Semi-tractability of optimal stopping problems via a weighted stochastic mesh algorithm

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Abstract

In this article we propose a Weighted Stochastic Mesh (WSM) algorithm for approximating the value of discrete and continuous time optimal stopping problems. In this context we consider tractability of such problems via a useful notion of semi-tractability and the introduction of a tractability index for a particular numerical solution algorithm. It is shown that in the discrete time case the WSM algorithm leads to semi-tractability of the corresponding optimal stopping problem in the sense that its complexity is bounded in order by $\varepsilon^{-4} \log^{d+2}(1/\varepsilon)$ with d being the dimension of the underlying Markov chain. Furthermore we study the WSM approach in the context of continuous time optimal stopping problems and derive the corresponding complexity bounds. Although we can not prove semi-tractability in this case, our bounds turn out to be the tightest ones among the complexity bounds known in the literature. We illustrate our theoretical findings by a numerical example.

Key Words: optimal stopping, American options, Monte Carlo algorithms, complexity

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1 Introduction

The theory of optimal stopping is concerned with the problem of choosing a time to take a particular action, in order to maximize an expected reward or minimize an expected cost. Such problems can be found in many areas of statistics, economics, and mathematical finance (e.g. the pricing problem of American options). Primal and dual approaches have been developed in the literature

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giving rise to regression-type Monte Carlo algorithms for high-dimensional optimal stopping problems. So far modern literature on numerical analysis of high-dimensional optimal stopping problems focused almost exclusively on convergence analysis of various simulation-based stochastic algorithms. However, comparing different algorithms based only on their convergence rates is not possible since these algorithms may have different costs. Therefore it is important to carry out complexity analysis of stochastic algorithms for optimal stopping problems. Such a complexity analysis will provide us with convergence rates of a stochastic algorithm in terms of its cost and hence can be viewed as a universal criteria for comparing different algorithms. Complexity analysis has played and is still playing an important role in numerical analysis of algorithms, see [15] and the references therein. The key numerical problem studied in this literature is the computation of integrals by means of deterministic and stochastic (randomised) algorithms. Optimal stopping problems can not be boiled down to the computation of a single integral but rather require computation of several nested integrals (dynamic programming principle). Hence the standard concepts and results from the existing complexity theory can not be directly transferred to complexity analysis of optimal stopping problems.

One of the most widely adopted regression algorithms by practitioners is the Longstaff & Schwartz (LS) algorithm. It is based on approximating conditional expectations using least-squares regression on a given basis of functions in each backward induction step. Longstaff and Schwartz [14] demonstrated the efficiency of their approach through a number of numerical examples, and in [6] and [20] general convergence properties of the method were established. In particular, it follows from Corollary 3.10 in [20] that for a fixed number L of stopping opportunities and a popular choice of polynomial basis functions of degree less or equal to m, the error of estimating the corresponding value function at one point is bounded by

$$\kappa \, 5^L \left(\sqrt{\frac{m^d}{N}} + \frac{1}{m^\alpha} \right), \tag{1}$$

where N is the number of paths used to perform regression, $\alpha \geq 1$ is related to smoothness of the corresponding conditional expectation operator, d is the dimension of the underlying state space, and κ is some constant independent of N, d, and α. On the other hand, due to the computation of a (random) pseudo-inverse at every stopping date, the computational cost of the least-squares MC algorithm is approximately proportional to $\kappa_1 N m^{2d} L$ where κ_1 is proportional to the cost of an elementary operation (multiplication for example). This leads to the following estimate for the complexity of the LS algorithm, that is, the amount of "elementary" operations needed to construct an approximation for the value function with accuracy ε.

Proposition 1 *For L stopping opportunities and underlying dimension d, the computational work for achieving an accuracy ε by the LS algorithm is bounded*

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by

$$\mathcal{C}_L\left(\varepsilon, d\right) = \kappa_1 \frac{L\, 5^{(\kappa_2 + L)(2 + 3d/\alpha)}}{\varepsilon^{2 + 3d/\alpha}} \tag{2}$$

with $\kappa_2 := \ln(2\kappa)/\ln 5$.

If we next consider a continuous time optimal stopping problem, then we need to approximate it by a discrete one with L stopping dates, and then let $L \to \infty$. For instance, let us assume that the error due to the time discretization is of order $L^{-\beta}$ for some $0 < \beta < 1$, independent of d. Then, for achieving an overall accuracy of order ε, we may take $L = \kappa_3 \varepsilon^{-1/\beta}$ for some $\kappa_3 > 0$, which gives by (2) the complexity bound

$$\mathcal{C}_\infty(\varepsilon, d) = \frac{\kappa_1 \kappa_3 5^{(\kappa_2 + \kappa_3 \varepsilon^{-1/\beta})(2 + 3d/\alpha)}}{\varepsilon^{2 + 3d/\alpha + 1/\beta}}. \tag{3}$$

It follows that the complexity of the LS algorithm for continuous time optimal stopping problems may even grow faster than $\exp(\varepsilon^{-1/\beta})$. Similar complexity bounds can be derived for other simulation based regression algorithms, such as the value iteration algorithm by Tsitsiklis & van Roy [19] (TV). See also for example, [8] for more general regression algorithms or [10] for a novel nested type MC approach with complexity which is independent of d but exponential in ε^{-1}, unfortunately.

An interesting question is whether the complexity bounds (2) and (3) for the discrete and continuous time stopping problems, respectively, are attained in worst cases. The appearance of $1/\varepsilon$ in the exponential in (3) and the number of exercise dates L in the exponential in (2), respectively, is of course due to the appearance of L in the exponential of the convergence estimate (1). In fact, the latter appearance is observed in all error bounds concerning regression-based backward dynamic programs for optimal stopping in the literature (e.g. [8], [20]). It also appears in a later result by [21], based on dependent samples, and in the convergence analysis by [4] in the context of optimal stopping of McKean-Vlasov processes. This factor seems to be unavoidable because at each backward step the projection error of the estimated continuation function needs to be bounded in relation to the projection error of the true continuation function. For details, see for instance [20], Theorem 3.1 versus Theorem 3.3, and [21], Theorem 5.1 versus Theorem 5.6. It should also be noted that if we discretize a continuous time optimal stopping problem, then conditional variance of the underlying process decreases from one exercise date to the next. However, this decrease is typically of order $1/L$ with L being the number of exercise dates and as such is not fast enough to remove exponential dependence on L in the above convergence estimates. Thus, in view of the above considerations, the complexity bounds (2) and (3) for the LS algorithm seem to be sharp in some sense, but, a rigorous proof of this assertion seems highly nontrivial and therefore beyond the scope of this article.

An important notion in complexity analysis is tractability of a numerical problem. A d-dimensional numerical problem, for example, computation of an

integral $\int_{[0,1]^d} f(x)\,dx$ is called tractable, if there is an algorithm to solve it with complexity $\mathcal{C}(\varepsilon, d)$ satisfying

$$\lim_{d+\varepsilon^{-1}\nearrow\infty} \frac{\log\mathcal{C}\,(\varepsilon,d)}{d+\varepsilon^{-1}} = 0. \tag{4}$$

Unfortunately in the case of optimal stopping problems this definition is not very meaningful and rather restrictive. It turns out that for all regression-type algorithms one has, already in the case of discrete time optimal stopping problems,

$$\limsup_{d+\varepsilon^{-1}\nearrow\infty} \frac{\log\mathcal{C}(\varepsilon,d)}{d+\varepsilon^{-1}} = \infty \tag{5}$$

(based on the convergence rates known in the literature), that is, any discrete time optimal stopping problem is intractable according to this definition. As an example consider again the LS algorithm in the case of analytic (hence infinitely smooth) continuation functions. Using results from [18], it can be shown that the error (cf. (1)) of the estimated value function in this case has the form:

$$5^L \left(\sqrt{\frac{m^d}{N}} + e^{-\theta m} \right), \quad \theta > 0,$$

where m is again the maximal polynomial degree. Similar to Proposition 1, it then follows that

$$\mathcal{C}_L(\varepsilon, d) = L\frac{5^{2L}}{\varepsilon^2} \left(\log \frac{5^{L/\theta}}{\varepsilon^{1/\theta}} \right)^{3d} \tag{6}$$

is an upper bound for the LS complexity and so we get (5) (take $\varepsilon^{-1} = d$ and let $d \nearrow \infty$). Thus even in the case of analytic continuation functions tractability of discrete time optimal stopping problems in the sense of (4) can not be established by the LS algorithm. However, the latter observation also applies to any other simulation based algorithm addressed in this paper, including the WSM algorithm that we are going to present below.

In fact, the problem with criterion (4) is that it puts too much weight on the dimension d on the one hand, but on the other hand, is too relaxed regarding the dependence of $\mathcal{C}(\varepsilon, d)$ on ε. For instance, it is not difficult to see that it renders a problem with an algorithmic complexity of order $d^2 \exp(\varepsilon^{-1}/\log\log\ldots\log\varepsilon^{-1})$ to be (weakly) tractable while an algorithm with complexity $2^d/\varepsilon$ is not. However, running an algorithm with a complexity growth of $\exp(\varepsilon^{-1}/\log\log\ldots\log\varepsilon^{-1})$, hence faster than any ε^{-k}, $k \in \mathbb{N}$, when $\varepsilon \searrow 0$, is in practice impossible (even when $d = 1$). In the setting of optimal stopping problems, the dimension d is typically fixed, though can be large. Therefore in our paper we propose a more meaningful definition of tractability based on the quantity called tractability index.

Definition 2   (i) *For an algorithm with complexity $\mathcal{C}\,(\varepsilon, d)$ the so-called tractability index is defined as*

$$\Gamma_{\mathcal{C}} \stackrel{def}{=} \limsup_{d\nearrow\infty} \limsup_{\varepsilon\searrow 0} \frac{\log\mathcal{C}\,(\varepsilon,d)}{d\log(1/\varepsilon)}. \tag{7}$$

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(ii) We call a problem *semi-tractable* if there is an algorithm to solve it which has zero-tractability index, that is,

$$\Gamma_{\mathcal{C}} = 0. \tag{8}$$

For example, it is easily seen from Proposition 1 that for continuation functions with smoothness α the discrete time LS algorithm has tractability index $\Gamma_{\mathcal{C}} = 3/\alpha$, whereas in the case of analytic continuation functions (6) implies

$$\frac{\log \mathcal{C}_L(\varepsilon, d)}{d \log(1/\varepsilon)} \lesssim \frac{\log L + \log \frac{1}{\varepsilon} + d \log \log \frac{1}{\varepsilon}}{d \log \frac{1}{\varepsilon}},$$

hence $\Gamma_{\mathcal{C}}^{\alpha=\infty} = 0$ and semi-tractability in our sense follows. However, we see from (3) (for $\alpha < \infty$) and a similar expression obtained from (6) in the case of analytic continuation functions, that the LS algorithm has tractability index ∞ for continuous time optimal stopping problems.

In this paper we introduce Weighted Stochastic Mesh (WSM) algorithm and show (under mild assumptions) that a discrete time optimal stopping problem may be computed by this algorithm with tractability index 0, and so is semi-tractable in the sense of Definition 2-(ii). Furthermore, we show that the continuous time stopping problems may be computed via this algorithm with finite tractability index equal to 2. The construction of this algorithm in the discrete time case follows closely the idea of the mesh method by Broadie and Glasserman [5]. By enhancing the latter method with a suitable regularization, we prove that the complexity of the resulting WSM algorithm satisfies (8) (under mild conditions), provided the transition densities of the underlying Markov chain are analytically known or can be well approximated. It turns out that for solving a continuous time optimal stopping problem we do not need to assume that the transition densities are known but can use Gaussian transition densities of the corresponding Euler scheme. This results in an algorithm with complexity of order $c^d \varepsilon^{-(2d+14)}$ for some constant $c > 1$. Although this does not imply semi-tractability of continuous time optimal stopping problems, the proposed algorithm is very simple and its complexity remains provably polynomial in ε^{-1} as opposed to the LS approach. In particular, it follows that the WSM algorithm for continuous time optimal stopping problems has tractability index 2, and as such has the smallest tractability index among the existing algorithms for continuous-time optimal stopping problems.

Let us remark that a complete convergence analysis as well as complexity analysis of the stochastic mesh method for optimal stopping in discrete and continuous time is still missing in the literature, apart from some preliminary results for the discrete case in Agarwal and Juneja [1]. But, [1] does not trace the dependence of the errors on the underlying dimension and the number of stopping times, and is moreover based on a rather restrictive assumption of a compact state space. Further, the WSM algorithm presented in this paper bears some similarities to the random grid algorithm of Rust [16]. However, the Rust [16] algorithm was constructed and studied for Markov Decision Problems in discrete time and is not directly applicable to optimal stopping problems.

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As such the corresponding convergence analysis in [16] differs in several respects. For example, it assumes a compact state space and Lipschitz continuity of transition densities with Lipschitz constant basically not depending on the dimension. (The latter assumption is violated by a d-dimensional Gaussian kernel with small enough variance, due to exponentially growing Lipschitz constants in d, for instance.)

The paper is organized as follows. A description of the proposed algorithm is given in Section 2. Section 2.2 is devoted to convergence and complexity analysis of our algorithm. In Section 3 we turn to continuous time optimal stopping problems. Section 4 concisely highlights the main achievements of the paper. Some numerical experiments are presented in Section 5 and all proofs are collected in Section 6.

2 Discrete time optimal stopping problems

We begin with the description of the WSM algorithm for discrete time optimal stopping problems. Let us assume a finite set of stopping dates $\{0, \ldots, L\}$, for some natural $L > 0$, and let $(Z_l, l = 0, \ldots, L)$ be a Markov chain in $\mathbb{R}^d$, adapted to a filtration $(\mathcal{F}_l, l = 0, \ldots, L)$. For a given set of nonnegative reward functions $g_l, l = 0, \ldots, L$, on $\mathbb{R}^d$, we then consider the discrete Snell envelope process:

$$U_l = U_l(Z_l) \stackrel{\text{def}}{=} \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}_l \left[g_\tau(Z_\tau) \right], \tag{9}$$

where $\mathcal{T}_{l,L}$ stands for the set of $\mathcal{F}$-stopping times with values in the set $\{l, \ldots, L\}$, and $\mathsf{E}_l := \mathsf{E}_{\mathcal{F}_l}$ stands for the $\mathcal{F}_l$-conditional expectation, and the measurable functions $U_l(\cdot)$ exist due to Markovianity of the process $(Z_l)_{l \geq 0}$.

For simplicity and without loss of generality we assume that the Markov chain $(Z_l)_{l \geq 0}$ is time homogeneous with l-steps transition density denoted by $p_l(y|x)$ and one-step density denoted by $p(y|x) = p_1(y|x)$, so that

$$\mathbb{P}\left[Z_{k+1} \in dy \,|\, Z_k = x \right] = p(y|x)dy, \qquad p(y|x) > 0,$$

for all $x, y \in \mathbb{R}^d$. Fix some $x_0 \in \mathbb{R}^d$ and assume that $Z_0 = x_0$. It is well known that the Snell envelope (9) satisfies the dynamic program principle,

$$U_L(Z_L) = g_L(Z_L), \tag{10}$$
$$U_l(Z_l) = \max\left\{ g_l(Z_l), \mathsf{E}\left[U_{l+1}(Z_{l+1}) \,|\, Z_l \right] \right\}, \quad l = 0, \ldots, L-1.$$

Next we fix some $R > 0$ and define a truncated version of the above dynamic program via

$$\widetilde{U}_L(Z_L) = g_L(Z_L) \cdot \mathbb{1}_{Z_L \in B_R}, \tag{11}$$
$$\widetilde{U}_l(Z_l) = \max\left\{ g_l(Z_l), \mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1}) \,\middle|\, Z_l \right] \right\} \cdot \mathbb{1}_{Z_l \in B_R}, \quad l = 0, \ldots, L-1,$$

where $B_R \stackrel{\text{def}}{=} \{z : |z - x_0| \leq R\}$. Thus, by construction, $\widetilde{U}_l$ vanishes outside the ball B_R. Also by construction it holds that

$$\|\widetilde{U}_l\|_\infty \leq G_R \stackrel{\text{def}}{=} \max_{0 \leq l \leq L} \sup_{z \in B_R} g_l(z), \qquad (12)$$

which is easily seen by backward induction. In view of (11) we may write

$$\mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1})\Big| Z_l = x\right] = \int \widetilde{U}_{l+1}(y) \frac{p(y|x)}{p_{l+1}(y|x_0)} p_{l+1}(y|x_0)\, dy.$$

Now assume that we have a set of trajectories $Z_l^{(n)}$, $l = 0, \ldots, L$, with $Z_0^{(n)} = x_0$, $n = 1, \ldots, N$, simulated according to the one-step transition density p, and consider the approximation:

$$\mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1})\Big| Z_l = x\right] \approx \frac{1}{N} \sum_{n=1}^{N} \widetilde{U}_{l+1}(Z_{l+1}^{(n)}) \frac{p(Z_{l+1}^{(n)}|x)}{p_{l+1}(Z_{l+1}^{(n)}|x_0)},$$

where in view of the Chapman-Kolmogorov equation

$$p_{l+1}(Z_{l+1}^{(n)}|x_0) = \int p(Z_{l+1}^{(n)}|z) p_l(z|x_0)\, dz \approx \frac{1}{N} \sum_{m=1}^{N} p(Z_{l+1}^{(n)}|Z_l^{(m)}).$$

Hence we have approximately

$$\mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1})\Big| Z_l = x\right] \approx \sum_{n=1}^{N} \widetilde{U}_{l+1}(Z_{l+1}^{(n)}) \frac{p(Z_{l+1}^{(n)}|x)}{\sum_{m=1}^{N} p(Z_{l+1}^{(n)}|Z_l^{(m)})}. \qquad (13)$$

We thus propose the following algorithm. We start with

$$\overline{U}_L(Z_L^{(n)}) \stackrel{\text{def}}{=} g_L(Z_L^{(n)}) \mathbb{1}_{Z_L^{(n)} \in B_R}$$

for $n = 1, \ldots, N$. Once $\overline{U}_{l+1}$ is constructed on the grid for $0 < l+1 \leq L$, we set

$$\overline{U}_l(Z_l^{(r)}) \stackrel{\text{def}}{=} \max\left\{ g_l(Z_l^{(r)}), \sum_{n=1}^{N} \overline{U}_{l+1}^{(n)}(Z_{l+1}^{(n)}) \frac{p(Z_{l+1}^{(n)}|Z_l^{(r)})}{\sum_{m=1}^{N} p(Z_{l+1}^{(n)}|Z_l^{(m)})} \right\} \mathbb{1}_{Z_l^{(r)} \in B_R},$$
$$\qquad (14)$$

for $r = 1, \ldots, N$. By construction, each function $\overline{U}_l$ vanishes outside the ball B_R. Working all the way down to $l = 0$ results in the approximation:

$$\overline{U}_0 = \max\left[g_0(x_0), \sum_{n=1}^{N} \overline{U}_1^{(n)}(Z_1^{(n)}) \frac{p(Z_1^{(n)}|x_0)}{\sum_{m=1}^{N} p(Z_1^{(n)}|x_0)} \right]$$

for U_0. As such the presented algorithm is closely related to the mesh method of Broadie and Glasserman [5] apart from truncation at level R and a special choice of weights.

2.1 Cost estimation

Let us estimate the cost of carrying out the backward dynamic program (14). One needs to compute $p(Z_{l+1}^{(n)}|Z_l^{(m)})$ for all $l = 1, \ldots, L$, $n, m = 1, \ldots, N$. This can be done at a cost of order $N^2 L c_f^{(d)}$, where $c_f^{(d)}$ is the cost of evaluating a (typical) function of $2d$ arguments. In the typical situation $c_f^{(d)}$ is proportional to d. The evaluation of

$$\frac{1}{N} \sum_{m=1}^{N} p(Z_{l+1}^{(n)}|Z_l^{(m)})$$

for $l = 1, \ldots, L$, $n = 1, \ldots, N$, has a cost of order $N^2 L c_*$ with c_* being the cost of an elementary numerical operation, which is negligible if $c_* \ll c_f^{(d)}$. So the overall cost of carrying out the backward dynamic program (14) is of order $N^2 L c_f^{(d)}$.

2.2 Error and complexity analysis

In this section we analyze convergence of the WSM estimate (14) to the solution of the discrete optimal stopping problem (9) for $l = 0$ and a fixed $x_0 \in \mathbb{R}^d$ as $N \to \infty$. Let us first bound a distance between U_l and $\widetilde{U}_l$, $l = 0, \ldots, L$.

Proposition 3 *With*

$$\varepsilon_{l,R} \overset{def}{=} \int_{|x - x_0| > R} U_l(x) p_l(x|x_0)\, dx$$

$l = 0, \ldots, L$, *it holds that*

$$\int \left| U_l(x) - \widetilde{U}_l(x) \right| p_l(x|x_0)\, dx \leq \sum_{j=l}^{L} \varepsilon_{j,R}. \tag{15}$$

Proposition 4 *Suppose that*

$$\max_{0 \leq l \leq L} g_l(x) \leq c_g(1 + |x|), \quad x \in \mathbb{R}^d \tag{16}$$

and that

$$\mathsf{E}\left[\max_{l \leq l' \leq L} |Z_{l'}| \,\Big|\, Z_l = x \right] \leq c_Z(1 + |x|), \quad x \in \mathbb{R}^d. \tag{17}$$

Suppose further that for some $\varkappa$, $\alpha > 0$, and $l = 1, \ldots, L$,

$$0 < p_l(y|x) \leq \frac{\varkappa}{(2\pi\alpha l)^{d/2}} e^{-\frac{|x-y|^2}{2\alpha l}} \tag{18}$$

for all $x, y \in \mathbb{R}^d$. One then has

$$\int \left| U_l(x) - \widetilde{U}_l(x) \right| p_l(x|x_0)\, dx$$

$$\leq L c_g \varkappa \left(1 + c_Z + c_Z |x_0| + c_Z \sqrt{d\alpha L} \right) 2^{d/4} e^{-\frac{R^2}{8\alpha L}}. \tag{19}$$

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Next we control the discrepancy between $\overline{U}_0$ and $\widetilde{U}_0$.

Proposition 5 *With*

$$F_R^2 \overset{def}{=} \max_{1 \le l \le L} \int \int_{|y-x_0| \le R} \frac{p^2(y|x)}{p_{l+1}(y|x_0)} \, p_l(x|x_0) \, dxdy, \qquad (20)$$

and N such that $(1 + F_R)/\sqrt{N} < 1$, it holds that

$$\mathsf{E}\left[\left|\overline{U}_0 - \widetilde{U}_0\right|\right] \le \left(3 + \sqrt{2}\right) LG_R \frac{1 + F_R}{\sqrt{N}}.$$

Corollary 6 *Under the assumptions of Proposition 4, we have for (20) the estimate*

$$F_R^2 \le \frac{\varkappa}{(2\pi\alpha)^{d/2}} \mathtt{Vol}(B_R) = \frac{\varkappa R^d}{(2\alpha)^{d/2}\Gamma\left(1 + d/2\right)} \le \varkappa \left(e/\alpha\right)^{d/2} R^d d^{-d/2},$$

where the last inequality follows from $\Gamma\left(1 + a\right) \ge a^a e^{-a}$ for any $a \ge 1/2$. Then by combining (19) with Proposition 5 we obtain the error estimate,

$$\mathsf{E}\left[\left|U_0 - \overline{U}_0\right|\right] \le Lc_g\varkappa \left(1 + c_Z + c_Z\left|x_0\right| + c_Z\sqrt{d\alpha L}\right) 2^{d/4} e^{-\frac{R^2}{8\alpha L}}$$

$$+ \left(3 + \sqrt{2}\right) Lc_g(1 + R)\frac{1 + \varkappa^{1/2}\left(e/\alpha\right)^{d/4} R^{d/2}d^{-d/4}}{\sqrt{N}}. \qquad (21)$$

Proposition 7 *Under the assumptions of Proposition 4 the complexity of the WSM algorithm is bounded from above by*

$$\mathcal{C}(\varepsilon, d) = c_1\alpha^2 c_g^4\varkappa^2 c_f^{(d)} c_2^d L^{d+7}\varepsilon^{-4}$$

$$\times \log^{d+2}\left[\frac{L\left(1 + c_Z + c_Z\left|x_0\right|\right) e^{\frac{c_Z\sqrt{\alpha L}}{1 + c_Z + c_Z|x_0|}} 2^{3/4}\left(c_g\varkappa \vee 1\right)}{\varepsilon}\right], \qquad (22)$$

where $c_1 > 0$ and $c_2 > 1$ are natural constants and $c_f^{(d)}$ stands for the cost of computing the transition density $p_l(y|x)$ at one point (x, y).

Corollary 8 *For a fixed $L > 0$ the discrete time optimal stopping problem (9) with g and $(Z_l)_{l \ge 0}$ satisfying (16), (17) and (18) is semi-tractable, provided that the complexity of computing the transition density $p_l(y|x)$ at one point (x, y) is at most polynomial in d. Different approximation algorithms for discrete time optimal stopping problems can be compared using the tractability index (7). For example, it follows from (2) that the tractability index of the LS approach is equal to $3/\alpha$. If the continuation functions are analytic, then the tractability index for the LS approach becomes zero. Moreover from inspection of Theorem 2.4 in [3], we see that the Quantization Tree Method (QTM) has tractability index 2.*

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LS	WSM	QTM
$3/\alpha$	0	2

Table 1: Tractability index Γ of different algorithms for discrete time optimal stopping problems

2.3 Approximation of the transition density

A crucial condition for semi-tractability in the discrete exercise case is the availability of the transition density $p(y|x)$ of the chain $(Z_l)_{l\geq 0}$ in a closed (or cheaply computable) form. However, we can show that, if a sequence of approximating densities $p^n(y|x)$, $n \in \mathbb{N}$, converging to $p(y|x)$ can be constructed in such a way that

$$\left| \frac{p^n(y|z) - p(y|z)}{p^n(y|z)} \right| \lesssim \frac{(1 + |y - x_0|^m + |z - x_0|^m)^n}{n!}, \quad y, z \in B_{R_n} \qquad (23)$$

for some $m \in \mathbb{N}$ and a sequence $R_n \nearrow \infty$, $n \nearrow \infty$, then under proper assumptions on the growth of R_n and the cost of computing p^n (in fact it should be at most polynomial in d), one can derive a complexity bound $\mathcal{C}(\varepsilon, d)$ satisfying

$$\lim_{\varepsilon \searrow 0} \frac{\log \mathcal{C}(\varepsilon, d)}{\log \frac{1}{\varepsilon}} \text{ is finite and does not depend on } d \ .$$

The proof involves a (rather straightforward) extension of the present one based on exact transition densities. But, on the one hand, one of the main results in this paper, tractability index 2 of the continuous time stopping problem, does not rely on transition density approximation, and on the other hand, such a proof would entail a notational blow up and might detract the reader from the main lines, therefore the details are omitted.

To construct a sequence of approximations $p^n(y|z)$ satisfying the assumption (23), one can use various small-time expansions for transition densities of stochastic processes, see, for example, [2] and [13]. Let us exemplify this type of approximation in the case of one-dimensional diffusion processes of the form:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x_0,$$

where b is a bounded function, twice continuously differentiable, with bounded derivatives and σ is a function with three continuous and bounded derivatives such that there exist two positive constants $\sigma_\circ, \sigma^\circ$ with $\sigma_\circ \leq \sigma(x) \leq \sigma^\circ$. Consider a Markov chain $(Z_l)_{l\geq 0}$ defined as a time discretization of $(X_t)_{t\geq 0}$, that is, $Z_l \stackrel{\text{def}}{=} X_{\Delta l}$, $l = 0, 1, 2, \ldots$ for some $\Delta > 0$. Under the above conditions the following representation for the (one-step) transition density p of the chain Z is proved in [9] (see also [7] for more general setting):

$$p(y|x) = \frac{1}{\sqrt{2\pi\Delta}} \frac{1}{\sigma(y)} \exp\left(-\frac{(s(x) - s(y))^2}{2\Delta}\right) U_\Delta(s(x), s(y)), \quad x, y \in \mathbb{R},$$

with $U_\Delta(x,y) = R_\Delta(x,y)\exp\left[\int_0^x \bar{b}(z)\,dz - \int_0^y \bar{b}(z)\,dz\right]$,

$$R_\Delta(x,y) = \mathsf{E}\left[\exp\left(-\Delta\int_0^1 \bar{\rho}(x+z(y-x)+\sqrt{\Delta}B_z)\,dz\right)\right], \qquad (24)$$

where B_z is a standard Brownian bridge, $s(x) = \int_0^x \frac{dy}{\sigma(y)}$, $g = s^{-1}$ and

$$\bar{\rho} = (\bar{b}^2 + \bar{b}')/2 \quad \text{with} \quad \bar{b} = (b/\sigma)\circ g - \sigma'\circ g/2.$$

Note that the expectation in (24) is taken with respect to the known distribution of the Brownian bridge B_z. By expanding the exponent in (24) into Taylor series, we get for Δ small enough

$$p(x|y) = \frac{1}{\sqrt{2\pi\Delta}}\frac{1}{\sigma(y)}\exp\left(-\frac{(s(x)-s(y))^2}{2\Delta}\right)$$
$$\times \exp\left[\int_0^x \bar{b}(z)\,dz - \int_0^y \bar{b}(z)\,dz\right]\sum_{k=0}^\infty \frac{\Delta^k}{k!}c_k(x,y)$$

with

$$c_k(x,y) = (-1)^k\mathsf{E}\left[\left(\int_0^1 \bar{\rho}(x+z(y-x)+\sqrt{\Delta}B_z)\,dz\right)^k\right].$$

If $\bar{\rho}$ is uniformly bounded by a constant $D > 0$, then the above series converges uniformly in x and y for all Δ small enough. Set

$$p^n(x|y) = \frac{1}{\sqrt{2\pi\Delta}}\frac{1}{\sigma(y)}\exp\left(-\frac{(s(x)-s(y))^2}{2\Delta}\right)$$
$$\times \exp\left[\int_0^x \bar{b}(z)\,dz - \int_0^y \bar{b}(z)\,dz\right]\left\{\sum_{k=0}^n \frac{\Delta^k}{k!}c_k(x,y)\right\}.$$

It obviously holds $p^n(y|x) > 0$ for $\Delta < \Delta_0(D)$ and

$$\left|\frac{p^n(y|z)-p(y|z)}{p^n(y|z)}\right| \leq \frac{(\Delta D)^n}{(1-\Delta D\exp(\Delta D))} \qquad (25)$$

uniformly for all $x,y \in \mathbb{R}$. Hence the assumption (23) is satisfied with $m = 0$, provided that $\Delta < \Delta_0$ for some Δ_0 depending only on D. Similarly if $\bar{\rho} \leq 0$, then (23) holds. To sample from p^n we can use the well-known acceptance rejection method which does not require the exact knowledge of a scaling factor $\int p^n(y|x)\,dy$.

3 Continuous time optimal stopping for diffusions

In this section we consider diffusion processes of the form

$$dX_s^i = b_i(X_s)\,ds + \sum_{j=1}^m \sigma_{ij}(X_s)\,dW_s^j, \quad X_0^i = x_0^i, \quad i = 1,\ldots,d, \qquad (26)$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, are Lipschitz continuous and $W = (W^1, \ldots, W^m)$ is a m-dimensional standard Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. As usual, the (augmented) filtration generated by $(W_s)_{s \geq 0}$ is denoted by $(\mathcal{F}_s)_{s \geq 0}$. We are interested in solving optimal stopping problems of the form:

$$U_t^\star = \operatorname*{esssup}_{\tau \in \mathcal{T}_{t,T}} \mathsf{E}[e^{-r(\tau - t)} f(X_\tau) | \mathcal{F}_t], \tag{27}$$

where f is a given real valued function on $\mathbb{R}^d$, $r \geq 0$, and $\mathcal{T}_{t,T}$ stands for the set of stopping times τ taking values in $[t, T]$. The problem (27) is related to the so-called free boundary problem for the corresponding partial differential equation. Let us introduce the differential operator L_t :

$$L_t u(t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j}(t, x) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x^i}(t, x),$$

where

$$a_{ij}(x) = \sum_{k=1}^d \sigma_{ik}(x) \sigma_{jk}(x).$$

We denote by $X_s^{t,x}$ (or $X^{t,x}(s)$), $s \geq T$, the solution of (26) starting at moment t from x : $X_t^{t,x} = x$. Denote by $u(t, x)$ a regular solution of the following system of partial differential inequalities:

$$\frac{\partial u}{\partial t} + L_t u - ru \leq 0, \ u \geq f, \quad (t, x) \in [0, T) \times \mathbb{R}^d, \tag{28}$$

$$\left(\frac{\partial u}{\partial t} + L_t u - ru \right)(f - u) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d,$$

$$u(T, x) = f(x), \quad x \in \mathbb{R}^d,$$

then under some mild conditions (see, e.g. [11])

$$u(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathsf{E}[e^{-r(\tau - t)} f(X_\tau^{t,x})] \quad , \ (t, x) \in [0, T] \times \mathbb{R}^d, \tag{29}$$

that is, $u(t, x) = U_t^\star(x)$.

With this notation established, it is worth discussing the main issue that we are going to address in this section. Our goal is to estimate $u(t, x)$ at a given point (t_0, x_0) with accuracy less than ε by an algorithm with complexity $\mathcal{C}^\star(\varepsilon, d)$ which is polynomial in $1/\varepsilon$. As already mentioned in the introduction some well known algorithms such as the regression ones fail to achieve this goal (at least according to the existing complexity bounds in the literature).

Let us introduce the Snell envelope process:

$$U_t^\star \stackrel{\text{def}}{=} \operatorname{esssup}_{\tau \in \mathcal{T}_{t,T}} \mathsf{E}_{\mathcal{F}_t} [g(\tau, X_\tau)], \tag{30}$$

where (somewhat more general than in (27)) g is a given nonnegative function on $\mathbb{R}_{\geq 0} \times \mathbb{R}^d$. In the first step we perform a time discretization by introducing

a finite set of stopping dates $t_l = lh$, $l = 1, \ldots, L$, with $h = T/L$ and L some natural number, and next consider the discretized Snell envelope process:

$$U_{t_l}^{\circ}(X_{t_l}) \overset{\text{def}}{=} \underset{\tau \in \mathcal{T}_{l,L}}{\text{esssup}} \, \mathsf{E}_{\mathcal{F}_{t_l}} \left[g(\tau, X_\tau) \right],$$

where $\mathcal{T}_{l,L}$ stands for the set of stopping times with values in the set $\{t_l, \ldots, t_L\}$. Note that the measurable functions $U_{t_l}^{\circ}(\cdot)$ exist due to Markovianity of the process X. The error due to the time discretization is well studied in the literature. We will rely on the following result which is implied by Thm. 2.1 in [3] for instance.

Proposition 9 *Let $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous and $p \geq 1$. Then one has that*

$$\max_{l=0,\ldots,L} \left\| U_{t_l}^{\star}(X_{t_l}) - U_{t_l}^{\circ}(X_{t_l}) \right\|_p \leq \frac{c_\circ e^{C_\circ T}(1 + |x_0|)}{L},$$

where the constants $c_\circ, C_\circ > 0$ depend on the Lipschitz constants for b, σ, and g, respectively.

In order to achieve an acceptable discretization error we choose a sufficiently large L, and then concentrate on the computation of U°.

In the next step we approximate the underlying process X using some strong discretization scheme on the time grid $t_i = iT/L$, $i = 0, \ldots, L$, yielding an approximation $\overline{X}$. It is assumed that the one step transition densities of this scheme are explicitly known. The simplest and the most popular scheme is the Euler scheme,

$$\overline{X}_{t_{l+1}}^i = \overline{X}_{t_l}^i + b_i(\overline{X}_{t_l})\, h + \sum_{j=1}^{m} \sigma_{ij}(\overline{X}_{t_l}) \left(W_{t_{l+1}}^j - W_{t_l}^j \right), \quad \overline{X}_0^i = x_0^i, \qquad (31)$$

$i = 1, \ldots, d$, which in general has strong convergence order $1/2$, and the one-step transition density of the chain $(\overline{X}_{t_{l+1}})_{l \geq 0}$ is given by

$$\bar{p}_h(y|x) \overset{\text{def}}{=} \frac{1}{\sqrt{(2\pi h)^d |\Sigma|}} \exp\left[-\frac{1}{2} h^{-1}(y - x - b(x)h)^\top \Sigma^{-1}(y - x - b(x)h) \right] \tag{32}$$

with $\Sigma = \sigma\sigma^\top \in \mathbb{R}^{d \times d}$ and $h = T/L$. Now we will turn to the discrete time optimal stopping problem with possible stopping times $\{t_l = lh, l = 0, \ldots, L\}$. To this end we introduce the discrete time Markov chain $Z_l \overset{\text{def}}{=} \overline{X}_{t_l}$ adapted to the filtration $(\mathcal{F}_l) \overset{\text{def}}{=} (\mathcal{F}_{t_l})$, and $g_l(x) \overset{\text{def}}{=} g(t_l, x)$ (while abusing notation slightly) and consider the discretized Snell envelope process

$$U_{t_l}(\overline{X}_{t_l}) \overset{\text{def}}{=} \underset{\tau \in \mathcal{T}_{l,L}}{\text{esssup}} \, \mathsf{E}_{\mathcal{F}_{t_l}} \left[g(\tau, \overline{X}_\tau) \right] = \underset{\iota \in \mathcal{I}_{l,L}}{\text{esssup}} \, \mathsf{E}_{\mathcal{F}_l} \left[g_\iota(Z_\iota) \right] \overset{\text{def}}{=} U_l(Z_l), \qquad (33)$$

where $\mathcal{I}_{l,L}$ stands for the set of stopping indices with values in $\{l, \ldots, L\}$, and the measurable functions $U_{t_l}(\cdot)$ (or $U_l(\cdot)$) exist due to Markovianity of the process $\overline{X}$ (or Z). The distance between U and U° is controlled by the next proposition.

Proposition 10 *There exists a constant $C^{Euler} > 0$ depending on the Lipschitz constants of b, σ, and g, such that*

$$\max_{l=0,\ldots,L} \mathsf{E}\left[\left|U^\circ_{t_l}(X_{t_l}) - U_{t_l}(\overline{X}_{t_l})\right|\right] \leq C^{Euler}\sqrt{h}.$$

Thus, combining Proposition 9 and Proposition 10 yields:

Corollary 11 *If $\overline{X}$ is constructed by the Euler scheme with time step size $h = T/L$, where L is the number of discretization steps, then under the conditions of Proposition 9 and Proposition 10 we have that*

$$\mathsf{E}\left[\left|U^\star_0(x_0) - U_0(x_0)\right|\right] \lesssim C^{Euler}\sqrt{h} \quad for\ h \to 0, \tag{34}$$

where $\lesssim$ stands for inequality up to constant depending on $c_\circ, C_\circ$ and C^{Euler}.

Since the transition densities of the Euler scheme are explicitly known (see (32)), the WSM algorithm can be directly used for constructing an approximation $\overline{U}_0(x_0)$ based on the paths of the Markov chain (Z_l). To derive the complexity bounds of the resulting estimate, we shall make the following assumptions.

(AG) Suppose that $c_g > 0$ is such that

$$g(t, x) \leq c_g\left(1 + |x|\right) \quad \text{for all } 0 \leq t \leq T,\ x \in \mathbb{R}^d. \tag{35}$$

(AX) Assume that there exists a constant $c_{\overline{X}} > 0$ such that for all $0 \leq l \leq L$,

$$\mathsf{E}_{\mathcal{F}_{t_l}}\left[\sup_{l \leq l' \leq L} \left|\overline{X}_{l'h}\right| \left| \overline{X}_{lh} = x\right.\right] \leq c_{\overline{X}}\left(1 + |x|\right), \quad x \in \mathbb{R}^d, \tag{36}$$

uniformly in L (hence h). This assumption is satisfied under Lipschitz conditions on the coefficients of the SDE (26), and can be proved using the Burkholder-Davis-Gundy inequality and the Gronwall lemma.

(AP) Assume furthermore that $\left(\overline{X}_{lh},\ l = 0, \ldots, L\right)$ is time homogeneous with transition densities $\overline{p}_{lh}(y|x)$ that satisfy the Aronson type inequality: there exist positive constants $\overline{\varkappa}$ and $\overline{\alpha}$ such that for any $x, y \in \mathbb{R}^d$ and any $l > 0$, it holds that
$$0 < \overline{p}_{lh}(y|x) \leq \frac{\overline{\varkappa}}{(2\pi\overline{\alpha}lh)^{d/2}} e^{-\frac{|x-y|^2}{2\overline{\alpha}lh}}.$$

This assumption holds if the coefficients in (26) are bounded and σ is uniformly elliptic.

The next proposition provides complexity bounds for the WSM algorithm in the case of continuous time optimal stopping problems.

Proposition 12 *Assume that the assumptions (AG), (AX) and (AP) hold, then*

- *the cost of computing $U_0(x_0)$ in (33) for a fixed $L > 0$ with precision $\varepsilon > 0$ via the WSM algorithm is bounded from above by*

$$\mathcal{C}(\varepsilon, d) = c_1 \overline{\alpha}^2 c_g^4 \varkappa^2 c_f^{(d)} c_2^d \frac{T^{d+7}}{h^{d+5}}$$

$$\times \varepsilon^{-4} \log^{d+2} \left[\frac{\frac{T}{h} \left(1 + c_{\bar{X}} + c_{\bar{X}} |x_0|\right) e^{\frac{c_{\bar{X}} \sqrt{\overline{\alpha}T}}{1 + c_{\bar{X}} + c_{\bar{X}} |x_0|}} 2^{3/4} \left(c_g \varkappa \vee 1\right)}{\varepsilon} \right]. \quad (37)$$

- *the cost of computing $U_0^\star(x_0)$ with an accuracy $\varepsilon > 0$ via the WSM algorithm is bounded from above by*

$$\mathcal{C}^\star(\varepsilon, d) = c_1 \overline{\alpha}^2 c_g^4 \varkappa^2 c_f^{(d)} c_2^d \frac{T^{d+7}}{\varepsilon^{2d+14}}$$

$$\times \log^{d+2} \left[\frac{T \left(1 + c_{\bar{X}} + c_{\bar{X}} |x_0|\right) e^{\frac{c_{\bar{X}} \sqrt{\overline{\alpha}T}}{1 + c_{\bar{X}} + c_{\bar{X}} |x_0|}} 2^{3/4} \left(c_g \varkappa \vee 1\right)}{\varepsilon} \right]. \quad (38)$$

The first statement follows directly from Proposition 7 by taking in (22), $\alpha = \overline{\alpha}h$, $c_Z = c_{\bar{X}}$, and $L = T/h$. Then by setting $h \asymp \varepsilon^2$ we obtain (38) (with possibly modified natural constants c_1, c_2).

Discussion As can be seen from (38),

$$\Gamma_{\mathrm{WSM}} = \lim_{d \nearrow \infty} \lim_{\varepsilon \searrow 0} \frac{\log \mathcal{C}^\star(\varepsilon, d)}{d \log \varepsilon^{-1}} = 2 \quad (39)$$

and this shows the efficiency of the proposed algorithm as compared to the existing algorithms for continuous time optimal stopping problems at least as far as the tractability index is concerned. Indeed, the only algorithm available in the literature with a provably finite limit of type (39) is the quantization tree method (QTM) of Bally, Pagès, and Printems [3]. Indeed, by tending the number of stopping times and the quantization number to infinity such that the corresponding errors in Thm. 2.4-b in [3] are balanced, we derive the following complexity upper bound

$$\mathcal{C}_{\mathrm{QTM}}^\star(\varepsilon, d) = O\left(\frac{1}{\varepsilon^{6d+6}}\right) \quad (40)$$

Hence $\Gamma_{\mathrm{QTM}} = 6$.

LS	WSM	QTM
∞	2	6

Table 2: Tractability index Γ of different algorithms for continuous time optimal stopping problems.

4   Summary

For discrete time optimal stopping problems we have established semi-tractability for the proposed WSM algorithm with respect to rather general Markov chains governed by certain transition kernels. In particular, apart from assumption (17) on the spatial decay of such kernels and some growth condition on the pay-off, no further smoothness assumptions are imposed. As a rule, if both the transition kernels and the rewards are non-smooth, the continuation functions may be smooth only up to some finite degree. (Examples may be easily constructed.)

In the most common case of infinitely smooth continuation functions, many regression algorithms including the LS and TV algorithms are also semi-tractable for discrete time optimal stopping problems. But when passing on to continuous stopping problems, the tractability index of the WSM method remains bounded (equal to two) while the tractability index of the regression methods tends to infinity.

5   Numerical experiments

In the following experiments[1] we illustrate the WSM algorithm in the case of continuous time optimal stopping problems. A lower bound for the value function $u(t_0, x_0)$ at a given point (t_0, x_0) via the WSM algorithm can be obtained using a suboptimal policy computed on an independent set of trajectories. This policy can be constructed either directly via (13) or by using an interpolation of the likelihood weights

$$\frac{p(Z_{l+1}^{(j)}|\cdot)}{\sum_{m=1}^{N} p(Z_{l+1}^{(j)}|Z_l^{(m)})}.$$

The fastest and simplest way to do this is to use the nearest neighbor interpolation based on training set of trajectories, in all experiments below the number of neighbors was set to 500.

5.1   American put option on a single asset

In order to illustrate the performance of the WSM algorithm in continuous time, we consider a financial problem of pricing American put option on a single log-

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[1]The data that support the findings of this study are available from the corresponding author upon reasonable request.

Brownian asset

$$X_t = X_0 \exp(\sigma W_t + (r - \sigma/2)t), \tag{41}$$

with r denoting the riskless rate of interest, assumed to be constant, and σ denoting the constant volatility. The payoff function is given by $g(x) = (K-x)^+$ and a fair price of the option is given by

$$U_0 = \sup_{\tau \in \mathcal{T}[0,T]} \mathsf{E}\left[e^{-r\tau} g(X_\tau)\right].$$

No closed-form solution for the price of this option is known, but there are various numerical methods which give accurate approximations to U_0. The parameter values used are $r = 0.08$, $\sigma = 0.20$, $\delta = 0$, $K = 100$, $T = 3$. An accurate estimate for the true price obtained via a binomial tree type algorithm is 6.9320 (see [12]). In Figure 1 we show lower bounds due to WSM, the least squares approach of Longstaff and Schwartz [14] (LS) and the value function regression algorithm of Tsitsiklis and Van Roy [19] (VF) as functions of the number of stopping times L forming a uniform grid on $[0, T]$. These lower bounds are constructed using a suboptimal stopping rule due to estimated continuation values evaluated on a new independent set of trajectories. The maximal degree of polynomials used as basis functions in LS and VF are indicated by the numbers (2 and 4) in the legend. As can be seen WSM lower bounds are more stable when L increases. The VF lower bounds seem to diverge as $L \to \infty$. A similar behavior of regression algorithms for increasing L was observed in [17].

5.2 American max-call on five assets

Here a model with $d = 5$ identically distributed assets is considered, where each underlying has dividend yield δ. The risk-neutral dynamic of assets is given by

$$\frac{dX_t^k}{X_t^k} = (r - \delta)dt + \sigma dW_t^k, \quad k = 1, \ldots, d,$$

where W_t^k, $k = 1, \ldots, d$, are independent one-dimensional Brownian motions and $r = 0.05$, $\delta = 0.1$, $\sigma = 0.2$. At any time $t \in [0, T]$ with $T = 3$ the holder of the option may exercise it and receive the payoff

$$g(X_t) = (\max(X_t^1, \ldots, X_t^d) - K)^+.$$

We consider the approximations of this continuous time American option by discrete time American options with stopping possibilities $t_i = iT/L$, $i = 0, \ldots, L$, and apply the WSM approach to construct a lower bound. The results for different L are presented in Figure 2 where also the related results for the LS algorithm with a polynomial basis of order 2 are reported.
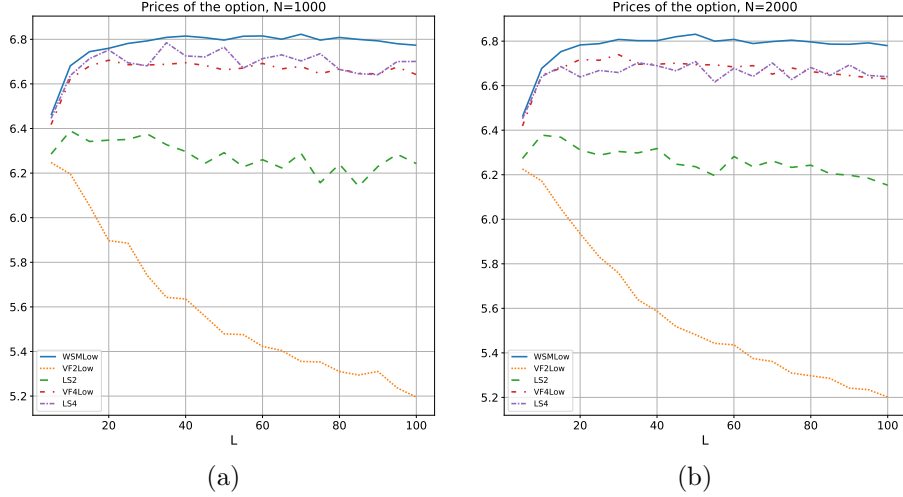
Figure 1: Lower bounds for the price of a one-dimensional American put option approximated using different methods and a uniform grid $t_k = kT/L, k = 0, \ldots, L$, of exercise dates. The numbers of training paths are $N_{\text{train}} = 1000$ (a) and $N_{\text{train}} = 2000$ (b), and the number of new trajectories used to construct lower bounds is $N_{\text{test}} = 20000$ in both cases. In LS and VF regression methods a polynomial basis of degree 2 and 4 is used.

6 Proofs

6.1 Proof of Proposition 1

For achieving a target accuracy of order ε it is reasonable to divide the error equally over the variance and the bias part of (1). One thus chooses m such that $\kappa 5^L/m^\alpha \approx \varepsilon/2$, that is, $m \approx (2\kappa 5^L/\varepsilon)^{1/\alpha}$, and then takes N such that $\kappa 5^L m^{d/2}/N^{1/2} \approx \varepsilon/2$, i.e. $N \approx (2\kappa)^2 5^{2L} m^d/\varepsilon^2$, yielding a computational work load $\mathcal{C}_L(\varepsilon, d) = \kappa_1 N m^{2d} L$ as stated.

6.2 Proof of Proposition 3

For $l = L$ the statement reads

$$\int \left| U_L(x) - \widetilde{U}_L(x) \right| p_L(x|x_0) dx = \int 1_{|x-x_0|>R} \, g(x) p_L(x|x_0) dx = \varepsilon_{L,R},$$

so then it is true. Suppose (15) is true for $0 < l + 1 \leq L$. Then, by using $|\max(a, b) - \max(a, c)| \leq |b - c|$ and the fact that $\widetilde{U}_l(x)$ vanishes for $|x - x_0| >$
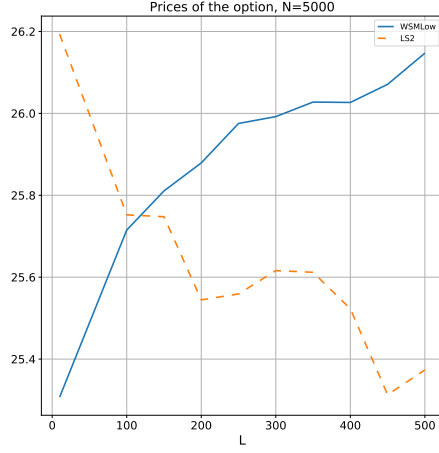
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Figure 2: Lower bounds for the price of a five-dimensional American put option approximated using different methods and a uniform grid $t_k = kT/L, k = 0, \ldots, L$, of exercise dates. The numbers of training paths is $N_{\text{train}} = 2000$. In the LS method a polynomial basis of degree 2 is used.

R,

$$
\begin{aligned}
\left| U_l(x) - \widetilde{U}_l(x) \right| &\leq 1_{|x-x_0| \leq R} \left| \max \left[g(x), \mathsf{E} \left[U_{l+1}(X_{l+1}) \middle| X_l = x \right] \right] \right. \\
&\quad \left. - \max \left[g(x), \mathsf{E} \left[\widetilde{U}_{l+1}(X_{l+1}) \middle| X_l = x \right] \right] \right| + 1_{|x-x_0| > R} U_l(x) \\
&\leq 1_{|x-x_0| \leq R} \mathsf{E} \left[\left| U_{l+1}(X_{l+1}) - \widetilde{U}_{l+1}(X_{l+1}) \right| \middle| X_l = x \right] + 1_{|x-x_0| > R} U_l(x).
\end{aligned}
$$

Hence we have by induction,

$$
\begin{aligned}
&\int \left| U_l(x) - \widetilde{U}_l(x) \right| p_l(x|x_0) dx \\
&\leq \int 1_{|x-x_0| > R} \mathsf{E} \left[\left| U_{l+1}(X_{l+1}) - \widetilde{U}_{l+1}(X_{l+1}) \right| \middle| X_l = x \right] p_l(x|x_0) dx + \varepsilon_{l,R} \\
&\leq \int \left| U_{l+1}(y) - \widetilde{U}_{l+1}(y) \right| p_{l+1}(y|x_0) dy + \varepsilon_{l,R} \\
&= \sum_{j=l+1}^{L} \varepsilon_{j,R} + \varepsilon_{l,R} = \sum_{j=l}^{L} \varepsilon_{j,R}.
\end{aligned}
$$

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6.3 Proof of Proposition 4

Combining the assumptions (16) and (17) yields

$$U_l(x) = \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}\left[g_\tau(Z_\tau) | \, Z_l = x \right]$$

$$\leq c_g \mathsf{E}\left[1 + \max_{l \leq l' \leq L} |Z_{l'}| \, \middle| \, Z_l = x \right]$$

$$\leq c_g \left(1 + c_Z \right) + c_g c_Z |x|.$$

By the estimate

$$\int_{|x-x_0|>R} e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx \leq e^{-\frac{R^2}{8\alpha l}} \left(4/3\right)^{d/2} \left(2\pi\alpha l\right)^{d/2},$$

and (using Cauchy-Schwarz) the estimate

$$\int_{|x-x_0|>R} |x - x_0| \, e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx \leq \sqrt{\int_{|x-x_0|>R} e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx} \sqrt{\int |x - x_0|^2 \, e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx}$$

$$\leq e^{-\frac{R^2}{8\alpha l}} 2^{d/4} (2\pi\alpha l)^{d/2} \sqrt{d\alpha l},$$

we get (note that $(4/3)^{1/2} < 2^{1/4}$)

$$\varepsilon_{l,R} \leq \frac{\varkappa}{(2\pi\alpha l)^{d/2}} \int_{|x-x_0|>R} \left(c_g \left(1 + c_Z \right) + c_g c_Z |x| \right) e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx$$

$$\leq \frac{\varkappa c_g \left(1 + c_Z + c_Z |x_0| \right)}{(2\pi\alpha l)^{d/2}} \int_{|x-x_0|>R} e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx$$

$$+ \frac{\varkappa c_g c_Z}{(2\pi\alpha l)^{d/2}} \int_{|x-x_0|>R} |x - x_0| \, e^{-\frac{|x-x_0|^2}{2\alpha l}} \, dx$$

$$\leq \varkappa c_g \left(1 + c_Z + c_Z |x_0| + c_Z \sqrt{d\alpha}\sqrt{l} \right) 2^{d/4} e^{-\frac{R^2}{8\alpha l}}$$

$$\equiv \left(A + B\sqrt{l} \right) c_g \varkappa e^{-\frac{R^2}{8\alpha l}},$$

for $l \geq 1$ ($\varepsilon_{0,R} = 0$ for $R > 0$). Now by (15), i.e. Proposition 3, we get

$$\int \left| U_l(x) - \widetilde{U}_l(x) \right| p_l(x|x_0) \, dx \leq L \left(A + B\sqrt{L} \right) c_g \varkappa e^{-\frac{R^2}{8\alpha L}},$$

whence the estimate (19).

6.4 Proof of Proposition 5

Let us write the sample based backward dynamic program (14) for step $l < L$ in the form

$$\overline{U}_l\left(Z_l^{(i)}\right) = \mathbb{1}_{\left|Z_l^{(i)}-x_0\right|\leq R} \max\left[g_l(Z_l^{(i)}), \sum_{j=1}^{N} \overline{U}_{l+1}(Z_{l+1}^{(j)})\overline{w}_{ij}\right] \qquad (42)$$

by defining the weights

$$w_{ij} := \frac{p(Z_{l+1}^{(j)}|Z_l^{(i)})}{\sum_{m=1}^{N} p(Z_{l+1}^{(j)}|Z_l^{(m)})}, \qquad (43)$$

where l is fixed and suppressed. Let us further abbreviate

$$\mathcal{E}[f](x) = \mathsf{E}\left[f(Z_{l+1})\mid Z_l = x\right] = \int f(y)p(y|x)dy$$

for a generic Borel function $f \geq 0$. Using

$$\widetilde{U}_l\left(Z_l^{(i)}\right) = \mathbb{1}_{\left|Z_l^{(i)}-x_0\right|\leq R} \max\left[g_l(Z_l^{(i)}), \mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(i)})\right],$$

(42) and $|\max(a,b) - \max(a,c)| \leq |b-c|$, we thus get

$$\left|\overline{U}_l - \widetilde{U}_l\right|_N := \frac{1}{N}\sum_{i=1}^{N} \left|\overline{U}_l(Z_l^{(i)}) - \widetilde{U}_l(Z_l^{(i)})\right| \leq$$

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left|Z_l^{(i)}-x_0\right|\leq R} \left|\sum_{j=1}^{N}\overline{U}_{l+1}(Z_{l+1}^{(j)})w_{ij} - \mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(i)})\right|$$

$$\leq \left|\overline{U}_{l+1} - \widetilde{U}_{l+1}\right|_N + \mathcal{R}_{l+1} \qquad (44)$$

with

$$\mathcal{R}_{l+1} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left|Z_l^{(i)}-x_0\right|\leq R} \left|\sum_{j=1}^{N}\widetilde{U}_{l+1}(Z_{l+1}^{(j)})w_{ij} - \mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(i)})\right|,$$

where we have used the fact that the weights in (43) sum up to one. One thus gets by iterating (44)

$$\left|\overline{U}_k - \widetilde{U}_k\right|_N \leq \sum_{l=k}^{L-1} \mathcal{R}_{l+1}, \qquad (45)$$

since $\overline{U}_L - \widetilde{U}_L = 0$. Let us now introduce

$$w_{ij}^{\circ} := \frac{1}{N}\frac{p(Z_{l+1}^{(j)}|Z_l^{(i)})}{p_{l+1}(Z_{l+1}^{(j)}|x_0)}, \qquad (46)$$

and consider the generic term

$$\mathcal{R}_{l+1} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left|Z_l^{(i)} - x_0\right| \le R} \left| \sum_{j=1}^{N} \widetilde{U}_{l+1}(Z_{l+1}^{(j)}) w_{ij} - \mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(i)}) \right|$$

$$\le \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left|Z_l^{(i)} - x_0\right| \le R} \sum_{j=1}^{N} \widetilde{U}_{l+1}(Z_{l+1}^{(j)}) \left| w_{ij} - w_{ij}^{\circ} \right|$$

$$+ \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\left|Z_l^{(i)} - x_0\right| \le R} \left| \sum_{j=1}^{N} \left(w_{ij}^{\circ} \widetilde{U}_{l+1}(Z_{l+1}^{(j)}) - \frac{1}{N} \mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(i)}) \right) \right|$$

$$=: \mathrm{Term}_1 + \mathrm{Term}_2.$$

We have
$$\mathsf{E}\left[\mathcal{R}_{l+1}\right] \le \mathsf{E}\left[\mathrm{Term}_1\right] + \mathsf{E}\left[\mathrm{Term}_2\right].$$

While the first term Term_1 is small as (w_{ij}) are close to (w_{ij}°), the second one Term_2 tends to 0 as $N \to \infty$ by the law of large numbers. Indeed, due to (12) one has

$$\mathrm{Term}_1 \le \frac{G_R}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{1}_{\left|Z_l^{(i)} - x_0\right| \le R} \mathbb{1}_{\left|Z_{l+1}^{(j)} - x_0\right| \le R} \left| w_{ij} - w_{ij}^{\circ} \right|,$$

and due to (43) and (46) we may write

$$\left| w_{ij} - w_{ij}^{\circ} \right| = \left| \frac{p(Z_{l+1}^{(j)}|Z_l^{(i)})}{\sum_{m=1}^{N} p(Z_{l+1}^{(j)}|Z_l^{(m)})} - \frac{1}{N} \frac{p(Z_{l+1}^{(j)}|Z_l^{(i)})}{p_{l+1}(Z_{l+1}^{(j)}|x_0)} \right|$$

$$= \frac{p(Z_{l+1}^{(j)}|Z_l^{(i)})}{\sum_{m=1}^{N} p(Z_{l+1}^{(j)}|Z_l^{(m)})} \left| 1 - \frac{\frac{1}{N} \sum_{m=1}^{N} p(Z_{l+1}^{(j)}|Z_l^{(m)})}{p_{l+1}(Z_{l+1}^{(j)}|x_0)} \right|$$

to obtain

$$\mathrm{Term}_1 \le \frac{G_R}{N} \sum_{j=1}^{N} \mathbb{1}_{\left|Z_{l+1}^{(j)} - x_0\right| \le R} \left| 1 - \frac{\frac{1}{N} \sum_{m=1}^{N} p(Z_{l+1}^{(j)}|Z_l^{(m)})}{p_{l+1}(Z_{l+1}^{(j)}|x_0)} \right|.$$

Obviously, the expectation of the random variable inside of the above sum is independent of j. So by taking $j = 1$ and splitting off the with $Z^{(1)}$ correlating term due to $m = 1$, one gets

$$\mathsf{E}\left[\mathrm{Term}_1\right] \le \frac{G_R}{N} \mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{(1)} - x_0\right| \le R} \left| \sum_{m=1}^{N} \left(1 - \frac{p(Z_{l+1}^{(1)}|Z_l^{(m)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)} \right) \right| \right]$$

$$\le \frac{G_R}{N} D_{R,l} + \frac{G_R}{N} \mathsf{E}\left[\left| \sum_{m=2}^{N} \mathbb{1}_{\left|Z_{l+1}^{(1)} - x_0\right| \le R} \left(1 - \frac{p(Z_{l+1}^{(1)}|Z_l^{(m)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)} \right) \right| \right]$$

with

$$D_{R,l} := \mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{(1)}-x_0\right|\leq R}\left|1-\frac{p(Z_{l+1}^{(1)}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)}\right|\right].$$

Now consider the i.i.d. random variables,

$$\eta_m^{(l+1)} := \mathbb{1}_{\left|Z_{l+1}^{(1)}-x_0\right|\leq R}\left(1-\frac{p(Z_{l+1}^{(1)}|Z_l^{(m)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)}\right), \quad m = 2, ..., N. \qquad (47)$$

It is easy to check by conditioning on $Z_{l+1}^{(1)}$ that they have zero mean. Then by applying Jensen's inequality to the square-root, using the independence of the random variables (47), and that the latter variables are identically distributed with zero mean, we derive

$$\mathsf{E}\left|\sum_{m=2}^{N}\eta_m^{(l+1)}\right| \leq \sqrt{\mathsf{E}\left(\sum_{m=2}^{N}\eta_m^{(l+1)}\right)^2} = E_{R,l}\sqrt{N}$$

with

$$E_{R,l}^2 := \mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{(1)}-x_0\right|\leq R}\left|1-\frac{p(Z_{l+1}^{(1)}|Z_l^{(2)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)}\right|^2\right].$$

Finally we get for Term_1,

$$\mathsf{E}[\text{Term}_1] \leq \frac{G_R D_{R,l}}{N} + \frac{G_R E_{R,l}}{\sqrt{N}}.$$

Concerning Term_2, let us write

$$\mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(i)}) = \int \widetilde{U}_{l+1}(y)\frac{p(y|Z_l^{(i)})}{p_{l+1}(y|x_0)}p_{l+1}(y|x_0)dy$$

$$= \mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1}^{0,x_0})\frac{p(Z_{l+1}^{0,x_0}|Z_l^{(i)})}{p_{l+1}(Z_{l+1}^{0,x_0}|x_0)}\right],$$

where Z^{0,x_0} is an independent dummy trajectory. We thus have

$$\mathsf{E}\left[\text{Term}_2\right] \leq \mathsf{E}\left[\mathbb{1}_{\left|Z_l^{(1)}-x_0\right|\leq R}\left|\left(w_{11}^\circ \widetilde{U}_{l+1}(Z_{l+1}^{(1)}) - \frac{1}{N}\mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(1)})\right)\right|\right]$$

$$+ \mathsf{E}\left[\left|\sum_{j=2}^{N}\zeta_j^{(l+1)}\right|\right],$$

where for $j = 2, ..., N$, the random variables

$$\zeta_j^{(l+1)} := \mathbb{1}_{\left|Z_l^{(1)}-x_0\right|\leq R}\left(w_{1j}^\circ \widetilde{U}_{l+1}(Z_{l+1}^{(j)}) - \frac{1}{N}\mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(1)})\right)$$

$$= \frac{\mathbb{1}_{\left|Z_l^{(1)}-x_0\right|\leq R}}{N}\left(\frac{p(Z_{l+1}^{(j)}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{(j)}|x_0)}\widetilde{U}_{l+1}(Z_{l+1}^{(j)}) - \mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1}^{0,x_0})\frac{p(Z_{l+1}^{0,x_0}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{0,x_0}|x_0)}\right]\right)$$

are i.i.d. with zero mean. We so have by the Jensen's inequality again,

$$\mathsf{E}\left[\left|\sum_{j=2}^{N}\zeta_j^{(l+1)}\right|\right] \le \sqrt{N\mathsf{Var}\left(\zeta_2^{(l+1)}\right)} \le F_{R,l}G_R/\sqrt{N},$$

where

$$F_{R,l}^2 = \mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{(2)}-x_0\right|\le R}\left|\frac{p(Z_{l+1}^{(2)}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{(2)}|x_0)}\right|^2\right] = \int\int_{|y-x_0|\le R}\frac{p^2(y|x)}{p_{l+1}(y|x_0)}p_l(x|x_0)\,dxdy.$$

Secondly, by (46) one has

$$\mathsf{E}\left[\mathbb{1}_{\left|Z_l^{(1)}-x_0\right|\le R}\left|\left(w_{11}^{\circ}\widetilde{U}_{l+1}(Z_{l+1}^{(1)}) - \frac{1}{N}\mathcal{E}[\widetilde{U}_{l+1}](Z_l^{(1)})\right)\right|\right]$$

$$\le \frac{1}{N}\mathsf{E}\left[\mathbb{1}_{\left|Z_l^{(1)}-x_0\right|\le R}\frac{p(Z_{l+1}^{(1)}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)}\widetilde{U}_{l+1}(Z_{l+1}^{(1)})\right]$$

$$+ \frac{1}{N}\mathsf{E}\left[\mathbb{1}_{\left|Z_l^{(1)}-x_0\right|\le R}\mathsf{E}\left[\widetilde{U}_{l+1}(Z_{l+1}^{0,x_0})\frac{p(Z_{l+1}^{0,x_0}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{0,x_0}|x_0)}\right]\right]$$

$$\le \frac{G_R}{N}\mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{(1)}-x_0\right|\le R}\frac{p(Z_{l+1}^{(1)}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)}\right]$$

$$+ \frac{G_R}{N}\mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{0,x_0}-x_0\right|\le R}\frac{p(Z_{l+1}^{0,x_0}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{0,x_0}|x_0)}\right] =: \frac{G_R}{N}H_{R,l},$$

where the latter inequality follows from (12) and the fact that $\widetilde{U}_{l+1}$ vanishes outside the ball B_R. Combining the above estimates, we get for Term$_2$

$$\mathsf{E}[\text{Term}_2] \le \frac{F_{R,l}G_R}{\sqrt{N}} + \frac{G_R}{N}H_{R,l}.$$

Thus we have expressed our bounds for $\mathsf{E}\left[\text{Term}_1\right]$ and $\mathsf{E}\left[\text{Term}_2\right]$ in terms of the quantities $D_{R,l}$, $E_{R,l}$, $F_{R,l}$, $H_{R,l}$, and G_R. Furthermore, it is easy to see that using (20)

$$D_{R,l} \le 1 + \mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{(1)}-x_0\right|\le R}\frac{p(Z_{l+1}^{(1)}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{(1)}|x_0)}\right]$$

$$= 1 + \int p_l(x|x_0)\,dx \int_{|y-x_0|\le R}\frac{p^2(y|x)}{p_{l+1}(y|x_0)}dy$$

$$\le 1 + F_R^2.$$

Similarly, it follows that $E_{R,l}^2 \le 2 + 2F_R^2$, and that $H_{R,l} \le 1 + F_R^2$ due to

$$\mathsf{E}\left[\mathbb{1}_{\left|Z_{l+1}^{0,x_0}-x_0\right|\le R}\frac{p(Z_{l+1}^{0,x_0}|Z_l^{(1)})}{p_{l+1}(Z_{l+1}^{0,x_0}|x_0)}\right] \le 1.$$

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By now taking the expectation in (45) and gathering all together we obtain,

$$\mathsf{E}\left[\left|\overline{U}_k - \widetilde{U}_k\right|_N\right] \leq (L-k)G_R\left(\frac{\sqrt{2+2F_R^2}+F_R}{\sqrt{N}} + \frac{2+2F_R^2}{N}\right). \qquad (48)$$

By next taking $k = 0$ and assuming that N is taken such that $(1+F_R)/\sqrt{N} < 1$, Proposition 5 follows.

6.5   Proof of Proposition 7

In order to achieve a required accuracy $\varepsilon > 0$, let us take R and N large enough such that both error terms in (21) are equal to $\varepsilon/2$. Hence, we first take

$$R_{\varepsilon,d} = (8\alpha L)^{1/2}\log^{1/2}\frac{Lc_g\varkappa\left(1+c_Z+c_Z\left|x_0\right|+c_Z\sqrt{d\alpha L}\right)2^{1+d/4}}{\varepsilon},$$

that is $R \nearrow \infty$ when $d + \varepsilon^{-1} \nearrow \infty$. Then take, with $\asymp$ denoting asymptotic equivalence for $R \nearrow \infty$ up to some natural constant,

$$N_\varepsilon \asymp L^2 c_g^2\varkappa\,(e/\alpha)^{d/2}\,d^{-d/2}R_\varepsilon^{d+2}\varepsilon^{-2} \asymp \alpha c_g^2\varkappa\,(8e/d)^{d/2}\,L^{d/2+3}$$

$$\times\,\varepsilon^{-2}\log^{d/2+1}\frac{L\left(1+c_Z+c_Z\left|x_0\right|+c_Z\sqrt{d\alpha L}\right)2^{1+d/4}c_g\varkappa}{\varepsilon}.$$

Thus, the computational work load (complexity) is given by

$$c_f^{(d)}N_\varepsilon^2 L \leq c_1\alpha^2 c_g^4\varkappa^2 c_f^{(d)}\,(8e/d)^d\,L^{d+7}$$

$$\times\,\varepsilon^{-4}\log^{d+2}\frac{L\left(1+c_Z+c_Z\left|x_0\right|+c_Z\sqrt{d\alpha L}\right)2^{1+d/4}c_g\varkappa}{\varepsilon} \qquad (49)$$

where c_1 is a natural constant. Now let us write

$$d^{-d}\log^{d+2}\frac{L\left(1+c_Z+c_Z\left|x_0\right|+c_Z\sqrt{d\alpha L}\right)2^{1+d/4}c_g\varkappa}{\varepsilon}$$

$$= d^2\log^{d+2}\left[\frac{L^{1/d}\left(1+c_Z+c_Z\left|x_0\right|+c_Z\sqrt{d\alpha L}\right)^{1/d}2^{1/d+1/4}\left(c_g\varkappa\right)^{1/d}}{\varepsilon^{1/d}}\right].$$

Then, using the elementary estimate $\left(a+b\sqrt{d}\right)^{1/d} \leq ae^{b/a}$, for $a,b > 0$, $d \geq 1$, and assuming that $\varepsilon < 1$, (49) implies (22).

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6.6 Proof of Proposition 10

On the one hand one has

$$U_{t_l}^\circ(X_{t_l}) - U_{t_l}(\overline{X}_{t_l}) = \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[g(\tau, X_\tau)\right] - \operatorname*{esssup}_{\overline{\tau} \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[g(\overline{\tau}, \overline{X}_{\overline{\tau}})\right]$$

$$\leq \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[g(\tau, X_\tau) - g(\tau, \overline{X}_\tau)\right]$$

$$\leq \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[\left|g(\tau, X_\tau) - g(\tau, \overline{X}_\tau)\right|\right],$$

and on the other one has similarly

$$U_{t_l}(\overline{X}_{t_l}) - U_{t_l}^\circ(X_{t_l}) = \operatorname*{esssup}_{\overline{\tau} \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[g(\overline{\tau}, \overline{X}_{\overline{\tau}})\right] - \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[g(\tau, X_\tau)\right]$$

$$\leq \operatorname*{esssup}_{\overline{\tau} \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[g(\overline{\tau}, \overline{X}_{\overline{\tau}}) - g(\tau, X_\tau)\right]$$

$$\leq \operatorname*{esssup}_{\tau \in \mathcal{T}_{l,L}} \mathsf{E}_{\mathcal{F}_{t_l}}\left[\left|g(\tau, X_\tau) - g(\tau, \overline{X}_\tau)\right|\right].$$

Hence we get

$$\mathsf{E}\left[\left|U_{t_l}^\circ(X_{t_l}) - U_{t_l}(\overline{X}_{t_l})\right|\right] \leq \mathsf{E}\left[\sup_{0 \leq s \leq T} \left|g(s, X_s) - g(s, \overline{X}_s)\right|\right]$$

$$\leq L_g \mathsf{E}\left[\sup_{0 \leq s \leq T} \left|X_s - \overline{X}_s\right|\right] \leq C^{\text{Euler}}\sqrt{h},$$

due to the strong order of the Euler scheme, with L_g being some Lipschitz constant for g.

Acknowledgements

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