

Regression methods for stochastic control problems

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Abstract

In this paper we develop several regression algorithms for solving general stochastic optimal control problems via Monte Carlo. This type of algorithms is particularly useful for problems with high-dimensional state space and complex dependence structure of the underlying Markov process with respect to some control. The main idea of the algorithms is to simulate a set of trajectories under some reference measure P^* and to use a dynamic program formulation combined with fast methods for approximating conditional expectations and functional optimizations on these trajectories. Theoretical properties of the presented algorithms are investigated and convergence to the optimal solution is proved under mild assumptions. Finally, we present numerical results showing the efficiency of regression algorithms in a case of a high-dimensional Bermudan basket options, in a model with a large investor and transaction costs.

1 Introduction

Since the appearance of the groundbreaking articles of Carriere (1996), Longstaff and Schwartz (2001), and Tsitsiklis and Van Roy (1999), regression methods have become an indispensable tool for solving high dimensional optimal stopping problems in the context of pricing American style derivatives. From a mathematical point of view any optimal stopping problem can be seen as a particular case of a more general stochastic control problem. Optimal stochastic control problems appear in a natural way in many areas of applied stochastics, in particular in mathematical finance. For instance, problems of portfolio optimization under market imperfections, optimal portfolio liquidation, superhedging, etc., do all come down

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to problems in stochastic optimal control. In fact, an active interplay between stochastic control and financial mathematics has been emerged in the last decades: While stochastic control has been a powerful tool for studying problems in finance on the one hand side, financial applications have been stimulating the development of several new methods in optimal stopping and control on the other hand, see for example besides the works mentioned above, Rogers (2002), Andersen and Broadie (2004), Broadie and Glasserman (2004), Haugh and Kogan (2004), Ibáñez (2004), Meinshausen and Hambly (2004), Belomestny and Milstein (2006), Bender and Schoenmakers (2006), Belomestny et al. (2007), Chen and Glasserman (2007), Kolodko and Schoenmakers (2006), Jamshidian (2007), Rogers (2007), and Carmona and Touzi (2008), and many others.

There are several approaches for solving stochastic optimal control problems. The most familiar approach requires consideration of all possible future evolutions of the process at each time that a control choice is to be made. This method is well developed and generally effective, but there are certainly problems (such as the optimal control of a diffusion in high dimensions) where the approach is impractical. In this paper we propose a Monte Carlo approach combined with fast approximation methods and methods of functional optimization. This approach is applicable to any discrete-time controlled Markov processes. The main idea of the method is to simulate a set of trajectories under some reference measure and then apply a dynamic programming formulation (Bellman principle) to compute recursively estimates for the optimal control process and the optimal stopping rule where the use of fast approximation methods allows for computing conditional expectations without nested simulations. We propose a number of regression procedures and prove the convergence of a value function estimate under some additional assumptions.

2 Basic setup

For our framework we adopt the discrete time setup as in Rogers (2007). On a filtered measurable space (Ω, \mathcal{F}) , with $\mathcal{F} := (\mathcal{F}_r)_{r=0,1,\dots,T}$, $T \in \mathbb{N}_+$, we consider an adapted control process $\mathbf{a} : \Omega \times \{0, \dots, T-1\} \rightarrow A$, control for short, where (A, \mathcal{B}) is a measurable space. We assume a given set of admissible controls which is denoted by \mathcal{A} . Given a control $\mathbf{a} = (a_0, a_1, \dots, a_{T-1}) \in \mathcal{A}$, we consider a *controlled Markov* process X valued in some measurable space (S, \mathcal{S}) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^{\mathbf{a}})$ with $X_0 = x_0$ a.s. and transition kernel of the following type,

$$\mathbb{P}^{\mathbf{a}}(X_{r+1} \in dy \mid X_r = x) = P^{a_r}(x, dy), \quad 0 \leq r < T.$$

So in particular it is assumed that the distribution of X_{r+1} conditional on \mathcal{F}_r is governed by a (one-step) transition kernel $P^{a_r}(X_r, dy)$ which is controlled

by a_r . In this setting we may consider the general optimal control problem

$$(2.1) \quad Y_0^* := \sup_{\mathbf{a} \in \mathcal{A}} E^{\mathbf{a}} \left[\sum_{r=0}^{T-1} f_r(X_r, a_r) \right],$$

for given functions f_r . The optimization problem (2.1) contains the standard optimal stopping problem

$$Y_0^* := \sup_{\tau} E [g_{\tau}(X_{\tau})],$$

as special case: Take $P^{\mathbf{a}}$ independent of \mathbf{a} , $f_r(x, a) = g_r(x)a$, and $\mathcal{A} = \mathcal{A}^{\text{stop}} = \{\mathbf{a} = (1_{\tau=0}, \dots, 1_{\tau=T}) : \tau, 0 \leq \tau \leq T, \text{ is a stopping time}\}$. Multiple stopping problems may be considered in a similar way by choosing a suitable \mathcal{A} . In this article however we choose \mathcal{A} to be the set of *all* adapted controls (as in Rogers 2008), while keepin the standard optimal stopping problem as a special case. This leads to our central goal solving the optimal control problem

$$(2.2) \quad Y_0^* = \sup_{\mathbf{a} \in \mathcal{A}, \tau \in \mathfrak{T}} E^{\mathbf{a}} \left[\sum_{r=0}^{\tau-1} f_r(X_r, a_r) + g_{\tau}(X_{\tau}) \right]$$

for a given set of measurable functions $f_r : S \times A \rightarrow \mathbb{R}$, $g_r : S \rightarrow \mathbb{R}$. For technical reasons f_r and g are assumed to be bounded from below. To exclude trivialities we further assume that

$$\sup_{\mathbf{a} \in \mathcal{A}} E^{\mathbf{a}} \left[\sum_{r=0}^{T-1} f_r(X_r, a_r) \right] < \infty, \quad \sup_{\mathbf{a} \in \mathcal{A}} E^{\mathbf{a}} [g_i(X_i)] < \infty, \quad i = 0, \dots, T.$$

The supremum in (2.2) is taken over $\mathbf{a} \in \mathcal{A}$ and all \mathcal{F} -stopping times with values in a subset $\mathfrak{T} \subset \{0, \dots, T\}$.

The optimal control problem (2.2) with $\mathfrak{T} = \{0, \dots, T\}$ will be the main object of our study. To this end we consider the process

$$(2.3) \quad Y_r^* = \sup_{\mathbf{a} \in \mathcal{A}_r, \tau \in \mathfrak{T}_r} E^{\mathbf{a}} \left[\sum_{s=r}^{\tau-1} f_s(X_s, a_s) + g_{\tau}(X_{\tau}) \middle| \mathcal{F}_r \right], \quad 0 \leq r \leq T$$

with $\mathfrak{T}_r := \{r, \dots, T\}$ and \mathcal{A}_r being the set of all adapted controls $\mathbf{a} : \Omega \times \{r, \dots, T-1\} \rightarrow A$. As a general result, there exists a vector $h^* = (h_0^*, \dots, h_T^*)$ of measurable functions on S , such that $Y_j^* = h_j^*(X_j)$ and h^* satisfies

$$(2.4) \quad \begin{aligned} h_r^*(x) &= \max [g_r(x), (\mathcal{L}h^*)_r(x)], \quad 0 \leq r < T, \\ h_T^*(x) &= g_T(x), \end{aligned}$$

where $\mathcal{L} : h \rightarrow \mathcal{L}h$ is a Bellman-type operator defined by

$$(\mathcal{L}h)_r(x) := \sup_{a \in A} [f_r(x, a) + P^a h_{r+1}(x)]$$

and

$$P^a h_{r+1}(x) := \int P^a(x, dy) h_{r+1}(y).$$

We now assume that there exists a reference measure P^* equivalent to $P^{\mathbf{a}}$, such that

$$P^a(x, dy) = \varphi(x, y, a) P^*(x, dy), \quad a \in A,$$

with $P^*(x, dy) := P^*(X_{r+1} \in dy \mid X_r = x)$ and function $\varphi(x, y, a)$ satisfying $\varphi > 0$ and $\int P^*(x, dy) \varphi(x, y, a) \equiv 1$. Then for any nonnegative measurable function $F : S^{T+1} \rightarrow \mathbb{R}_+$ it holds

$$(2.5) \quad \mathbb{E}^{\mathbf{a}}[F(X) \mid \mathcal{F}_j] = \mathbb{E}^*[F(X) \Lambda_{j,T}(\mathbf{a}, X) \mid \mathcal{F}_j],$$

where

$$\Lambda_{j,r}(\mathbf{a}, y) := \prod_{l=j}^{r-1} \varphi(y_l, y_{l+1}, a_l), \quad r = j+1, \dots, T, \quad y \in S^{T+1}.$$

If, moreover, F depends on X_0, \dots, X_r only, we have for $0 \leq j \leq r$,

$$\mathbb{E}^{\mathbf{a}}[F(X) \mid \mathcal{F}_j] = \mathbb{E}^*[F(X) \Lambda_{j,r}(\mathbf{a}, X) \mid \mathcal{F}_j],$$

and if F depends only on X_{j+1} it holds

$$(2.6) \quad \mathbb{E}^{\mathbf{a}}[F(X_{j+1}) \mid \mathcal{F}_j] = \mathbb{E}^*[F(X_{j+1}) \varphi(X_j, X_{j+1}, a_j) \mid \mathcal{F}_j].$$

3 Regression methods for control problems

The solution Y_0^* of the optimal control problem (2.2) can in principle be computed backwardly via the a dynamic programming principle (2.4). However, in particular if the space S is high-dimensional, an analytic computation of the conditional expectation

$$C_r(x, a) := \mathbb{E}^{\mathbf{a}}[h_r(X_{r+1}) \mid X_r = x] = \mathbb{E}^*[\varphi(X_r, X_{r+1}, a) h_{r+1}(X_{r+1}) \mid X_r = x],$$

where henceforth for notational convenience $h := h^*$, is usually difficult, even if h_{r+1} is explicitly known. On the other hand, a straightforward backward (approximative) construction of (2.4) by Monte Carlo simulation (under P^*) would lead to nested simulations where the degree of nesting explodes with the number of exercise dates. In the context of optimal stopping problems, much research was focused on the development of fast methods for computing approximations of C_r to resolve this issue. We will show that these methods can be extended to a more general optimal control problems.

From now on we assume that $S \subset \mathbb{R}^d$ for some $d > 0$. Suppose that h_{r+1} is estimated by \widehat{h}_{r+1} and that we want to approximate h_r via (2.4) and (2.5), hence

$$\begin{aligned}\widehat{h}_r(x) &:= \max \left[g_r(x), \sup_{a \in A} \left[f_r(x, a) + P^a \widehat{h}_{r+1}(x) \right] \right] \\ &= \max \left[g_r(x), \sup_{a \in A} \left\{ f_r(x, a) + \mathbb{E}^* \left[\varphi(X_r, X_{r+1}, a) \widehat{h}_{r+1}(X_{r+1}) \mid X_r = x \right] \right\} \right].\end{aligned}$$

Let

$$\left(\left(X_r^{(1)}, X_{r+1}^{(1)} \right), \dots, \left(X_r^{(M)}, X_{r+1}^{(M)} \right) \right)$$

be a Monte Carlo sample from the joint distribution of (X_r, X_{r+1}) under \mathbb{P}^* and suppose that, based on this Monte Carlo sample and an approximation \widehat{h}_{r+1} of h_{r+1} , an estimate $\widehat{C}_{r,M}(x, a)$ of the conditional expectation $C_r(x, a)$ is constructed for all $x \in S$ and $a \in A$. In this paper we consider a class of estimation methods where $\widehat{C}_{r,M}$ is of the form

$$(3.7) \quad \widehat{C}_{r,M}(x, a) = \sum_{m=1}^M w_{m,M}(x, X_r^{(\cdot)}) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m)})$$

where

$$w_{m,M}(x, X_r^{(\cdot)}) = w_{m,M}(x, X_r^{(1)}, \dots, X_r^{(M)})$$

are some coefficients which are to be specified by the method under consideration. It turns out that this class of approximation methods is very general and contains local and global regression methods. We discuss these two types of method in the next sections.

3.1 Algorithms based on local estimators

By introducing

$$d_r(x, a) := \int_S \varphi(x, y, a) h_{r+1}(y) p_r(x, y) dy, \quad p_r(x) := \int_S p_r(x, y) dy$$

with $p_r(x, y)$ being the joint density of (X_r, X_{r+1}) under \mathbb{P}^* , we may write

$$C_r(x, a) = d_r(a, x) / p_r(x).$$

So it is natural to estimate C_r as a ratio of estimates for p_r and d_r , respectively. With this goal in mind we consider for a Borel measurable kernel function $\Phi_M(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ the following estimators,

$$\begin{aligned}p_{r,M}(x) &:= M^{-1} \sum_{m=1}^M \Phi_M(x, X_r^{(m)}), \\ \widehat{d}_{r,M}(x, a) &:= M^{-1} \sum_{m=1}^M \Phi_M(x, X_r^{(m)}) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m)}),\end{aligned}$$

where $x \in \mathbb{R}^d$ and $a \in A$, and then consider for C_r the estimator

$$(3.8) \quad \widehat{C}_{r,M}(x, a) := \frac{\widehat{d}_{r,M}(x, a)}{p_{r,M}(x)} \\ =: \sum_{m=1}^M w_{m,M}(x, X_r^{(\cdot)}) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m)})$$

with weight coefficients defined by

$$w_{m,M}(x, y^{(\cdot)}) := w_{m,M}(x, y_1, y_2, \dots) := \frac{\Phi_M(x, y_m)}{\sum_{m'=1}^M \Phi_M(x, y_{m'})}.$$

If in (3.8) $p_{r,M} = 0$ we set $\widehat{C}_{r,M} := 0$. It is important to note that here $w_{m,M}$ are nonnegative weights summing up to one. The name ‘‘local’’ comes from the fact that in most cases the function $\Phi_M(x, y)$ converges (in some sense) to a delta function as $M \rightarrow \infty$. The class of local estimators is rather large and contains well known examples such as the Nadaraya-Watson and the k-nearest neighbors regression estimators.

Example 1. Let K be a measurable function on \mathbb{R}^d . Take

$$\Phi_M(x, y) = \delta_M^{-d} K((x - y)/\delta_M),$$

where $\{\delta_M\}$ is a sequence of positive numbers tending to zero. Then (3.8) yields the well-known Nadaraya-Watson regression estimator

$$(3.9) \quad \widehat{C}_{r,M}(x, a) = \frac{\sum_{m=1}^M K((x - X_r^{(m)})/\delta_M) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m)})}{\sum_{m=1}^M K((x - X_r^{(m)})/\delta_M)}.$$

Example 2. We can modify the estimator in Example 1 by specifying an increasing sequence (k_M) of natural numbers with $k_M \leq M$, and reducing the number of summands in (3.9) to k_M in the following way. Consider the first k_M nearest neighbors of x , say $X_r^{(m_1)}, \dots, X_r^{(m_{k_M})}$ in the Monte Carlo sample $X_r^{(1)}, \dots, X_r^{(M)}$, and define $R_M := \left\| x - X_r^{(m_{k_M})} \right\|_2$ to obtain the k_M -nearest neighbors regression estimator

$$(3.10) \quad \widehat{C}_{r,M}(x, a) = \frac{\sum_{n=1}^{k_M} \varphi(x, X_{r+1}^{(m_n)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m_n)}) K((x - X_r^{(m_n)})/R_M)}{\sum_{n=1}^{k_M} K((x - X_r^{(m_n)})/R_M)}.$$

Finally, after estimating $C_r(x, a)$ we construct

$$(3.11) \quad \widehat{a}_{r,M}(x) := \arg \sup_{a \in A} [f_r(x, a) + \widehat{C}_{r,M}(x, a)], \quad x \in S,$$

and estimate h_r by

$$(3.12) \quad \widehat{h}_{r,M}(x) := \max\{g_r(x), f_r(x, a_{r,M}(x)) + \widehat{C}_{r,M}(x, \widehat{a}_{r,M}(x))\}.$$

Starting with $\widehat{h}_{T,M}(x) = g_T(x)$ and working backwardly, we so obtain estimates for all h_r , $r = 0, \dots, T - 1$.

Remark 3. Local estimators have in some respects nice theoretical properties, for example, almost sure convergence to C_r under rather weak smoothness assumptions. Basically only local smoothness is required for this. A disadvantage of local estimators is their numerical complexity in general. For instance, if we want to compute $\widehat{C}_{r,M}(x, a)$ at M points in \mathbb{R}^d using the Nadaraya-Watson estimator (3.9), it will require M^2 operations. In the case of the k_M -nearest neighbors estimator, this number can be reduced to $M \log M$ using fast search algorithms.

3.2 Global regression estimators

As an alternative to local regression methods we now consider algorithms based on global regression. From a practical point of view global regression estimators are easier to implement in an efficient way than local estimators. Convergence proofs for global estimators are more delicate and usually impose rather strong assumptions on C_r and the underlying Markov process X_r . For the standard Bermudan stopping problem ($f_r \equiv 0$, $\varphi \equiv 1$) we refer to Clément, Lamberton and Protter (2002), Egloff (2005) and Egloff, Kohler and Todorovic (2007). The global regression procedures in the next two sections are in some sense a generalization of the methods of Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001), respectively, to optimal control problems.

3.2.1 Algorithms based on continuation functions

For a given Monte Carlo sample $\{X_r^{(m)}, 0 \leq r \leq T, 1 \leq m \leq M\}$ under the measure P^* and a system of basis functions $\psi := [\psi_1, \dots, \psi_K]^\top$ we consider for each $a \in A$ the regression problem

$$(3.13) \quad \widehat{\beta}_r(a) := \arg \min_{\beta \in \mathbb{R}^K} \sum_{m=1}^M \left(\psi^\top(X_r^{(m)})\beta - Y^{(m)}(a) \right)^2,$$

where

$$Y^{(m)}(a) := \varphi(X_r^{(m)}, X_{r+1}^{(m)}, a) \widehat{h}_{r+1}(X_{r+1}^{(m)})$$

and an estimate \widehat{h}_{r+1} of h_{r+1} is assumed to be already constructed. The solution of (3.13) is explicitly given by

$$(3.14) \quad \widehat{\beta}_r(a) = (F^\top F)^{-1} F^\top Y(a) =: F^\dagger Y(a),$$

where $F = (F_{mk}) = (\psi_k(X_r^{(m)}))$ is a $M \times K$ design matrix and $Y(a) := (Y^{(m)}(a))_{m=1, \dots, M}$. Note that the design matrix F does not depend on a .

We next consider

$$(3.15) \quad \widehat{a}_{r,M}(x) = \arg \max_{a \in A} \{f_r(x, a) + \widehat{C}_{r,M}(x, a)\},$$

where

$$\begin{aligned} \widehat{C}_{r,M}(x, a) &= \psi^\top(x) \widehat{\beta}_r(a) = \psi^\top(x) F^\dagger Y(a) \\ &= \sum_{m=1}^M w_{m,M}(x, X_r^{(\cdot)}) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1,M}(X_{r+1}^{(m)}) \end{aligned}$$

with coefficients $w_{m,M}$ given by

$$(3.16) \quad w_{m,M}(x, X_r^{(\cdot)}) = \psi^\top(x) \left((F^\top F)(X_r^{(\cdot)}) \right)^{-1} \psi(X_r^{(m)}).$$

In order to solve (3.15) one may, for instance, construct an approximation procedure for finding the a roots of the stationary point equation

$$\frac{\partial}{\partial a} \widehat{C}_{r,M}(x, a) = \frac{\partial}{\partial a} f_r(x, a) + \sum_{k=1}^K \psi_k(x) F^\dagger \frac{\partial}{\partial a} Y(a) = 0.$$

We proceed with a second regression problem, based on a new sample $\{\widetilde{X}_r^{(m)}, 0 \leq r \leq T, 1 \leq m \leq M\}$ under P^* :

$$(3.17) \quad \widetilde{\beta}_r = \arg \min_{\beta \in \mathbb{R}^K} \sum_{m=1}^M \left(\varphi(\widetilde{X}_r^{(m)}, \widetilde{X}_{r+1}^{(m)}, \widehat{a}_{r,M}(\widetilde{X}_r^{(m)})) \widehat{h}_{r+1}(\widetilde{X}_{r+1}^{(m)}) - \psi^\top(\widetilde{X}_r^{(m)}) \beta \right)^2$$

to end up with

$$(3.18) \quad \widehat{h}_{r,M}(x) = \max \left[g(x), f_r(x, \widehat{a}_{r,M}(x)) + \psi^\top(x) \widetilde{\beta}_r \right].$$

The second regression is needed to avoid matrix multiplication in (3.14) for each $\widetilde{X}_r^{(m)}$.

3.2.2 Algorithms based on backward construction of stopping time and control

In this section we present an algorithm where, instead of regressing continuation functions, the control and stopping times are backwardly constructed on a sample of simulated trajectories. This method relies on the following consistency theorem proved in Appendix.

Theorem 4. *The optimal stopping time $\tau^*(r)$ and the optimal control $\mathbf{a}^*(r)$ solving the problem*

$$Y_r^* = \sup_{\mathbf{a} \in \mathcal{A}_r, \tau \in \mathfrak{X}_r} \mathbb{E}^{\mathbf{a}} \left[\sum_{s=r}^{\tau-1} f_s(X_s, a_s) + g_\tau(X_\tau) \middle| \mathcal{F}_r \right],$$

satisfy the following consistency relations

$$\tau^*(r) > r \Rightarrow \tau^*(r) = \tau^*(r+1) \text{ and } a_j^*(r) = a_j^*(r+1)$$

for all j such that $r+1 \leq j < \tau^*(r+1)$.

Note that $a_j^*(r)$ is only defined for $r \leq j < \tau^*(r)$, i.e. the control $\mathbf{a}^*(r)$ is not defined if $\tau^*(r) = r$. Given a sample $(X_0^{(m)}, \dots, X_T^{(m)})$, $m = 1, \dots, M$, we construct estimates $\tau^{(m)}(r)$ and $a_j^{(m)}(r)$, $r \leq j < \tau^{(m)}(r)$ for stopping times and control processes respectively in the following way. At the terminal time we set

$$\tau^{(m)}(T) = T, \quad m = 1, \dots, M.$$

Let $\tau^{(m)}(r+1)$, $a_j^{(m)}(r+1)$, $r+1 \leq j < \tau^{(m)}(r+1)$ be constructed for $m = 1, \dots, M$, at time $r+1$, $0 \leq r < T$. Let $\psi := [\psi_1, \dots, \psi_K]^\top$ be a system of basis functions. For any $a \in A$ consider the regression problem

$$(3.19) \quad \widehat{\beta}(a) := \arg \min_{\beta \in \mathbb{R}^K} \sum_{m=1}^M \left(\psi^\top(X_r^{(m)})\beta - Y^{(m)}(a) \right)^2,$$

where

$$Y^{(m)}(a) = \varphi(X_r^{(m)}, X_{r+1}^{(m)}, a)Z_{r+1}^{(m)}$$

with

$$\begin{aligned} Z_{r+1}^{(m)} := & \sum_{l=r+1}^{\tau^{(m)}(r+1)-1} \Lambda_{r+1,l}(\mathbf{a}^{(m)}(r+1), X^{(m)}) f_l(X_l^{(m)}, a_l^{(m)}(r+1)) \\ & + \Lambda_{r+1, \tau^{(m)}(r+1)}(\mathbf{a}^{(m)}(r+1), X^{(m)}) g(X_{\tau^{(m)}(r+1)}^{(m)}). \end{aligned}$$

The solution of (3.19) is given by (3.14) and we can define an estimate $\widehat{C}_{r,M}(x, a) = \psi^\top(x)\widehat{\beta}(a)$ and then $\widehat{a}_{r,M}(x)$ as a solution of (3.15). Now we simulate a new set of trajectories

$$(\widetilde{X}_0^{(m)}, \dots, \widetilde{X}_T^{(m)}), \quad m = 1, \dots, M$$

under \mathbf{P}^* and define

$$\widetilde{\beta}_r := \arg \min_{\beta \in \mathbb{R}^K} \sum_{m=1}^M \left(\psi^\top(\widetilde{X}_r^{(m)})\beta - \varphi(\widetilde{X}_r^{(m)}, \widetilde{X}_{r+1}^{(m)}, a_{r,M}(\widetilde{X}_r^{(m)}))Z_{r+1}^{(m)} \right)^2.$$

Put $\widetilde{C}_{r,M}(x) = \psi^\top(x)\widetilde{\beta}_r$. By setting for $m = 1, \dots, M$,

$$\tau^{(m)}(r) = r, \quad \text{if } f_r(X_r^{(m)}, \widehat{a}_{r,M}(X_r^{(m)})) + \widetilde{C}_{r,M}(X_r^{(m)}) < g(X_r^{(m)}),$$

otherwise

$$\begin{aligned}\tau^{(m)}(r) &= \tau^{(m)}(r+1), & a^{(m)}(r) &= \widehat{a}_{r,M}(X_r^{(m)}), \\ a_j^{(m)}(r) &= a_j^{(m)}(r+1) & \text{for } r+1 \leq j < \tau^{(m)}(r+1),\end{aligned}$$

we so end up with a sequence of estimates

$$(3.20) \quad \widetilde{C}_{r,M}(x) := \sum_{k=1}^K \widetilde{\beta}_{r,k} \psi_k(x), \quad r = 0, \dots, T-1,$$

and a sequence of functions $\widehat{a}_{r,M}$, $r = 0, \dots, T-1$. Based on (3.20) one may take the (generally suboptimal) stopping rule

$$\tau_M := \inf\{0 \leq r \leq T : g(X_r) \geq f_r(X_r, \widehat{a}_{r,M}(X_r)) + \widetilde{C}_{r,M}(X_r)\}$$

and the (generally suboptimal) control process

$$a_M(X) = (\widehat{a}_{0,M}(X_0), \widehat{a}_{1,M}(X_1), \dots, \widehat{a}_{T-1,M}(X_{T-1}))$$

to construct a lower approximation for Y_0^* via a next Monte Carlo simulation.

4 Convergence analysis of regression methods

The issues of convergence for regression algorithms in the context of pricing Bermudan options have been already studied in several papers. Clément, Lamberton and Protter (2002) were first who proved the convergence of the Longstaff-Schwartz algorithm. Glasserman and Yu (2005) have shown that the number of Monte Carlo paths has to be exponential in the number of basis functions used for regression in order to ensure convergence. Recently, Egloff, Kohler and Todorovic (2007) have derived rates of convergence for continuation values estimates by the so called dynamic look-ahead algorithm (see also Egloff (2004)) that “interpolates” between Longstaff-Schwartz and Tsitsiklis-Roy algorithms.

In the case of general control problems the issue of convergence is much more delicate because along with the convergence of regression estimates $C_{r,M}$ we also need the convergence of control estimates $a_{r,M}$. The latter convergence can be ensured only if the first one is uniform on the set of all possible controls. This type of convergence can be proved only under some additional assumptions.

A convergence analysis can be divided into two parts. The first part is concerned with the convergence of a one step estimate

$$h_{r,M}(x) := \max \left[g_r(x), \sup_{a \in A} [f_r(x, a) + C_{r,M}(x, a)] \right],$$

based on the “pseudo” estimator

$$(4.21) \quad C_{r,M}(x, a) := \sum_{m=1}^M w_{m,M}(x, X_r^{(\cdot)}) \varphi(x, X_{r+1}^{(m)}, a) h_{r+1}(X_{r+1}^{(m)}),$$

i.e. (3.7) with \widehat{h}_{r+1} replaced by the exact solution h_{r+1} . In practice, however, one starts from $r = T$ and proceeds backwardly where at each step the previously constructed estimate \widehat{h}_{r+1} is used instead of h_{r+1} . Thus the second part of the convergence analysis consists of proving a “global” convergence of $\widehat{h}_{r,M}$ to h_r in a suitable sense, taking into account all errors from the previous steps. It turns out that the first type of convergence relies exclusively on the sort of regression estimate under consideration and can be established via standard results from the theory of empirical processes and regression analysis. In this paper we will carry out the second part of the convergence analysis assuming that $C_{r,M}$ is known to converge to C_r in a certain sense. In fact, the prove of the “global” convergence is more generic and involves only general properties of the weights in (3.7).

Theorem 5. *Suppose that starting with $\widehat{h}_{T,M} = h_T^*(x) = g_T(x)$, for each backward step $\widehat{h}_{r,M}$ is constructed from $\widehat{h}_{r+1,M}$ via (3.12) or (3.18) using a new independent sample of M trajectories. Suppose further that the function φ is bounded, that is $|\varphi| \leq A_\varphi$ for some constant A_φ . If*

$$\begin{aligned} & \mathbb{E} \left\{ \int_{\mathbb{R}^d} \|C_{r,M}(x, \cdot) - C_r(x, \cdot)\|_A^q p_r(x) dx \right\}^{1/q} \\ &= \mathbb{E} \left\{ \int_{\mathbb{R}^d} \left[\sup_{a \in A} |C_{r,M}(x, a) - C_r(x, a)| \right]^q p_r(x) dx \right\}^{1/q} \\ &= O(\varepsilon_M), \quad r = 0, \dots, T-1, \quad M \rightarrow \infty \end{aligned}$$

with some $q > 1$ and some sequence ε_M tending to 0, then it holds

$$\mathbb{E} \left\| \widehat{h}_{r,M} - h_r \right\|_{L_q(p_r)} = O \left(\lambda_{q,M}^{T-r} \varepsilon_M \right), \quad 0 \leq r \leq T$$

with

$$\lambda_{q,M} = \sup_{0 \leq r \leq T} \sum_{m=1}^M \|w_{m,M}(\cdot, \cdot)\|_{L_q(p_r \otimes_{i=1}^M p_r)}.$$

Corollary 6. *If all coefficients $w_{m,M}$ in (3.7) are nonnegative and sum up to 1 (e.g. in the case (3.8)), then $\lambda_{q,M} < M^{1-1/q}$ and*

$$\mathbb{E} \left\| \widehat{h}_{r,M} - h_r \right\|_{L_q(p_r)} = O \left(M^{(1-1/q)(T-r)} \varepsilon_M \right), \quad 0 \leq r \leq T.$$

In particular if $q = 1$ we have

$$\mathbb{E} \left\| \widehat{h}_{r,M} - h_r \right\|_{L_q(p_r)} = O(\varepsilon_M), \quad 0 \leq r \leq T.$$

Thus, in the case of nonnegative weights and $q = 1$ the “global” rates coincide with the rates of a particular regression estimator.

Proof. For $r = T$ the statement is trivial. As induction hypothesis we assume that

$$(4.22) \quad \mathbb{E} \left\| \widehat{h}_{r+1,M} - h_{r+1} \right\|_{L_q(p_{r+1})} = O \left(\lambda_{q,M}^{T-r-1} \varepsilon_M \right), \quad M \rightarrow \infty.$$

Based on a new sample $(X_r^{(m)}, X_{r+1}^{(m)})$, $m = 1, \dots, M$, independent of the samples needed for constructing the estimate $\widehat{h}_{r+1,M}$, we define

$$\begin{aligned} a_{r,M}(x) &:= \arg \sup_{a \in A} [f_r(x, a) + C_{r,M}(x, a)], \\ \widehat{a}_{r,M}(x) &:= \arg \sup_{a \in A} [f_r(x, a) + \widehat{C}_{r,M}(x, a)], \end{aligned}$$

where

$$\widehat{C}_{r,M}(x, a) := \sum_{m=1}^M w_{m,M}(x, X_r^{(\cdot)}) \varphi(x, X_{r+1}^{(m)}, a) \widehat{h}_{r+1,M}(X_{r+1}^{(m)}).$$

Observe that due to

$$\begin{aligned} & - \sup_{a \in A} \left| \widehat{C}_{r,M}(x, a) - C_{r,M}(x, a) \right| \\ & \leq f_r(x, \widehat{a}_{r,M}(x)) + \widehat{C}_{r,M}(x, \widehat{a}_{r,M}(x)) - \{f_r(x, a_{r,M}(x)) + C_{r,M}(x, a_{r,M}(x))\} \\ & \leq \sup_{a \in A} \left| \widehat{C}_{r,M}(x, a) - C_{r,M}(x, a) \right| \end{aligned}$$

the inequality

$$\left| \widehat{h}_{r,M}(x) - h_{r,M}(x) \right| \leq \sup_{a \in A} \left| \widehat{C}_{r,M}(x, a) - C_{r,M}(x, a) \right|$$

holds for all x and a , where

$$h_{r,M}(x) := \max\{g_r(x), f_r(x, a_{r,M}(x)) + C_{r,M}(x, a_{r,M}(x))\}.$$

Analogously one can show that

$$(4.23) \quad |h_r(x) - h_{r,M}(x)| \leq \sup_{a \in A} |C_r(x, a) - C_{r,M}(x, a)|.$$

On the other hand we have

$$(4.24) \quad \begin{aligned} & \widehat{C}_{r,M}(x, a) - C_{r,M}(x, a) = \\ & \sum_{m=1}^M w_{m,M}(x, X_r^{(\cdot)}) \varphi(x, X_{r+1}^{(m)}, a) (\widehat{h}_{r+1,M}(X_{r+1}^{(m)}) - h_{r+1}(X_{r+1}^{(m)})), \end{aligned}$$

hence

$$\begin{aligned} & \left| \widehat{h}_{r,M}(x) - h_{r,M}(x) \right| \\ & \leq A_\varphi \sum_{m=1}^M |w_{m,M}(x, X_r^{(\cdot)})| \left| \widehat{h}_{r+1,M}(X_{r+1}^{(m)}) - h_{r+1}(X_{r+1}^{(m)}) \right|, \quad x \in \mathbb{R}^d. \end{aligned}$$

Denote with \mathcal{G}_{r+1} the σ -algebra generated by the samples used from T down to $r+1$. The application of Hölder's and Jensen inequality leads to

$$\begin{aligned} & \mathbb{E} \left\| \widehat{h}_{r,M} - h_{r,M} \right\|_{L_q(p_r)} \\ & \leq A_\varphi \mathbb{E} \sum_{m=1}^M \mathbb{E}^{\mathcal{G}_{r+1}} \left[\left| \widehat{h}_{r+1,M}(X_{r+1}^{(m)}) - h_{r+1}(X_{r+1}^{(m)}) \right| \left\| w_{m,M}(\cdot, X_r^{(\cdot)}) \right\|_{L_q(p_r)} \right] \\ & \leq A_\varphi \mathbb{E} \left\{ \left[\mathbb{E}^{\mathcal{G}_{r+1}} \left| \widehat{h}_{r+1,M}(X_{r+1}^{(1)}) - h_{r+1}(X_{r+1}^{(1)}) \right|^q \right]^{1/q} \right. \\ & \quad \left. \times \sum_{m=1}^M \left[\mathbb{E}^{\mathcal{G}_{r+1}} \left\| w_{m,M}(\cdot, X_r^{(\cdot)}) \right\|_{L_q(p_r)}^{\frac{q}{q-1}} \right]^{1-1/q} \right\} \\ & \leq A_\varphi \mathbb{E} \left\| \widehat{h}_{r+1,M} - h_{r+1} \right\|_{L_q(p_{r+1})} \sum_{m=1}^M \left(\int p_r(x) \mathbb{E} \left| w_{m,M}(x, X_r^{(\cdot)}) \right|^q dx \right)^{\frac{1}{q}} \\ & = A_\varphi \mathbb{E} \left\| \widehat{h}_{r+1,M} - h_{r+1} \right\|_{L_q(p_{r+1})} \sum_{m=1}^M \left\| w_{m,M}(\cdot, \cdot) \right\|_{L_q(p_r \otimes_{i=1}^M p_r)}. \end{aligned}$$

The induction assumption (4.22) implies now that

$$\mathbb{E} \left\| \widehat{h}_{r,M} - h_{r,M} \right\|_{L_q(p_r)} = O(\varepsilon_M \lambda_{q,M}^{T-r}).$$

Note that by letting $q \downarrow 1$, the last estimate holds true for $q = 1$ as well. Further we have

$$\mathbb{E} \left\| \widehat{h}_{r,M} - h_r \right\|_{L_q(p_r)} \leq \mathbb{E} \left\| \widehat{h}_{r,M} - h_{r,M} \right\|_{L_q(p_r)} + \mathbb{E} \|h_{r,M} - h_r\|_{L_q(p_r)}.$$

Hence due to (4.23)

$$\begin{aligned} & \mathbb{E} \|h_{r,M} - h_r\|_{L_q(p_r)} \\ & \leq \left\{ \int_{\mathbb{R}^d} \|C_r(x, \cdot) - C_{r,M}(x, \cdot)\|_A^q p_r(x) dx \right\}^{1/q} \\ & = O(\varepsilon_M), \quad M \rightarrow \infty. \end{aligned}$$

□

5 Upper bounds

For computing upper bounds for solutions of control problems we extend the approach in Rogers (2007) to problem (2.2). In fact, the following theorem is a straightforward generalization of Theorem 1 in Rogers (2007).

Theorem 7. *Let Y_r^* be the solution of the optimal control problem (2.3), then the following representation holds*

$$Y_r^* = \inf_{h \in \mathcal{H}} \left\{ h_r(X_r) + \mathbf{E}^* \left[\sum_{j=r}^{T-1} W_{r,j} \left((\mathcal{L}h)_j(X_j) - h_j(X_j) \right)^+ + \max_{r \leq i \leq T} W_{r,i} (g_i(X_i) - h_i(X_i))^+ \middle| \mathcal{F}_r \right] \right\},$$

where $W_{r,j} = \sup_{\mathbf{a} \in \mathcal{A}} [\Lambda_{r,j}(\mathbf{a}, X)]$ and \mathcal{H} is the space of bounded measurable vector functions $h = (h_0, \dots, h_T)$ on S^{T+1} .

Proof. For any $h = (h_0, \dots, h_T) \in \mathcal{H}$ and $\mathbf{a} \in \mathcal{A}$ let consider a martingale M_r from the Doob decomposition of $h_r(X_r)$:

$$M_{r+1}^{\mathbf{a}} - M_r^{\mathbf{a}} = h_{r+1}(X_{r+1}) - \mathbf{E}^{\mathbf{a}} [h_{r+1}(X_r) | \mathcal{F}_r],$$

with $M_0^{\mathbf{a}} = 0$, i.e.,

$$M_r^{\mathbf{a}} = \sum_{j=0}^{r-1} (M_{j+1}^{\mathbf{a}} - M_j^{\mathbf{a}}) = \sum_{j=0}^{r-1} (h_{j+1}(X_j) - P^{a_j} h_{j+1}(X_j)).$$

We then have

$$\begin{aligned} Y_r^* &= \inf_h \sup_{\substack{\mathbf{a} \in \mathcal{A}_r \\ \tau, \tau \geq r}} \mathbf{E}^{\mathbf{a}} \left[\sum_{j=r}^{\tau-1} f_j(X_j, a_j) + g_{\tau}(X_{\tau}) - \sum_{j=r}^{\tau-1} (h_{j+1}(X_j) - P^{a_j} h_{j+1}(X_j)) \middle| \mathcal{F}_r \right] \\ &\leq \inf_h \left\{ h_r(X_r) + \sup_{\mathbf{a} \in \mathcal{A}_r} \mathbf{E}^* \left[\sum_{j=r}^{i-1} \Lambda_{r,j}(\mathbf{a}, X) (f_j(X_j, a_j) + P^{a_j} h_{j+1}(X_j) - h_j(X_j)) \right. \right. \\ &\quad \left. \left. + \Lambda_{r,i}(\mathbf{a}, X) (g_i(X_i) - h_i(X_i)) \middle| \mathcal{F}_r \right] \right\} \\ &\leq \inf_h \left\{ h_r(X_r) + \mathbf{E}^* \left[\sum_{j=r}^{T-1} \sup_{\mathbf{a} \in \mathcal{A}_r} \Lambda_{r,j}(\mathbf{a}, X) \left((\mathcal{L}h)_j(X_j) - h_j(X_j) \right)^+ \right. \right. \\ &\quad \left. \left. + \max_{i \geq r} \sup_{\mathbf{a} \in \mathcal{A}_r} \Lambda_{r,i}(\mathbf{a}, X) (g_i(X_i) - h_i(X_i))^+ \middle| \mathcal{F}_r \right] \right\}. \end{aligned}$$

For $h = h^*$ it holds $\max [g_i, (\mathcal{L}h^*)_i] = h_i^*$, and $h_T^*(x) = g_T(x)$, so we finally have identity. \square

6 Numerical experiment

Now we illustrate our algorithms by pricing a Bermudan basket option in a model, where asset prices can be influenced by an investor holding large amounts of shares of the asset. The large investor can increase (or decrease) an asset price by buying (or selling) assets.

We consider a Bermudan put option on a basket of d assets with payoff

$$g(X_r) := \left(\frac{1}{d} \sum_{i=1}^d X_r^{(i)} - K \right)^+$$

(K is the strike price), which can be exercised at times $r = 1, \dots, T$. We assume that a large investor buys $a_r \times 100\%$ ($0 \leq a \leq 1$) of each asset at time r , and that the asset dynamics from time r to $r + 1$ depend on a_r via the Markovian model

$$X_{r+1}^{(i)} = X_r^{(i)} \exp \left(-\frac{\sigma^2}{2} \delta_r + \sigma \sqrt{\delta_r} \zeta_{r,i} \right) \gamma(a_r), \quad X_0^{(i)} = x_0, \quad i = 1, \dots, d,$$

where $\zeta_{r,i}$ are i.i.d. standard gaussian random variables, $\gamma : [0, 1] \rightarrow \mathbb{R}_+$ is some function, and δ_r is a time scaling parameter. In this case we have,

$$\begin{aligned} P^{a_r}(x, dy) &= \frac{1}{y \sigma^d \sqrt{2\pi \delta_r}^d} \exp \left(-\frac{\sum_{j=1}^d (\ln \frac{y_j}{x_j} + \frac{\sigma^2}{2} \delta_r - \ln \gamma(a_r))^2}{2\sigma^2 \delta_r} \right) dy \\ &= \exp \left(\ln \gamma(a_r) \frac{\sum_{j=1}^d \ln \frac{y_j}{x_j} - d \frac{\sigma^2}{2} \delta_r}{\sigma^2 \delta_r} - \frac{d \ln^2 \gamma(a_r)}{2\sigma^2 \delta_r} \right) \\ &\quad \times \frac{1}{y \sigma^d \sqrt{2\pi \delta_r}^d} \exp \left(-\frac{\sum_{j=1}^d (\ln \frac{y_j}{x_j} - \frac{\sigma^2}{2} \delta_r)^2}{2\sigma^2 \delta_r} \right) dy. \end{aligned}$$

As reference measure we take the one corresponding to $\gamma \equiv 1$, hence

$$P^a(x, dy) = \varphi(x, y; a) P^*(x, dy),$$

with

$$\varphi(x, y; a) = \exp \left(\ln \gamma(a) \frac{\sum_{j=1}^d \ln \frac{y_j}{x_j} - d \frac{\sigma^2}{2} \delta_r}{\sigma^2 \delta_r} - \frac{d \ln^2 \gamma(a)}{2\sigma^2 \delta_r} \right).$$

In our particular example we take $\gamma(a) = \exp(a/20)$. So the large investor may push the asset price about 5% upwards by buying more shares. Further we assume that, up to the call date, the investor pays at each step transaction costs according to

$$(6.25) \quad f_r(X, a) = -\frac{2^{r+1}a}{x_0 d} (X^{(1)} + \dots + X^{(d)}).$$

As a matter of fact, increasing transaction costs may shift the optimal stopping time from the terminal exercise date, thus leading to a nontrivial optimal exercise policy. The price of the Bermudan basket option is finally given by (2.2) with $g_r \equiv g$ and f_r given by (6.25).

We now study a numerical example with $d = 5$, $T = 3$, $\delta_r \equiv 1$, $x_0 = 100$, $K = 90$, $\sigma = 0.2$, and construct lower bounds for the option price based on local regression (Section 3.1) and the global regression method in Section 3.2.1. Due to a (suboptimal) stopping time and a (suboptimal) control based on the k -nearest neighbor estimator (3.10) and corresponding estimator (3.11) we construct a lower bound denoted by $Y_{0,M}^{knn,low}$ (with the M from (3.10)). On the other hand, due to a (suboptimal) stopping time and a (suboptimal) control based on (3.15) and the global regression estimate (3.17) we obtain a lower bound denoted by $Y_{0,M}^{gr,low}$ (with the M from (3.17)). Furthermore, we simulate upper bounds for the option price based on the dual representation in Theorem 7, using approximative value functions (3.12) and (3.18) for the local and global regression method, denoted by $Y_{0,M}^{knn,up}$ and $Y_{0,M}^{gr,up}$ respectively. For the upper bounds we simulate 50 (“outer”) trajectories where on each trajectory the conditional expectations in $(\mathcal{L}h)_r$ are estimated using 10000 independent (“inner”) trajectories. The lower bounds are simulated using 50000 Monte Carlo trajectories in the final simulation, see Tables 1,2.

For the optimal choice of k_M in the nearest neighbors estimator (for given M) one needs to balance between the variance and the bias of the estimator (3.10). Moreover, it turns out that it can be advantageous to take k_M depending on x . To illustrate this we plot in Figure 1 the root-mean-square errors of $\widehat{C}_{2,10000}^{knn}(x, 1)$ and $\widehat{C}_{2,50000}^{knn}(x, 1)$ relative to the “exact” values $C_2(x, 1)$ (computed using 10^6 Monte Carlo trajectories) for

$$x^{(i)} = x_0 \exp(-\sigma^2 + \sigma\sqrt{\delta_0}\zeta_{0,i} + \sigma\sqrt{\delta_1}\zeta_{1,i})$$

with $\zeta_{r,i} \equiv 0$ (left figure), $\zeta_{r,i} \equiv 1.5$ (right figure), and different number of nearest neighbors. Here the best value of k for the “central” point x is about $0.1 \times M$, and the error does not exceed 5% for $M = 10000$. However, the error becomes rather large if x lies in a region with small concentration of pre-simulated regression points (the optimal k_M is about 10 in the right-hand figure). Thus, the performance of the k -nearest neighbor estimator can be improved by choosing k_M adaptively depending on x .

The global regression estimator provides better lower and upper bounds for the option price than the local regression estimators, see Table 2. The gap between lower and upper bound for the best choice of base functions does not exceed 4% (relative to the lower estimate), while for the local regression estimator the gap is larger than 15%.

k	$Y_{0,10000}^{knn,low}$ (SD)	$Y_{0,10000}^{knn,up}$ (SD)	$Y_{0,50000}^{knn,low}$ (SD)	$Y_{0,50000}^{knn,up}$ (SD)
10	13.94(0.06)	20.94(0.23)	13.82(0.06)	21.22(0.27)
20	14.10(0.06)	18.89(0.20)	14.20(0.06)	18.41(0.16)
50	14.08(0.06)	16.74(0.09)	14.33(0.06)	17.08(0.14)
100	14.13(0.05)	16.59(0.14)	14.19(0.05)	16.68(0.13)
500	14.17(0.05)	16.73(0.14)	14.17(0.05)	16.48(0.13)
1000	13.56(0.05)	17.04(0.13)	14.06(0.05)	16.27(0.11)

Table 1: Lower and upper bounds obtained via local regression k -nearest neighbor estimators.

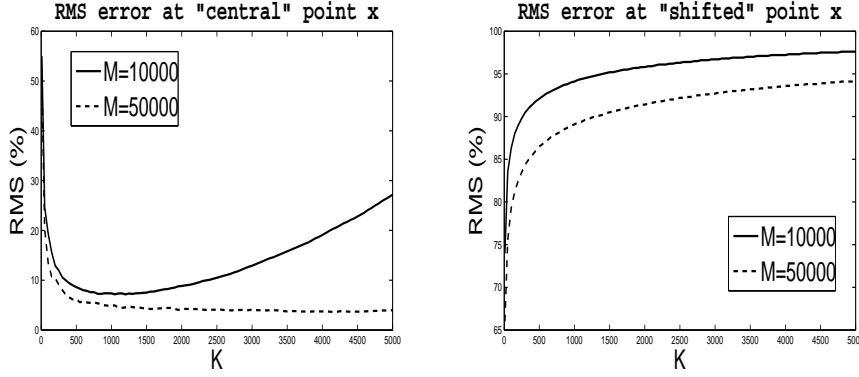


Figure 1: Root-mean-square error (in %) of the estimators $\hat{C}_{2,10000}^{knn}(x, 0)$ and $\hat{C}_{2,50000}^{knn}(x, 0)$ for different number k of nearest neighbors at two points $x^{(i)} = x_0 \exp(-\sigma^2 + \sigma\sqrt{\delta_0}\zeta_{0,i} + \sigma\sqrt{\delta_1}\zeta_{1,i})$ with $\zeta_{r,i} \equiv 0$ (left) and $\zeta_{r,i} \equiv 1.5$ (right).

7 Appendix

7.1 Proof of Theorem 4

Proof. The statements hold trivially true for $r = T$. For $r < T$ we have

$$\begin{aligned}
\mathbf{1}_{\tau^*(r) > r} Y_r^* &= \mathbf{1}_{\tau^*(r) > r} \sup_{\mathbf{a} \in \mathcal{A}_r, \tau \in \mathfrak{I}_r} \mathbf{E}^{\mathbf{a}} \left[\sum_{j=r}^{\tau-1} f_j(X_j, a_j) + g_\tau(X_\tau) \middle| \mathcal{F}_r \right] \\
&= \mathbf{1}_{\tau^*(r) > r} \sup_{\tau \in \mathfrak{I}_{r+1}} \mathbf{E}_r^{\mathbf{a}^*(r)} \left[\sum_{j=r}^{\tau-1} f_j(X_j, a_j) + g_\tau(X_\tau) \middle| \mathcal{F}_r \right] \\
&= \mathbf{1}_{\tau^*(r) > r} f_r(X_r, a_r^*(r)) + \\
&\quad + \mathbf{1}_{\tau^*(r) > r} \sup_{\tau \in \mathfrak{I}_{r+1}} \mathbf{E}_r^{\mathbf{a}^*(r)} \mathbf{E}_{r+1}^{(a_{r+1}^*(r), \dots)} \left[\sum_{j=r+1}^{\tau-1} f_j(X_j, a_j^*(r)) + g_\tau(X_\tau) \middle| \mathcal{F}_{r+1} \right]
\end{aligned}$$

Table 2: Lower and upper bound via global regression estimator.

base functions	$Y_{0,200000}^{gr,low}$ (SD)	$Y_{0,200000}^{gr,up}$ (SD)
up to 2nd degree polynomials on $g_r(X_r)$	15.15(0.06)	15.75(0.10)
up to 3th degree polynomials on $g_r(X_r)$	15.10(0.07)	15.62(0.07)
up to 4th degree polynomials on $g_r(X_r)$	15.13(0.07)	15.70(0.09)
$1, X_r^{(1)}, \dots, X_r^{(5)}, g_r(X_r)$	15.01(0.07)	15.76(0.08)
up to 2nd degree polynomials on $X_r^{(1)}, \dots, X_r^{(5)}, g_r(X_r)$	15.09(0.06)	15.55(0.07)

$$\begin{aligned}
&\leq 1_{\tau^*(r)>r} f_r(X_r, a_r^*(r)) + \\
&1_{\tau^*(r)>r} \mathbf{E}_r^{a^*(r)} \sup_{\mathbf{a} \in \mathcal{A}_{r+1}, \tau \in \mathfrak{T}_{r+1}} \mathbf{E}^{(a_{r+1}(r), \dots)} \left[\sum_{j=r+1}^{\tau-1} f_j(X_j, a_j) + g_\tau(X_\tau) \middle| \mathcal{F}_{r+1} \right] \\
&= 1_{\tau^*(r)>r} f_r(X_r, a_r^*(r)) + 1_{\tau^*(r)>r} \mathbf{E}^{(a^*(r), a_{r+1}^*(r+1), \dots)} \times \\
&\times \left[\sum_{j=r+1}^{\tau^*(r+1)-1} f_j(X_j, a_j^*(r+1)) + g_{\tau^*(r+1)}(X_{\tau^*(r+1)}) \middle| \mathcal{F}_r \right] \\
&= 1_{\tau^*(r)>r} f_r(X_r, a_r^*(r)) + 1_{\tau^*(r)>r} \mathbf{E}_r^{a^*(r)} Y_{r+1}^* = 1_{\tau^*(r)>r} Y_r^*,
\end{aligned}$$

due to the Bellman property. Hence

$$\begin{aligned}
1_{\tau^*(r)>r} Y_r^* &= 1_{\tau^*(r)>r} f_r(X_r, a_r^*(r)) \\
&+ 1_{\tau^*(r)>r} \mathbf{E}_r^{(a^*(r), a_{r+1}^*(r+1), \dots)} \times \\
&\times \left[\sum_{j=r+1}^{\tau^*(r+1)-1} f_j(X_j, a_j^*(r+1)) + g_{\tau^*(r+1)}(X_{\tau^*(r+1)}) \middle| \mathcal{F}_r \right]
\end{aligned}$$

from which the consistency relations follow. \square

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