}),\, n \geq 1\}$ *of distributions on* R_+^Λ *obeys the LDP with some rate function.*

Remark 2.3 *Equivalently,* $\{\mathcal{E}_n\,, n \geq 1\}$ *obeys the LDP if* $\mathcal{L}(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}) \stackrel{l.d.}{\to} \mathbf{V}_\Lambda,\ \Lambda \in \mathcal{A}(\Theta)$, *where* $\mathbf{V}_\Lambda$ *is a deviability on* R_+^Λ .

We next study consequences of the definition and, particularly, prove that the definitions of the LDP for the dominated and general cases are consistent. We start by giving another characterisation of the LDP. Let $\mathcal{H}_\Lambda$ denote the set of all non-negative, continuous and positively homogeneous functions on R_+^Λ : $h \in \mathcal{H}_\Lambda$ if $h(z_\Lambda) \geq 0$, h is continuous and $h(\lambda z_\Lambda) = \lambda h(z_\Lambda)$ for all $z_\Lambda \in R_+^\Lambda$ and $\lambda \geq 0$. We say that a deviability $\mathbf{V}_\Lambda$ has support in S_Λ if $\mathbf{V}_\Lambda(z_\Lambda) = 0$ for $z_\Lambda \notin S_\Lambda$.

Lemma 2.2 *Let* $\Lambda \in \mathcal{A}(\Theta)$. *Then* $\mathcal{L}\left(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}\right) \stackrel{l.d.}{\to} \mathbf{V}_\Lambda$ *if and only if* $\mathbf{V}_\Lambda$ *has support in* S_Λ *and*

$$\lim_{n\to\infty} E_{n,\Lambda}^{1/n} h^n(\mathbf{Z}_{n,\Lambda}) = \sup_{z_\Lambda \in R_+^\Lambda} h(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda) \quad \textit{for every } h \in \mathcal{H}_\Lambda\,.$$

In particular, if $\mathcal{L}\left(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}\right) \stackrel{l.d.}{\to} \mathbf{V}_\Lambda$ *then, for all* $\theta \in \Lambda$,

$$(R) \qquad \sup\nolimits_{z_\Lambda \in R_+^\Lambda} \pi_\theta z_\Lambda \mathbf{V}_\Lambda(z_\Lambda) = 1\,.$$

Proof. Let $\mathcal{L}\left(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}\right) \stackrel{l.d.}{\to} \mathbf{V}_\Lambda$. We have, using the equivalence of LD convergence and the LDP on R_+^Λ, that, for $\varepsilon > 0$,

$$\varliminf_{n\to\infty} P_{n,\Lambda}^{1/n}(|\|\mathbf{Z}_{n,\Lambda}\| - 1| > \varepsilon) \geq \sup_{z_\Lambda:\, |\|z_\Lambda\|-1|>\varepsilon} \mathbf{V}_\Lambda(z_\Lambda).$$

Inequalities (2.16) imply that the left-hand side equals zero. Since ε is arbitrary, $\mathbf{V}_\Lambda$ has support in S_λ. The claimed limit follows by the definition of LD convergence since, by (2.16), $h(\mathbf{Z}_{n,\Lambda}) = \widehat{h}(\mathbf{Z}_{n,\Lambda})$ $P_{n,\Lambda}$–almost surely, where $\widehat{h}(z_\Lambda) = h(z_\Lambda)[(2 - \|z_\Lambda\|/\Lambda) \wedge 1 \vee 0]$, and the latter function is non-negative, bounded and continuous.

For the converse, pick a non-negative continuous bounded function f on R_+^Λ . We need to prove that

$$\lim_{n\to\infty} E_{n,\Lambda}^{1/n} f^n(\mathbf{Z}_{n,\Lambda}) = \sup_{z_\Lambda \in R_+^\Lambda} f(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda). \tag{2.17}$$

We define a function $\widetilde{f}$ by

$$\widetilde{f}(z_\Lambda) = \begin{cases} \|z_\Lambda\| f\left(\dfrac{z_\Lambda}{\|z_\Lambda\|}\right), & \text{if } \ ||z_\Lambda|| > 0\,, \\ 0\,, & \text{if } \ ||z_\Lambda|| = 0\,. \end{cases}$$

Note that f and $\widetilde{f}$ coincide on S_Λ and, since $\mathbf{V}_\Lambda$ is supported by S_Λ, we can change f to $\widetilde{f}$ on the right-hand side of (2.17). The continuity of f and the inequalities (2.16) easily imply that the random variables $f(\mathbf{Z}_{n,\Lambda})$ and $\widetilde{f}(\mathbf{Z}_{n,\Lambda})$ are uniformly bounded and

$$\lim_{n\to\infty} \left| E_{n,\Lambda}^{1/n} f^n(\mathbf{Z}_{n,\Lambda}) - E_{n,\Lambda}^{1/n} \widetilde{f}^n(\mathbf{Z}_{n,\Lambda}) \right| = 0.$$

Since $\widetilde{f} \in \mathcal{H}_\Lambda$, taking $h = \widetilde{f}$ in the conditions of the lemma, we get

$$\lim_{n\to\infty} E_{n,\Lambda}^{1/n} \widetilde{f}^n(\mathbf{Z}_{n,\Lambda}) = \sup_{z_\Lambda \in R_+^\Lambda} \widetilde{f}(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda),$$

concluding the proof of (2.17).

Property (R) follows by taking $h(z_\Lambda) = \pi_\theta z_\Lambda$. $\square$

We now show that if $\Lambda \subset \Lambda' \in \mathcal{A}(\Theta)$ then the deviability $\mathbf{V}_\Lambda$ is a sort of projection of the deviability $\mathbf{V}_{\Lambda'}$, the property being inherited from corresponding probabilities. Recall notations $\pi_{\Lambda'\Lambda}$ and π_Λ for the projections from $R_+^{\Lambda'}$ onto R_+^Λ and R_+^Θ onto R_+^Λ, respectively, and let $\Pi_{\Lambda'\Lambda}$ and Π_Λ stand for normalised projections:

$$\begin{aligned} \Pi_{\Lambda'\Lambda} z_{\Lambda'} &= \pi_{\Lambda'\Lambda} z_{\Lambda'} / \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda, \quad z_{\Lambda'} \in R_+^{\Lambda'}, \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda > 0, \\ \Pi_\Lambda z_\Theta &= \pi_\Lambda z_\Theta / \|\pi_\Lambda z_\Theta\|_\Lambda, \quad z_\Theta \in R_+^\Theta, \|\pi_\Lambda z_\Theta\|_\Lambda > 0. \end{aligned}$$

Also we adhere to the convention that $\sup_\emptyset = 0$.

Lemma 2.3 *Let $\Lambda \subset \Lambda' \in \mathcal{A}(\Theta)$. If $\mathcal{L}(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}) \overset{l.d.}{\to} \mathbf{V}_\Lambda$ and $\mathcal{L}\left(\mathbf{Z}_{n,\Lambda'}|P_{n,\Lambda'}\right) \overset{l.d.}{\to} \mathbf{V}_{\Lambda'}$ then the following conditions hold:*

(C) $$\sup_{z_\Lambda \in R_+^\Lambda} h(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_{\Lambda'} \in R_+^{\Lambda'}} h(\pi_{\Lambda'\Lambda} z_{\Lambda'})\mathbf{V}_{\Lambda'}(z_{\Lambda'}), \ h \in \mathcal{H}_\Lambda;$$

(S) $$\mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_{\Lambda'} \in \Pi_{\Lambda'\Lambda}^{-1} z_\Lambda} \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \mathbf{V}_{\Lambda'}(z_{\Lambda'}), \ z_\Lambda \in R_+^\Lambda,$$

where $\Pi_{\Lambda'\Lambda}^{-1} z_\Lambda = \{z_{\Lambda'} \in R_+^{\Lambda'} : \Pi_{\Lambda'\Lambda} z_{\Lambda'} = z_\Lambda\}$.

Proof. Define

$$\mathbf{Z}_{n,\Lambda;\Lambda'} = \left(\frac{dP_{n,\Lambda}}{dP_{n,\Lambda'}}\right)^{1/n}.$$

By (2.14),

$$\pi_{\Lambda'\Lambda}\mathbf{Z}_{n,\Lambda'} = \mathbf{Z}_{n,\Lambda}\mathbf{Z}_{n,\Lambda;\Lambda'} \qquad P_{n,\Lambda'}\text{–almost surely},$$

and, since $h \in \mathcal{H}_\Lambda$, we have that

$$E_{n,\Lambda}^{1/n} h^n(\mathbf{Z}_{n,\Lambda}) = E_{n,\Lambda'}^{1/n} \left[h(\mathbf{Z}_{n,\Lambda'})\mathbf{Z}_{n,\Lambda;\Lambda'}\right]^n = E_{n,\Lambda'}^{1/n} h^n(\pi_{\Lambda',\Lambda}\mathbf{Z}_{n,\Lambda'}).$$

Applying Lemma 2.2 to the leftmost and rightmost sides we obtain (C).

Now, (S), for a given $\widehat{z}_\Lambda \in S_\Lambda$, can formally be obtained by substituting $\widehat{h}(z_\Lambda) = \mathbf{1}(z_\Lambda = \|z_\Lambda\|\widehat{z}_\Lambda)\|z_\Lambda\|$ into (C) and using the fact that $\mathbf{V}_\Lambda$ has support in S_Λ. However, the function $\widehat{h}$ is not continuous, so we approximate it with a sequence of continuous functions $h_k \in \mathcal{H}_\Lambda$, $k \geq 1$, as follows. Let

$$h_k(z_\Lambda) = (\|z_\Lambda\| - k\|z_\Lambda - \widehat{z}_\Lambda\|z_\Lambda\|\,\|)^+.$$

Since the h_k are from $\mathcal{H}_\Lambda$, they satisfy (C). Also $h_k(z_\Lambda) \downarrow \widehat{h}(z_\Lambda)$ as $k \to \infty$. ¿From the fact that the $h_k(z_\Lambda)$ are continuous and $\mathbf{V}_\Lambda$ and $\mathbf{V}_{\Lambda'}$ are deviabilities, it is not difficult to check by using Dini's theorem (for a proof see, e.g., Lemmas A.1 and A.4 in Puhalskii, 1995b) that one can take limit as $k \to \infty$ in (C) for the h_k, as required. □

Remark 2.4 *Property (S) implies that (C) holds for non-continuous positively homogeneous non-negative functions too.*

In analogy with statistical decision theory, Strasser, 1985, we further call a family of deviabilities $\{V_\Lambda, \Lambda \in \mathcal{A}(\Theta)\}$, where V_Λ is defined on R_+^Λ, *conical* if it satisfies (C). If, in addition, $V_\Lambda(z_\Lambda) = 0$ for all $z_\Lambda \notin S_\Lambda$, the family is called *standard*. The proof of Lemma 2.3 shows that a family is standard if it meets (S).

The next result is of particular importance for the minimax theorem below. It states that every standard family of deviabilities admits an extension to a function on R_+^Θ which preserves the conical property.

Lemma 2.4 *For every standard family of deviabilities $\{\mathbf{V}_\Lambda, \Lambda \in \mathcal{A}(\Theta)\}$, there exists a function $\mathbf{V}_\Theta$ on R_+^Θ such that the following conditions hold:*

(i) the function $\mathbf{V}_\Theta$ is upper semicontinuous, assumes values in $[0,1]$, $\sup_{z_\Theta \in R_+^\Theta} \mathbf{V}_\Theta(z_\Theta) = 1$ and $\mathbf{V}_\Theta(z_\Theta) = 0$ if $z_\Theta \notin S_\Theta$;

(ii) for all $\Lambda \in \mathcal{A}(\Theta)$ and $h \in \mathcal{H}_\Lambda$,

$$\sup_{z_\Lambda \in R_+^\Lambda} h(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_\Theta \in R_+^\Theta} h(\pi_\Lambda z_\Theta)\mathbf{V}_\Theta(z_\Theta);$$

(iii) for all $z_\Lambda \in R_+^\Lambda$,

$$\mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_\Theta \in \Pi_\Lambda^{-1} z_\Lambda} \|\pi_\Lambda z_\Theta\|_\Lambda \mathbf{V}_\Theta(z_\Theta)\,,$$

where $\Pi_\Lambda^{-1} z_\Lambda = \{z_\Theta \in R_+^\Theta \,:\, \Pi_\Lambda z_\Theta = z_\Lambda\}$.

We relegate the proof to the appendix.

We conclude the section by showing consistency of the above definitions of the LDP for statistical experiments.

Lemma 2.5 *Let* $\{\mathcal{E}_n, P_n, n \geq 1\}$ *be a sequence of dominated statistical experiments. If it obeys the dominated LDP, then it obeys the LDP. More specifically, denoting by* V_Θ *the deviability on* R_+^Θ *that is the LD limit of* $\mathcal{L}(Z_{n,\Theta}|P_n)$ *as* $n \to \infty$*, we have that* $\mathcal{L}(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda}) \overset{l.d.}{\to} \mathbf{V}_\Lambda, \Lambda \in \mathcal{A}(\Theta)$*, where*

$$\mathbf{V}_\Lambda(z_\Lambda) = \begin{cases} \sup_{z_\Theta \in \Pi_\Lambda^{-1} z_\Lambda} \|\pi_\Lambda z_\Theta\| V_\Theta(z_\Theta), & \text{if } z_\Lambda \in S_\Lambda\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Also, denoting by $\mathbf{V}_\Theta$ *the extension of the standard family* $\{\mathbf{V}_\Lambda, \Lambda \in \mathcal{A}(\Theta)\}$ *defined in Lemma 2.4, we have that, for every* $\Lambda \in \mathcal{A}(\Theta)$ *and* $h \in \mathcal{H}(\Lambda)$*,*

$$\sup_{z_\Theta \in R_+^\Theta} h(\pi_\Lambda z_\Theta) V_\Theta(z_\Theta) = \sup_{z_\Theta \in R_+^\Theta} h(\pi_\Lambda z_\Theta) \mathbf{V}_\Theta(z_\Theta).$$

Proof.

We first prove that, for all $\Lambda \in \mathcal{A}(\Theta)$ and $h \in \mathcal{H}_\Lambda$,

$$\lim_{n\to\infty} E_{n,\Lambda}^{1/n} h^n(\mathbf{Z}_{n,\Lambda}) = \sup_{z_\Theta \in R_+^\Theta} h(\pi_\Lambda z_\Theta) V_\Theta(z_\Theta). \tag{2.18}$$

Since by (2.1) and (2.14),

$$Z_{n,\theta} = \mathbf{Z}_{n,\theta;\Lambda} \left(\frac{dP_{n,\Lambda}}{dP_n} \right)^{1/n} \qquad P_n\text{–almost surely}\ , \quad \theta \in \Lambda,$$

and h is positively homogeneous, we have that

$$E_{n,\Lambda}^{1/n} h^n(\mathbf{Z}_{n,\Lambda}) = E_n^{1/n} h^n(\mathbf{Z}_{n,\Lambda}) \frac{dP_{n,\Lambda}}{dP_n} = E_n^{1/n} h^n(\pi_\Lambda Z_{n,\Theta}). \tag{2.19}$$

Now using the assumed LD convergence $\mathcal{L}(Z_{n,\Theta}|P_n) \overset{l.d.}{\to} V_\Theta$ we want to prove that

$$\lim_{n\to\infty} E_n^{1/n} h^n(\pi_\Lambda Z_{n,\Theta}) = \sup_{z_\Theta \in R_+^\Theta} h(\pi_\Lambda z_\Theta) V_\Theta(z_\Theta), \tag{2.20}$$

which by (2.19) would yield (2.18). The function h being non-negative and continuous but not bounded, (2.20) would follow if the uniform exponential integrability condition introduced in (2.6) holds:

$$\lim_{H\to\infty} \overline{\lim_{n\to\infty}} E_n^{1/n} h^n(\pi_\Lambda Z_{n,\Theta}) 1(h(\pi_\Lambda Z_{n,\Theta}) > H) = 0. \tag{2.21}$$

It is here that we need condition (U). Let $h^* = \sup_{z_\Lambda \in S_\Lambda} h(z_\Lambda)$ which is finite by the continuity of h. Since $h \in \mathcal{H}_\Lambda$, it follows that $h(Z_{n,\Lambda}) \leq h^* \|Z_{n,\Lambda}\|$, so, in view of condition (U),

$$\begin{aligned} \overline{\lim_{n\to\infty}} E_n^{1/n} h^n(\pi_\Lambda Z_{n,\Theta}) 1(h(\pi_\Lambda Z_{n,\Theta}) > H) &\leq \overline{\lim_{n\to\infty}} \sum_{\theta\in\Lambda} E_n^{1/n} h^{*n} Z_{n,\theta}^n 1(h^* Z_{n,\theta} > H) \\ &\leq \overline{\lim_{n\to\infty}} h^* \sum_{\theta\in\Lambda} P_{n,\theta}^{1/n}(h^* Z_{n,\theta} > H) \to 0 \text{ as } H \to \infty, \end{aligned}$$

where the last convergence follows by Chebyshev's inequality. So, (2.20) and hence (2.18) have been proved.

Since by the definition of $\mathbf{V}_\Lambda$,

$$\sup_{z_\Lambda \in R_+^\Lambda} h(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_\Theta \in R_+^\Theta} h(\pi_\Lambda z_\Theta) V_\Theta(z_\Theta), \tag{2.22}$$

Lemma 2.2 implies that the proof of the first claim of the lemma is completed by checking that $\mathbf{V}_\Lambda$ is a deviability on R_+^Λ.

Limit (2.21), in view of the LD convergence of $\mathcal{L}(Z_{n,\Theta}|P_n)$ to V_Θ, implies that (use property (2.7))

$$\lim_{H\to\infty} \sup_{z_\Theta \in R_+^\Theta} \|\pi_\Lambda z_\Theta\|_\Lambda \, 1(\|\pi_\Lambda z_\Theta\|_\Lambda > H) V_\Theta(z_\Theta) = 0.$$

Therefore, for every $\varepsilon > 0$ there exists H_ε such that

$$\{z_\Theta \in R_+^\Theta : \|\pi_\Lambda z_\Theta\|_\Lambda V_\Theta(z_\Theta) \ge \varepsilon\} \subset \{z_\Theta \in R_+^\Theta : V_\Theta(z_\Theta) \ge \frac{\varepsilon}{H_\varepsilon}\}$$

so that the set on the left is compact. Since also $\|\pi_\Lambda z_\Theta\|_\Lambda \ge \varepsilon$ when $\|\pi_\Lambda z_\Theta\|_\Lambda V_\Theta(z_\Theta) \ge \varepsilon$, and Π_Λ is continuous on $\{z_\Theta \in R_+^\Theta : \|\pi_\Lambda z_\Theta\|_\Lambda \ge \varepsilon\}$, it follows that the set $\Pi_\Lambda\{z_\Theta \in R_+^\Theta : \|\pi_\Lambda z_\Theta\|_\Lambda V_\Theta(z_\Theta) \ge \varepsilon\}$ is compact. Since, for $a > 0$,

$$\{z_\Lambda \in R_+^\Lambda : \mathbf{V}_\Lambda(z_\Lambda) \ge a\} = \bigcap_{n=1}^\infty \Pi_\Lambda \left\{ z_\Theta \in R_+^\Theta : \|\pi_\Lambda z_\Theta\|_\Lambda V_\Theta(z_\Theta) \ge a\left(1 - \frac{1}{n+1}\right)\right\},$$

we conclude that the sets $\{z_\Lambda \in R_+^\Lambda : \mathbf{V}_\Lambda(z_\Lambda) \ge a\}$ are compact. Thus, we are left to check that

$$\sup_{z_\Lambda \in R_+^\Lambda} \mathbf{V}_\Lambda(z_\Lambda) = 1. \tag{2.23}$$

By (2.18) with $h(z_\Lambda) = \pi_\theta z_\Lambda,\ \theta \in \Lambda$,

$$\sup_{z_\Theta \in R_+^\Theta} \pi_\theta z_\Theta V_\Theta(z_\theta) = 1,$$

hence,

$$\sup_{z_\Theta \in R_+^\Theta} \|\pi_\Lambda z_\Theta\|_\Lambda V_\Theta(z_\theta) = \sup_{\theta\in\Lambda} \sup_{z_\Theta \in R_+^\Theta} \pi_\theta z_\Theta V_\Theta(z_\theta) = 1,$$

and (2.23) follows by the definition of $\mathbf{V}_\Lambda$.

The second claim of the lemma follows by (2.22) and Lemma 2.4. The lemma is proved. □

Remark 2.5 *Equality (2.22) implies that projections* $V_\Lambda, \Lambda \in \mathcal{A}(\Theta)$, *of* V_Θ *defined by*

$$V_\Lambda(z_\Lambda) = \sup_{z_\Theta \in \pi_\Lambda^{-1} z_\Lambda} V_\Theta(z_\Theta)$$

constitute a family of deviabilites with properties (C) *and* (R).

3 A Minimax Theorem

We start the section by showing that, in analogy with the classical asymptotic theory of statistical experiments (Strasser, 1985), the LDP for statistical experiments allows us to obtain an asymptotic lower bound for appropriately defined risks which, in fact, has been the purpose of introducing the concept of the LDP for sequences of statistical experiments. We next prove that under additional conditions the bound is tight and study the problem of constructing decisions attaining it.

We consider a sequence of statistical experiments $\{\mathcal{E}_n\,, n \geq 1\}$, where $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta},\, \theta \in \Theta)$, and assume that it obeys the LDP. The associated deviabilities are denoted by $\mathbf{V}_\Lambda\,, \Lambda \in \mathcal{A}(\Theta)$, and $\mathbf{V}_\Theta$ denotes the extension defined in Lemma 2.4.

We introduce some more notation common for statistical decision theory, see, e.g., Strasser, 1985. We denote by $\mathcal{D}$ a Hausdorff topological space with the Borel σ–field which we take as a decision space; $W_\theta = (W_\theta(r), r \in \mathcal{D})$, $\theta \in \Theta$, are, for each θ, non-negative and lower semicontinuous functions on $\mathcal{D}$ which play the role of loss functions; $\mathcal{R}_n$ denotes the set of all measurable mappings $\rho_n : \Omega_n \to \mathcal{D}$, i.e., $\mathcal{R}_n$ is the set of all decision functions with values in $\mathcal{D}$. We define the large deviation (LD) risk of a decision $\rho_n \in \mathcal{R}_n$ in the experiment $\mathcal{E}_n$ by

$$R_n(\rho_n) = \sup_{\theta\in\Theta} E_{n,\theta}^{1/n} W_\theta^n(\rho_n). \tag{3.1}$$

Obviously, this is an analogue of the risk in minimax decision theory, cf. Strasser, 1985.

Recall that a function $f : U \to R$ on a topological space U is level-compact if it is bounded from below and the sets $\{u \in U : f(u) \leq \alpha\}$ are compact for all $\alpha < \sup_{u\in U} f(u)$, Strasser, 1985, Definition 6.3. Obviously, if U is Hausdorff, a level-compact function is lower semicontinuous and the supremum of a family of level-compact functions is level-compact. For the sequel, it is also worth mentioning that level-compact functions attain infimums on closed sets.

Theorem 3.1 *Let the sequence $\{\mathcal{E}_n\,, n \geq 1\}$ obey the LDP. Assume that the functions $W_\theta, \theta \in \Theta$, are level-compact. Then*

$$\varliminf_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R_n(\rho_n) \geq R^*,$$

where

$$R^* = \sup_{z_\Theta\in R_+^\Theta} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta} W_\theta(r) z_\theta \mathbf{V}_\Theta(z_\Theta).$$

In particular, if $\{\mathcal{E}_n, P_n, n \geq 1\}$ obeys the dominated LDP and V_Θ is the associated deviability then the lower bound can be rewritten as

$$R^* = \sup_{z_\Theta\in R_+^\Theta} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta} W_\theta(r) z_\theta V_\Theta(z_\Theta). \tag{3.2}$$

If, moreover, conditions (Y) and (U) hold then

$$R^* = \sup_{y\in\mathcal{Y}} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta} W_\theta(r)\,\mathfrak{z}_\theta(y)V(y).$$

Proof Let $\Lambda \in \mathcal{A}(\Theta)$. We first prove that

$$\varliminf_{n\to\infty} \inf_{\rho_n} \sup_{\theta\in\Lambda} E_{n,\theta}^{1/n} W_\theta^n(\rho_n) \geq \sup_{z_\Lambda\in R_+^\Lambda} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Lambda} W_\theta(r) z_\theta \mathbf{V}_\Lambda(z_\Lambda). \tag{3.3}$$

Let $\{\rho_n, n\geq 1\}$ be an arbitrary sequence of decisions. We have, by the definition of $\mathbf{Z}_{n,\Lambda}$ (see (2.14)),

$$\begin{aligned}
\varliminf_{n\to\infty} \sup_{\theta\in\Lambda} E_{n,\theta}^{1/n} W_\theta^n(\rho_n) &= \varliminf_{n\to\infty} \sup_{\theta\in\Lambda} E_{n,\Lambda}^{1/n} W_\theta^n(\rho_n)\mathbf{Z}_{n,\theta;\Lambda}^n \\
&\geq \varliminf_{n\to\infty} \left[\frac{1}{|\Lambda|} E_{n,\Lambda} \sum_{\theta\in\Lambda} W_\theta^n(\rho_n)\mathbf{Z}_{n,\theta;\Lambda}^n\right]^{1/n} \\
&\geq \varliminf_{n\to\infty} E_{n,\Lambda}^{1/n} \sup_{\theta\in\Lambda} W_\theta^n(\rho_n)\mathbf{Z}_{n,\theta;\Lambda}^n \\
&\geq \varliminf_{n\to\infty} E_{n,\Lambda}^{1/n} w^n(\mathbf{Z}_{n,\Lambda}),
\end{aligned}$$

where

$$w(z_\Lambda) = \inf_{r\in\mathcal{D}} \sup_{\theta\in\Lambda} W_\theta(r) z_\theta, \quad z_\Lambda = (z_\theta, \theta\in\Lambda) \in R_+^\Lambda.$$

Since the set Λ is finite and the functions W_θ are level-compact, it is not difficult to see that the function $w(\cdot)$ is lower semicontinuous (cf. Aubin, 1984, Proposition 1.7). So by the LD convergence of $\mathcal{L}(\mathbf{Z}_{n,\Lambda}|P_{n,\Lambda})$ to $\mathbf{V}_\Lambda$,

$$\varliminf_{n\to\infty} E_{n,\Lambda}^{1/n} w^n(\mathbf{Z}_{n,\Lambda}) \geq \sup_{z_\Lambda\in R_+^\Lambda} w(z_\Lambda)\mathbf{V}_\Lambda(z_\Lambda)$$

implying (3.3).

Since the function $w(\cdot)$ belongs to $\mathcal{H}_\Lambda$, an application of Lemma 2.4(ii) yields

$$\sup_{z_\Lambda\in R_+^\Lambda} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Lambda} W_\theta(r) z_\theta \mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_\Theta\in R_+^\Theta} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Lambda} W_\theta(r) z_\theta \mathbf{V}_\Theta(z_\Theta),$$

so by (3.3)

$$\varliminf_{n\to\infty} \inf_{\rho_n} \sup_{\theta\in\Lambda} E_{n,\theta}^{1/n} W_\theta^n(\rho_n) \geq \sup_{z_\Theta\in R_+^\Theta} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Lambda} W_\theta(r) z_\theta \mathbf{V}_\Theta(z_\Theta).$$

Now the proof of the lower bound is completed by observing that, for every $z_\Theta = (z_\theta, \theta \in \Theta) \in R_+^\Theta$,

$$\sup_{\Lambda\in\mathcal{A}(\Theta)} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Lambda} W_\theta(r) z_\theta = \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta} W_\theta(r) z_\theta \tag{3.4}$$

(for a proof see Lemma A.3 in the appendix or Aubin and Ekeland, 1984, Theorem 6, Section 2, Chapter 6).

If $\{\mathcal{E}_n, P_n, n \geq 1\}$ obeys the dominated LDP, then by Lemma 2.5

$$\sup_{z_\Theta \in R_+^\Theta} \inf_{r \in \mathcal{D}} \sup_{\theta \in \Lambda} W_\theta(r) z_\theta V_\Theta(z_\Theta) = \sup_{z_\Theta \in R_+^\Theta} \inf_{r \in \mathcal{D}} \sup_{\theta \in \Lambda} W_\theta(r) z_\theta \mathbf{V}_\Theta(z_\Theta),$$

and representation (3.2) follows by (3.4). The last representaion for R^* in the statement of the theorem follows since by Lemma 2.1 $V_\Theta = V \circ \mathfrak{z}_\Theta^{-1}$.

□

Remark 3.1 *Note that the proof only uses what is known as a lower bound in the LDP.*

Remark 3.2 *Now we are in a position to explain why we consider condition (U) in the definition of the dominated LDP to be important. Assume that $\{\mathcal{E}_n, n \geq 1\}$ is a dominated family with dominating measures P_n such that, for a deviability V_Θ on R_+^Θ, we have the LD convergence $\mathcal{L}(Z_{n,\Theta}|P_n) \stackrel{l.d.}{\to} V_\Theta$. The proof of Theorem 3.1 with $\mathbf{V}_\Theta$ replaced by V_Θ and $\mathbf{V}_\Lambda$ replaced by $V_\Theta \circ \pi_\Lambda^{-1}$ (which would not use condition (U)) would still give the right-hand side of (3.2) as a lower bound. However these lower bounds can generally be different for different sequences of dominating measures. The role of condition (U) is to eliminate this possibility by making sure that equality (3.2) holds so that the lower bounds do not depend on a choice of dominating measures.*

In applications, as we will see, the assumption that the loss functions are level-compact is normally met. However, in the appendix we give a variant of Theorem 3.1 for more general loss functions. As in the classical theory, tackling this case requires considering generalised decisions, cf. Strasser, 1985.

We now turn to the question of tightness of the above lower bound and start with defining the concept of large deviation efficiency. Say that a sequence of decisions $\{\rho_n^*, n \geq 1\}$ is *large deviation (LD) efficient* if, for any other sequence of decisions $\{\rho_n\}$,

$$\overline{\lim_{n\to\infty}} \, (R_n(\rho_n^*) - R_n(\rho_n)) \leq 0.$$

Theorem 3.1 implies that to construct LD efficient decisions one can apply an approach similar to the one used in the classical asymptotic decision theory. Indeed, by Theorem 3.1, if the $W_\theta, \theta \in \Theta$, are level-compact, then, for any sequence of decisions $\{\rho_n, n \geq 1\}$,

$$\underline{\lim}_{n\to\infty} \, R_n(\rho_n) \geq R^*.$$

Now if a sequence $\{\rho_n^*, n \geq 1\}$ is such that $R_n(\rho_n^*) \to R^*$ as $n \to \infty$, it is obviously LD efficient.

Further, motivated by applications, we assume that the sequence $\{\mathcal{E}_n, n \geq 1\}$ is dominated and conditions (Y) and (U) hold. Then, by Theorem 3.1, the asymptotic minimax risk can be written as

$$R^* = \sup_{y \in \mathcal{Y}} \inf_{r \in \mathcal{D}} \sup_{\theta \in \Theta} W_\theta(r) \mathfrak{z}_\theta(y) V(y). \tag{3.5}$$

Representation (3.5) prompts considering for each $y \in \mathcal{Y}$ the subproblem

$$(Q) \qquad Q^*(y) = \inf_{r \in \mathcal{D}} \sup_{\theta \in \Theta} W_\theta(r) \mathfrak{z}_\theta(y).$$

Since the functions W_θ are level-compact for each $\theta \in \Theta$, it follows that, given $y \in \mathcal{Y}$, we can find $r^*(y) \in \mathcal{D}$ that delivers the infimum in (Q). The value $r^*(y)$ can be viewed as "the best decision if the value of Y_n is y". Hence, provided the function $r^*(y) : \mathcal{Y} \to \mathcal{D}$ is Borel measurable, the decisions $r^*(Y_n)$ are natural candidates for the LD efficient decisions. Unfortunately, we cannot prove this without requiring that $Q^*(y)$ be continuous (or upper semicontinuous) which usually is not fulfilled in applications. The reason for the latter, as in condition (Y) above, is that the $\mathfrak{z}_\theta(y)$ typically are not continuous as maps from $\mathcal{Y}$ into R_+. Therefore, as in condition (Y), we invoke the idea of regularisation. We require that there exist functions $\mathfrak{z}_{\theta,\delta}(y)$ such that functions $Q_\delta(y)$ defined by

$$(Q_\delta) \qquad Q_\delta(y) = \inf_{r \in \mathcal{D}} \sup_{\theta \in \Theta} W_\theta(r) \mathfrak{z}_{\theta,\delta}(y), \ y \in \mathcal{Y},$$

are continuous in y, on the one hand, and approximate $Q^*(y)$ for small δ, on the other hand. A rigourous formulation is given by condition $(\sup Y)$ which strengthens condition (Y) to the effect that the requirements of (Y) hold uniformly in $\theta \in \Theta$. This way of handling the technical difficulties does not allow us, however, to get LD efficient decisions: as the next theorem shows, in general we are only able to obtain decisions whose asymptotic risk is arbitrarily close to the lower bound. Still we succeed in proving that the lower bound of Theorem 3.1 is tight and LD efficient decisions exist. We next state the condition. Recall that $Z_{n,\theta} = (dP_{n,\theta}/dP_n)^{1/n}$.

$(\sup Y)$ There exist statistics $Y_n : \Omega_n \to \mathcal{Y}$ with values in a metric space $\mathcal{Y}$ with the Borel σ-field, functions $\mathfrak{z}_\theta : \mathcal{Y} \to R_+$, $\theta \in \Theta$, and $\mathfrak{z}_{\theta,\delta} : \mathcal{Y} \to R_+$, $\theta \in \Theta, \delta > 0$, such that

$(Y.1)$ the sequence $\{\mathcal{L}(Y_n | P_n), n \geq 1\}$ LD converges to a deviability $V(y)$, $y \in \mathcal{Y}$;

$(\sup Y.2)$ for the uniform topology on R_+^Θ, the functions $\mathfrak{z}_{\Theta,\delta} = (\mathfrak{z}_{\theta,\delta}, \theta \in \Theta) : \mathcal{Y} \to R_+^\Theta$, $\delta > 0$, are Borel measurable and continuous V–a.e.;

$$(\sup Y.3) \qquad \lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \sup_{\theta \in \Theta} P_n^{1/n} \left(|Z_{n,\theta} - \mathfrak{z}_{\theta,\delta}(Y_n)| > \varepsilon \right) = 0 \text{ for all } \varepsilon > 0;$$

(sup Y.4) $$\lim_{\delta\to 0}\sup_{\theta\in\Theta}\sup_{y\in\Phi_V(a)}|\mathfrak{z}_{\theta,\delta}(y)-\mathfrak{z}_\theta(y)|=0 \text{ for all } a>0.$$

In the next theorem, condition $(\sup Y)$ is used together with condition $(\sup U)$ which strengthens (U):

$(\sup U)$ $$\lim_{H\to\infty}\ \overline{\lim_{n\to\infty}}\ \sup_{\theta\in\Theta} E_n^{1/n} Z_{n,\theta}^n 1(Z_{n,\theta}>H)=0.$$

Theorem 3.2 *Let a sequence of dominated experiments* $\{\mathcal{E}_n,\, P_n,\, n\geq 1\}$ *satisfy conditions* $(\sup Y)$ *and* $(\sup U)$. *Let the function* $W_\theta(r)$ *be bounded in* (θ,r) *and level-compact in* r *for each* $\theta\in\Theta$. *Assume that there exist Borel functions* $r_\delta(y):\mathcal{Y}\to\mathcal{D}$ *such that the infimum in* (Q_δ) *is attained at* $r_\delta(y)$, *and denote* $\rho_{n,\delta}=r_\delta(Y_n)$.

Then

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ R_n(\rho_{n,\delta})=\lim_{\delta\to 0}\ \underline{\lim}_{n\to\infty}\ R_n(\rho_{n,\delta})=R^*$$

so that

$$\lim_{n\to\infty}\ \inf_{\rho_n\in\mathcal{R}_n} R_n(\rho_n)=R^*.$$

In particular, for some sequence ρ_n^*,

$$\lim_{n\to\infty} R_n(\rho_n^*)=R^*.$$

Proof. Since $(\sup Y)$ implies (Y), by Lemma 2.1, $\mathcal{L}(Z_{n,\Theta}|P_n)\overset{l.d.}{\to} V_\Theta=V\circ\mathfrak{z}_\Theta^{-1}$, so by Theorem 3.1, for each δ,

$$\underline{\lim}_{n\to\infty}\ R_n(\rho_{n,\delta})\geq R^*.$$

The proof of the first set of equalities would be over if

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ R_n(\rho_{n,\delta})\leq R^*. \tag{3.6}$$

Let C be an upper bound for W: $W_\theta(r)\leq C$. Since

$$R_n(\rho_{n,\delta})=\sup_{\theta\in\Theta}E_{n,\theta}^{1/n}W_\theta^n(\rho_{n,\delta})=\sup_{\theta\in\Theta}E_n^{1/n}W_\theta^n(\rho_{n,\delta})Z_{n,\theta}^n,$$

we have that, for any $H>0$,

$$R_n(\rho_{n,\delta})\leq\sup_{\theta\in\Theta}E_n^{1/n}W_\theta^n(\rho_{n,\delta})(Z_{n,\theta}\wedge H)^n+C\sup_{\theta\in\Theta}E_n^{1/n}Z_{n,\theta}^n 1(Z_{n,\theta}>H).$$

The second term on the right tends to 0 as $n\to\infty$ and $H\to\infty$ by condition $(\sup U)$, so the required would follow by

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ \sup_{\theta\in\Theta}E_n^{1/n}W_\theta^n(\rho_{n,\delta})(Z_{n,\theta}\wedge H)^n\leq R^*. \tag{3.7}$$

Since

$$
\begin{aligned}
&|\sup_{\theta\in\Theta} E_n^{1/n} W_\theta^n(\rho_{n,\delta})(Z_{n,\theta}\wedge H)^n - \sup_{\theta\in\Theta} E_n^{1/n} W_\theta^n(\rho_{n,\delta})(\mathfrak{z}_{\theta,\delta}(Y_n)\wedge H)^n| \\
&\qquad \le C \sup_{\theta\in\Theta} E_n^{1/n}(|Z_{n,\theta} - \mathfrak{z}_{\theta,\delta}(Y_n)|\wedge H)^n,
\end{aligned}
$$

condition $(\sup Y.3)$ implies that

$$
\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ |\sup_{\theta\in\Theta} E_n^{1/n} W_\theta^n(\rho_{n,\delta})(Z_{n,\theta}\wedge H)^n - \sup_{\theta\in\Theta} E_n^{1/n} W_\theta^n(\rho_{n,\delta})(\mathfrak{z}_{\theta,\delta}(Y_n)\wedge H)^n| = 0. \quad (3.8)
$$

Next, using the definitions of Q_δ and $\rho_{n,\delta}$ and the inequality $W_\theta(r)\le C$, we get

$$
\begin{aligned}
\sup_{\theta\in\Theta} E_n^{1/n} W_\theta^n(\rho_{n,\delta})(\mathfrak{z}_{\theta,\delta}(Y_n)\wedge H)^n &\le E_n^{1/n}\left(\sup_{\theta\in\Theta}(W_\theta^n(\rho_{n,\delta}(y))\mathfrak{z}_{\theta,\delta}(Y_n))\wedge CH\right)^n \\
&= E_n^{1/n}\left(Q_\delta(Y_n)\wedge CH\right)^n. \qquad (3.9)
\end{aligned}
$$

The last two expectations in (3.9) are well defined since the assumptions of the theorem imply that $Q_\delta(y) = \sup_{\theta\in\Theta} W_\theta(r_\delta(y))\mathfrak{z}_{\theta,\delta}(y)$ is a Borel function.

By the boundedness of $W_\theta(r)$ and $(\sup Y.2)$, the function $Q_\delta(y)$ is V–a.e. continuous. Since $\mathcal{L}(Y_n|P_n) \overset{l.d.}{\to} V$, we get

$$
\lim_{n\to\infty} E_n^{1/n}\left(Q_\delta(Y_n)\wedge CH\right)^n = \sup_{y\in\mathcal{Y}}(Q_\delta(y)\wedge CH)\,V(y). \qquad (3.10)
$$

By (Q), (Q_δ) and the inequality $W_\theta(r)\le C$, we have that

$$
\begin{aligned}
&|\sup_{y\in\mathcal{Y}}(Q_\delta(y)\wedge CH)\,V(y) - \sup_{y\in\mathcal{Y}}(Q^*(y)\wedge CH)\,V(y)| \\
&\qquad \le C\sup_{y\in\mathcal{Y}}\sup_{\theta\in\Theta}(|\mathfrak{z}_{\theta,\delta}(y) - \mathfrak{z}_\theta(y)|\wedge H)\,V(y),
\end{aligned}
$$

and $(\sup Y.4)$ easily implies that the right-hand side tends to 0 as $\delta\to 0$. Thus,

$$
\begin{aligned}
\lim_{\delta\to 0}\sup_{y\in\mathcal{Y}}(Q_\delta(y)\wedge CH)\,V(y) &= \sup_{y\in\mathcal{Y}}(Q^*(y)\wedge CH)\,V(y) \\
&\le \sup_{y\in\mathcal{Y}} Q^*(y)V(y) = R^*, \qquad (3.11)
\end{aligned}
$$

where the last equality follows by (3.5) and (Q). Putting together (3.8)–(3.11) proves (3.7) and hence (3.6).

The second claim of the theorem follows by (3.6) and a string of inequalities the first of which is Theorem 3.1:

$$
R^* \le \underline{\lim}_{n\to\infty}\ \inf_{\rho_n} R_n(\rho_n) \le \overline{\lim_{n\to\infty}}\ \inf_{\rho_n} R_n(\rho_n) \le \overline{\lim_{n\to\infty}}\ R_n(\rho_{n,\delta}).
$$

□

Remark 3.3 *Obviously, $r_\delta(y)$ chosen so that*

$$\sup_{\theta\in\Theta} W_\theta(r_\delta(y))\mathfrak{z}_{\theta,\delta}(y) \geq Q_\delta(y) - \epsilon_\delta,$$

where $\epsilon_\delta \to 0$ as $\delta \to 0$, would work too.

Remark 3.4 *If condition $(\sup Y)$ holds with $\mathfrak{z}_{\theta,\delta}(y) = \mathfrak{z}_\theta(y)$, then the $r_\delta(y)$ in the theorem do not depend on δ and the decisions $\rho_n^* := \rho_{n,\delta}$ are LD efficient.*

Kelley, 1957) so that

Remark 3.5 *As with condition (Y), in applications it is more convenient to deal with a logarithmic form of condition $(\sup Y)$. Specifically, defining $\Xi_{n,\Theta}$ and $\Phi'_I(a)$ as in Remark 2.2, let us introduce condition $(\sup Y')$:*

$(\sup Y')$ *there exist statistics $Y_n : \Omega_n \to \mathcal{Y}$ with values in a metric space $\mathcal{Y}$ with the Borel σ–field, functions $\zeta_\theta : \mathcal{Y} \to R$, $\theta \in \Theta$, and $\zeta_{\theta,\delta} : \mathcal{Y} \to R$, $\theta \in \Theta, \delta > 0$, such that*

$(Y'.1)$ *the sequence $\{\mathcal{L}(Y_n|P_n), n \geq 1\}$ obeys the LDP with rate function $I(y)$, $y \in \mathcal{Y}$;*

$(\sup Y'.2)$ *for the uniform topology on R^Θ, the functions $\zeta_{\Theta,\delta} = (\zeta_{\theta,\delta}, \theta \in \Theta) : \mathcal{Y} \to R^\Theta$, $\delta > 0$, are Borel measurable and continuous at each point y such that $I(y) < \infty$;*

$(\sup Y'.3)$ $\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \sup_{\theta\in\Theta} P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(Y_n)| > \varepsilon) = 0$ *for all $\varepsilon > 0$;*

$(\sup Y'.4)$ $\lim_{\delta\to 0} \sup_{\theta\in\Theta} \sup_{y\in\Phi'_I(a)} |\zeta_{\theta,\delta}(y) - \zeta_\theta(y)| = 0$ *for all $a \geq 0$.*

Then condition $(\sup Y)$ is implied by condition $(\sup Y')$. Similarly, condition $(\sup U)$ follows from the condition

$(\sup U')$ $$\lim_{H\to\infty} \overline{\lim_{n\to\infty}} \sup_{\theta\in\Theta} E_n^{1/n} \exp(n\Xi_{n,\theta}) 1(\Xi_{n,\theta} > H) = 0.$$

We further refer to the decisions $\rho_{n,\delta}$ as nearly LD efficient.

4 Asymptotic LD Risks and Efficient Decisions for Hypothesis Testing and Estimation Problems

This section specifies the asymptotic minimax bound of Theorem 3.1 and (nearly) LD efficient decisions for some typical statistical set-ups by considering hypothesis testing and estimation with Bahadur–type criteria. We consider indicator loss functions, i.e.,

$$W_\theta(r) = 1(r \notin A_\theta),\ r \in \mathcal{D}, \theta \in \Theta,$$

where A_θ are closed subsets of $\mathcal{D}$. Then the LD risk of a decision ρ_n in the nth experiment is

$$R_n(\rho_n) = \sup_{\theta\in\Theta} P_{n,\theta}^{1/n}(\rho_n \notin A_\theta).$$

For applications, it is convenient to introduce the logarithmic risk

$$R_n'(\rho_n) = \sup_{\theta\in\Theta} \frac{1}{n}\ln P_{n,\theta}(\rho_n \notin A_\theta). \tag{4.1}$$

Accordingly, we consider the logarithm of the lower bound R^*:

$$R'^* = \sup_{\zeta_\Theta\in R^\Theta} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta: A_\theta\not\ni r} (\zeta_\theta - \mathbf{I}_\Theta(\zeta_\Theta)),$$

where $\mathbf{I}_\Theta(\zeta_\Theta) = -\log \mathbf{V}_\Theta(z_\Theta)$ for $z_\Theta = (\exp(\zeta_\theta), \theta\in\Theta)$, $\zeta_\Theta = (\zeta_\theta, \theta\in\Theta)$.

Theorem 3.1 then yields the following result.

Theorem 4.1 *Assume that the $A_\theta, \theta\in\Theta$, are compact. If the sequence $\{\mathcal{E}_n, n\ge 1\}$ obeys the LDP then*

$$\varliminf_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R_n'(\rho_n) \ge R'^*.$$

Let us assume now that the sequence $\{\mathcal{E}_n, n\ge 1\}$ is dominated and conditions (Y') and (U') hold. According to Remark 2.2 and Theorem 3.1, we then have that

$$R'^* = \sup_{y\in\mathcal{Y}} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta\,:\,A_\theta\not\ni r} (\zeta_\theta(y) - I(y)). \tag{4.2}$$

Similarly, subproblems (Q) and (Q_δ) of Section 3 take the form

(Q') $$Q'^*(y) = \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta\,:\,A_\theta\not\ni r} \zeta_\theta(y)\,,\ y\in\mathcal{Y}\,,$$

and

(Q_δ') $$Q'_\delta(y) = \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta\,:\,A_\theta\not\ni r} \zeta_{\theta,\delta}(y)\,,\ y\in\mathcal{Y}\,.$$

Obviously,

$$R'^* = \sup_{y\in\mathcal{Y}} (Q'^*(y) - I(y)).$$

Let the infimum in (Q_δ') be attained at some point $r_\delta'(y)$ which is the case, e.g., if the $A_\theta\,, \theta\in\Theta\,$, are compact. We denote $\rho_{n,\delta}' = r_\delta'(Y_n)\,$.

Combining Theorem 4.1 and Theorem 3.2, and taking into account Remarks 2.2 and 3.5, we obtain the following theorem.

Theorem 4.2 *Assume that $\{\mathcal{E}_n, P_n, n\ge 1\}$ is a dominated sequence of statistical experiments and the $A_\theta, \theta\in\Theta$, are compact.*

1. *If conditions* (Y') *and* (U') *hold then*

$$\underline{\lim}_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R'_n(\rho_n) \geq R'^*.$$

2. *Let the functions* r'_δ, $\delta > 0$, *which map* $\mathcal{Y}$ *into* $\mathcal{D}$, *be Borel measurable. If conditions* $(\sup Y')$ *and* $(\sup U')$ *hold then*

$$\lim_{\delta\to 0} \overline{\lim}_{n\to\infty} R'_n(\rho'_{n,\delta}) = \lim_{\delta\to 0} \underline{\lim}_{n\to\infty} R'_n(\rho'_{n,\delta}) = R'^*$$

so that

$$\lim_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R'_n(\rho_n) = R'^*.$$

4.1 Hypothesis Testing

Let Θ_0 and Θ_1 be non-intersecting subsets of the parameter set Θ: $\Theta_0 \subset \Theta, \Theta_1 \subset \Theta, \Theta_0 \cap \Theta_1 = \emptyset$. We want to test the hypothesis $H_0 : \theta \in \Theta_0$ versus the alternative $H_1 : \theta \in \Theta_1$.

The decision space $\mathcal{D}$ consists of two points: $\mathcal{D} = \{0, 1\}$. We endow it with the discrete topology and, for any decision (test) ρ, we treat the event $\{\rho = 0\}$ (respectively, $\{\rho = 1\}$) as accepting (respectively, rejecting) the null hypothesis.

An associated loss function $W_\theta(r)$ is the indicator of the wrong choice:

$$W_\theta(r) = 1(\theta \notin \Theta_r),\ r = 0, 1, \tag{4.3}$$

and the logarithmic risk $R'(\rho_n)$ of a decision ρ_n in (4.1) takes the form

$$R^T_n(\rho_n) = \max\left\{\sup_{\theta\in\Theta_0} \frac{1}{n}\ln P_{n,\theta}(\rho_n = 1), \sup_{\theta\in\Theta_1} \frac{1}{n}\ln P_{n,\theta}(\rho_n = 0)\right\}. \tag{4.4}$$

Denoting the corresponding asymptotic minimax risk R'^* by T^*, we have by (4.2) that

$$T^* = \sup_{y\in\mathcal{Y}} \min\left\{\sup_{\theta\in\Theta_0} (\zeta_\theta(y) - I(y)), \sup_{\theta\in\Theta_1} (\zeta_\theta(y) - I(y))\right\}. \tag{4.5}$$

For the sequel, it is more convenient to use another representation for T^* which is

$$T^* \quad = \quad \sup_{\theta\in\Theta_0,\, \theta'\in\Theta_1} S(\theta, \theta'), \tag{4.6}$$

where

$$S(\theta, \theta') = \sup_{y\in\mathcal{Y}} \min\{\zeta_\theta(y) - I(y), \zeta_{\theta'}(y) - I(y)\}. \tag{4.7}$$

Next, subproblem (Q'_δ) for this case is

$$T'_\delta(y) = \min_{r=0,1} \sup_{\theta\in\Theta_{1-r}} \zeta_{\theta,\delta}(y),\ y \in \mathcal{Y}.$$

It has the solution

$$r_\delta^T(y) = 1\left(\sup_{\theta\in\Theta_0} \zeta_{\theta,\delta}(y) < \sup_{\theta\in\Theta_1} \zeta_{\theta,\delta}(y)\right),$$

which leads us to tests of the form

$$\rho_{n,\delta}^T = 1\left(\sup_{\theta\in\Theta_0} \zeta_{\theta,\delta}(Y_n) < \sup_{\theta\in\Theta_1} \zeta_{\theta,\delta}(Y_n)\right). \tag{4.8}$$

In the case of two simple hypotheses θ_0 and θ_1, the tests reduce to a regularisation of the Neyman–Pearson test:

$$\rho_{n,\delta}^T = 1\left(\zeta_{\theta_0,\delta}(Y_n) < \zeta_{\theta_1,\delta}(Y_n)\right).$$

Applying Theorem 4.2, we get the following theorem.

Theorem 4.3 *Let Θ_0 and Θ_1 be non-intersecting subsets of Θ. If a sequence of dominated experiments $\{\mathcal{E}_n,\, P_n,\, n\ge 1\}$ satisfies conditions (Y') and (U') then*

$$\varliminf_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R_n^T(\rho_n) \ge T^*.$$

If conditions $(\sup Y')$ and $(\sup U')$ hold then

$$\lim_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R_n^T(\rho_n) = T^*,$$

and the tests $\rho_{n,\delta}^T$ are nearly LD efficient:

$$\lim_{\delta\to 0} \varlimsup_{n\to\infty} R_n^T(\rho_{n,\delta}^T) = \lim_{\delta\to 0} \varliminf_{n\to\infty} R_n^T(\rho_{n,\delta}^T) = T^*.$$

4.2 Parameter Estimation

Let Θ be a subset of a normed space $\mathcal{B}$ with norm $\|\cdot\|$. We are interested in estimating a parameter θ under the Bahadur–type loss function

$$W_\theta(r) = 1(\|r-\theta\| > c) \tag{4.9}$$

for a given positive c. The logarithmic risk of an estimator ρ_n is

$$R_n^E(\rho_n) = \sup_{\theta\in\Theta} \frac{1}{n} \ln P_{n,\theta}(\|\rho_n-\theta\| > c). \tag{4.10}$$

We assume that the decision space $\mathcal{D}$ is either a compact subset of $\mathcal{B}$ with the induced topology or a closed convex subset of $\mathcal{B}$ with the weak topology; in the latter case, $\mathcal{B}$ is assumed to be a reflexive Banach space. For both cases, the functions $W_\theta, \theta\in\Theta$, are level-compact on $\mathcal{D}$.

In this set-up, we denote the asymptotic minimax risk R'^* from (4.2) by E^*:

$$E^* = \sup_{y\in\mathcal{Y}} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta:\,\|r-\theta\|>c} (\zeta_\theta(y) - I(y)), \tag{4.11}$$

and the corresponding subproblem (Q'_δ) is

$$(E_\delta) \qquad E_\delta(y) = \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta\,:\,||r-\theta||>c} \zeta_{\theta,\delta}(y), \;\; y \in \mathcal{Y}.$$

We next describe solutions to (E_δ). Consider a real–valued function $f(\theta), \theta \in \Theta$, and let

$$A(h) = \{\theta \in \Theta : \; f(\theta) > h\}, \; h \in R, \tag{4.12}$$

$$r(h) = \inf_{r\in\mathcal{D}} \sup_{\theta\in A(h)} ||r-\theta||, \; h \in R, \tag{4.13}$$

$$h_c = \inf(h : \; r(h) \leq c).$$

We assume that $h_c < \infty$ (e.g., $f(\theta)$ is bounded). Note that, for both definitions of $\mathcal{D}$, the infimum in (4.13) is attained since the functions $r \to \|r-\theta\|$ from $\mathcal{D}$ to R_+ are level-compact for all $\theta \in \Theta$.

Lemma 4.1 *The set $D_c = \{r \in \mathcal{D} : \sup_{\theta\in A(h_c)} \|r-\theta\| \leq c\}$ is nonempty and consists of all $r_c \in \mathcal{D}$ at which $\inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta:||r-\theta||>c} f(\theta)$ is attained. Also the latter infimum equals h_c.*

Proof. Since the function $(r,h) \to \sup_{\theta\in A(h)} ||r-\theta||$ is decreasing in h and level-compact in $r \in \mathcal{D}$, the function $r(h)$ is decreasing and right-continuous. Hence, $r(h_c) \leq c$ and, since $\inf_{r\in\mathcal{D}} \sup_{\theta\in A(h_c)} ||r-\theta|| = r(h_c)$ and the infimum is attained, the set D_c is nonempty.

Now let $r_c \in D_c$. By definition, $\|r_c-\theta\| \leq c$ for all $\theta \in \Theta$ such that $f(\theta) > h_c$. Hence,

$$\sup_{\theta\in\Theta:\,||r_c-\theta||>c} f(\theta) \leq h_c. \tag{4.14}$$

On the other hand, if $h < h_c$, then $r(h) > c$ which implies that, for every $r \in \mathcal{D}$, $\sup_{\theta\in A(h)} \|r-\theta\| > c$ or, equivalently, there exists θ such that $f(\theta) > h$ and $\|r-\theta\| > c$ so that $\inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta:||r-\theta||>c} f(\theta) \geq h$. Since h is arbitrarily close to h_c, we conclude that

$$\inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta:||r-\theta||>c} f(\theta) \geq h_c$$

which by (4.14) proves that $\inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta:||r-\theta||>c} f(\theta) = h_c$ and r_c delivers the infimum.

Finally, if $r \notin D_c$ then $\sup_{\theta\in A(h_c)} \|r-\theta\| > c$, i.e., there exists θ such that $\|r-\theta\| > c$ and $f(\theta) > h_c$ which yields the inequality $\sup_{\theta\in\Theta:\|r-\theta\|>c} f(\theta) > h_c$. $\square$

Remark 4.1 *Informally, $r(h)$ is the smallest radius of the balls that contain all the θ with $f(\theta) > h$, and h_c is the lowest level h for which there exists a ball of radius c with this property. The lemma, particularly, states that h_c is the infimum over all the balls of radius c of the largest values of $f(\theta)$ outside the balls. For a one–dimensional parameter θ, the construction in the lemma chooses the largest level set of the function f contained in an interval of length $2c$, and the r_c are the centres of the intervals.*

Let $r_c(f)$ denote an element of the set D_c in the lemma and, taking $f(\theta) = \zeta_{\theta,\delta}(y)$, let $r^E_{\delta,c}(y) = r_c(\zeta_{\Theta,\delta}(y))$. We assume that the functions $r^E_{\delta,c}(y) : \mathcal{Y} \to \mathcal{D}$ are Borel measurable. We can then define the estimators

$$\rho^E_{n,\delta} = r^E_{\delta,c}(Y_n). \tag{4.15}$$

Motivated by Remark 4.1, we call these estimators *interval-median.*

A version of Theorem 4.2 for this case is the next theorem.

Theorem 4.4 *Assume that either $\mathcal{B}$ is a normed space and $\mathcal{D}$ is its compact subset with the induced topology, or $\mathcal{B}$ is a reflexive Banach space and $\mathcal{D}$ is its closed convex subset with the weak topology. Let $\Theta \subset \mathcal{B}$.*
If a sequence of dominated experiments $\{\mathcal{E}_n,\, P_n,\, n \geq 1\}$ satisfies conditions (Y') and (U') then

$$\varliminf_{n\to\infty} \inf_{\rho_n \in \mathcal{R}_n} R^E_n(\rho_n) \geq E^*.$$

If conditions $(\sup Y')$ and $(\sup U')$ hold then

$$\lim_{n\to\infty} \inf_{\rho_n \in \mathcal{R}_n} R^E_n(\rho_n) = E^*,$$

and the interval–median estimators $\rho^E_{n,\delta} = r^E_{\delta,c}(Y_n)$ are nearly LD efficient:

$$\lim_{\delta\to 0} \varlimsup_{n\to\infty} R^E_n(\rho^E_{n,\delta}) = \lim_{\delta\to 0} \varliminf_{n\to\infty} R^E_n(\rho^E_{n,\delta}) = E^*.$$

Remark 4.2 *If $\mathcal{B}$ is a separable reflexive Banach space then the Borel σ–fields for the strong and weak topologies coincide, hence the condition of measurability of $r^E_{\delta,c}$ does not depend on which topology on $\mathcal{B}$ has been chosen.*

4.3 Estimation of Linear Functionals

Let Θ be a subset of a vector space and $L(\cdot)$ a linear functional on the vector space. Consider the problem of estimating $L(\theta)$. We take $\mathcal{D} = R$, the real line. As above, we consider Bahadur–type criteria: the loss function is

$$W_\theta(r) = 1(|r - L(\theta)| > c),\ \theta \in \Theta, r \in R,$$

where $c > 0$ is fixed, and the risk of an estimator ρ_n is given by

$$R^F_n(\rho_n) = \sup_{\theta\in\Theta} \frac{1}{n} \ln P_{n,\theta}\left(|\rho_n - L(\theta)| > c\right). \tag{4.16}$$

The asymptotic minimax lower bound R'^* assumes the form

$$F^* = \sup_{y\in\mathcal{Y}} \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta\,:\,|r-L(\theta)|>c} (\zeta_\theta(y) - I(y)), \tag{4.17}$$

and subproblem (Q'_δ) becomes

$$(F_\delta) \qquad F_\delta(y) = \inf_{r\in\mathcal{D}} \sup_{\theta\in\Theta\,:\,|r-L(\theta)|>c} \zeta_{\theta,\delta}(y),\ y\in\mathcal{Y}\,.$$

Associated solutions $r'_\delta(y)$ can be constructed along the same lines as for the parameter-estimation problem. Specifically, fixing y and δ, let us denote $f(\theta)=\zeta_{\theta,\delta}(y)$ and let, for $h\in R$ and $A(h)$ from (4.12), denote by $L\circ A(h)$ the image of $A(h)$ on the real line for the mapping L:

$$L\circ A(h) = \{L(\theta):\theta\in A(h)\}\,.$$

Let $B(h)$ be the smallest closed interval in R containing $L\circ A(h)$. Set further, denoting by $d(B(h))$ the length of $B(h)$,

$$h_{c,L} \;=\; \inf\{h:\, d(B(h))\le 2c\}\,.$$

Finally, consider the intervals $B_{c,L}$ of the length $2c$ that contain $B(h_{c,L})$ (note that $d(B(h_{c,L}))\le 2c$), and let $D_{c,L}$ be the set of the centres of all such intervals. The argument of the proof of Lemma 4.1 yields the following lemma.

Lemma 4.2 *The set $D_{c,L}$ is nonempty and consists of all $r_{c,L}\in\mathcal{D}$ at which $\inf_{r\in\mathcal{D}}\sup_{\theta\in\Theta:|r-L(\theta)|>c} f(\theta)$ is attained. Also the latter infimum equals $h_{c,l}$.*

To emphasise dependence on f, let us denote the elements of $D_{c,L}$ by $r_{c,L}(f)$. By the lemma, $r^F_{\delta,c}(y)=r_{c,L}(\zeta_{\Theta,\delta}(y))$ solves (F_δ). Assuming that the $r^F_{\delta,c}(y)$ are Borel functions from $\mathcal{Y}$ into R, we introduce estimators $\rho^F_{n,\delta}$ of $L(\theta)$ by

$$\rho^F_{n,\delta} = r_{c,L}(\zeta_{\Theta,\delta}(Y_n)), \tag{4.18}$$

and call them also interval–median. Applying Theorem 4.2 we get the following result.

Theorem 4.5 *If a sequence of dominated experiments $\{\mathcal{E}_n,\,P_n,\,n\ge 1\}$ satisfies conditions (Y') and (U') then*

$$\varliminf_{n\to\infty}\ \inf_{\rho_n\in\mathcal{R}_n} R^F_n(\rho_n)\ge F^*.$$

If conditions $(\sup Y')$ and $(\sup U')$ hold then

$$\lim_{n\to\infty}\ \inf_{\rho_n\in\mathcal{R}_n} R^F_n(\rho_n) = F^*,$$

and the interval–median estimators $\rho^F_{n,\delta}=r_{c,L}(\zeta_{\Theta,\delta}(Y_n))$ are nearly LD efficient:

$$\lim_{\delta\to 0}\ \varlimsup_{n\to\infty}\ R^F_n(\rho^F_{n,\delta}) = \lim_{\delta\to 0}\ \varliminf_{n\to\infty}\ R^F_n(\rho^F_{n,\delta}) = F^*.$$

We conclude the section by giving a more explicit representation for F^*.

Lemma 4.3 *Under the above notation and conditions,*

$$F^* = \sup_{\theta,\theta' : |L(\theta-\theta')|>2c} S(\theta,\theta'),$$

where $S(\theta,\theta')$ is defined by (4.7):

$$S(\theta,\theta') = \sup_{y\in\mathcal{Y}} \min\left\{\zeta_\theta(y) - I(y), \zeta_{\theta'}(y) - I(y)\right\}.$$

Proof. We fix $y \in \mathcal{Y}$ with $I(y) < \infty$, set $f(\theta) = \zeta_\theta(y)$ and define $h_{c,L}$ as above. We show that

$$h_{c,L} = \sup_{\theta,\theta' : |L(\theta-\theta')|>2c} \min\left\{f(\theta), f(\theta')\right\}.$$

By (4.17) and Lemma 4.2, this implies the claim.

Since $d(B(h)) \le 2c$ for $h > h_{c,L}$, we have that if $\theta,\theta' \in \Theta$ are such that $|L(\theta-\theta')| > 2c$ then $\min(f(\theta), f(\theta')) \le h_{c,L}$. Conversely, if $h < h_{c,L}$ then $d(B(h)) > 2c$, hence there exist $\theta,\theta' \in \Theta$ such that $L(\theta-\theta') > 2c$ and $f(\theta) > h, f(\theta') > h$ which, by the arbitrariness of $h < h_{c,L}$, ends the proof. $\square$

Remark 4.3 *The latter case of functional estimation includes the case of the estimation of a one–dimensional parameter θ if we take $L(\theta) = \theta$, so the result of Lemma 4.3 can be used for evaluating E^* from (4.11) too.*

5 Statistical Applications

In this section, we go back to the statistical models introduced in Section 2 and apply to them the general results of Sections 3 and 4. We first verify the LDP for the models by checking conditions (Y') and (U'). This is done under weaker assumptions than in Section 2. After that we give conditions that imply $(\sup Y')$ and $(\sup U')$. Next, considering certain hypothesis testing and estimation problems for the models, we calculate the asymptotic minimax risks and indicate (nearly) LD efficient decisions.

Each of the subsections below uses its own notation. We mention it if different subsections reuse certain symbols for the same objects. For the reader's convenience, we repeat the main points of the analysis of the models in Section 2 and recall the models themselves. Also we implicitly assume that the functions we choose as estimators are properly measurable.

5.1 Gaussian Observations

We observe a sample of n independent real-valued random variables $\mathbf{X}_n = (X_{1,n}, \dots, X_{n,n})$ normally distributed with $\mathcal{N}(\theta, 1)$, $\theta \in \Theta \subset R$. For this model, $\Omega_n = R^n$ and $P_{n,\theta} = (\mathcal{N}(\theta,1))^n$, $\theta \in \Theta$. We take $P_{n,0}$ as a dominating measure P_n. Then

$$\frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(\mathbf{X}) = \frac{1}{n} \sum_{k=1}^{n} (\theta X_k - \frac{1}{2}\theta^2), \ \mathbf{X} = (X_1, \dots, X_n) \in R^n.$$

Thus, it is natural to take

$$Y_n = \frac{1}{n} \sum_{k=1}^{n} X_{k,n}, \ n \geq 1,$$

so that

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n} (\mathbf{X}_n) = \theta Y_n - \frac{1}{2}\,\theta^2.$$

Then $\{\mathcal{L}\left(Y_n | P_n\right), n \geq 1\}$ obeys the LDP in R with rate function $I^N(y) = y^2/2, y \in R$ (see, e.g., Freidlin and Wentzell, 1979). This checks condition $(Y'.1)$.

We next take

$$\zeta_\theta(y) = \zeta_{\theta,\delta}(y) = \theta y - \frac{1}{2}\,\theta^2. \tag{5.1}$$

Conditions $(Y'.2)$–$(Y'.4)$ are then obvious. Condition (U') follows by Chebyshev's inequality since

$$E_n^{1/n} \exp(n\Xi_{n,\theta}) 1(\Xi_{n,\theta} > H) \leq e^{-H} E_n^{1/n} \exp(2n\Xi_{n,\theta}) \to e^{-H} e^{\theta^2}.$$

By Remark 2.2, the sequence $\{\mathcal{E}_n\,, n \geq 1\}$ obeys the LDP. Moreover, condition $(\sup Y)$ trivially holds. If, in addition, Θ is bounded, it readily follows that condition $(\sup U')$ is met as well.

We now turn to hypothesis testing and estimation problems and begin with calculating, for $\theta, \theta' \in \Theta$, the value of the function $S(\theta, \theta')$ from (4.7).

Lemma 5.1 *For all* $\theta, \theta' \in \Theta$,

$$S(\theta, \theta') := \sup_{y \in R} \min\left\{\zeta_\theta(y) - I^N(y), \zeta_{\theta'}(y) - I^N(y)\right\} = -\frac{(\theta - \theta')^2}{8}.$$

Proof. By (5.1) and the definition of I^N, $\zeta_\theta(y) - I(y) = -(y-\theta)^2/2$, so

$$S(\theta, \theta') = \sup_{y \in R} \min\left\{-\frac{(y-\theta)^2}{2}, -\frac{(y-\theta')^2}{2}\right\} = -\frac{(\theta-\theta')^2}{8}.$$

□

5.1.1 Testing $\theta = 0$ versus $|\theta| \geq 2c$

Assume that Θ contains 0 as an internal point. We test the simple hypothesis $H_0 : \theta = 0$ versus the two-sided alternative $H_1 : |\theta| \geq 2c$ with some $c > 0$ such that the interval $[-2c, 2c]$ is contained in Θ. The logarithmic risk of a test ρ_n is given by (see (4.4))

$$R_n^T(\rho_n) = \max\left\{\frac{1}{n} \ln P_{n,0}(\rho_n = 1), \frac{1}{n} \sup_{|\theta| \geq 2c} \ln P_{n,\theta}(\rho_n = 0)\right\}.$$

Now, using (4.6) with $\Theta_0 = \{0\}$ and $\Theta_1 = \{\theta \in \Theta : |\theta| \geq 2c\}$ and Lemma 5.1, we readily get

$$T^* = \sup_{|\theta'| \geq 2c} S(0, \theta') = -\frac{c^2}{2}.$$

Next, by Theorem 4.3 and Remark 3.4, LD efficient tests ρ_n^T can be taken in the form

$$\rho_n^T = 1\left(\sup_{|\theta| \geq 2c} \zeta_\theta(Y_n) > \zeta_0(Y_n)\right) = 1\left(\sup_{|\theta| \geq 2c} (\theta Y_n - \frac{\theta^2}{2}) > 0\right) = 1(|Y_n| > c).$$

Applying Theorem 4.3 and Remark 3.4, we arrive at the following result.

Proposition 5.1 *Let* $[-2c, 2c] \subset \Theta$. *Then*

$$\varliminf_{n \to \infty} \inf_{\rho_n} R_n^T(\rho_n) \geq -\frac{c^2}{2}.$$

If Θ *is bounded then*

$$\lim_{n \to \infty} \inf_{\rho_n} R_n^T(\rho_n) = -\frac{c^2}{2},$$

and the tests ρ_n^T *are LD efficient:*

$$\lim_{n \to \infty} R_n^T(\rho_n^T) = -\frac{c^2}{2}.$$

5.1.2 Parameter Estimation

Now we consider the problem of estimating the parameter θ. We take the real line as a decision space $\mathcal{D}$. Recall (see (4.10)) that, for a given $c>0$, the risk of an estimator ρ_n is defined by

$$R_n^E(\rho_n)=\sup_{\theta\in\Theta}\frac{1}{n}\ln P_{n,\theta}(|\rho_n-\theta|>c).$$

In view of Remark 4.3, the asymptotic minimax risk E^* is given by Lemma 4.3:

$$E^*=\sup_{\theta,\theta'\in\Theta\,:\,|\theta-\theta'|>2c}S(\theta,\theta').$$

Lemma 5.1 implies that if Θ contains an interval of the length greater than $2c$, then $E^*=-c^2/2$. An application of Theorem 4.4 and Remark 3.4 yields the following result.

Proposition 5.2 *Let Θ contain an interval of the length greater than $2c$. Then*

$$\varliminf_{n\to\infty}\inf_{\rho_n}R_n^E(\rho_n)\geq-\frac{c^2}{2}.$$

If Θ is bounded then

$$\lim_{n\to\infty}\inf_{\rho_n}R_n^E(\rho_n)=-\frac{c^2}{2},$$

and the interval–median estimators $\rho_n^E=r_c(\zeta_\Theta(Y_n))$ (see Section 4.2) are LD efficient:

$$\lim_{n\to\infty}R_n^E(\rho_n^E)=-\frac{c^2}{2}.$$

Remark 5.1 *It is easy to see that the estimator $\rho_n^E=r_c(\zeta_n(Y_n))$ coincides with Y_n if $Y_n-c\in\Theta$ and $Y_n+c\in\Theta$. Direct calculations show that the estimators $\widehat{\rho}_n=Y_n$ are also LD efficient, i.e., $\lim_n R_n^E(\widehat{\rho}_n)=-c^2/2$. The latter estimator is of simpler structure and does not depend on either c or Θ. However, the ρ_n^E seem to perform better at points outside or close to the boundary of Θ. In particular, if $Y_n\notin\Theta$ then $\widehat{\rho}_n\notin\Theta$ whereas, for Θ convex, ρ_n^E always belongs to Θ.*

5.2 An Independent-Identically-Distributed Sample

We observe an independent-identically-distributed sample $\mathbf{X}_n=(X_{1,n},\dots,X_{n,n})$ from a distribution $P_\theta,\theta\in\Theta$, on the real line. We assume that the family $\mathcal{P}=\{P_\theta,\theta\in\Theta\}$ is dominated by a probability measure P, i.e., $P_\theta\ll P$, $\theta\in\Theta$. This model is described by dominated experiments $\mathcal{E}_n=(\Omega_n,\mathcal{F}_n;P_{n,\theta},\,\theta\in\Theta)$ with $\Omega_n=R^n$, $\mathcal{F}_n=\mathcal{B}(R^n)$, $P_{n,\theta}=P_\theta^n$, $\theta\in\Theta$, and $P_n=P^n$.

Assume that the family $\mathcal{P}$ satisfies the following regularity conditions:

$(R.1)$ *the densities $dP_\theta/dP(x)$, $\theta\in\Theta$, are continuous and positive functions of $x\in R$;*

$(R.2)$ $\displaystyle\int_R \left(\frac{dP_\theta}{dP}(x)\right)^\gamma P(dx) < \infty\,,\ \theta \in \Theta\,,$ *for all* $\gamma \in R.$

We have that

$$\Xi_{n,\theta} = \frac{1}{n}\ln\frac{dP_{n,\theta}}{dP_n}(\mathbf{X}_n) = \sum_{k=1}^n \frac{1}{n}\ln\frac{dP_\theta}{dP}(X_{k,n}) = \int_R \ln\frac{dP_\theta}{dP}(x)F_n(dx),$$

where

$$F_n(x) = \frac{1}{n}\sum_{k=1}^n 1(X_{k,n} \le x), \qquad x \in R, \tag{5.2}$$

are empirical distribution functions.

We take the latter as statistics Y_n in condition (Y). The underlying space $\mathcal{Y}$ is the space of cumulative distribution functions on R which we denote by $\mathcal{F}$ and endow with the topology of weak convergence of associated probability measures. By Sanov's theorem (Sanov, 1957, Deuschel and Stroock, 1989, 3.2.17), the sequence $\{\mathcal{L}(Y_n|P_n)\,, n \ge 1\}$ obeys the LDP with rate function $I^S(F) = K(F,P)\,,\ F \in \mathcal{F}$, where $K(F,P)$ is the Kullback–Leibler information:

$$K(F,P) = \begin{cases} \displaystyle\int_R \frac{dF}{dP}(x)\ln\frac{dF}{dP}(x)\,P(dx), & \text{if } F \ll P, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.3}$$

This checks condition $(Y'.1)$. The verification of the rest of condition (Y') is more intricate than in the previous example.

Denote for $\theta \in \Theta$, $x \in R$ and $\delta > 0$,

$$\begin{aligned} L_\theta(x) &= \ln\frac{dP_\theta}{dP}(x), \\ L_{\theta,\delta}(x) &= L_\theta(x) \wedge \delta^{-1} \vee (-\delta^{-1}) \end{aligned}$$

and let

$$\zeta_{\theta,\delta}(F) = \int_R L_{\theta,\delta}(x)F(dx), \quad F \in \mathcal{F}.$$

By $(R.1)$, the functions $\zeta_{\theta,\delta}$ are continuous on $\mathcal{F}$, so $(Y'.2)$ holds.

We check $(Y'.3)$. Condition $(R.2)$ implies that, for all $\gamma > 0$,

$$\lim_{\delta\to 0}\int_R \left[\exp\left(\gamma\left|L_\theta(x) - L_{\theta,\delta}(x)\right|\right) - 1\right] P(dx) = 0. \tag{5.4}$$

Then, for $\gamma > 0, \varepsilon > 0$, with the use of Chebyshev's inequality,

$$\begin{aligned} P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(F_n)| > \varepsilon) &\le P_n^{1/n}\left(\int_R \left|L_\theta(x) - L_{\theta,\delta}(x)\right| F_n(dx) > \varepsilon\right) \\ &\le \exp(-\gamma\varepsilon)E_n^{1/n}\exp\left(n\gamma\int_R \left|L_\theta(x) - L_{\theta,\delta}(x)\right| F_n(dx)\right) \\ &= \exp(-\gamma\varepsilon)\int_R \exp\left(\gamma\left|L_\theta(x) - L_{\theta,\delta}(x)\right|\right) P(dx). \end{aligned}$$

By (5.4), it then follows that

$$\varlimsup_{\delta\to 0}\ \varlimsup_{n\to\infty}\ P_n^{1/n}(|\Xi_{n,\theta}-\zeta_{\theta,\delta}(F_n)|>\varepsilon)\le \exp(-\gamma\varepsilon).$$

Since γ is arbitrary, $(Y'.3)$ follows.

We next check $(Y'.4)$ with

$$\zeta_\theta(F)=\begin{cases}\displaystyle\int_R L_\theta(x)F(dx), & \text{if } I^S(F)<\infty,\\ 0, & \text{otherwise.}\end{cases}\tag{5.5}$$

To begin, we show that the ζ_θ are well defined. Since the functions $x\ln x-x+1$ and $\exp x-1$ are convex conjugates (Rockafellar, 1970), by the Young–Fenchel inequality (Rockafellar, 1970, Krasnoselskii and Rutickii, 1961), for $F\ll P$,

$$\begin{aligned}\int_R\left|L_\theta(x)\frac{dF}{dP}(x)\right|P(dx) &\le \int_R[\exp(|L_\theta(x)|)-1]\,P(dx)\\ &\quad+\int_R\left(\frac{dF}{dP}(x)\ln\frac{dF}{dP}(x)-\frac{dF}{dP}(x)+1\right)P(dx)\\ &\le 1+\int_R\left(\frac{dP_\theta}{dP}(x)\right)^{-1}P(dx)+I^S(F).\end{aligned}$$

In view of $(R.2)$, this proves that the ζ_θ are well defined.

Now, for F with $I^S(F)<\infty$, we have, for $\gamma>0$, using the Young–Fenchel inequality again,

$$\begin{aligned}\gamma|\zeta_{\theta,\delta}(F)-\zeta_\theta(F)| &\le \int_R\gamma\,|L_{\theta,\delta}(x)-L_\theta(x)|\,F(dx)\\ &\le \int_R[\exp(\gamma\,|L_{\theta,\delta}(x)-L_\theta(x)|)-1]\,P(dx)\\ &\quad+\int_R\left(\frac{dF}{dP}(x)\ln\frac{dF}{dP}(x)-\frac{dF}{dP}(x)+1\right)P(dx)\\ &= \int_R[\exp(\gamma\,|L_{\theta,\delta}(x)-L_\theta(x)|)-1]\,P(dx)+I^S(F).\end{aligned}$$

Hence, by (5.4)

$$\varlimsup_{\delta\to 0}\ \sup_{F\in\Phi'_{I^S}(a)}|\zeta_{\theta,\delta}(F)-\zeta_\theta(F)|\le\frac{a}{\gamma},$$

and letting $\gamma\to\infty$, we arrive at $(Y'.4)$. Remark 2.2 then implies that the LDP holds for $\{\mathcal{L}(\Xi_{n,\Theta}|P_n),n\ge 1\}$.

It remains to check (U'). Using once again Chebyshev's inequality, we obtain, for $H>0$,

$$\begin{aligned}E_n^{1/n}\exp(n\Xi_{n,\theta})1(\Xi_{n,\theta}>H) &\le \exp(-H)E_n^{1/n}\exp(2n\Xi_{n,\theta})\\ &= \exp(-H)\int_R\left(\frac{dP_\theta}{dP}(x)\right)^2P(dx),\end{aligned}$$

and the required follows by condition $(R.2)$.

Conditions (Y') and (U') have been checked, and thus the LDP holds.

Remark 5.2 *It is possible to do without condition* $(R.1)$. *Then the functions* $L_{\theta,\delta} = (L_{\theta,\delta}(x), x \in R), \delta > 0, \theta \in \Theta$, *should be chosen bounded, continuous and so that (5.4) holds. The existence of such functions follows from* $(R.2)$.

To check $(\sup Y')$ and $(\sup U')$, we assume that stronger versions of conditions $(R.1)$ and $(R.2)$ hold:

$(\sup R.1)$ *the functions* $dP_\theta/dP(x)$, $\theta \in \Theta$, *are positive and equicontinuous at each* $x \in R$*;*

$(\sup R.2)$ $\sup\limits_{\theta\in\Theta} \int_R \left(\frac{dP_\theta}{dP}(x)\right)^\gamma P(dx) < \infty$ *for all* $\gamma \in R$.

Defining ζ_θ, $\zeta_{\theta,\delta}$, L_θ and $L_{\theta,\delta}$ as above, we have, by $(\sup R.2)$, that for all $\gamma > 0$

$$\lim_{\delta\to 0} \sup_{\theta\in\Theta} \int_R \left[\exp\left(\gamma \left|L_\theta(x) - L_{\theta,\delta}(x)\right|\right) - 1\right] P(dx) = 0.$$

The latter equality enables us to check conditions $(\sup Y'.3)$ and $(\sup Y'.4)$ in the same way as conditions $(Y'.3)$ and $(Y'.4)$. Condition $(\sup U')$ is also checked analogously to condition (U'), with the use of $(\sup R.2)$. Condition $(Y'.1)$ has already been checked.

It remains to check $(\sup Y'.2)$. We show that the functions $(\zeta_{\theta,\delta}(F), \theta \in \Theta)$ are continuous in F for the uniform topology on R_+^Θ which obviously implies $(\sup Y'.2)$. Since the weak topology on $\mathcal{F}$ is metrisable, it is enough to check sequential continuity. Let $F^{(n)}$ weakly converge to F as $n \to \infty$. Then the definition of the $L_{\theta,\delta}$ and $(\sup R.1)$ imply that the $L_{\theta,\delta}(x), \theta \in \Theta$, for δ fixed, are uniformly bounded and equicontinuous at each $x \in R$ so that (see, e.g., Billingsley, 1968, Problem 8, §2)

$$\sup_{\theta\in\Theta} \left| \int_R L_{\theta,\delta}(x) F^{(n)}(dx) - \int_R L_{\theta,\delta}(x) F(dx) \right| \to 0$$

checking $(\sup Y'.2)$. Conditions $(\sup Y')$ and $(\sup U')$ have been checked.

We now proceed to considering concrete statistical problems for the model. For this we need the following result by Chernoff, 1952, see also Kullback, 1959.

Lemma 5.2 *Let* $\mathcal{P}$ *be the space of probability measures on a Polish space* E *with the Borel* σ*–field, and let measures* $P, Q \in \mathcal{P}$ *be dominated by a measure* μ *and have respective densities* $p(x)$ *and* $q(x)$. *Then*

$$\inf_{F\in\mathcal{P}} \max\left\{K(F,P), K(F,Q)\right\} = C(P,Q),$$

where $K(F,P)$ *is the Kullback–Leibler information (5.3) and* $C(P,Q)$ *is Chernoff's function:*

$$C(P,Q) = -\inf_{\gamma\in[0,1]} \ln \int_E p^\gamma(x)\, q^{1-\gamma}(x)\, \mu(dx).$$

We next apply Lemma 5.2 to calculating the function $S(\theta,\theta')$ from (4.7).

Lemma 5.3 *For* $\theta,\theta'\in\Theta$,

$$S(\theta,\theta') := \sup_{F\in\mathcal{F}} \min\left\{\zeta_\theta(F) - I^S(F), \zeta_{\theta'}(F) - I^S(F)\right\} = -C(P_\theta,P_{\theta'}).$$

Proof. Let $I^S(F)<\infty$. Then $F\ll P$ and, since the densities $dP_\theta/dP(x)$, $\theta\in\Theta$, are positive, we also have that $F\ll P_\theta$ and P–almost surely

$$\frac{dF}{dP} = \frac{dF}{dP_\theta}\,\frac{dP_\theta}{dP}.$$

Therefore, by the definitions of ζ_θ and I^S,

$$\begin{aligned}\zeta_\theta(F) - I^S(F) &= \int_R \ln\frac{dP_\theta}{dP}(x) F(dx) - \int_R \ln\frac{dF}{dP} F(dx)\\ &= -\int_R \ln\frac{dF}{dP_\theta} F(dx) = -K(F,P_\theta),\end{aligned}$$

and the result follows by Lemma 5.2. □

We now give an application to hypothesis testing problems. Consider the tests from (4.8):

$$\rho^T_{n,\delta} = 1\left(\sup_{\theta\in\Theta_0}\zeta_{\theta,\delta}(F_n) < \sup_{\theta\in\Theta_1}\zeta_{\theta,\delta}(F_n)\right).$$

As above, the risk $R^T_n(\rho_n)$ of a test ρ_n is defined by (4.4). By (4.6) and Lemma 5.3,

$$T^* = -\inf_{\theta\in\Theta_0,\,\theta'\in\Theta_1} C(P_\theta,P_{\theta'}),$$

so Theorem 4.3 yields the following.

Proposition 5.3 *Let* Θ_1 *and* Θ_2 *be non-intersecting subsets of* Θ.
If conditions $(R.1)$ *and* $(R.2)$ *hold then*

$$\varliminf_{n\to\infty} \inf_{\rho_n} R^T_n(\rho_n) \ge -\inf_{\theta\in\Theta_0,\,\theta'\in\Theta_1} C(P_\theta,P_{\theta'}).$$

If conditions $(\sup R.1)$ *and* $(\sup R.2)$ *hold then*

$$\lim_{n\to\infty} \inf_{\rho_n} R^T_n(\rho_n) = -\inf_{\theta\in\Theta_0,\,\theta'\in\Theta_1} C(P_\theta,P_{\theta'}),$$

and the tests $\rho^T_{n,\delta}$ *are nearly LD efficient, i.e.,*

$$\begin{aligned}\lim_{\delta\to 0}\varlimsup_{n\to\infty} R^T_n(\rho^T_{n,\delta}) &= \lim_{\delta\to 0}\varliminf_{n\to\infty} R^T_n(\rho^T_{n,\delta})\\ &= -\inf_{\theta\in\Theta_0,\,\theta'\in\Theta_1} C(P_\theta,P_{\theta'}).\end{aligned}$$

In a similar manner one can tackle estimation problems for θ or linear functionals of θ.

5.3 "Signal + White Noise"

We observe a real–valued stochastic process $X_n = (X_n(t), t \in [0,1])$ obeying the stochastic differential equation

$$dX_n(t) = \theta(t)dt + \frac{1}{\sqrt{n}} dW(t),\ 0 \le t \le 1, \tag{5.6}$$

where $W = (W(t), t \in [0,1])$ is a standard Wiener process and $\theta(\cdot)$ is an unknown continuous function.

This model is described by statistical experiments $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta},\, \theta \in \Theta)$, where $\Omega_n = C[0,1]$, the space of continuous functions on $[0,1]$ with the uniform metric, $\Theta \subset C[0,1]$ and $P_{n,\theta}$ is the distribution of X_n on $C[0,1]$ for θ. We take $P_n = P_{n,0}$, where $P_{n,0}$ corresponds to the zero function $\theta(\cdot) \equiv 0$. Then $P_{n,\theta} \ll P_n$ and, moreover, by Girsanov's formula, P_n–almost surely,

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) = \int_0^1 \theta(t) dX_n(t) - \frac{1}{2} \int_0^1 \theta^2(t) dt. \tag{5.7}$$

So, to check condition (Y'), we take $Y_n = X_n$ and $\mathcal{Y} = C[0,1]$.

Let $C_0[0,1]$ be the subset of $C[0,1]$ of the functions $x(\cdot)$ that are absolutely continuous with respect to Lebesgue measure and equal to 0 at 0. Since the sequence $\{\mathcal{L}(X_n | P_n), n \ge 1\}$ obeys the LDP in $C[0,1]$ with rate function

$$I^W(x(\cdot)) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 (\dot{x}(t))^2\, dt, & \text{if } x(\cdot) \in C_0[0,1], \\ \infty, & \text{otherwise}, \end{cases} \tag{5.8}$$

where $\dot{x}(t)$ denotes the derivative of $x(\cdot) \in C[0,1]$ at t (see, e.g., Freidlin and Wentzell, 1979), condition $(Y'.1)$ holds.

We next take

$$\zeta_{\theta,\delta}(x(\cdot)) = \int_0^1 \theta_\delta(t)\, dx(t) - \frac{1}{2} \int_0^1 \theta^2(t)\, dt,\ x(\cdot) \in C[0,1], \tag{5.9}$$

where

$$\theta_\delta(t) = \sum_{k=0}^{[1/\delta]} \theta(k\delta) 1(t \in [k\delta, (k+1)\delta)),\ t \in [0,1], \tag{5.10}$$

the first integral on the right of (5.9) being understood as a finite sum.

By the continuity of $\theta(\cdot)$,

$$\lim_{\delta \to 0} \int_0^1 (\theta(t) - \theta_\delta(t))^2\, dt = 0. \tag{5.11}$$

The $\zeta_{\theta,\delta}$ are obviously continuous in $x(\cdot) \in C[0,1]$, so $(Y'.2)$ holds. Next, by (5.7) and (5.9), we have, for $\varepsilon > 0$ and $\gamma > 0$, in view of Chebyshev's inequality,

$$P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(X_n)| > \varepsilon) \le P_n^{1/n}\left(\left|\int_0^1 (\theta(t) - \theta_\delta(t)) \frac{1}{\sqrt{n}}\, dW(t)\right| > \varepsilon\right)$$
$$\le 2e^{-\gamma\varepsilon} \exp\left(\frac{\gamma^2}{2} \int_0^1 (\theta(t) - \theta_\delta(t))^2\, dt\right),$$

and by (5.11)

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(X_n)| > \varepsilon) \le 2\exp(-\gamma\varepsilon)$$

which proves $(Y'.3)$ by the arbitrariness of γ.

For condition $(Y'.4)$, we take

$$\zeta_\theta(x(\cdot)) = \begin{cases} \displaystyle\int_0^1 \theta(t)\dot{x}(t)\, dt - \frac{1}{2}\int_0^1 \theta^2(t)\, dt, & \text{if } I^W(x(\cdot)) < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

The ζ_θ are well defined, since, by the Cauchy–Schwarz inequality and (5.8), if $x(\cdot)$ is absolutely continuous then

$$\int_0^1 |\theta(t)\dot{x}(t)|\, dt \le \left(\int_0^1 \theta^2(t)\, dt\right)^{1/2} (2I^W(x(\cdot)))^{1/2}.$$

Moreover, if $I^W(x(\cdot)) < \infty$ then

$$\begin{aligned} |\zeta_{\theta,\delta}(x(\cdot)) - \zeta_\theta(x(\cdot))| &\le \int_0^1 |\theta_\delta(t) - \theta(t)||\dot{x}(t)|\, dt \\ &\le \left(\int_0^1 (\theta_\delta(t) - \theta(t))^2\, dt\right)^{1/2} \left(\int_0^1 (\dot{x}(t))^2\, dt\right)^{1/2}, \end{aligned}$$

so

$$\sup_{x(\cdot)\in\Phi'_{I^W}(a)} |\zeta_{\theta,\delta}(x(\cdot)) - \zeta_\theta(x(\cdot))| \le (2a)^{1/2} \left(\int_0^1 (\theta_\delta(t) - \theta(t))^2\, dt\right)^{1/2},$$

and the latter goes to 0 as $\delta \to 0$ by (5.11). Condition (Y') has been verified.

It remains to check (U'). Using the model equation (5.6), (5.7) and Chebyshev's inequality once again, we have that

$$E_n^{1/n} \exp(n\Xi_{n,\theta}) 1(\Xi_{n,\theta} > H) \le \exp(-H) E_n^{1/n} \exp(2n\Xi_{n,\theta})$$
$$= \exp(-H) \exp\left(\int_0^1 \theta^2(t) dt\right) \to 0 \qquad \text{as } H \to \infty.$$

Conditions (Y') and (U') have been checked.

Remark 5.3 *The condition of continuity of the functions $\theta(\cdot)$ can be weakened to the condition*

$$\int_0^1 \theta^2(t)\,dt < \infty.$$

The functions θ_δ should then be chosen as step functions for which (5.11) holds.

For conditions $(\sup Y')$ and $(\sup U')$, we require that the functions $\theta(\cdot)$ belong to a compact in $C[0,1]$. More specifically, for fixed $\beta \in (0,1]$, $M > 0$ and $K > 0$, we introduce the Hölder class

$$\Sigma(\beta, M) = \{\theta(\cdot) : |\theta(t) - \theta(s)| \le M|t-s|^\beta, \quad \text{for all } s,t \in [0,1]\}, \tag{5.12}$$

define $\Sigma_K(\beta, M)$ to be the subset of $\Sigma(\beta, M)$ of functions θ such that $\sup_{t\in[0,1]}|\theta(t)| \le K$ and assume that $\Theta \subset \Sigma_K(\beta, M)$. By the Arzelà–Ascoli theorem, the set $\Sigma_K(\beta, M)$ is compact in $C[0,1]$. Also

$$\sup_{\theta(\cdot)\in\Sigma_K(\beta,M)} \int_0^1 \theta^2(t)\,dt < \infty \tag{5.13}$$

and

$$\lim_{\delta\to 0} \sup_{\theta(\cdot)\in\Sigma_K(\beta,M)} \int_0^1 (\theta(t) - \theta_\delta(t))^2\,dt = 0. \tag{5.14}$$

Now conditions $(\sup Y'.3)$ and $(\sup Y'.4)$ are checked as conditions $(Y'.3)$ and $(Y'.4)$, respectively, with the use of (5.14) in place of (5.11). Condition $(\sup Y'.2)$ follows by the uniform boundedness of functions from $\Sigma_K(\beta, M)$ which implies that $x(\cdot) \to (\zeta_{\theta,\delta}(x(\cdot)), \theta \in \Sigma_K(\beta, M))$ is a continuous map from $C[0,1]$ into R_+^Θ with the uniform topology.

Finally, condition $(\sup U')$ follows in analogy with condition (U') with the use of (5.13). This completes verification of conditions $(\sup Y')$ and $(\sup U')$.

We now calculate the function $S(\theta, \theta')$ for the model.

Lemma 5.4 *For all $\theta, \theta' \in C[0,1]$,*

$$\begin{aligned} S(\theta,\theta') &:= \sup_{x(\cdot)\in C[0,1]} \min\{\zeta_\theta(x(\cdot)) - I^W(x(\cdot)), \zeta_{\theta'}(x(\cdot)) - I^W(x(\cdot))\} \\ &= -\frac{1}{8}\int_0^1 [\theta(t) - \theta'(t)]^2 dt. \end{aligned}$$

Proof. Since by the definitions of I^W and ζ_θ, for $x(\cdot)$ with $I^W(x(\cdot)) < \infty$,

$$\zeta_\theta(x(\cdot)) - I^W(x(\cdot)) = -\frac{1}{2}\int_0^1 (\dot{x}(t) - \theta(t))^2\,dt\,,$$

we get, by the inequality $\max(a^2,b^2)\ge (a-b)^2/4$,

$$S(\theta,\theta') = -\inf_{x(\cdot)\in C[0,1]} \max\left\{\frac{1}{2}\int_0^1 [\dot{x}(t)-\theta(t)]^2dt, \frac{1}{2}\int_0^1 [\dot{x}(t)-\theta'(t)]^2dt\right\}$$
$$\le -\frac{1}{8}\int_0^1 [\theta(t)-\theta'(t)]^2dt.$$

On the other hand, for $x(\cdot)$ with $\dot{x}(t)=[\theta(t)+\theta'(t)]/2$, we have that

$$\frac{1}{2}\int_0^1 [\dot{x}(t)-\theta(t)]^2dt = \frac{1}{2}\int_0^1 [\dot{x}(t)-\theta'(t)]^2dt = \frac{1}{8}\int_0^1 [\theta(t)-\theta'(t)]^2dt,$$

and the required follows. □

Now we apply these formulae and the general results from Section 4 to two statistical problems concerning the value of the function $\theta(\cdot)$ at an internal point t_0 of $[0,1]$.

5.3.1 Testing $\theta(t_0)=0$ versus $|\theta(t_0)|\ge 2c$

Given $c>0$, denote $\Theta_0=\{\theta\in\Theta:\ \theta(t_0)=0\}$, $\Theta_1=\{\theta\in\Theta:\ |\theta(t_0)|\ge 2c\}$ and define the risk $R_n^T(\rho_n)$ of a test ρ_n by (4.4). Introduce

$$t^* = (c/M)^{1/\beta}. \tag{5.15}$$

Proposition 5.4 *Let c,β,M,K and t_0 be such that $[t_0-t^*,t_0+t^*]\subset[0,1]$ and $K\ge 2c$.*
If $\Theta=\Sigma(\beta,M)$ then

$$\underline{\lim}_{n\to\infty}\ \inf_{\rho_n} R_n^T(\rho_n) \ge -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

If $\Theta=\Sigma_K(\beta,M)$ then

$$\lim_{n\to\infty}\ \inf_{\rho_n} R_n^T(\rho_n) = -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta},$$

and the tests $\rho_{n,\delta}^T$ from (4.8) are nearly LD efficient, i.e.,

$$\lim_{\delta\to 0}\ \overline{\lim}_{n\to\infty}\ R_n^T(\rho_{n,\delta}^T) = \lim_{\delta\to 0}\ \underline{\lim}_{n\to\infty}\ R_n^T(\rho_{n,\delta}^T) = -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

Proof. By Theorem 4.3, we need only to calculate T^* from (4.6). Denote

$$\theta^*(t) = [c-M|t-t_0|^\beta]^+, \tag{5.16}$$

where $a^+=\max(a,0)$. If $\theta\in\Theta_0$ and $\theta'\in\Theta_1$ then the inequality $|\theta(t_0)-\theta'(t_0)|\ge 2c$ and the Hölder constraints (5.12) imply that $|\theta(t)-\theta'(t)|\ge 2[c-M|t-t_0|^\beta]^+=2\theta^*(t)$, and hence

$$\int_0^1 (\theta(t)-\theta'(t))^2dt \ge \int_0^1 4(\theta^*(t))^2dt.$$

This yields, by Lemma 5.4,

$$S(\theta,\theta') \le -\frac{1}{8}4\int_0^1 (\theta^*(t))^2\,dt = -\int_0^{t^*} (c-Mt^\beta)^2 dt$$
$$= -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

On the other hand, evidently, $c-\theta^* \in \Theta_0$, $c+\theta^* \in \Theta_1$ and $S(c-\theta^*, c+\theta^*) = -\frac{1}{2}\int\limits_0^1 (\theta^*(t))^2\,dt$ so that

$$T^* = \sup_{\theta\in\Theta_0,\theta'\in\Theta_1} S(\theta,\theta') = -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

□

5.3.2 Estimating $\theta(t_0)$

Treating $\theta(t_0)$ as a linear functional of $\theta(\cdot)$, we define the risk of an estimator ρ_n of $\theta(t_0)$ by

$$R_n^F(\rho_n) = \sup_{\theta\in\Theta}\frac{1}{n}\ln P_{n,\theta}(|\rho_n-\theta(t_0)|>c).$$

Proposition 5.5 *Let c,β,M,K and t_0 be such that $[t_0-t^*, t_0+t^*]\subset[0,1]$ and $K>c$. If $\Theta=\Sigma(\beta,M)$ then*

$$\varliminf_{n\to\infty}\ \inf_{\rho_n} R_n^F(\rho_n) \ge -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

If $\Theta=\Sigma_K(\beta,M)$ then

$$\lim_{n\to\infty}\ \inf_{\rho_n} R_n^F(\rho_n) = -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta},$$

and the interval–median estimators $\rho_{n,\delta}^F$ from (4.18) are nearly LD efficient, i.e.,

$$\lim_{\delta\to0}\ \varlimsup_{n\to\infty}\ R_n^T(\rho_{n,\delta}^F) = \lim_{\delta\to0}\ \varliminf_{n\to\infty}\ R_n^T(\rho_{n,\delta}^F) = -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

Proof. By Theorem 4.5 and Lemma 4.3,

$$\varliminf_{n\to\infty}\ \inf_{\rho_n} R_n^F(\rho_n) \ge F^* = \sup_{\theta,\theta':|\theta(t_0)-\theta'(t_0)|>2c} S(\theta,\theta').$$

Repeating the above calculation for the testing problem, we obtain with $\theta^*(t)$ from (5.16)

$$F^* = S(\theta^*,-\theta^*) = -\frac{2\beta^2c^2}{(\beta+1)(2\beta+1)}\left(\frac{c}{M}\right)^{1/\beta}.$$

□

Remark 5.4 *The latter problem has been studied by Korostelev, 1993, who suggests different upper estimators, namely, the kernel estimators*

$$\widehat{\rho}_n = \int K(t_0 - t) dX_n(t)$$

with the kernel $K(t) = (\beta+1)/(2c\beta)\,(M/c)^{1/\beta}\,[c - M|t - t_0|]^+$. *These estimators have proved to be asymptotically efficient in the sense that* $R_n^T(\widehat{\rho}_n) \to F^*$ *as* $n \to \infty$.

5.4 Gaussian Regression

We consider the regression model

$$X_{k,n} = \theta(t_{k,n}) + \xi_{k,n}, \qquad t_{k,n} = \frac{k}{n}, \quad k = 1, \ldots, n, \tag{5.17}$$

where errors $\xi_{k,n}$ are independent standard normal and $\theta(\cdot)$ is an unknown continuous function.

In this model, $\Omega_n = R^n$, $\Theta \subset C[0,1]$ and $P_{n,\theta}$ is the distribution of $\mathbf{X}_n = (X_{1,n}, \ldots, X_{n,n})$ for $\theta(\cdot)$. As above, we take $P_n = P_{n,0}$. Then

$$\begin{aligned} \Xi_{n,\theta} &= \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) \\ &= \frac{1}{n} \sum_{k=1}^{n} \theta(t_{k,n}) X_{k,n} - \frac{1}{2n} \sum_{k=1}^{n} \theta^2(t_{k,n}) \\ &= \int_0^1 \theta(t)\, dX_n(t) - \frac{1}{2n} \sum_{k=1}^{n} \theta^2\left(t_{k,n}\right), \end{aligned} \tag{5.18}$$

where

$$X_n(t) = \frac{1}{n} \sum_{k=1}^{[nt]} X_{k,n}, \qquad 0 \le t \le 1.$$

This prompts taking the process $X_n = (X_n(t)\,,\ t \in [0,1])$ as a statistic Y_n in condition (Y'). We define $\mathcal{Y}$ to be the space of right-continuous with left-hand limits functions on $[0,1]$ with the uniform metric.

Since the $X_{k,n}$ are $\mathcal{N}(0,1)$–distributed under P_n, the sequence $\{\mathcal{L}\left(X_n|P_n\right), n \ge 1\}$ obeys the LDP with I^W from (5.8), Mogulskii, 1976. This checks condition $(Y'.1)$.

Next, we define $\zeta_{\theta,\delta}(x(\cdot))$ as in Subsection 5.3, i.e.,

$$\zeta_{\theta,\delta}(x(\cdot)) = \int_0^1 \theta_\delta(t) dx(t) - \frac{1}{2} \int_0^1 \theta^2(t) dt,\ x(\cdot) \in \mathcal{Y}, \tag{5.19}$$

where

$$\theta_\delta(t) = \sum_{k=0}^{[1/\delta]} \theta(k\delta) 1(t \in [k\delta, (k+1)\delta))\,,\ t \in [0,1].$$

Note that the $\zeta_{\theta,\delta}$ are measurable with respect to the Borel σ–field on $\mathcal{Y}$ and continuous at $x(\cdot)$ with $I^W(x(\cdot)) = \infty$ since they are continuous at continuous functions and $I^W(x(\cdot)) = \infty$ when $x(\cdot)$ is not absolutely continuous. This checks condition $(Y'.2)$.

Now, by (5.18) and (5.19),

$$P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(X_n)| > \varepsilon) \le 1\left(\left|\int_0^1 \theta^2(t)\,dt - \frac{1}{n}\sum_{k=1}^n \theta^2(k/n)\right| > \varepsilon/4\right) + P_n^{1/n}\left(\left|\int_0^1 (\theta(t) - \theta_\delta(t))\,dX_n(t)\right| > \varepsilon/2\right).$$

The first term on the right is zero for all n large enough by the continuity of $\theta(\cdot)$. The second is not greater than

$$e^{-\gamma\varepsilon/2} E_n^{1/n} \exp\left(n\gamma\left|\int_0^1 (\theta(t) - \theta_\delta(t))\,dX_n(t)\right|\right) \le 2e^{-\gamma\varepsilon/2}\exp\left(\frac{\gamma^2}{2n}\sum_{k=1}^n (\theta(k/n)) - \theta_\delta(k/n))^2\right).$$

Since the $\theta(\cdot)$ are continuous and the $\theta_\delta(\cdot)$ are step functions,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n (\theta(k/n) - \theta_\delta(k/n))^2 = \int_0^1 (\theta(t) - \theta_\delta(t))^2\,dt,$$

and the latter goes to 0 as $\delta \to 0$. Since γ is arbitrary, condition $(Y'.3)$ follows.

Conditions $(Y'.4)$ and (U') are checked as for the "signal + white noise" model (with the same choice of ζ_θ).

Remark 5.5 *As in the "signal + white noise" model, instead of continuity of $\theta(\cdot)$, we could require that it be square integrable on $[0,1]$.*

To get nearly LD efficient decisions, we assume that the $\theta(\cdot)$ belong to the class $\Sigma_K(\beta, M)$ defined above. Conditions $(\sup Y'.2)$, $(\sup Y'.3)$, $(\sup Y'.4)$ and $(\sup U')$ are checked as for the "signal + white noise" model if we in addition take into account that

$$\lim_{n\to\infty} \sup_{\theta(\cdot)\in\Sigma_K(\beta,M)} \int_0^1 (\theta([nt] + 1/n) - \theta(t))^2\,dt = 0.$$

Condition $(\sup Y'.2)$ is obvious.

Since here we have the same functions $I^W(x)$ and $\zeta_\theta(x)$ as for the "signal + white noise" model, the statistical problems of Subsection 5.3 are solved in the same way.

5.5 Non-Gaussian Regression

We consider the regression model (5.17) but now assume that independent identically distributed errors $\xi_{k,n}$ have a distribution P on the real line with a probability density function $p(x)$ with respect to Lebesgue measure. An unknown regression function $\theta(\cdot)$ is again assumed to be continuous, so $\Theta \subset C[0,1]$.

Next, we assume that the density $p(x)$ obeys the following condition, cf. conditions $(R.1)$ and $(R.2)$ for the model of an independent-identically-distributed sample:

(P) *the density* $p(x)$ *is positive and continuous, and the function*

$$H_\gamma(s) = \int_R p^\gamma(x) p^{1-\gamma}(x-s)\,dx$$

is bounded over s *from bounded domains for all* $\gamma \in R$.

As above, for a regression function $\theta(\cdot)$, we denote by $P_{n,\theta}$ the distribution of $X_n = (X_{1,n},\ldots,X_{n,n})$. We have, with $P_n = P_{n,0}$,

$$\Xi_{n,\theta} = \frac{1}{n}\ln\frac{dP_{n,\theta}}{dP_n}(X_n) = \frac{1}{n}\sum_{k=1}^n \ln\frac{p(X_{k,n}-\theta(k/n))}{p(X_{k,n})}.$$

As in the case of an independent-identically-distributed sample, this representation suggests taking for Y_n an empirical process $F_n = F_n(x,t)$, $x \in R$, $t \in [0,1]$, defined by $F_n(x,0) = 0$ and

$$F_n(x,t) = \frac{1}{n}\sum_{k=1}^{[nt]} 1(X_{k,n} \le x), \qquad 0 < t \le 1. \tag{5.20}$$

Then

$$\Xi_{n,\theta} = \int_0^1 \int_R \ln\frac{p(x-\theta(t))}{p(x)} F_n(dx,dt). \tag{5.21}$$

We define $\mathcal{Y}$ as the space of cumulative distribution functions $F = F(x,t)$, $x \in R$, $t \in [0,1]$, on $R\times[0,1]$ with the weak topology. Let $\mathcal{Y}_0$ be the subset of $\mathcal{Y}$ of absolutely continuous with respect to Lebesgue measure on $R\times[0,1]$ functions $F(x,t)$ with densities $p_t(x)$ satisfying the condition $\int_R p_t(x)\,dx = 1$, $t \in [0,1]$. As it follows from Dembo and Zajic, 1995 or Puhalskii, 1996[Theorem 1], the sequence $\{\mathcal{L}(F_n|P_n), n \ge 1\}$ obeys the LDP in $\mathcal{Y}$ with rate function $I^{SK}(F)$ given by

$$I^{SK}(F) = \begin{cases} \displaystyle\int_0^1 \int_R \ln\frac{p_t(x)}{p(x)}\, p_t(x)\,dx\,dt, & \text{if } F \in \mathcal{Y}_0,\\ \infty, & \text{otherwise.}\end{cases}$$

This checks $(Y'.1)$.

To define $\zeta_{\theta,\delta}(F)$, introduce the functions

$$\begin{aligned} L_\theta(x,t) &= \ln\frac{p(x-\theta(t))}{p(x)},\\ L_{\theta,\delta}(x,t) &= L_\theta(x,t) \vee (-\delta^{-1}) \wedge \delta^{-1},\ x \in R,\ t \in [0,1]. \end{aligned}$$

The functions $L_{\theta,\delta}$ are bounded, continuous and, in view of (P), satisfy the relations

$$\lim_{\delta\to 0}\int_0^1\int_R [\exp(\gamma\,|L_\theta(x,t)-L_{\theta,\delta}(x,t)|)-1]\,p(x)\,dx\,dt=0,\quad \gamma>0, \tag{5.22}$$

and, for every $\gamma>0$,

$$\overline{\lim_{n\to\infty}}\int_0^1\int_R\left[\exp\left(\gamma\left|L_\theta\left(x,\frac{[nt]}{n}\right)-L_{\theta,\delta}\left(x,\frac{[nt]}{n}\right)\right|\right)-1\right]p(x)\,dx\,dt\to 0 \tag{5.23}$$

as $\delta\to 0$. We set

$$\zeta_{\theta,\delta}(F)=\int_0^1\int_R L_{\theta,\delta}(x,t)\,F(dx,dt). \tag{5.24}$$

Then condition $(Y'.2)$ holds by the definition of the topology on $\mathcal{Y}$ and choice of the $L_{\theta,\delta}$.

For condition $(Y'.3)$, write, for $\gamma>0$, using Chebyshev's inequality, and (5.20), (5.21) and (5.24),

$$\begin{aligned}&\frac{1}{n}\ln P_n(|\Xi_{n,\theta}-\zeta_{\theta,\delta}(F_n)|>\varepsilon)\\ &\quad\le\frac{1}{n}\ln P_n\left(\int_0^1\int_R|L_\theta(x,t)-L_{\theta,\delta}(x,t)|\;F_n(dx,dt)>\varepsilon\right)\\ &\quad\le-\gamma\varepsilon+\frac{1}{n}\sum_{k=1}^n\ln\int_R\exp(\gamma\,|L_\theta(x,k/n)-L_{\theta,\delta}(x,k/n)|)\,p(x)\,dx.\end{aligned}$$

Limit (5.23) yields

$$\overline{\lim_{\delta\to 0}}\;\overline{\lim_{n\to\infty}}\;\frac{1}{n}\ln P_n(|\Xi_{n,\theta}-\zeta_{\theta,\delta}(F_n)|>\varepsilon)\le-\gamma\varepsilon$$

which proves $(Y'.3)$ since γ is arbitrary.

For condition $(Y'.4)$, we take

$$\zeta_\theta(F)=\begin{cases}\displaystyle\int_0^1\int_R L_\theta(x,t)\,F(dx,dt), & \text{if } I^{SK}(F)<\infty,\\ 0, & \text{otherwise.}\end{cases}$$

The ζ_θ are well defined since, by the Young–Fenchel inequality, if $F(x,t)=\int_0^t\int_{-\infty}^x p_s(y)dy\,ds$ then

$$\begin{aligned}\int_0^1\int_R|L_\theta(x,t)|\,\frac{p_t(x)}{p(x)}\,p(x)\,dx\,dt&\le\int_0^1\int_R[\exp(|L_\theta(x,t)|)-1]\,p(x)\,dx\,dt\\ &+\int_0^1\int_R\left(\frac{p_t(x)}{p(x)}\ln\frac{p_t(x)}{p(x)}-\frac{p_t(x)}{p(x)}+1\right)p(x)\,dx\,dt\\ \le 1+\int_0^1\int_R p^2(x)(p(x-\theta(t)))^{-1}\,dx\,dt+I^{SK}(F)\end{aligned}$$

which is finite when $I^{SK}(F)<\infty$ by condition (P).

Next, once again by the Young–Fenchel inequality, we have, for $\gamma > 0$,

$$\gamma|\zeta_{\theta,\delta}(F) - \zeta_\theta(F)| \leq \int_0^1 \int_R \gamma\,|L_{\theta,\delta}(x,t) - L_\theta(x,t)|\,F(dx,dt)$$
$$\leq \int_0^1 \int_R [\exp(\gamma\,|L_{\theta,\delta}(x,t) - L_\theta(x,t)|) - 1]\,p(x)\,dx\,dt + I^{SK}(F),$$

so by (5.22)

$$\overline{\lim_{\delta\to 0}} \sup_{F\in\Phi'_{I^{SK}}(a)} |\zeta_{\theta,\delta}(F) - \zeta_\theta(F)| \leq \frac{a}{\gamma}$$

which proves $(Y'.4)$ since γ is arbitrary.

Condition (U') is checked as in the case of an independent-identically-distributed sample with the use of condition (P).

We now check conditions $(\sup Y')$ and $(\sup U')$. For this, we assume that the $\theta(\cdot)$ are again from the set $\Sigma_K(\beta, M)$ defined in Subsection 5.3. Then limits (5.22) and (5.23) hold uniformly over $\theta \in \Sigma_K(\beta, M)$ which allows us to check $(\sup Y'.3)$, $(\sup Y'.4)$ and $(\sup U')$ analogously to $(Y'.3)$, $(Y'.4)$ and (U'), respectively. Condition $(\sup Y'.2)$ follows from the fact that the $L_{\theta,\delta}(x,t), \theta \in \Sigma_K(\beta, M)$, are equicontinuous at each (x,t) and uniformly bounded, so the $(\zeta_{\theta,\delta}, \theta \in \Theta) : \mathcal{Y} \to R_+^\Theta$ are continuous for the uniform topology on R_+^Θ.

We now calculate the function $S(\theta,\theta')$, $\theta,\theta' \in \Theta$. This is carried out with the use of a generalisation of Chernoff's result in Lemma 5.2 which we state and prove next. Let E be a Polish space with the Borel σ–field $\mathcal{E}$ and $\mathcal{P}(E)$, the space of probability measures on $(E,\mathcal{E})$. As above, for $F, P \in \mathcal{P}(E)$, we denote by $K(F,P)$ the Kullback–Leibler information:

$$K(F,P) = \begin{cases} \displaystyle\int_E \log \frac{dF}{dP}(x) F(dx), & \text{if } F \ll P,\\ \infty, & \text{otherwise.}\end{cases}$$

Recall that $K(F,P)$, for P fixed, is convex and is a rate function in F for the weak topology on $\mathcal{P}(E)$, Deuschel and Stroock, 1989, 3.2.17.

If the role of E is taken over by $E \times [0,1]$ with the product topology, then given a probability Borel measure ν on $[0,1]$, we denote by $\mathcal{P}_\nu(E\times[0,1])$ the subset of $\mathcal{P}(E\times[0,1])$ of measures F such that $F(E \times [0,t]) = \nu([0,t])$, $t \in [0,1]$.

Our version of Chernoff's result is the following lemma.

Lemma 5.5 *Let E be a Polish space. Let probability measures $P, Q \in \mathcal{P}(E \times [0,1])$ be dominated by the product measure $\mu \times \nu$, where μ and ν are Borel measures on E and $[0,1]$, respectively, with $\nu([0,1]) = 1$.*

Then

$$\inf_{F\in\mathcal{P}_\nu(E\times[0,1])} \max\{K(F,P), K(F,Q)\}$$

$$= - \inf_{\gamma\in[0,1]} \int_0^1 \ln \left[\int_E p_t^\gamma(x)\, q_t^{1-\gamma}(x)\, \mu(dx) \right] \nu(dt),$$

where $p_t(x)$ and $q_t(x)$ are the respective densities of P and Q relatively to $\mu \times \nu$.

Proof. Obviously,

$$\max\{K(F,P), K(F,Q)\} = \sup_{\gamma\in[0,1]} (\gamma K(F,P) + (1-\gamma)K(F,Q)). \tag{5.25}$$

Let $\mathcal{P}(E \times [0,1])$ be endowed with the weak topology. Since $K(F,P)$ is convex and is a rate function in F, we deduce that the function $\gamma K(F,P) + (1-\gamma)K(F,Q)$, $\gamma \in [0,1], F \in \mathcal{P}_\nu(E \times [0,1])$, meets the conditions of a minimax theorem (see, e.g., Aubin and Ekeland, 1984, Theorem 7, Section 2, Chapter 6). Hence,

$$\begin{aligned} &\inf_{F\in\mathcal{P}_\nu(E\times[0,1])} \sup_{\gamma\in[0,1]} (\gamma K(F,P) + (1-\gamma)K(F,Q)) \\ &= \sup_{\gamma\in[0,1]} \inf_{F\in\mathcal{P}_\nu(E\times[0,1])} (\gamma K(F,P) + (1-\gamma)K(F,Q)). \end{aligned} \tag{5.26}$$

The latter infimum can equivalently be taken over F dominated by P and Q, and hence by $\mu \times \nu$. Denote by $f_t(x)$ the density of F with respect to $\mu \times \nu$. Since, by the definition of $\mathcal{P}_\nu(E \times [0,1])$,

$$F(E \times [0,t]) = \int_0^t \int_E f_t(x)\, \mu(dx)\, \nu(dt) = \nu([0,t]),\ t \in [0,1],$$

we have that

$$\int_E f_t(x)\, \mu(dx) = 1 \qquad \nu\text{–almost everywhere.} \tag{5.27}$$

Next, by the definition of the Kullback–Leibler information,

$$\gamma K(F,P) + (1-\gamma)K(F,Q) = \int_0^1 \int_E \log \frac{f_t(x)}{p_t^\gamma(x) q_t^{1-\gamma}(x)} f_t(x)\, \mu(dx)\, \nu(dt), \tag{5.28}$$

where $0/0 = 0, 0\log 0 = 0$. Since the function $x \log x$, $x \geq 0$, is convex, an application of Jensen's unequality and (5.27) gives that ν–almost everywhere in $t \in [0,1]$

$$\int_E \log \frac{f_t(x)}{p_t^\gamma(x) q_t^{1-\gamma}(x)} f_t(x)\, \mu(dx) \geq -\log \int_E p_t^\gamma(x) q_t^{1-\gamma}(x)\, \mu(dx)\,.$$

On the other hand, taking

$$f_t(x) = p_t^\gamma(x) q_t^{1-\gamma}(x) \left(\int_E p_t^\gamma(x) q_t^{1-\gamma}(x)\, \mu(dx) \right)^{-1} \tag{5.29}$$

we get equality above. Since the measure F with the density defined by (5.29) belongs to $\mathcal{P}_\nu(E \times [0,1])$, we obtain by (5.28) that

$$\begin{aligned} &\inf_{F\in\mathcal{P}_\nu(E\times[0,1])} [\gamma K(F,P) + (1-\gamma)K(F,Q)] \\ &= - \int_0^1 \log \left[\int_E p_t^\gamma(x) q_t^{1-\gamma}(x)\, \mu(dx) \right] \nu(dt) \end{aligned}$$

which, by (5.25) and (5.26), concludes the proof. □

Remark 5.6 *Chernoff's result follows when ν is a Dirac measure.*

Now we apply Lemma 5.5 to evaluating the function $S(\theta, \theta')$.

Lemma 5.6 *For all $\theta, \theta' \in \Theta$,*

$$S(\theta, \theta') = \inf_{\gamma \in [0,1]} \int_0^1 \log H_\gamma(\theta'(t) - \theta(t))\, dt.$$

Proof. We have, for $F \in \mathcal{Y}_0$ with $I^{SK}(F) < \infty$,

$$\zeta_\theta(F) - I^{SK}(F) = -K(F, \overline{P}_\theta),$$

where $\overline{P}_\theta(dx, dt) = p(x - \theta(t))\, dx\, dt$, and the claim follows by (4.7) and Lemma 5.5 with $E = R$, $\mu(dx) = dx$, $\nu(dt) = dt$, $P = \overline{P}_\theta$ and $Q = \overline{P}_{\theta'}$. □

The latter result enables us to calculate asymptotic minimax risks for various statistical problems. To compare with the Gaussian case, let us consider the same statistical problems as in Subsection 5.4 dealing with the value of $\theta(t_0)$ for a given t_0. Sets $\Sigma(\beta, M)$ and $\Sigma_K(\beta, M)$ are defined as above.

5.5.1 Testing $\theta(t_0) = 0$ versus $|\theta(t_0)| \geq 2c$

Given $c > 0$, let $\Theta_0 = \{\theta \in \Theta : \theta(t_0) = 0\}$, $\Theta_1 = \{\theta \in \Theta : |\theta(t_0)| \geq 2c\}$ and define the risk $R_n^T(\rho_n)$ of a test ρ_n by (4.4). Recall that t^* was defined in (5.15).

Proposition 5.6 *Let c, β, M, K and t_0 be such that $[t_0 - t^*, t_0 + t^*] \subset [0,1]$ and $K \geq 2c$. Let the measure P satisfy condition (P) and the function $H_\gamma(s)$ monotonously decrease in $s \geq 0$ for each $\gamma \in [0,1]$.*
If $\Theta = \Sigma(\beta, M)$ then

$$\varliminf_{n\to\infty} \inf_{\rho_n} R_n^T(\rho_n) \geq \inf_{\gamma \in [0,1]} 2 \int_0^{t^*} \log H_\gamma(2(c - Mt^\beta))\, dt.$$

If $\Theta = \Sigma_K(\beta, M)$ then

$$\lim_{n\to\infty} \inf_{\rho_n} R_n^T(\rho_n) = \inf_{\gamma \in [0,1]} 2 \int_0^{t^*} \log H_\gamma(2(c - Mt^\beta))\, dt,$$

and the tests $\rho_{n,\delta}^T$ from (4.8) are nearly LD efficient, i.e.,

$$\begin{aligned}\lim_{\delta\to 0} \varlimsup_{n\to\infty} R_n^T(\rho_{n,\delta}^T) &= \lim_{\delta\to 0} \varliminf_{n\to\infty} R_n^T(\rho_{n,\delta}^T) \\ &= \inf_{\gamma \in [0,1]} 2 \int_0^{t^*} \log H_\gamma(2(c - Mt^\beta))\, dt.\end{aligned}$$

Proof. By Theorem 4.3 we need only to evaluate T^* from (4.6). A straightforward calculation using Lemma 5.6 and the monotonicity of $H_\gamma(s)$ shows that

$$T^* := \sup_{\theta\in\Theta_0,\,\theta'\in\Theta_1} S(\theta,\theta') = \inf_{\gamma\in[0,1]} 2\int_0^1 \log H_\gamma(2\theta^*(t))\,dt,$$

where $\theta^*(t) = [c - M|t-t_0|^\beta]^+$. The claim follows. □

5.5.2 Estimating $\theta(t_0)$

For the problem of estimating $\theta(t_0)$, the risk of an estimator ρ_n is defined by

$$R_n^F(\rho_n) = \sup_{\theta\in\Theta} \frac{1}{n} \ln P_{n,\theta}(|\rho_n - \theta(t_0)| > c).$$

Proposition 5.7 *Let the conditions of Proposition 5.6 hold.*
If $\Theta = \Sigma(\beta, M)$ then

$$\varliminf_{n\to\infty} \inf_{\rho_n} R_n^F(\rho_n) \ge \inf_{\gamma\in[0,1]} 2\int_0^{t^*} \log H_\gamma(2(c - Mt^\beta))\,dt.$$

If $\Theta = \Sigma_K(\beta, M)$ then

$$\lim_{n\to\infty} \inf_{\rho_n} R_n^F(\rho_n) = \inf_{\gamma\in[0,1]} 2\int_0^{t^*} \log H_\gamma(2(c - Mt^\beta))\,dt,$$

and the interval–median estimators $\rho_{n,\delta}^F$ from (4.18) are nearly LD efficient, i.e.,

$$\begin{aligned}\lim_{\delta\to 0} \varlimsup_{n\to\infty} R_n^F(\rho_{n,\delta}^F) &= \lim_{\delta\to 0} \varliminf_{n\to\infty} R_n^F(\rho_{n,\delta}^F)\\ &= \inf_{\gamma\in[0,1]} 2\int_0^{t^*} \log H_\gamma(2(c - Mt^\beta))\,dt.\end{aligned}$$

Proof. By Theorem 4.5 and Remark 4.3 it suffices to calculate the asymptotic minimax risk given by Lemma 4.3:

$$F^* = \sup_{\theta,\theta'\in\Theta\,:\,|\theta(t_0)-\theta'(t_0)|>2c} S(\theta,\theta')$$

which is done as for the "signal + white noise" model. □

Remark 5.7 *The latter problem of estimating $\theta(t_0)$ has been considered by Korostelev and Spokoiny, 1995 under the assumption that $\ln p(x)$ is concave upwards, and by Korostelev and Leonov, 1995 who study the double asymptotics as $n\to\infty$ and then $c\to 0$.*

5.6 The Change–Point Model

Let us observe a sample $X_n = (X_{1,n}, \dots, X_{n,n})$ of real–valued random variables, where, for some $k_n \ge 1$, the observations $X_{1,n}, \dots, X_{k_n,n}$ are independent identically distributed with a distribution P_0 and the observations $X_{k_n+1,n}, \dots, X_{n,n}$ are independent identically distributed with a distribution P_1. We assume that P_0 and P_1 are known and k_n is unknown. Let us also assume that $k_n = [n\theta]$, where $\theta \in \Theta = [0,1]$. For this model, $\Omega_n = R^n$ and $P_{n,\theta}$ denotes the distribution of X_n for θ.

Let a probability measure P dominate P_0 and P_1, and

$$f_0(x) = \frac{dP_0}{dP}(x),\ f_1(x) = \frac{dP_1}{dP}(x),\ \ x \in R,$$

be respective densities. We assume that $f_0(x)$ and $f_1(x)$ are positive and continuous and

$$\int_R f_0^\gamma(x)\, P(dx) < \infty,\ \int_R f_1^\gamma(x)\, P(dx) < \infty \quad \text{for all } \gamma \in R. \tag{5.30}$$

Introducing $P_n = P^n$, we have

$$\Xi_{n,\theta} = \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n) = \frac{1}{n} \sum_{i=1}^{[n\theta]} \ln f_0(X_{i,n}) + \frac{1}{n} \sum_{i=[n\theta]+1}^{n} \ln f_1(X_{i,n})$$

so that defining an empirical process by

$$F_n(x,t) = \frac{1}{n} \sum_{i=1}^{[nt]} 1(X_{i,n} \le x),\ x \in R, t \in [0,1],$$

we obtain the representation

$$\Xi_{n,\theta} = \int_0^\theta \int_R \ln f_0(x)\, F_n(dx,dt) + \int_\theta^1 \int_R \ln f_1(x)\, F_n(dx,dt).$$

We define statistics Y_n and a space $\mathcal{Y}$ as for the non-Gaussian-regression model. Let $\mathcal{Y}_P$ consist of the functions $F \in \mathcal{Y}$ that are absolutely continuous with respect to the measure $P(dx) \times dt$ with densities $p_t(x)$ such that $\int_R p_t(x) P(dx) = 1\,, t \ge 0$. As for the non-Gaussian-regression model, condition $(Y'.1)$ holds with

$$I_P^{SK}(F) = \begin{cases} \displaystyle\int_0^1 \int_R p_t(x) \ln p_t(x) P(dx)\, dt, & \text{if } F \in \mathcal{Y}_P, \\ \infty, & \text{otherwise.} \end{cases}$$

We next take, for $F(\cdot,\cdot) \in \mathcal{Y}$,

$$\zeta_{\theta,\delta}(F) = \int_0^1 \int_R L_{0,\delta}(x) g_\delta(\theta - t)\, F(dx,dt) + \int_0^1 \int_R L_{1,\delta}(x) g_\delta(t-\theta)\, F(dx,dt)\,,$$

where

$$
\begin{aligned}
L_{i,\delta}(x) &= \ln p_i(x) \wedge \delta^{-1} \vee (-\delta^{-1}), \qquad i = 0,1, \\
g_\delta(t) &= 0 \vee \bigl(1/2 + \delta^{-2} t\bigr) \wedge 1.
\end{aligned}
$$

The functions $L_{i,\delta}$ and g_δ are bounded, continuous and

$$
\lim_{\delta \to 0} \int_R \left[\exp\left(\gamma |\ln p_i(x) - L_{i,\delta}(x)|\right) - 1\right] P(dx) = 0,\ i = 0,1,\ \gamma > 0. \tag{5.31}
$$

The $\zeta_{\theta,\delta}$ are easily seen to be continuous, so $(Y'.2)$ holds.

For $(Y'.3)$, write, by Chebyshev's inequality, for $\gamma > 0, \varepsilon > 0$ and $n \ge \delta^{-2}$,

$$
\begin{aligned}
&P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(F_n)| > \varepsilon) \\
&\quad \le P_n^{1/n}\left(\int_0^1 \int_R |\ln f_0(x) - L_{0,\delta}(x)| F_n(dx,dt) + 2\delta > \frac{\varepsilon}{2}\right) \\
&\qquad + P_n^{1/n}\left(\int_0^1 \int_R |\ln f_1(x) - L_{1,\delta}(x)| F_n(dx,dt) + 2\delta > \frac{\varepsilon}{2}\right) \\
&\quad \le \exp(-\gamma\varepsilon/2)\exp(2\gamma\delta)\,[E\exp(\gamma|\ln f_0(X_{1,n}) - L_{0,\delta}(X_{1,n})|) \\
&\qquad + E\exp(\gamma|\ln f_1(X_{1,n}) - L_{1,\delta}(X_{1,n})|)]\,,
\end{aligned}
$$

so

$$
\begin{aligned}
&\overline{\lim_{n\to\infty}}\ P_n^{1/n}(|\Xi_{n,\theta} - \zeta_{\theta,\delta}(F_n)| > \varepsilon) \\
&\quad \le \exp(-\gamma\varepsilon/2)\exp(2\gamma\delta)\left(\int_R \exp(\gamma|\ln f_0(x) - L_{0,\delta}(x)|)P(dx)\right. \\
&\qquad \left. + \int_R \exp(\gamma|\ln f_1(x) - L_{1,\delta}(x)|)P(dx)\right),
\end{aligned}
$$

and, by (5.31), this goes to $2\exp(-\gamma\varepsilon/2)$ as $\delta \to 0$. Since γ is arbitrary, condition $(Y'.3)$ is checked.

To check $(Y'.4)$, we take

$$
\zeta_\theta(F) = \begin{cases} \displaystyle\int_0^\theta \int_R \ln f_0(x) F(dx,dt) + \int_\theta^1 \int_R \ln f_1(x) F(dx,dt), & \text{if } I_P^{SK}(F) < \infty, \\ 0, & \text{otherwise.} \end{cases}
$$

The facts that the ζ_θ are well defined and $(Y'.4)$ holds are proved as for the non-Gaussian-regression model with the use of (5.30). Condition (U') also is easily checked.

Remark 5.8 *The continuity condition on $f_0(x)$ and $f_1(x)$ can be omitted. One should then choose the $L_{i,\delta}$ bounded, continuous and satisfying (5.31).*

Next, the argument used for (Y') and (U') checks also conditions $(\sup Y')$ and $(\sup U')$ (the verification of $(\sup Y'.2)$ uses the fact that the function $g_\delta(t-\theta)$ is equicontinuous for $\theta \in [0,1]$ at each $t \in [0,1]$).

The next step is evaluating $S(\theta,\theta')$ for $\theta,\theta' \in [0,1]$.

Lemma 5.7 *For all* $\theta,\theta' \in [0,1]$,

$$S(\theta,\theta') = -|\theta-\theta'|\, C(P_0,P_1).$$

Proof. In a manner similar to the case of non–Gaussian regression, we have, for any $F \in \mathcal{Y}_P, I_P^{SK}(F) < \infty$ with $F(dx,dt) = p_t(x)\, P(dx)\, dt$,

$$\zeta_\theta(F) - I_P^{SK}(F) = -\int_0^\theta \int_R \ln \frac{p_t(x)}{p_0(x)} p_t(x)\, P(dx)\, dt$$
$$-\int_\theta^1 \int_R \ln \frac{p_t(x)}{p_1(x)} p_t(x)\, P(dx)\, dt = -K(F,\overline{P}_\theta),$$

where $\overline{P}_\theta(dx,dt) = (f_0(x)1(t \le \theta) + f_1(x)1(t > \theta))\, P(dx)\, dt$. The claim follows by (4.7), Lemma 5.5 with $E = R, \mu(dx) = P(dx)$, $\nu(dt) = dt$, $P = \overline{P}_\theta$ and $Q = \overline{P}_{\theta'}$ and the definition of Chernoff's function in Lemma 5.2. □

We apply this result and the general theorems from Section 4 to the problem of estimating the parameter θ. The risk of an estimator ρ_n is defined in a standard way:

$$R_n^F(\rho_n) = \sup_{\theta \in [0,1]} \frac{1}{n} \ln P_{n,\theta}(|\rho_n - \theta| > c). \tag{5.32}$$

Proposition 5.8 *For each* $c < 1/2$,

$$\lim_{n\to\infty} \inf_{\rho_n} R_n^F(\rho_n) = -2c\, C(P_0,P_1).$$

If $\rho_{n,\delta}^F$ *are the interval–median estimators from (4.18) then*

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\; R_n^F(\rho_{n,\delta}^F) = \lim_{\delta\to 0} \underline{\lim_{n\to\infty}}\; R_n^F(\rho_{n,\delta}^F) = -2c\, C(P_0,P_1).$$

Proof. We apply Theorem 4.5. One needs only to calculate the minimax risk F^*. Using Lemmas 4.3 and 5.7, we obtain

$$F^* = \sup_{\theta,\theta' : |\theta-\theta'| > 2c} S(\theta,\theta') = -2c\, C(P_0,P_1).$$

□

Remark 5.9 *The same result has been obtained by Korostelev, 1995 who uses another kind of an upper estimator. The construction is based on considering the concave hull of a sample path of the likelihood process. By Lemma 4.2 this estimator is a particular case of the interval–median estimators* $\rho_{n,\delta}^F$.

5.7 Regression with Random Design

We consider the model

$$X_{k,n} = \theta(t_{k,n}) + \xi_{k,n}, \qquad k = 1, \ldots, n, \tag{5.33}$$

where real–valued errors $\xi_{k,n}$ are independent with a common distribution P having a density $p(x)$ that obeys condition (P) of Subsection 5.5, and design points $t_{k,n}$ are real-valued independent random variables with a common distribution Π and are independent of the $\xi_{k,n}$. We impose a standard condition on the design measure Π.

(Π) *The measure* Π *is compactly supported and has a positive density with respect to Lebesgue measure on the support.*

We denote the support by D. An unknown regression function $\theta(\cdot)$ is assumed to be continuous. In this model, $P_{n,\theta}$ is the joint distribution of $X_n = (X_{1,n}, \ldots, X_{n,n})$ and $t_n = (t_{1,n}, \ldots, t_{n,n})$ for θ.

Let us take for Y_n the joint empirical distribution function F_n of X_n and t_n:

$$F_n(A, B) = \frac{1}{n} \sum_{k=1}^{n} 1(X_{k,n} \in A, t_{k,n} \in B) \tag{5.34}$$

for Borel sets $A \subset R$, $B \subset D$. We take $\mathcal{Y}$ to be the space of distributions on $R \times D$ with the weak topology. Let also $P_n = P_{n,0} = (P \times \Pi)^n$.

With these definitions,

$$\begin{aligned}
\Xi_{n,\theta} &= \frac{1}{n} \ln \frac{dP_{n,\theta}}{dP_n}(X_n, t_n) \\
&= \frac{1}{n} \sum_{k=1}^{n} \ln \frac{p(X_{k,n} - \theta(t_{k,n}))}{p(X_{k,n})} \\
&= \int_D \int_R \ln \frac{p(x - \theta(t))}{p(x)} F_n(dx, dt).
\end{aligned}$$

Let $\mathcal{Y}_1$ be the subset of the set $\mathcal{Y}$ of the cumulative distribution functions on R^2 that are absolutely continuous with respect to Lebesgue measure on R^2 and have support in $R \times D$.

Under P_n, the random pairs $(X_{k,n}, t_{k,n})$ are independent identically distributed with the distribution $P \times \Pi$, and hence, by Sanov's theorem, the LDP holds for the F_n with rate function $I^{SS}(F)$ defined by

$$I^{SS}(F) = \begin{cases} \displaystyle\int_D \int_R \ln \frac{p(x,t)}{p(x)\pi(t)} p(x,t)\, dx\, dt, & \text{if } F \in \mathcal{Y}_1, \\ \infty, & \text{otherwise.} \end{cases}$$

Here $F(dx,dt)=p(x,t)dxdt$. This checks $(Y'.1)$.

Set next, for $F\in\mathcal{Y}$,

$$\zeta_\theta(F) = \begin{cases} \int_D\int_R \ln\frac{p(x-\theta(t))}{p(x)}F(dx,dt), & \text{if } I^{SS}(F)<\infty,\\ 0, & \text{otherwise,}\end{cases}$$

$$\zeta_{\theta,\delta}(F) = \int_D\int_R \left[\ln\frac{p(x-\theta(t))}{p(x)}\right]\wedge\delta^{-1}\vee(-\delta^{-1})\,F(dx,dt).$$

With this notation, the rest of condition (Y') and condition (U') are checked in analogy with the case of non-Gaussian regression. This proves the LDP for the model.

For conditions $(\sup Y')$ and $(\sup U')$, we again assume that $\theta\in\Sigma_K(\beta,M)$, where the set $\Sigma_K(\beta,M)$ was defined above. The conditions are then checked as for the non-Gaussian-regression model.

Now we calculate the function $S(\theta,\theta')$ from (4.7). Recall that the function $H_\gamma(s)$ was defined in condition (P).

Lemma 5.8 *Under conditions* (P) *and* (Π),

$$S(\theta,\theta')=\inf_{\gamma\in[0,1]}\log\int_D H_\gamma(\theta'(t)-\theta(t))\pi(t)\,dt.$$

Proof. Given $F\in\mathcal{Y}_1$ with $I^{SS}(F)<\infty$, we easily get

$$\zeta_\theta(F)-I^{SS}(F)=-K(F,\overline{P}_\theta),$$

where $\overline{P}_\theta(dx,dt)=p(x-\theta(t))\pi(t)\,dx\,dt$, and the claim follows by (4.7) and Lemma 5.2 with $E=R\times D$, $\mu(dx,dt)=dx\,dt, P=\overline{P}_\theta$ and $Q=\overline{P}_{\theta'}$. □

Now we consider the same two statistical problems as in Subsection 5.5 and compare the results for the cases of random and deterministic design.

5.7.1 Testing $\theta(t_0)=0$ versus $|\theta(t_0)|\ge 2c$

Given $t_0\in D$ and $c>0$, consider the hypothesis testing problem: $\theta(t_0)=0$ versus $|\theta(t_0)|\ge 2c$. The risk $R_n^T(\rho_n)$ of a test ρ_n, as well as the sets $\Sigma(\beta,M)$ and $\Sigma_0(\beta,M)$, and t^* are defined as above.

Proposition 5.9 *Let* $D=[0,1]$. *Let* c,β,M,K *and* t_0 *be such that* $[t_0-t^*,t_0+t^*]\subset[0,1]$ *and* $K\ge 2c$. *Let conditions* (P) *and* (Π) *hold and the function* $H_\gamma(s)$ *monotonously decrease in* $s\ge 0$ *for each* $\gamma\in[0,1]$.
If $\Theta=\Sigma(\beta,M)$ *then*

$$\varliminf_{n\to\infty}\inf_{\rho_n}R_n^T(\rho_n)\ge T^*,$$

where

$$T^* = \inf_{\gamma\in[0,1]} \log\left(1 + \int_{t_0-t^*}^{t_0+t^*} \left[H_\gamma(2(c - M|t-t_0|^\beta)) - 1\right] \pi(t)\,dt\right).$$

If $\Theta = \Sigma_K(\beta, M)$ *then*

$$\lim_{n\to\infty} \inf_{\rho_n} R_n^T(\rho_n) = T^*,$$

and the tests $\rho_{n,\delta}^T$ *from (4.8) are nearly LD efficient, i.e.,*

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\ R_n^T(\rho_{n,\delta}^T) = \lim_{\delta\to 0} \underline{\lim_{n\to\infty}}\ R_n^T(\rho_{n,\delta}^T) = T^*.$$

Proof. Theorem 4.3 reduces the proof to calculating T^* from (4.6). Using the result of Lemma 5.8 and proceeding in analogy with the case of deterministic design, we conclude that

$$\begin{aligned} T^* &= S(c-\theta^*, c+\theta^*) \\ &= \inf_{\gamma\in[0,1]} \log\left(\int_0^{t_0-t^*} \pi(t)\,dt + \int_{t_0-t^*}^{t_0+t^*} H_\gamma(2(c-M|t-t_0|^\beta))\pi(t)\,dt + \int_{t_0+t^*}^{1} \pi(t)\,dt\right). \end{aligned}$$

Now the claim follows by the equality $\int_D \pi(t)dt = 1$. □

5.7.2 Estimating $\theta(t_0)$

As above, when estimating $\theta(t_0)$, we define the risk of an estimator ρ_n by

$$R_n^F(\rho_n) = \sup_{\theta\in\Sigma_K(\beta,M)} \frac{1}{n} \ln P_{n,\theta}(|\rho_n - \theta(t_0)| > c).$$

Proposition 5.10 *Let the conditions of Proposition 5.9 hold.*
If $\Theta = \Sigma(\beta, M)$ *then*

$$\underline{\lim_{n\to\infty}} \inf_{\rho_n} R_n^F(\rho_n) \ge F^*,$$

where

$$F^* = \inf_{\gamma\in[0,1]} \log\left(1 + \int_{t_0-t^*}^{t_0+t^*} \left[H_\gamma(2(c - M|t-t_0|^\beta)) - 1\right] \pi(t)\,dt\right).$$

If $\Theta = \Sigma_K(\beta, M)$ *then*

$$\lim_{n\to\infty} \inf_{\rho_n} R_n^F(\rho_n) = F^*,$$

and the interval–median estimators $\rho_{n,\delta}^F$ *from (4.18) are nearly LD efficient, i.e.,*

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\ R_n^F(\rho_{n,\delta}^F) = \lim_{\delta\to 0} \underline{\lim_{n\to\infty}}\ R_n^F(\rho_{n,\delta}^F) = F^*.$$

Proof. By Theorem 4.5 it suffices to calculate the asymptotic minimax risk F^* from Lemma 4.3 which is done in analogy with the proof of Proposition 5.9. □

Remark 5.10 *If we consider the uniform random design on* $[0,1]$*, i.e., take* $\pi(t) = 1$*, Jensen's inequality easily implies that its asymptotic minimax risks are not greater than the corresponding risks for regression with deterministic design (see Subsection 5.5). This fact also follows from comparing Lemma 5.2 and Lemma 5.5.*

Remark 5.11 *The problem of estimating* $\theta(t_0)$ *for the uniform random design has been considered by Korostelev, 1995 who studies the double asymptotics as* $n \to \infty$ *and then* $c \to 0$*.*

A Appendix

A.1 Proof of Lemma 2.4

Let $\{\mathbf{V}_\Lambda,\ \Lambda \in \mathcal{A}(\Theta)\}$ be a standard family of deviabilities so that for all $\Lambda \subset \Lambda' \in \mathcal{A}(\Theta)$ and $z_\Lambda \in S_\Lambda$,

$$\mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_{\Lambda'} \in \Pi^{-1}_{\Lambda'\Lambda} z_\Lambda} \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \mathbf{V}_{\Lambda'}(z_{\Lambda'})\,. \tag{1.1}$$

We define

$$\mathbf{V}_\Theta(z_\Theta) = \begin{cases} \inf_{\Lambda \in \mathcal{A}(\Theta)} \|\pi_\Lambda z_\Theta\|_\Lambda^{-1} \mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta)\,, & z_\Theta \in S_\Theta\,, \\ 0\,, & \text{otherwise}, \end{cases} \tag{1.2}$$

where we set $\mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta) = 1$ and $\|\pi_\Lambda z_\Theta\|_\Lambda^{-1} \mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta) = \infty$ when $\|\pi_\Lambda z_\Theta\|_\Lambda = 0$.

The functions $\|\pi_\Lambda z_\Theta\|_\Lambda^{-1} \mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta)\,, \Lambda \in \mathcal{A}(\Theta)$, are easily seen to be upper semicontinuous on S_Θ, so $(\mathbf{V}_\Theta(z_\Theta)\,, z_\Theta \in R_+^\Theta)$ is upper semicontinuous as the infimum of a family of upper semicontinuous functions. Moreover, since, for every $z_\Theta \in S_\Theta$ and $\varepsilon > 0$, there exists $\Lambda \in \mathcal{A}(\Theta)$ such that $\|\pi_\Lambda z_\Theta\|_\Lambda > 1 - \varepsilon$, and since $\mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta) \le 1$, we conclude that $\mathbf{V}_\Theta(z_\Theta) \le 1$. Since *(ii)* obviously follows by *(iii)*, we are left to prove *(iii)* and the equality

$$\sup_{z_\Theta \in S_\Theta} \mathbf{V}_\Theta(z_\Theta) = 1. \tag{1.3}$$

We begin with *(iii)*. Let us fix Λ and z_Λ assuming that $z_\Lambda \in S_\Lambda$. Definition (1.2) implies that

$$\mathbf{V}_\Lambda(z_\Lambda) \ge \sup_{z_\Theta \in \Pi_\Lambda^{-1} z_\Lambda} \|\pi_\Lambda z_\Theta\|_\Lambda \mathbf{V}_\Theta(z_\Theta)\,,$$

so we need to prove that

$$\mathbf{V}_\Lambda(z_\Lambda) \le \sup_{z_\Theta \in \Pi_\Lambda^{-1} z_\Lambda} \|\pi_\Lambda z_\Theta\|_\Lambda \mathbf{V}_\Theta(z_\Theta)\,. \tag{1.4}$$

We, first, note that (1.2) and (1.1) imply that

$$\mathbf{V}_\Theta(z_\Theta) = \inf_{\substack{\Lambda' \in \mathcal{A}(\Theta) \\ \Lambda' \supset \Lambda}} \|\pi_{\Lambda'} z_\Theta\|_{\Lambda'}^{-1} \mathbf{V}_{\Lambda'}(\Pi_{\Lambda'} z_\Theta)\,,\ z_\Theta \in S_\Theta\,. \tag{1.5}$$

Indeed, by (1.1), if $\Lambda \subset \Lambda' \in \mathcal{A}(\Theta)$ and $z_\Theta \in S_\Theta$ is such that $\|\pi_\Lambda z_\Theta\|_\Lambda > 0$ then

$$\mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta) \ge \|\pi_{\Lambda'\Lambda} \Pi_{\Lambda'} z_\Theta\|_\Lambda \mathbf{V}_{\Lambda'}(\Pi_{\Lambda'} z_\Theta)\,,$$

and hence, since $\pi_{\Lambda'\Lambda}\Pi_{\Lambda'} z_\Theta = \pi_\Lambda z_\Theta / \|\pi_{\Lambda'} z_\Theta\|_{\Lambda'}$,

$$\|\pi_{\Lambda'} z_\Theta\|_{\Lambda'}^{-1} \mathbf{V}_{\Lambda'}(\Pi_{\Lambda'} z_\Theta) \le \|\pi_\Lambda z_\Theta\|_\Lambda^{-1} \mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta)$$

which, in view of (1.2), proves (1.5).

Next, we obviously can assume that $a := \mathbf{V}_\Lambda(z_\Lambda) > 0$. For $\Lambda' \supset \Lambda, \Lambda' \in \mathcal{A}(\Theta)$, introduce the sets

$$A_{\Lambda'} = \{z_{\Lambda'} \in S_{\Lambda'} : \Pi_{\Lambda'\Lambda} z_{\Lambda'} = z_\Lambda \text{ and } \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \mathbf{V}_{\Lambda'}(z_{\Lambda'}) = a\}. \tag{1.6}$$

We show that $A_{\Lambda'}$ is nonempty. Since $\mathbf{V}_{\Lambda'}(z_{\Lambda'}) \leq 1$, the supremum on the right of (1.1) can equivalently be taken over the set $\Pi^{-1}_{\Lambda'\Lambda} z_\Lambda \cap \{\|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \geq a/2\}$. This set is closed since the projection $\Pi_{\Lambda'\Lambda}$ is continuous in restriction to the set $\{z_{\Lambda'} : \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \geq a/2\}$. Since $\mathbf{V}_{\Lambda'}$ is a deviability, it attains supremums on closed sets, so the supremum on the right of (1.1) is attained which is equivalent to $A_{\Lambda'}$ being nonempty. Next, $A_{\Lambda'}$ is closed and hence compact since $\mathbf{V}_{\Lambda'}$ is upper semicontinuous and, by (1.1) and the definition of a, $\|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \mathbf{V}_{\Lambda'}(z_{\Lambda'}) = a$ if and only if $\|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \mathbf{V}_{\Lambda'}(z_{\Lambda'}) \geq a$.

Now we introduce for each $\Lambda' \in \mathcal{A}(\Theta)$, $\Lambda' \supset \Lambda$,

$$\mathbf{A}_{\Lambda'} = \{z_\Theta \in [0,1]^\Theta : \Pi_{\Lambda'} z_\Theta \in A_{\Lambda'} \text{ and } \|\pi_{\Lambda'} z_\Theta\|_{\Lambda'} \geq a\}.$$

These sets are easily seen to be nonempty (e.g., if $z_{\Lambda'} \in A_{\Lambda'}$ then $z_\Theta = (z_\theta, \theta \in \Theta)$, defined by $(z_\theta, \theta \in \Lambda') = z_{\Lambda'}$ and $z_\theta = 0, \theta \notin \Lambda'$, belongs to $\mathbf{A}_{\Lambda'}$) and compact for the Tihonov topology on $[0,1]^\Theta$ (the latter holds because $\Pi_{\Lambda'}$ is continuous in restriction to the set $\{z_\Theta : \|\pi_{\Lambda'} z_\Theta\|_{\Lambda'} \geq a\}$ and $A_{\Lambda'}$ is closed).

We next show that, for every elements Λ' and Λ'' of $\mathcal{A}(\Theta)$ containing Λ, the sets $\mathbf{A}_{\Lambda'}$ and $\mathbf{A}_{\Lambda''}$ have a nonempty intersection. Indeed, let $\Lambda''' = \Lambda' \cup \Lambda''$ and $z_\Theta \in [0,1]^\Theta$ be such that $z_\Theta \in \mathbf{A}_{\Lambda'''}$ and $\|\pi_{\Lambda'''} z_\Theta\| = 1$ (such a z_Θ obviously exists). We prove that $z_\Theta \in \mathbf{A}_{\Lambda'}$ and $z_\Theta \in \mathbf{A}_{\Lambda''}$.

Denote $z_{\Lambda'''} = \Pi_{\Lambda'''} z_\Theta$, $z_{\Lambda'} = \Pi_{\Lambda'} z_\Theta$, the latter being well defined since the definitions of $\mathbf{A}_{\Lambda'''}$ and $A_{\Lambda'''}$ imply that $\|\pi_\Lambda z_\Theta\|_\Lambda \geq a$. First, note that

$$\Pi_{\Lambda'\Lambda} z_{\Lambda'} = \Pi_\Lambda z_\Theta = \Pi_{\Lambda'''\Lambda} z_{\Lambda'''} = z_\Lambda, \tag{1.7}$$

where the last equality follows by the fact that $z_{\Lambda'''} \in A_{\Lambda'''}$. This and (1.1) yield, in view of the equality $\Pi_{\Lambda'''\Lambda'} z_{\Lambda'''} = z_{\Lambda'}$,

$$\mathbf{V}_\Lambda(z_\Lambda) \geq \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \mathbf{V}_{\Lambda'}(z_{\Lambda'}), \tag{1.8}$$

$$\mathbf{V}_{\Lambda'}(z_{\Lambda'}) \geq \|\pi_{\Lambda'''\Lambda'} z_{\Lambda'''}\|_{\Lambda'} \mathbf{V}_{\Lambda'''}(z_{\Lambda'''}). \tag{1.9}$$

Next, by the definitions of $z_{\Lambda'''}$ and $z_{\Lambda'}$,

$$\|\pi_{\Lambda'''\Lambda} z_{\Lambda'''}\|_\Lambda = \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \cdot \|\pi_{\Lambda'''\Lambda'} z_{\Lambda'''}\|_{\Lambda'}$$

so that, by (1.8) and (1.9),

$$\mathbf{V}_\Lambda(z_\Lambda) \geq \|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \cdot \|\pi_{\Lambda'''\Lambda'} z_{\Lambda'''}\|_{\Lambda'} \mathbf{V}_{\Lambda'''}(z_{\Lambda'''}) = \|\pi_{\Lambda'''\Lambda} z_{\Lambda'''}\|_\Lambda \mathbf{V}_{\Lambda'''}(z_{\Lambda'''}).$$

Since $z_{\Lambda'''} \in A_{\Lambda'''}$, we actually have equality here and hence in (1.8) and (1.9). The first of them and (1.7) prove that $z_{\Lambda'} \in A_{\Lambda'}$. Equalities in (1.8) and (1.9) together imply, since $\mathbf{V}_{\Lambda'''}(z_{\Lambda'''}) \le 1$ and $\|\pi_{\Lambda'\Lambda} z_{\Lambda'}\|_\Lambda \le 1$ that $\|\pi_{\Lambda'''\Lambda'} z_{\Lambda'''}\|_{\Lambda'} \ge \mathbf{V}_{\Lambda'}(z_{\Lambda'}) \ge \mathbf{V}_\Lambda(z_\Lambda) = a$; since also $\|\pi_{\Lambda'''} z_\Theta\|_{\Lambda'''} = 1$, we get

$$\|\pi_{\Lambda'} z_\Theta\|_{\Lambda'} = \|\pi_{\Lambda'''} z_\Theta\|_{\Lambda'''} \cdot \|\pi_{\Lambda'''\Lambda'} z_{\Lambda'''}\|_{\Lambda'} \ge a.$$

This concludes the proof of the inclusion $z_\Theta \in \mathbf{A}_{\Lambda'}$. The inclusion $z_\Theta \in \mathbf{A}_{\Lambda''}$ is proved by the same argument.

Thus, finite intersections of the compacts $\mathbf{A}_{\Lambda'}, \Lambda' \supset \Lambda$, are nonempty, hence $\cap_{\Lambda' \supset \Lambda} \mathbf{A}_{\Lambda'} \neq \emptyset$. Pick z_Θ from this intersection and let $\widehat{z}_\Theta = z_\Theta / \|z_\Theta\|_\Theta$. We prove that

$$\Pi_\Lambda \widehat{z}_\Theta = z_\Lambda \tag{1.10}$$

and

$$\mathbf{V}_\Theta(\widehat{z}_\Theta) = \|\pi_\Lambda \widehat{z}_\Theta\|_\Lambda^{-1} \mathbf{V}_\Lambda(z_\Lambda) \tag{1.11}$$

which yields (1.4) since $\widehat{z}_\Theta \in S_\Theta$. Let $\Lambda' \in \mathcal{A}(\Theta)$ with $\Lambda \subset \Lambda'$. Since $\Pi_{\Lambda'} \widehat{z}_\Theta = \Pi_{\Lambda'} z_\Theta \in A_{\Lambda'}$, it follows by the definition of $A_{\Lambda'}$ that $\Pi_\Lambda \widehat{z}_\Theta = \Pi_{\Lambda'\Lambda} \Pi_{\Lambda'} \widehat{z}_\Theta = z_\Lambda$ checking (1.10), also

$$\mathbf{V}_\Lambda(z_\Lambda) = a = \|\pi_{\Lambda'\Lambda} \Pi_{\Lambda'} \widehat{z}_\Theta\|_\Lambda \mathbf{V}_{\Lambda'}(\Pi_{\Lambda'} \widehat{z}_\Theta) = \frac{\|\pi_\Lambda \widehat{z}_\Theta\|_\Lambda}{\|\pi_{\Lambda'} \widehat{z}_\Theta\|_{\Lambda'}} \mathbf{V}_{\Lambda'}(\Pi_{\Lambda'} \widehat{z}_\Theta),$$

so

$$\|\pi_\Lambda \widehat{z}_\Theta\|_\Lambda^{-1} \mathbf{V}_\Lambda(z_\Lambda) = \|\pi_{\Lambda'} \widehat{z}_\Theta\|_{\Lambda'}^{-1} \mathbf{V}_{\Lambda'}(\Pi_{\Lambda'} \widehat{z}_\Theta).$$

In view of (1.5), this implies (1.11), and (1.4) follows. Assertion *(iii)* has been proved.

Finally, according to *(iii)*,

$$1 = \sup_{z_\Lambda \in S_\Lambda} \mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_\Theta \in S_\Theta} \|\pi_\Lambda z_\Theta\|_\Lambda \mathbf{V}_\Theta(z_\Theta) \le \sup_{z_\Theta \in S_\Theta} \mathbf{V}_\Theta(z_\Theta)$$

proving (1.3). □

Remark A.1 *Equality (1.5) shows that* $\mathbf{V}_\Theta$ *can equivalently be defined as*

$$\mathbf{V}_\Theta(z_\Theta) = \lim_{\Lambda \in \mathcal{A}(\Theta)} \mathbf{V}_\Lambda(\Pi_\Lambda z_\Theta), \; z_\Theta \in S_\Theta,$$

where the limit is with respect to the partial ordering by inclusion: $\Lambda \le \Lambda'$ *if* $\Lambda \subset \Lambda'$.

A.2 A Minimax Theorem for Non-Level-Compact Loss Functions

This subsection contains a minimax theorem for generalised risks and non-level-compact loss functions. We assume the setting described at the beginning of Section 3 and start by introducing an extension of the space of decisions, cf. Strasser, 1985.

Denote by $\mathcal{C}_+(\mathcal{D})$ the set of all non-negative bounded continuous functions on $\mathcal{D}$, and let $\mathbf{B}(\mathcal{D})$ be the set of all functionals $b : \mathcal{C}_+(\mathcal{D}) \to R_+$ with the following properties:

(1) $b(\mathbf{0}) = 0, b(\mathbf{1}) = 1$, where $\mathbf{0}$ (respectively, $\mathbf{1}$) denotes the element of $\mathcal{C}_+(\mathcal{D})$ identically equal to 0 (respectively, 1);

(2) $b(f) \le b(g)$ if $f \le g,\ f, g \in \mathcal{C}_+(\mathcal{D})$;

(3) $b(\lambda f) = \lambda b(f),\ f \in \mathcal{C}_+(\mathcal{D}), \lambda \in R_+$;

(4) $b(f+g) \le b(f) + b(g),\ f, g \in \mathcal{C}_+(\mathcal{D})$.

Also let $\mathbf{B}_1(\mathcal{D})$ be the subset of those $b \in \mathbf{B}(\mathcal{D})$ for which, in addition,

(5) $b(f \vee g) = b(f) \vee b(g),\ f, g \in \mathcal{C}_+(\mathcal{D})$,

where $f \vee g$ denotes the maximum of f and g.

We endow $\mathbf{B}(\mathcal{D})$ with the weak topology which is the topology induced by the Tihonov (product) topology on $R_+^{\mathcal{C}_+(\mathcal{D})}$, i.e., a net $\{b_\sigma, \sigma \in \Sigma\}$ of elements of $\mathbf{B}(\mathcal{D})$, where Σ is a directed set, converges to $b \in \mathbf{B}(\mathcal{D})$ if $\lim_{\sigma \in \Sigma} b_\sigma(f) = b(f)$ for all $f \in \mathcal{C}_+(\mathcal{D})$. Obviously, $\mathbf{B}(\mathcal{D})$ is closed in $R_+^{\mathcal{C}_+(\mathcal{D})}$.

We extend the domain of the functionals b to the set $\underline{\mathcal{C}}_+(\mathcal{D})$ of lower semicontinuous non-negative functions on $\mathcal{D}$ by letting

$$b(g) = \sup\{b(f) :\ f \le g, f \in \mathcal{C}_+(\mathcal{D})\},\ g \in \underline{\mathcal{C}}_+(\mathcal{D}). \tag{1.12}$$

It is easily seen that the map $b \to b(g)$ is lower semicontinuous on $\mathbf{B}(\mathcal{D})$ for each $g \in \underline{\mathcal{C}}_+(\mathcal{D})$.

Next, let us denote by $\mathcal{B}_n$ the set of all random elements on $(\Omega_n, \mathcal{F}_n)$ with values in $\mathbf{B}(\mathcal{D})$. We call the elements of $\mathcal{B}_n$ generalised decision functions (or generalised decisions). Given loss functions $W_\theta, \theta \in \Theta$, which are lower semicontinuous by definition, and a generalised decision $\beta_n \in \mathcal{B}_n$, we define $\beta_n(W_\theta^n)$ according to (1.12), and define the LD risk $B_n(\beta_n)$ of a generalised decision $\beta_n \in \mathcal{B}_n$ in the experiment $\mathcal{E}_n = (\Omega_n, \mathcal{F}_n; P_{n,\theta},\ \theta \in \Theta)$ by

$$B_n(\beta_n) = \sup_{\theta \in \Theta} E_{n,\theta}^{1/n} \beta_n(W_\theta^n). \tag{1.13}$$

Theorem A.1 *Let $\{\mathcal{E}_n, n \ge 1\}$ satisfy the LDP. Then*

$$\varliminf_{n \to \infty} \inf_{\beta_n \in \mathcal{B}_n} B_n(\beta_n) \ge B^*,$$

where

$$B^* = \sup_{z_\Theta \in R_+^\Theta} \inf_{b \in \mathbf{B}_1(\mathcal{D})} \sup_{\theta \in \Theta} b(W_\theta) z_\theta \mathbf{V}_\Theta(z_\Theta).$$

For a proof, we need to study properties of $\mathbf{B}(\mathcal{D})$ and $\mathbf{B}_1(\mathcal{D})$.

Lemma A.1 *Let $f_1, f_2, \dots, f_k \in \mathcal{C}_+(\mathcal{D})$ and $\{b_n, n \geq 1\}$ be a sequence of elements of $\mathbf{B}(\mathcal{D})$. Then there exists $b \in \mathbf{B}_1(\mathcal{D})$ such that $b(f_i)$ is an accumulation point of the sequence $\{b_n^{1/n}(f_i^n), n \geq 1\}$ for $i = 1, \dots, k$.*

Proof. Let $||\cdot||$ denote the uniform norm on $\mathcal{C}_+(\mathcal{D})$. Define $\mathcal{C}_{1,+}(\mathcal{D})$ as the subset of $\mathcal{C}_+(\mathcal{D})$ of functions f with $||f|| \leq 1$. Introduce the functionals $\overline{b}_n(f) = b_n^{1/n}(f^n), f \in \mathcal{C}_{1,+}(\mathcal{D})$. Then the set $B = \{\overline{b}_n, n \geq 1\}$ is contained in the set $[0,1]^{\mathcal{C}_{1,+}(\mathcal{D})}$. By Tihonov's theorem, $[0,1]^{\mathcal{C}_{1,+}(\mathcal{D})}$ with the product topology is compact, and hence B is relatively compact. Let $\widetilde{b}$ denote some its accumulation point. We extend $\widetilde{b}$ to a functional on $\mathcal{C}_+(\mathcal{D})$ by letting $\widetilde{b}(\lambda f) = \lambda \widetilde{b}(f), \lambda > 0, f \in \mathcal{C}_{1,+}(\mathcal{D})$. Since $b_n \in \mathbf{B}(\mathcal{D})$, it is easy to see that $\widetilde{b} \in \mathbf{B}(\mathcal{D})$. Also, since the topology on $\mathbf{B}(\mathcal{D})$ is the restriction of the product topology on $R_+^{\mathcal{C}_+(\mathcal{D})}$, it follows that $\widetilde{b}$ is an accumulation point of $\{\overline{b}_n, n \geq 1\}$, where the $\overline{b}_n$ are extended to functionals on $\mathcal{C}_+(\mathcal{D})$ by letting $\overline{b}_n(\lambda f) = \lambda \overline{b}_n(f), \lambda > 0, f \in \mathcal{C}_{1,+}(\mathcal{D})$. This implies, by the definition of the $\overline{b}_n$, that $\widetilde{b}(f_i)$ is an accumulation point of $\{b_n^{1/n}(f_i^n), n \geq 1\}$ for $i = 1, \dots, k$.

We end the proof by showing that $\widetilde{b} \in \mathbf{B}_1(\mathcal{D})$. Let $f, g \in \mathcal{C}_+(\mathcal{D})$. Then, since $\widetilde{b}$ is an accumulation point of $\{\overline{b}_n, n \geq 1\}$, it follows that $\widetilde{b}(f)$, $\widetilde{b}(g)$ and $\widetilde{b}(f \vee g)$ are respective accumulation points of $\{\overline{b}_n(f), n \geq 1\}$, $\{\overline{b}_n(g), n \geq 1\}$ and $\{\overline{b}_n(f \vee g), n \geq 1\}$. Hence, by the definition of the $\overline{b}_n$, for a subsequence (n'), we have that $b_{n'}^{1/n'}(f^{n'}) \to \widetilde{b}(f)$, $b_{n'}^{1/n'}(g^{n'}) \to \widetilde{b}(g)$ and $b_{n'}^{1/n'}((f \vee g)^{n'}) \to \widetilde{b}(f \vee g)$. By properties (2) and (4) of $\mathbf{B}(\mathcal{D})$,

$$b_n^{1/n}(f^n) \vee b_n^{1/n}(g^n) \leq b_n^{1/n}((f \vee g)^n) \leq 2^{1/n} \left[b_n^{1/n}(f^n) \vee b_n^{1/n}(g^n) \right],$$

and we conclude that $\widetilde{b}(f \vee g) = \widetilde{b}(f) \vee \widetilde{b}(g)$. $\square$

Lemma A.2 *The set $\mathbf{B}_1(\mathcal{D})$ is compact.*

Proof. An argument similar to the one used in Lemma A.1 shows that the set of functionals $\{(b(f), f \in \mathcal{C}_{1,+}(\mathcal{D})), b \in \mathbf{B}_1(\mathcal{D})\}$ is closed in $[0,1]^{\mathcal{C}_{1,+}(\mathcal{D})}$ and hence compact which obviously is equivalent to the compactness of $\mathbf{B}_1(\mathcal{D})$. $\square$

The next lemma is motivated by and extends Aubin, 1984, Proposition 8.2.

Lemma A.3 *Let T be an arbitrary set and U a topological space. Assume that a real–valued function $g(t,u), t \in T, u \in U$, has the properties:*

(a) $g(t,u)$ is level-compact in $u \in U$ for every $t \in T$,

(b) for every $t_1, t_2 \in T$, there exists $t_3 \in T$ such that $g(t_3,u) \geq g(t_1,u) \vee g(t_2,u)$ for all $u \in U$.

Then

$$\sup_{t \in T} \inf_{u \in U} g(t,u) = \inf_{u \in U} \sup_{t \in T} g(t,u).$$

Remark A.2 *Condition (a) holds when $g(t,u)$ is lower semicontinuous in u and U is a compact topological space.*

Remark A.3 *If T is a directed set, condition (b) holds when $g(t,u)$ is increasing in t for all u, i.e., $g(t_1,u) \le g(t_2,u)$, $u \in U$, for $t_1 \le t_2$ (the latter inequality is with respect to the order on T).*

Proof. We proceed analogously to Aubin, 1984. Pick $\alpha > \sup_{t\in T}\inf_{u\in U} g(t,u)$. We need to prove that

$$\alpha \ge \inf_{u\in U}\sup_{t\in T} g(t,u). \tag{1.14}$$

Let $T_0 = \{t \in T : \sup_{u\in U} g(t,u) > \alpha\}$. If T_0 is empty, the proof is over. So we assume that $T_0 \ne \emptyset$. By the choice of α, the sets $A_t = \{u \in U : g(t,u) \le \alpha\}$ are nonempty for all $t \in T$, and they are, moreover, compact for all $t \in T_0$, since the $g(t,u), u \in U$, are level-compact. Condition (b) implies that, whatever $t_1, t_2 \in T$, there exists $t_3 \in T$ such that $A_{t_1} \bigcap A_{t_2} \supset A_{t_3} \ne \emptyset$, which shows that finite intersections of the compacts $A_t, t \in T_0$, are nonempty, and hence $\bigcap_{t\in T_0} A_t \ne \emptyset$. The latter is equivalent to

$$\alpha \ge \inf_{u\in U}\sup_{t\in T_0} g(t,u).$$

Since by the definition of T_0, $\alpha \ge \sup_{t\in T\setminus T_0} g(t,u)$, $u \in U$, (1.14) is proved. □

Proof of Theorem A.1 We need to prove that, for an arbitrary sequence $\beta_n, n \ge 1$, of generalised decisions,

$$\varliminf_{n\to\infty} B_n(\beta_n) \ge B^*. \tag{1.15}$$

The argument is similar to the one in the proof of Theorem 3.1. Let $f_\theta(r)$, $\theta \in \Theta$, be some non-negative, bounded and continuous in $r \in \mathcal{D}$ functions. Fix a nonempty $\Lambda \in \mathcal{A}(\Theta)$. We have, by the definition of $\mathbf{Z}_{n,\Lambda}$ (see (2.14)),

$$\begin{aligned}
\varliminf_{n\to\infty} \sup_{\theta\in\Lambda} E_{n,\theta}^{1/n}\beta_n(f_\theta^n) &= \varliminf_{n\to\infty} \sup_{\theta\in\Lambda} E_{n,\Lambda}^{1/n}\beta_n(f_\theta^n)\mathbf{Z}_{n,\theta;\Lambda}^n \\
&\ge \varliminf_{n\to\infty} \left[\frac{1}{|\Lambda|}E_{n,\Lambda}\sum_{\theta\in\Lambda}\beta_n(f_\theta^n)\mathbf{Z}_{n,\theta;\Lambda}^n\right]^{1/n} \\
&\ge \varliminf_{n\to\infty} E_{n,\Lambda}^{1/n}\sup_{\theta\in\Lambda}\beta_n(f_\theta^n)\mathbf{Z}_{n,\theta;\Lambda}^n \\
&\ge \varliminf_{n\to\infty} E_{n,\Lambda}^{1/n}u_n^n(\mathbf{Z}_{n,\Lambda}),
\end{aligned} \tag{1.16}$$

where

$$u_n(z_\Lambda) = \inf_{b\in\mathbf{B}(\mathcal{D})}\sup_{\theta\in\Lambda} b^{1/n}(f_\theta^n)z_\theta, \quad z_\Lambda = (z_\theta, \theta \in \Lambda) \in R_+^\Lambda. \tag{1.17}$$

Note that the $u_n(z_\Lambda)\,, n = 1, 2, \ldots$, are upper semicontinuous (recall that Λ is finite) and hence measurable so that the expectations on the rightmost side of (1.16) are well defined.

Let us introduce

$$u(z_\Lambda) = \inf_{b \in \mathbf{B}_1(\mathcal{D})} \sup_{\theta \in \Lambda} b(f_\theta) z_\theta, \quad z_\Lambda \in R_+^\Lambda, \tag{1.18}$$

and prove that

$$\varliminf_{n \to \infty} \; u_n(z_\Lambda(n)) \geq u(z_\Lambda), \quad z_\Lambda \in R_+^\Lambda, \tag{1.19}$$

for each sequence $z_\Lambda(n) \to z_\Lambda$.

Let $b_n \in \mathbf{B}(\mathcal{D})$ be such that

$$\varliminf_{n \to \infty} \; u_n(z_\Lambda(n)) = \varliminf_{n \to \infty} \; \sup_{\theta \in \Lambda} b_n^{1/n}(f_\theta^n) z_\theta(n). \tag{1.20}$$

By Lemma A.1 and since Λ is finite, there exists $\widetilde{b} \in \mathbf{B}_1(\mathcal{D})$ such that $\widetilde{b}(f_\theta)$ is an accumulation point of $\{b_n^{1/n}(f_\theta^n), n \geq 1\}$ for all $\theta \in \Lambda$. Therefore, we have, for a subsequence (n'),

$$\lim_{n'} b_{n'}^{1/n'}(f_\theta^{n'}) = \widetilde{b}(f_\theta), \quad \theta \in \Lambda,$$

$$\lim_{n'} \sup_{\theta \in \Lambda} b_{n'}^{1/n'}(f_\theta^{n'}) z_\theta(n') = \varliminf_{n \to \infty} \; \sup_{\theta \in \Lambda} b_n^{1/n}(f_\theta^n) z_\theta(n).$$

Since Λ is finite and $z_\Lambda(n') \to z_\Lambda$, we conclude that

$$\varliminf_{n \to \infty} \; \sup_{\theta \in \Lambda} b_n^{1/n}(f_\theta^n) z_\theta(n) = \sup_{\theta \in \Lambda} \widetilde{b}(f_\theta) z_\theta$$

which, in view of (1.20) and (1.18), proves (1.19).

By (1.19) and the LD convergence of $\{\mathcal{L}\left(\mathbf{Z}_{n,\Lambda} | P_{n,\Lambda}\right), n \geq 1\}$ to $\mathbf{V}_\Lambda$, we have (see Varadhan, 1984; Chaganty, 1993; Puhalskii, 1995a)

$$\varliminf_{n \to \infty} \; E_{n,\Lambda}^{1/n} u_n^n(\mathbf{Z}_{n,\Lambda}) \geq \sup_{z_\Lambda \in R_+^\Lambda} u(z_\Lambda) \mathbf{V}_\Lambda(z_\Lambda). \tag{1.21}$$

Since by (1.18) $u \in \mathcal{H}_\Lambda$, property (ii) of $\mathbf{V}_\Theta$ in Lemma 2.4 yields

$$\sup_{z_\Lambda \in R_+^\Lambda} u(z_\Lambda) \mathbf{V}_\Lambda(z_\Lambda) = \sup_{z_\Theta \in R_+^\Theta} u(\pi_\Lambda z_\Theta) \mathbf{V}_\Theta(z_\Theta).$$

Relations (1.16) and (1.21) imply then that

$$\varliminf_{n \to \infty} \; \sup_{\theta \in \Lambda} E_{n,\theta}^{1/n} \beta_n(f_\theta^n) \geq \sup_{z_\Theta \in R_+^\Theta} u(\pi_\Lambda z_\theta) \mathbf{V}_\Theta(z_\Theta),$$

so, by the definition of the function u in (1.18),

$$\varliminf_{n\to\infty} \sup_{\theta\in\Lambda} E_{n,\theta}^{1/n}\beta_n(f_\theta^n) \geq \sup_{z_\Theta\in R_+^\Theta} \inf_{b\in\mathbf{B}_1(\mathcal{D})} \sup_{\theta\in\Lambda} b(f_\theta)z_\theta\mathbf{V}_\Theta(z_\Theta).$$

Hence, since $\Lambda\in\mathcal{A}(\Theta)$ and $\beta_n(f)$ are increasing in f from $\mathcal{C}_+(\mathcal{D})$, it follows that

$$\varliminf_{n\to\infty} \sup_{\theta\in\Theta} E_{n,\theta}^{1/n}\beta_n(W_\theta^n) \geq \sup_{z_\Theta\in R_+^\Theta} \sup_{\substack{\Lambda\in\mathcal{A}(\Theta)\\ f_\Theta\in C_W}} \inf_{b\in\mathbf{B}_1(\mathcal{D})} \sup_{\theta\in\Lambda} b(f_\theta)z_\theta\mathbf{V}_\Theta(z_\Theta),$$

where $C_W = \{f_\Theta = (f_\theta,\theta\in\Theta)\in\mathcal{C}_+(\mathcal{D})^\Theta:\ f_\theta\leq W_\theta,\theta\in\Theta\}$. Thus, (1.15) and the theorem would follow if, for every $z_\Theta=(z_\theta,\theta\in\Theta)\in R_+^\Theta$,

$$\sup_{\substack{\Lambda\in\mathcal{A}(\Theta)\\ f_\Theta\in C_W}} \inf_{b\in\mathbf{B}_1(\mathcal{D})} \sup_{\theta\in\Lambda} b(f_\theta)z_\theta = \inf_{b\in\mathbf{B}_1(\mathcal{D})} \sup_{\theta\in\Theta} b(W_\theta)z_\theta. \tag{1.22}$$

Fixing z_Θ, introduce, for $\Lambda\in\mathcal{A}(\Theta), f_\Theta\in\mathcal{C}_+(\mathcal{D})^\Theta, b\in\mathbf{B}_1(\mathcal{D})$,

$$g((\Lambda,f_\Theta),b) = \sup_{\theta\in\Lambda} b(f_\theta)z_\theta.$$

We check that $g((\Lambda,f_\Theta),b)$ satisfies the conditions of Lemma A.3. Supply the set $\mathcal{A}(\Theta)\times C_W$ with the natural order: $(\Lambda,f_\Theta)\leq(\Lambda',f'_\Theta)$ if $\Lambda\subset\Lambda'$ and $f_\theta\leq f'_\theta,\theta\in\Theta$. It is easily seen that $\mathcal{A}(\Theta)\times C_W$ is a directed set and $g((\Lambda,f_\Theta),b)$ is increasing in (Λ,f_Θ) for each b. Also, since Λ is finite, the definition of the topology on $\mathbf{B}(\mathcal{D})$ implies that $g((\Lambda,f_\Theta),b)$ is continuous in b for each (Λ,f_Θ). Therefore, since $\mathbf{B}_1(\mathcal{D})$ is compact by Lemma A.2, $g((\Lambda,f_\Theta),b)$ is level-compact in b. Thus, by Lemma A.3,

$$\sup_{(\Lambda,f_\Theta)\in\mathcal{A}(\Theta)\times C_W} \inf_{b\in\mathbf{B}_1(\mathcal{D})} g((\Lambda,f_\Theta),b) = \inf_{b\in\mathbf{B}_1(\mathcal{D})} \sup_{(\Lambda,f_\Theta)\in\mathcal{A}(\Theta)\times C_W} g((\Lambda,f_\Theta),b).$$

Recalling the definition of g and using the fact that by (1.12)

$$b(W_\theta) = \sup\{b(f_\theta):\ f_\theta\leq W_\theta, f_\theta\in\mathcal{C}_+(\mathcal{D})\},\quad \theta\in\Theta,$$

we get (1.22). □

It is interesting to relate Theorem A.1 with Theorem 3.1. Let us associate with each $r\in\mathcal{D}$ an element b_r of $\mathbf{B}_1(\mathcal{D})$ defined by

$$b_r(f) = f(r),\ f\in\mathcal{C}_+(\mathcal{D}). \tag{1.23}$$

Then $b_{\rho_n}\in\mathcal{B}_n$ when $\rho_n\in\mathcal{B}_n$. Therefore, in view of extension (1.12) and definitions (3.1) and (1.13), $B_n(b_{\rho_n})\leq R_n(\rho_n)$, so

$$\varliminf_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} R_n(\rho_n) \geq \varliminf_{n\to\infty} \inf_{\rho_n\in\mathcal{R}_n} B_n(b_{\rho_n}) \geq \varliminf_{n\to\infty} \inf_{\beta_n\in\mathcal{B}_n} B_n(\beta_n).$$

Similarly, $R^*\geq B^*$ so that Theorem 3.1 follows from Theorem A.1 if $B^*=R^*$. The next lemma establishes conditions for the latter.

Lemma A.4 *If the loss functions W_θ are such that*

$$W_\theta = \sup\{f_\theta : f_\theta \le W_\theta, f_\theta \in \mathcal{C}_+(\mathcal{D}), f_\theta \text{ are level-compact }\}, \theta \in \Theta,$$

then $R^* = B^*$.

Remark A.4 *The conditions of the lemma hold when the W_θ are level-compact and $\mathcal{D}$ is locally compact (cf. Strasser, 1985, Theorem 6.4). So, if $\mathcal{D}$ is locally compact, Theorem A.1 implies Theorem 3.1.*

The proof is preceded by two lemmas. We first derive a maxitive analogue of the partition of the unity (cf. Strasser, 1985, Lemma 6.6).

Lemma A.5 *Let $f_1, \dots, f_k \in \mathcal{C}_+(\mathcal{D})$. For every $\varepsilon > 0$, there exist $h_1, \dots, h_m \in \mathcal{C}_+(\mathcal{D})$ with the following properties:*

1^0 $\max_{1\le j\le m} h_j(r) = 1, r \in \mathcal{D}$,

2^0 $\max_{1\le i\le k} |f_i(r_1) - f_i(r_2)| \le \varepsilon$ *for all r_1 and r_2 such that $h_j(r_1) > 0$ and $h_j(r_2) > 0$ for some $j = 1, \dots, m$.*

Proof. The argument is similar to that in Strasser, 1985. Assume first that $k = 1$ and $\sup_{r\in\mathcal{D}} f_1(r) = 1$. Choose m such that $3/m \le \varepsilon$ and define, for $x \ge 0$,

$$g_j(x) = (x - (j-2))^+ \wedge (j + 1 - x)^+ \wedge 1,\ 1 \le j \le m.$$

Let

$$h_j(r) = g_j(mf_1(r)),\ 1 \le j \le m, r \in \mathcal{D}.$$

It is readily seen, since $g_j(x) = 1$ when $j - 1 \le x \le j$ and $0 \le f_1(r) \le 1$, that $\max_{1\le j\le m} h_j(r) = 1, r \in \mathcal{D}$.

Next, since, given $j = 1, \dots, m$, we have $g_j(x) = 0$ when $x \notin [(j-2)^+, j+1]$, it follows that if $h_j(r_1) > 0$ and $h_j(r_2) > 0$, then $|mf_1(r_1) - mf_1(r_2)| \le 3$, i.e., $|f_1(r_1) - f_1(r_2)| \le 3/m \le \varepsilon$ as required.

Now, if $\sup_{r\in\mathcal{D}} f_1(r) = a > 0$, then the h_j chosen as above for f_1/a and ε/a satisfy 1^0 and 2^0.

Finally, if $k > 1$, choose, for each $i = 1, \dots, k$, functions $h_{i,j}, 1 \le j \le m_i$, that satisfy 1^0 and 2^0. Then the functions

$$h_{j_1,\dots,j_k}(r) = \prod_{i=1}^{k} h_{i,j_i}(r),\ 1 \le j_i \le m_i, r \in \mathcal{D},$$

meet the required for all i with $m = m_1 \dots m_k$. $\square$

Denote by T_1 the set of non-negative (upper semicontinuous) functions of finite support $(t(r), r \in \mathcal{D})$ such that $\sup_{r\in\mathcal{D}} t(r) = 1$. Define $\mathbf{B}_2(\mathcal{D})$ as the set of those $b \in \mathbf{B}_1(\mathcal{D})$ that can be represented as $b(f) = \sup_{r\in\mathcal{D}} f(r)t(r)$, $f \in \mathcal{C}_+(\mathcal{D})$, for some $(t(r), r \in \mathcal{D}) \in T_1$. The next lemma parallels Strasser, 1985, Theorem 42.5.

Lemma A.6 *The set $\mathbf{B}_2(\mathcal{D})$ is dense in $\mathbf{B}_1(\mathcal{D})$ for the weak topology.*

Proof. We proceed as in the proof of Strasser, 1985, Theorem 42.5. Fix $b \in \mathbf{B}_1(\mathcal{D})$ and $f_1, \ldots, f_k \in \mathcal{C}_+(\mathcal{D})$. We have to check that for any $\varepsilon > 0$ there exists $\widetilde{b} \in \mathbf{B}_2(\mathcal{D})$ such that $|b(f_i) - \widetilde{b}(f_i)| \le \varepsilon$, $1 \le i \le k$.

Let functions h_j, $1 \le j \le m$, be as in Lemma A.5. Obviously we can assume that they are not identically equal to 0. For each $j = 1, \ldots, m$, choose r_j such that $h_j(r_j) > 0$. By the definition of the h_j, we have that, on the one hand,

$$|f_i(r)h_j(r) - f_i(r_j)h_j(r)| \le \varepsilon,\ 1 \le i \le k, r \in \mathcal{D},$$

and, on the other hand,

$$f_i(r) = \max_{1\le j\le m} f_i(r)h_j(r),\ 1 \le i \le k, r \in \mathcal{D}.$$

Hence,

$$|f_i(r) - \max_{1\le j\le m} f_i(r_j)h_j(r)| \le \max_{1\le j\le m} |f_i(r)h_j(r) - f_i(r_j)h_j(r)| \le \varepsilon,\qquad 1 \le i \le k, r \in \mathcal{D}.$$

Properties (1), (3) and (4) of $\mathbf{B}(\mathcal{D})$ then yield

$$|b(f_i) - b(\max_{1\le j\le m} f_i(r_j)h_j)| \le \varepsilon,\ 1 \le i \le k. \tag{1.24}$$

Now, since $b \in \mathbf{B}_1(\mathcal{D})$ and by property (3) again,

$$b(\max_{1\le j\le m} f_i(r_j)h_j) = \max_{1\le j\le m} f_i(r_j)b(h_j),\ 1 \le i \le k. \tag{1.25}$$

Define

$$t(r) = \begin{cases} \max_{l: r_l = r_j} b(h_l), & \text{if } r = r_j \text{ for some } j = 1, \ldots, m, \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$\widetilde{b}(f) = \sup_{r\in\mathcal{D}} f(r)t(r),\ f \in \mathcal{C}_+(\mathcal{D}).$$

By properties (1) and (5) of $\mathbf{B}_1(\mathcal{D})$ and the choice of the h_j,

$$\sup_{r\in\mathcal{D}} t(r) = \max_{1\le j\le m} b(h_j) = b(\max_{1\le j\le m} h_j) = b(\mathbf{1}) = 1,$$

so $(t(r)) \in T_1$.

Also by the definitions of $t(r)$ and $\widetilde{b}$, the right-hand side of (1.25) equals $\widetilde{b}(f_i)$, and (1.25) and (1.24) yield the required. $\square$

Proof of Lemma A.4 Since $R^* \geq B^*$, we prove the opposite inequality. Let $f_\theta, \theta \in \Theta$, belong to $\mathcal{C}_+(\mathcal{D})$, be level-compact and $f_\theta \leq W_\theta, \theta \in \Theta$. By the definition of B^*,

$$B^* \geq \sup_{z_\Theta \in R_+^\Theta} \inf_{b \in \mathbf{B}_1(\mathcal{D})} \sup_{\theta \in \Lambda} b(f_\theta) z_\theta \mathbf{V}_\Theta(z_\Theta), \ \Lambda \in \mathcal{A}(\Theta). \tag{1.26}$$

By Lemma A.6 and the definition of $\mathbf{B}_2(\mathcal{D})$, for $z_\Theta \in R_+^\Theta, \Lambda \in \mathcal{A}(\Theta)$,

$$\begin{aligned} \inf_{b \in \mathbf{B}_1(\mathcal{D})} \sup_{\theta \in \Lambda} b(f_\theta) z_\theta = \inf_{b \in \mathbf{B}_2(\mathcal{D})} \sup_{\theta \in \Lambda} b(f_\theta) z_\theta \\ = \inf_{(t(r)) \in T_1} \sup_{r \in \mathcal{D}} \sup_{\theta \in \Lambda} t(r) f_\theta(r) z_\theta = \inf_{r \in \mathcal{D}} \sup_{\theta \in \Lambda} f_\theta(r) z_\theta. \end{aligned}$$

Since the f_θ are level-compact, an application of Lemma A.3 shows, in analogy with the end of the proof of Theorem A.1, that the supremum of the latter quantity over the f_θ and $\Lambda \in \mathcal{A}(\Theta)$ equals $\inf_{r \in \mathcal{D}} \sup_{\theta \in \Theta} W_\theta(r) z_\theta$ which by (1.26) proves that $B^* \geq R^*$. $\square$

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