Lithuanian Mathematical Journal, Vol. 37, No. 3, 1997

STABILITY AND CONVERGENCE OF DUFORT–FRANKEL-TYPE DIFFERENCE SCHEMES FOR A NONLINEAR SCHRÖDINGER-TYPE EQUATION

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Abstract—We consider a first boundary problem for the nonlinear Schrödinger equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} + f(u, u^*)u.$$

The convergence of a three-layer explicit difference scheme in the C and W_2^1 norms is proved. The stability of the scheme with respect to the initial data in the same norms is proved. To justify the convergence and stability we use grid analogues of the energy-preservation laws and grid multiplicative inequalities. The relation $2|a|\tau/h^2 \leqslant \nu < 1$ is assumed for the grid stepsizes.

INTRODUCTION

We consider a first boundary problem for a nonlinear Schrödinger equation. Such equations appear in nonlinear optics models [1, 2]; they describe energy transfer models in molecular systems [3, 4]; they are used in quantum mechanics, seismology, plasma physics, theories of vortex motion and superconductivity, and other domains of natural sciences.

Many works are devoted to numerical solution of nonlinear Schrödinger equations for both initial and boundary problems. Some authors use the finite-difference method [5–13]; others prefer finite-element methods [14–17].

Among the difference methods, those having grid analogues of energy-preservation laws are of special interest (see [12]). This property is possessed, for example, by the Crank–Nicolson difference schemes that were considered in detail in [8–11]; in particular, the unconditional convergence and stability were proved there. Unfortunately, these schemes are implicit.

On the other hand, explicit schemes often appear to be unstable as, for example, Euler schemes [5]. Some modifications of the latter are conditionally stable. So is the three-layer explicit difference scheme of DuFort–Frankel. For Schrödinger equations, these schemes were presented in [5–7]. For approximating these schemes the condition $\tau/h \to 0$ is required, where τ and h denote the space and time steps of the grid, respectively.

In [6, 7], linear Schrödinger equations were considered and the stability of schemes was proved. In [5], nonlinear equations were already considered, and a certain grid analogue of the energy-preservation law in the space L_2 was obtained. The convergence and stability of the schemes, however, were not proved there. Thus, our paper improves the results of [5–7].

In our paper, in the cubic nonlinearity case we deduce analogues of energy-preservation laws in the spaces L_2 and W_2^1, while in the general case we deduce *a priori* estimates of a new type, presented in [8–11]. We also prove the convergence and stability of difference schemes in the spaces C and W_2^1 under the condition $\tau/h^2 \leqslant \nu < 1/2|a|$.

In Sec. 1, we state the problem and prove grid analogues of one embedding theorem and of a multiplicative inequality. In Sec. 2, we deduce grid analogues of energy-preservation laws for a cubic Schrödinger equation.

Vilnius University, Naugarduko 24, 2600 Vilnius, Lithuania. Translated from Lietuvos Matematikos Rinkinys, Vol. 37, No. 3, pp. 334–352, July–September, 1997. Original article submitted October 30, 1996.

0363-1672/97/3703-0249$18.00

In Sec. 3, we prove the convergence and stability of the difference scheme for a cubic equation in the spaces L_2 and C. In Sec. 4, we show the convergence and stability of the difference scheme in the spaces C and W_2^1 for a more general nonlinearity.

1. FORMULATION OF THE PROBLEM. AUXILIARY STATEMENTS

We consider the first boundary problem for the cubic Schrödinger equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} - i\lambda|u|^2 u, \qquad (x,t) \in Q, \tag{1.1}$$

with zero boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,T], \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad x \in [0,1]. \tag{1.3}$$

Here $i = \sqrt{-1}$, $Q = (0,1) \times (0,T)$, $a \neq 0$, and λ are real numbers, and $u(x,t)$ is a complex function.

We define the inner product of two functions $v(x)$ and $w(x)$ by

$$(v, w) = \int_0^1 v(x)w^*(x)\,dx.$$

By L_p and W_2^1 we denote the Sobolev spaces with the norms

$$\|v\|_{L_p} = \left(\int_0^1 |v(x)|^p\,dx\right)^{1/p}, \qquad \|v\|_{W_2^1} = \left(\|v\|_{L_2}^2 + \left\|\frac{\partial v}{\partial x}\right\|_{L_2}^2\right)^{1/2}.$$

Here $w^*(x)$ is the complex conjugate of a function $w(x)$.

The existence of a solution of problem (1.1)–(1.3) was considered in [18, 19].

It is well known that, for all $t \in [0,T]$, a solution of problem (1.1)–(1.3) satisfies the following energy-preservation laws:

$$\|u(t)\|_{L_2} = \|u(0)\|_{L_2}, \tag{1.4}$$

$$\left\|\frac{\partial u}{\partial x}(t)\right\|_{L_2}^2 + (\lambda/2a)\|u(t)\|_{L_4}^4 = \left\|\frac{\partial u}{\partial x}(0)\right\|_{L_2}^2 + (\lambda/2a)\|u(0)\|_{L_4}^4. \tag{1.5}$$

Introduce a uniform grid in the domain $\overline{Q}$ with steps τ and h: $\overline{Q}_h = \bar\omega_h * \bar\omega_\tau$ and $Q_h = \omega_h * \omega_\tau$. We suppose that $\tau = T/M$, $t_j = j\tau$, $h = 1/N$, $x_j = jh$, $\bar\omega_\tau = \{t_j;\, j = 0,\dots,M\}$, $\omega_\tau = \{t_j;\, j = 1,\dots,M-1\}$, $\bar\omega_h = \{x_j;\, j = 0,\dots,N\}$, and $\omega_h = \{x_j;\, j = 1,\dots,N-1\}$.

We will use grid analogues of Sobolev spaces L_{ph}, W_{2h}^1. By $C(\overline{Q}_h)$ we denote the analogue of the space $C(\overline{Q})$. Introduce the inner products

$$(u, v) = \sum_{j=1}^{N-1} u_j v_j^* h, \qquad (u, v] = \sum_{j=1}^{N} u_j v_j^* h.$$

Define the norms

$$\|u\|_{L_{ph}}^p = \sum_{j=1}^{N-1} |u_j|^p h, \qquad \|u]|^2 = (u, u],$$

$$\|u\|^2 = (u, u), \qquad \|u\|_{W_{2h}^1}^2 = \|u\|^2 + \|u_{\bar x}]|^2.$$

Denote $p = p_i^j = p(x_i, t_j)$, $\hat{p} = p_i^{j+1}$, $\check{p} = p_i^{j-1}$, $\dot{p} = (\check{p} + \hat{p})/2$, $p_+ = p_{i+1}^j$, $p_- = p_{i-1}^j$, $\bar{p} = p_- + p_+$, $p_t = (\hat{p} - \check{p})/2\tau$, $p_{\bar{x}} = (p - p_-)/h$, $p_{x\nwarrow} = (\check{p} - p_-)/h$, and $p_{x\swarrow} = (p - \check{p}_-)/h$.

We now prove grid analogues of one embedding theorem and of a multiplicative inequality. Similar estimates for grid functions on the time layer were considered and applied in [11].

LEMMA 1.1. *Let $v_0 = \hat{v}_0 = v_N = \hat{v}_N = 0$. Then the following estimate holds:*

$$\max\{\|\hat{v}\|_{C_h}, \|v\|_{C_h}\} \leqslant 0.5(\|\hat{v}_{x\nwarrow}]| + \|\hat{v}_{x\swarrow}]|). \tag{1.6}$$

Proof. Denote $\hat{v}_i$ by $v_{1;i}$ and v_i by $v_{-1;i}$. Then for all $i = 1, \dots, N-1$ we can write

$$|v_{1;i}| = \left|\sum_{k=-i}^{-1}\left(v_{(-1)^{k+1};i+k+1} - v_{(-1)^k;i+k}\right)\right| \leqslant \sum_{k=-i}^{-1}\left|v_{(-1)^{k+1};i+k+1} - v_{(-1)^k;i+k}\right|$$

and

$$|v_{1;i}| = \left|\sum_{k=0}^{N-i-1}\left(v_{(-1)^{k+1};i+k+1} - v_{(-1)^k;i+k}\right)\right| \leqslant \sum_{k=0}^{N-i-1}\left|v_{(-1)^{k+1};i+k+1} - v_{(-1)^k;i+k}\right|.$$

Summing these two inequalities, we get

$$2|v_{1;i}| \leqslant \sum_{k=0}^{N-i-1}\left|v_{(-1)^{k+1};i+k+1} - v_{(-1)^k;i+k}\right| = \sum_{k=1}^{N}\left|v_{(-1)^{k-i};k} - v_{(-1)^{k-i-1};k-1}\right|.$$

Similarly,

$$2|v_{-1;i}| \leqslant \sum_{k=1}^{N}\left|v_{(-1)^{k-i+1};k} - v_{(-1)^{k-i};k-1}\right|.$$

Summing the last two inequalities and noting that $\max\{\|\hat{v}\|_{C_h}, \|v\|_{C_h}\} \leqslant \max_i(|v_i| + |\hat{v}_i|)$, we get

$$2\max\{\|\hat{v}\|_{C_h}, \|v\|_{C_h}\} \leqslant \sum_{k=1}^{N}|\hat{v}_{k;x\nwarrow}|h + \sum_{k=1}^{N}|\hat{v}_{k;x\swarrow}|h.$$

Now estimate (1.6) follows by the Cauchy inequality. The lemma is proved.

Similarly, we prove a grid analogue of the multiplicative inequality.

LEMMA 1.2. *Let $v_0 = \hat{v}_0 = v_N = \hat{v}_N = 0$. Then the following estimate holds:*

$$\max\{\|\hat{v}\|_{C_h}^2, \|v\|_{C_h}^2\} \leqslant 0.5(\|\hat{v}\| + \|v\|)(\|\hat{v}_{x\nwarrow}]| + \|\hat{v}_{x\swarrow}]|). \tag{1.7}$$

Proof. For all $i = 1, \dots, N-1$, we have

$$|v_{1;i}|^2 = \left||v_{(-1)^0;i}|^2 - |v_{(-1)^{-i};i-i}|^2\right| = \left|\sum_{k=-i}^{-1}\left(|v_{(-1)^{k+1};i+k+1}|^2 - |v_{(-1)^k;i+k}|^2\right)\right|$$
$$\leqslant \sum_{k=-i}^{-1}\left||v_{(-1)^{k+1};i+k+1}|^2 - |v_{(-1)^k;i+k}|^2\right|.$$

In the same way we have

$$|v_{1;i}|^2 \leqslant \sum_{k=0}^{N-i-1}\left||v_{(-1)^{k+1};i+k+1}|^2 - |v_{(-1)^k;i+k}|^2\right|.$$

Summing these two inequalities, we get

$$2|v_{1;i}|^2 \leqslant \sum_{k=1}^{N} \left| |v_{(-1)^{k-i};k}|^2 - |v_{(-1)^{k-i-1};k-1}|^2 \right|.$$

Similarly,

$$2|v_{-1;i}|^2 \leqslant \sum_{k=1}^{N} \left| |v_{(-1)^{k-i+1};k}|^2 - |v_{(-1)^{k-i};k-1}|^2 \right|.$$

Summing the last two inequalities, we have

$$2\left(|v_{-1;i}|^2 + |v_{1;i}|^2\right) \leqslant \sum_{k=1}^{N} \left| |v_{-1;k}|^2 - |v_{-1;k-1}|^2 \right| + \sum_{k=1}^{N} \left| |v_{1;k}|^2 - |v_{-1;k-1}|^2 \right|.$$

Note that

$$\sum_{k=1}^{N} \left| |v_{(-1)^j;k}|^2 - |v_{(-1)^{j+1};k-1}|^2 \right| = \sum_{k=1}^{N} \left| \frac{|v_{(-1)^j;k}| - |v_{(-1)^{j+1};k-1}|}{h} \right| (|v_{(-1)^j;k}| + |v_{(-1)^{j+1};k-1}|)h$$
$$\leqslant \left(\sum_{k=1}^{N} \left| \frac{|v_{(-1)^j;k}| - |v_{(-1)^{j+1};k-1}|}{h} \right|^2 h \right)^{1/2} (\|v_{-1}\| + \|v_1\|).$$

Hence estimate (1.7) follows. The lemma is proved.

The following embedding theorems are known [21]:

$$\|v\|_{L_{ph}} \leqslant \|v\|_{C_h} \leqslant 0.5\|v_{\bar{x}}]| \leqslant 0.5\|v\|_{W^1_{2h}}. \tag{1.8}$$

We further also use the following grid analogue of the Gronwall inequality:

$$Y_j \leqslant \left(\overline{Y}_0 + 2et_j \max_{0\leqslant l<j}\{b_l\}\right) \exp(4dt_j). \tag{1.9}$$

Here $Y \geqslant 0$ and $b \geqslant 0$ are defined on the grid $\bar{\omega}_\tau$, $Y_0 \leqslant \overline{Y}_0$; $e \geqslant 0$, $0 < \tau d \leqslant 1/2$. For all $j = 1, \ldots, M$, the following estimate also holds:

$$Y_j \leqslant \overline{Y}_0 + \tau d \sum_{l=0}^{j-1} (Y_l + Y_{l+1}) + \tau e \sum_{l=0}^{j-1} b_l.$$

The latter is proved, for example, in [20].

2. DIFFERENCE SCHEME. GRID LAWS OF ENERGY-PRESERVATION

Consider the following DuFort–Frankel-type difference scheme related to problem (1.1)–(1.3):

$$p_t = ia\frac{\bar{p} - 2\dot{p}}{h^2} - i\lambda|p|^2\dot{p}, \qquad (x,t) \in Q_h, \tag{2.1}$$

$$p(x_0,t) = p(x_N,t) = 0, \qquad t \in \bar{\omega}_\tau, \tag{2.2}$$

$$p(x,0) = u_0(x), \qquad x \in \bar{\omega}_h. \tag{2.3}$$

We find a solution on the first layer t_1 using any two-layer scheme.

In [5], a partial case of a grid analogue of law (1.4) was proved for (2.1)–(2.3). Here we prove a grid analogue of the same law, and also an analogue of law (1.5).

LEMMA 2.1. *For a solution of scheme* (2.1)–(2.3), *the following energy-preservation law holds*:

$$\begin{aligned} &\|p(t_{j+1})\|^2 + \|p(t_j)\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar p(t_j), p(t_{j+1})) \\ &= \|p(t_1)\|^2 + \|p(t_0)\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar p(t_0), p(t_1)), \qquad j = 1, \dots, M-1. \end{aligned} \tag{2.4}$$

If the grid steps satisfy the relation

$$0 < 2|a|\tau/h^2 \leqslant \nu < 1, \tag{2.5}$$

then the following estimate holds:

$$\|p(t_{j+1})\|^2 + \|p(t_j)\|^2 \leqslant \mu\big(\|p(t_1)\|^2 + \|p(t_0)\|^2\big). \tag{2.6}$$

Here $\mu = (1+\nu)/(1-\nu)$.

Proof. Take the inner products of both sides of Eq. (2.1) with $4\tau\dot p$. Take the real part of the equality

$$\operatorname{Re}(\hat p - \check p, \hat p + \check p) = -\frac{4a\tau}{h^2}\operatorname{Im}(\bar p, \dot p) + \frac{8a\tau}{h^2}\operatorname{Im}\|\dot p\|^2 + 4\lambda\tau\operatorname{Im}\sum_{i=1}^{N-1}|p_i|^2|\dot p_i|^2 h.$$

Consequently,

$$\|\hat p\|^2 - \|\check p\|^2 + \frac{4a\tau}{h^2}\operatorname{Im}(\bar p, \dot p) = 0$$

and

$$\|\hat p\|^2 - \|p\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar p, \hat p) = \|p\|^2 - \|\check p\|^2 - \frac{2a\tau}{h^2}\operatorname{Im}(\bar p, \check p).$$

Note that $\operatorname{Im}(\bar p, \check p) = -\operatorname{Im}(\check p, \bar p)$ and

$$(\check p, \bar p) = \sum_{i=1}^{N-1}\check p_i(p_{i-1}^* + p_{i+1}^*)h = \sum_{i=0}^{N-2}\check p_{i+1}p_i^* h + \sum_{i=2}^{N}\check p_{i-1}p_i^* h = (\bar{\check p}, p).$$

Hence the following equality holds:

$$\|\hat p\|^2 + \|p\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar p, \hat p) = \|p\|^2 + \|\check p\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{\check p}, p).$$

Summing the equalities over time layers yields (2.4).

Now estimate $|\operatorname{Im}(\bar p(t_k), p(t_{k+1}))|$:

$$\begin{aligned} |\operatorname{Im}(\bar p(t_k), p(t_{k+1}))| &\leqslant |(p_+(t_k), p(t_{k+1}))| + |(p_-(t_k), p(t_{k+1}))| \\ &\leqslant 2\|p(t_k)\|\|p(t_{k+1})\| \leqslant \|p(t_k)\|^2 + \|p(t_{k+1})\|^2. \end{aligned}$$

From condition (2.5) it follows that

$$\left|\frac{2a\tau}{h^2}\operatorname{Im}(\bar p(t_k), p(t_{k+1}))\right| \leqslant \nu(\|p(t_k)\|^2 + \|p(t_{k+1})\|^2).$$

Hence (2.6) follows. The lemma is proved.

We now prove a grid analogue of law (1.5).

LEMMA 2.2. *For a solution of scheme* (2.1)–(2.3), *the following equality holds*:

$$\begin{aligned}&\|p_{x\nwarrow}(t_{j+1})]|^2 + \|p_{x\swarrow}(t_{j+1})]|^2 + \frac{\lambda}{a}\|p(t_{j+1})p(t_j)\|^2 \\ &= \|p_{x\nwarrow}(t_1)]|^2 + \|p_{x\swarrow}(t_1)]|^2 + \frac{\lambda}{a}\|p(t_1)p(t_0)\|^2, \qquad j = 1, \dots, M-1.\end{aligned} \tag{2.7}$$

Proof. Take the inner products of both sides of Eq. (2.1) with $\hat{p} - \check{p}$. The imaginary part of the equality obtained is

$$\frac{1}{2\tau}\mathrm{Im}\|\hat{p} - \check{p}\|^2 = \frac{a}{h^2}\mathrm{Re}(\bar{p} - 2\dot{p}, \hat{p} - \check{p}) - \frac{\lambda}{2}\mathrm{Re}(|p|^2(\hat{p} + \check{p}), \hat{p} - \check{p}).$$

Hence it follows that

$$\frac{2}{h^2}\mathrm{Re}(2\dot{p} - \bar{p}, \hat{p} - \check{p}) + \frac{\lambda}{a}(\|p\hat{p}\|^2 - \|\check{p}p\|^2) = 0.$$

Note that

$$\begin{aligned}\frac{2}{h^2}\mathrm{Re}(2\dot{p} - \bar{p}, \hat{p} - \check{p}) &= \frac{2}{h^2}(\mathrm{Re}(\hat{p} + \check{p}, \hat{p} - \check{p}) - \mathrm{Re}(\bar{p}, \hat{p} - \check{p})) \\ &= \frac{2}{h^2}(\|\hat{p}\|^2 - \|\check{p}\|^2 - \mathrm{Re}(\bar{p}, \hat{p}) + \mathrm{Re}(\check{p}, \bar{p})).\end{aligned}$$

Since $(\check{p}, \bar{p}) = (\bar{\check{p}}, p)$, we have

$$\frac{2}{h^2}\mathrm{Re}(2\dot{p} - \bar{p}, \hat{p} - \check{p}) = \frac{1}{h^2}(2(\|\hat{p}\|^2 - \mathrm{Re}(\bar{p}, \hat{p}) + \|p\|^2) - 2(\|p\|^2 - \mathrm{Re}(\bar{\check{p}}, p) + \|\check{p}\|^2)).$$

Using condition (2.2), we note that

$$\begin{aligned}2(\|\hat{p}\|^2 - \mathrm{Re}(\bar{p}, \hat{p}) + \|p\|^2) &= (\|p\|^2 - 2\mathrm{Re}(p_-, \hat{p}) + \|\hat{p}\|^2) + (\|p\|^2 - 2\mathrm{Re}(p_+, \hat{p}) + \|\hat{p}\|^2) \\ &= \sum_{j=1}^{N}(|p_{j-1}|^2 - 2\mathrm{Re}(p_{j-1}\hat{p}_j^*) + |\hat{p}_j|^2)h + \sum_{j=0}^{N-1}(|p_{j+1}|^2 - 2\mathrm{Re}(p_{j+1}\hat{p}_j^*) + |\hat{p}_j|^2)h \\ &= \sum_{j=1}^{N}|p_{j-1} - \hat{p}_j|^2 h + \sum_{j=0}^{N-1}|p_{j+1} - \hat{p}_j|^2 h.\end{aligned}$$

Consequently,

$$\frac{2}{h^2}(\|\hat{p}\|^2 - \mathrm{Re}(\bar{p}, \hat{p}) + \|p\|^2) = \|\hat{p}_{x\nwarrow}]|^2 + \|\hat{p}_{x\swarrow}]|^2.$$

This leads to the equality

$$\|\hat{p}_{x\nwarrow}]|^2 + \|\hat{p}_{x\swarrow}]|^2 + \frac{\lambda}{a}\|p\hat{p}\|^2 = \|p_{x\nwarrow}]|^2 + \|p_{x\swarrow}]|^2 + \frac{\lambda}{a}\|\check{p}p\|^2.$$

Summing the equalities over time layers yields (2.7). The lemma is proved.

CONVERGENCE AND STABILITY OF THE DIFFERENCE SCHEME

Suppose the solution $u(x, t)$ of problem (1.1)–(1.3) is sufficiently smooth for an approximation of the equation. More exactly, we assume that it is of class $C_3^4(Q)$, i.e., it has continuous partial derivatives of the third order with respect to t and derivatives of the fourth order with respect to x.

Let $\Phi(t_j)$ denote the approximation error. One easily checks that this error has the order $O(\tau^2 + h^2 + (\tau/h)^2)$. Thus, for justifying the scheme one needs $\tau/h \to 0$.

Furthermore, because of the smoothness of the solution the following conditions are satisfied:

$$\max_{t_j \in \omega_\tau}\{\|\Phi(t_j)\|_{L_{2h}}\} \to 0, \qquad \tau, h \to 0, \tag{3.1}$$

and

$$\begin{aligned} M_1 &= \max\left\{\max_{(h,\tau)\in\Delta, t\in\bar{\omega}_\tau} \|u(t)\|_{W^1_{2h}};\ \max_{t\in[0;T]} \|u(t)\|_{W^1_2}\right\} < \infty, \\ M_2 &= \max\left\{\max_{(h,\tau)\in\Delta, t\in\bar{\omega}_\tau} \left\|\frac{\hat{u}-u}{\tau}(t)\right\|;\ \max_{t\in[0;T]} \left\|\frac{\partial u}{\partial t}(t)\right\|_{L_2}\right\} < \infty, \end{aligned} \tag{3.2}$$

where $\Delta = (0; h_0] \times (0; \tau_0]$, and τ_0 and h_0 are some small positive constants.

Hence by the embedding theorem $W^1_{2h} \to C_h$ we have the following estimate:

$$\max_{(h,\tau)\in\Delta} \max_{(x,t)\in\overline{Q}_h} |u(x,t)| = \|u\|_{C(\overline{Q}_h)} \leqslant 0.5M_1. \tag{3.3}$$

The solution error satisfies the boundary problem

$$\varepsilon_t = \frac{ia}{h^2}(\bar{\varepsilon} - 2\dot{\varepsilon}) + \Psi + \Phi, \qquad (x,t) \in Q_h, \tag{3.4}$$

$$\varepsilon(x,0) = 0, \quad x \in \bar{\omega}_h, \qquad \varepsilon(x_0,t) = \varepsilon(x_N,t) = 0, \quad t \in \bar{\omega}_\tau. \tag{3.5}$$

Here

$$\Psi = -i\lambda\left(|u|^2\dot{u} - |p|^2\dot{p}\right).$$

We will also assume that for the solution error the following is true on the first layer:

$$\|\varepsilon(t_1)\|_{W^1_{2h}} \longrightarrow 0, \qquad \frac{1}{h}\|\varepsilon(t_1)\| \longrightarrow 0, \qquad \tau, h \to 0. \tag{3.6}$$

Suppose $\lambda/a \geqslant 0$. We will prove an auxiliary lemma.

LEMMA 3.1. *Let $\lambda/a \geqslant 0$ and $u(x,t) \in C^4_3(Q)$. Let conditions* (2.5) *and* (3.6) *be satisfied. Then there exist constants $\tau_0 > 0$ and $h_0 > 0$ such that, for all $\tau \leqslant \tau_0$ and $h \leqslant h_0$, the following estimates hold for a solution of problem* (2.1)–(2.3):

$$\max_{1\leqslant j\leqslant M} \|p_{x\nwarrow}(t_j)]| + \|p_{x\swarrow}(t_j)]|) \leqslant M_3, \tag{3.7}$$

$$\|p\|_{C(\overline{Q}_h)} = \max_{(x_i,t_j)\in\overline{Q}_h} |p(x_i,t_j)| \leqslant 0.5M_3. \tag{3.8}$$

Here $M_3 = M_3(a, \lambda, M_1, \nu)$.

Proof. Note that

$$
\begin{aligned}
&\|\hat{p}_{x\nwarrow}]|^2 + \|\hat{p}_{x\swarrow}]|^2 \\
&= \frac{1}{h^2}\sum_{i=1}^{N}\big(|p_i|^2 - 2\mathrm{Re}(p_i\hat{p}_{i-1}^*) + |\hat{p}_{i-1}|^2 + |\hat{p}_i|^2 - 2\mathrm{Re}(\hat{p}_i p_{i-1}^*) + |p_{i-1}|^2\big)h \\
&= \frac{1}{h^2}\sum_{i=1}^{N}\big(|p_i|^2 - 2\mathrm{Re}(p_i p_{i-1}^*) + |p_{i-1}|^2 + |\hat{p}_i|^2 - 2\mathrm{Re}(\hat{p}_i\hat{p}_{i-1}^*) + |\hat{p}_{i-1}|^2\big)h \qquad (3.9)\\
&+ \frac{1}{h^2}\sum_{i=1}^{N} 2\mathrm{Re}\big((\hat{p}_i - p_i)(\hat{p}_{i-1}^* - p_{i-1}^*)\big)h \\
&= \|\hat{p}_{\bar{x}}]|^2 + \|p_{\bar{x}}]|^2 + \frac{2\tau^2}{h^2}\mathrm{Re}\Big(\frac{\hat{p}-p}{\tau}, \frac{\hat{p}_- - p_-}{\tau}\Big).
\end{aligned}
$$

Using (3.6), (3.9), (3.2), and (1.8), we can estimate from above the right side of equality (2.7) for sufficiently small τ and h:

$$
\|p_{x\nwarrow}(t_1)]|^2 + \|p_{x\swarrow}(t_1)]|^2 + \frac{\lambda}{a}\|p(t_1)p(t_0)\|^2 \leqslant 2M_1^2 + \frac{\nu\tau}{|a|}M_2^2 + \frac{\lambda}{16a}M_1^4 \leqslant 0.5M_3^2;
$$

here $\tau \leqslant |a|M_1^2/\nu M_2^2$, $M_3^2 = 2M_1^2(3+\lambda M_1^2/16a)$. Hence (3.7) follows. Now (3.8) follows from (1.6) and (3.7). The lemma is proved.

We now prove the convergence of the scheme in the C norm.

THEOREM 3.1. *Let $\lambda/a \geqslant 0, u(x,t) \in C_3^4(Q)$ and let conditions* (2.5) *and* (3.6) *be satisfied. Then the solution of problem* (2.1)–(2.3) *converges to the solution of problem* (1.1)–(1.3) *in the space $C(\overline{Q}_h)$. There exist constants τ_0 and h_0 such that, for all $\tau \leqslant \tau_0$ and $h \leqslant h_0$, we have the estimate*

$$
\|\varepsilon\|_{C(\overline{Q}_h)}^2 \leqslant c_1\|\varepsilon(t_1)\| + c_2 \max_{1\leqslant j\leqslant M-1}\{\|\Phi(t_j)\|\}. \qquad (3.10)
$$

Here $c_i = c_i(a, \lambda, \nu, M_1, T)$, $i = 1, 2$.

Proof. Taking the inner product of both sides of Eq. (3.4) with $4\tau\dot{\varepsilon}$, similarly to Lemma 2.1, we obtain the equality

$$
\|\hat{\varepsilon}\|^2 + \|\varepsilon\|^2 + \frac{2a\tau}{h^2}\,\mathrm{Im}(\bar{\varepsilon}, \hat{\varepsilon}) = \|\varepsilon\|^2 + \|\check{\varepsilon}\|^2 + \frac{2a\tau}{h^2}\,\mathrm{Im}(\bar{\check{\varepsilon}}, \check{\varepsilon}) + 4\tau\mathrm{Re}(\Psi, \dot{\varepsilon}) + 4\tau\mathrm{Re}(\Phi, \dot{\varepsilon}).
$$

Now using (3.3) and (3.8) we can investigate the last two summands on the right side of the inequality. We have

$$
\begin{aligned}
4\tau\mathrm{Re}(\Psi, \dot{\varepsilon}) &= 4\lambda\tau\,\mathrm{Im}(|u|^2\dot{u} - |p|^2\dot{p}, \dot{\varepsilon}) = 4\lambda\tau\,\mathrm{Im}\big(((|u|^2 - |p|^2)\dot{u}, \dot{\varepsilon}) + (|p|^2\dot{\varepsilon}, \dot{\varepsilon})\big) \\
&\leqslant |\lambda|\tau|((|u| + |p|)(|\hat{u}| + |\check{u}|)|\varepsilon|, (|\hat{\varepsilon}| + |\check{\varepsilon}|))| \\
&\leqslant 0.5|\lambda|(M_1 + M_3)M_1\tau\|\varepsilon\|(\|\hat{\varepsilon}\| + \|\check{\varepsilon}\|) \\
&\leqslant d_1\tau\big((\|\varepsilon\|^2 + \|\hat{\varepsilon}\|^2) + (\|\check{\varepsilon}\|^2 + \|\varepsilon\|^2)\big);
\end{aligned}
$$

here $d_1 = 0.25|\lambda|(M_1 + M_3)M_1$.

For the other summand, we have

$$
4\tau\,\mathrm{Re}(\Phi, \dot{\varepsilon}) \leqslant 2\tau|(\Phi, (\hat{\varepsilon} + \check{\varepsilon}))| \leqslant 2\tau\|\Phi\|^2 + \tau\big(\|\hat{\varepsilon}\|^2 + \|\check{\varepsilon}\|^2\big).
$$

Using these estimates one can write

$$\|\hat{\varepsilon}\|^2 + \|\varepsilon\|^2 + \frac{2a\tau}{h^2}\,\mathrm{Im}(\bar{\varepsilon}, \hat{\varepsilon})$$
$$\leqslant \|\varepsilon\|^2 + \|\check{\varepsilon}\|^2 + \frac{2a\tau}{h^2}\,\mathrm{Im}(\bar{\check{\varepsilon}}, \varepsilon) + 2\tau\|\Phi\|^2 + (d_1+1)\tau\big((\|\varepsilon\|^2 + \|\hat{\varepsilon}\|^2) + (\|\check{\varepsilon}\|^2 + \|\varepsilon\|^2)\big).$$

Summing the inequalities over layers from t_1 to t_j, we obtain the estimate

$$\|\varepsilon(t_{j+1})\|^2 + \|\varepsilon(t_j)\|^2 + \frac{2a\tau}{h^2}\,\mathrm{Im}(\bar{\varepsilon}(t_j), \varepsilon(t_{j+1}))$$
$$\leqslant \|\varepsilon(t_1)\|^2 + \|\varepsilon(t_0)\|^2 + \frac{2a\tau}{h^2}\,\mathrm{Im}(\bar{\varepsilon}(t_0), \varepsilon(t_1)) + 2\tau\sum_{k=1}^{j}\|\Phi(t_k)\|^2$$
$$+ (d_1+1)\tau\sum_{k=1}^{j}\big((\|\varepsilon(t_{k+1})\|^2 + \|\varepsilon(t_k)\|^2) + (\|\varepsilon(t_k)\|^2 + \|\varepsilon(t_{k-1})\|^2)\big).$$

Using condition (2.5), similarly to Lemma 2.1, we obtain

$$\|\varepsilon(t_{j+1})\|^2 + \|\varepsilon(t_j)\|^2$$
$$\leqslant \mu(\|\varepsilon(t_1)\|^2 + \|\varepsilon(t_0)\|^2) + d_2\tau\sum_{k=1}^{j}\|\Phi(t_k)\|^2 + d_3\tau\sum_{k=1}^{j}\big((\|\varepsilon(t_{k+1})\|^2 + \|\varepsilon(t_k)\|^2) + (\|\varepsilon(t_k)\|^2 + \|\varepsilon(t_{k-1})\|^2)\big);$$

here $d_2 = 2/(1-\nu)$, $d_3 = (d_1+1)/(1-\nu)$.

Now apply the grid Gronwall inequality (1.9) with $d = d_3$, $e = d_2$, $b_i = \|\Phi(t_{i+1})\|^2$, $Y_j = \|\varepsilon(t_{j+1})\|^2 + \|\varepsilon(t_j)\|^2$, and $\overline{Y}_0 = \mu(\|\varepsilon(t_1)\|^2 + \|\varepsilon(t_0)\|^2)$. Hence, for every $\tau \leqslant 1/2d_3$, we have the estimate

$$\|\varepsilon(t_{j+1})\|^2 + \|\varepsilon(t_j)\|^2 \leqslant \exp(4d_3T)\Big(\mu(\|\varepsilon(t_1)\|^2 + \|\varepsilon(t_0)\|) + 2d_2T\max_{1\leqslant i\leqslant M-1}\big\{\|\Phi(t_i)\|^2\big\}\Big).$$

Recalling that $\varepsilon(t_0) = 0$, we can write that, for all $j = 0, \ldots, M$,

$$\|\varepsilon(t_j)\|^2 \leqslant d_4\|\varepsilon(t_1)\|^2 + d_5\max_{1\leqslant i\leqslant M-1}\big\{\|\Phi(t_j)\|^2\big\}.$$

Here $d_4 = \mu\exp(4d_3T)$, $d_5 = 2d_2T\exp(4d_3T)$.

By conditions (3.1) and (3.6), the right side of this inequality converges to zero as $\tau, h \to 0$. Thus, we have obtained the convergence of difference scheme (2.1)–(2.3) in the L_2 norm.

Using multiplicative inequality (1.7), we get

$$\|\hat{\varepsilon}\|^2_{C(\overline{Q}_h)} \leqslant \max_{t_j\in\bar{\omega}_\tau}\{\|\varepsilon(t_j)\|\}\max_{1\leqslant j\leqslant M}\{\|u_{x\nwarrow}(t_j)]| + \|u_{x\swarrow}(t_j)]|\} + \max_{t_j\in\bar{\omega}_\tau}\{\|\varepsilon(t_j)\|\}\max_{1\leqslant j\leqslant M}\{\|p_{x\nwarrow}(t_j)]| + \|p_{x\swarrow}(t_j)]|\}.$$

For sufficiently small τ, h, and $\tau \leqslant 5|a|M_1^2/2\nu M_2^2$, by equality (3.9) and condition (3.2) we have

$$\|\hat{u}_{x\nwarrow}]| + \|\hat{u}_{x\swarrow}]| \leqslant \left(4M_1^2 + \frac{2\tau\nu M_2^2}{|a|}\right)^{1/2} \leqslant 3M_1.$$

Thus by (3.7) and the estimate just obtained we get the inequality

$$\|\hat{\varepsilon}\|^2_{C(\overline{Q}_h)} \leqslant (3M_1 + M_3)\max_{t_j\in\bar{\omega}_\tau}\{\|\varepsilon(t_j)\|\},$$

where the right side converges to zero.

Hence inequality (3.10) follows with the constants $c_1 = (3M_1 + M_3)\sqrt{d_4}$ and $c_2 = (3M_1 + M_3)\sqrt{d_5}$. Since $M_3 = M_3(a, \lambda, M_1, \nu)$, we have that $c_i = c_i(a, \lambda, \nu, M_1, T)$, $i = 1, 2$. Theorem 3.1 is proved.

We now prove the stability of difference scheme (2.1)–(2.3) with respect to initial data in the C norm.

Let $u_1(x,t)$, $u_2(x,t)$ and p_1, p_2 be solutions of problems (1.1)–(1.3) and (2.1)–(2.3) with the initial data $u_{10}(x)$ and $u_{20}(x)$, respectively.

THEOREM 3.2. *Let the hypotheses of Theorem* 3.1 *be satisfied. Then there exist constants* τ_0 *and* h_0 *such that, for* $\tau \leqslant \tau_0$ *and* $h \leqslant h_0$, *the following estimate holds*:

$$\|p_1 - p_2\|^2_{C(\overline{Q}_h)} \leqslant c_3 \|u_{10} - u_{20}\|. \tag{3.11}$$

Here $c_3 = c_3(a, \lambda, \nu, T, \max_{t\in[0,T]}\{\|u_1(t)\|_{W_2^1}, \|u_2(t)\|_{W_2^1}\})$.

Proof. Denote $z = p_1 - p_2$. Then

$$z_t = \frac{ia}{h^2}(\bar{z} - 2\dot{z}) - i\lambda\big(|p_1|^2\dot{p}_1 - |p_2|^2\dot{p}_2\big), \qquad (x,t) \in Q_h,$$

$$z(x,0) = u_{10}(x) - u_{20}(x), \quad x \in \bar{\omega}_h, \qquad z(x_0,t) = z(x_N,t) = 0, \quad t \in \bar{\omega}_\tau.$$

Taking the inner product of both sides of the equation with $4\tau\dot{z}$, similarly to Theorem 3.1, we get the inequality

$$\begin{aligned}&\|z(t_{j+1})\|^2 + \|z(t_j)\|^2\\ &\leqslant \mu\big(\|z(t_1)\|^2 + \|z(t_0)\|^2\big) + d_1\tau\sum_{k=1}^{j}\big((\|z(t_{k+1})\|^2 + \|z(t_k)\|^2) + (\|z(t_k)\|^2 + \|z(t_{k-1})\|^2)\big);\end{aligned}$$

here $d_1 = (\lambda/(1-\nu))\|p_1\|_{C(\overline{Q}_h)}(\|p_1\|_{C(\overline{Q}_h)} + \|p_2\|_{C(\overline{Q}_h)})$. By (3.8), the quantity d_1 is bounded for sufficiently small τ and h.

By Gronwall inequality (1.9), we have the estimate

$$\max_{t_j\in\bar{\omega}_\tau}\|z(t_j)\|^2 \leqslant d_3(\|z(t_1)\|^2 + \|z(t_0)\|^2).$$

Here $d_3 = d_3(a, \lambda, T, \|p_1\|_{C(\overline{Q}_h)}, \|p_2\|_{C(\overline{Q}_h)}, \nu)$.

For sufficiently small τ, the condition $\|z(t_1)\| \leqslant 2\|z(t_0)\|$ is satisfied. The dependence of norms $\|p_1\|_{C(\overline{Q}_h)}$ and $\|p_2\|_{C(\overline{Q}_h)}$ on the quantities $\max_{t\in[0,T]}\|u_1(t)\|_{W_2^1}$ and $\max_{t\in[0,T]}\|u_2(t)\|_{W_2^1}$ can be proved as in Lemma 3.1. Hence using the last two estimates we obtain the stability in L_2:

$$\max_{t_j\in\bar{\omega}_\tau}\|z(t_j)\| \leqslant d_4\|z(t_0)\|;$$

here $d_4 = d_4(a, \lambda, T, \nu, \max_{t\in[0,T]}\{\|u_1(t)\|_{W_2^1}, \|u_2(t)\|_{W_2^1}\})$.

Using multiplicative inequality (1.7), similarly to Theorem 3.1, one can deduce (3.11). The theorem is proved.

4. GENERAL CASE

Consider the equation

$$\frac{\partial u}{\partial t} = ia\frac{\partial^2 u}{\partial x^2} + f(u, u^*)u, \tag{4.1}$$

where $f(u,u^*)$ is a polynomial with respect its arguments u and u^*. Introduce a continuous nondecreasing function $\varphi(u)$ satisfying the conditions

$$|f(u,u^*)| \leqslant \varphi(|u|), \qquad |D^j f(u,u^*)u| \leqslant \varphi(|u|), \qquad |j| = 1, 2; \tag{4.2}$$

here j is a two-dimensional vector, $|j| = j_1 + j_2$, $D^j = \partial^{|j|}/\partial u^{j_1}\partial u^{*j_2}$.

We relate to Eq. (4.1) the difference scheme

$$p_t = ia\frac{\bar{p} - 2\dot{p}}{h^2} + f(p, p^*)\dot{p}, \qquad (x, t) \in Q_h. \tag{4.3}$$

For the nonlinear grid function $f(v, v^*)\dot{v}$, the following estimates hold:

$$|f(v, v^*)\dot{v}, \dot{v}| \leqslant 0.5\varphi(\|v\|_{C_h})\big(\|\hat{v}\|^2 + \|\check{v}\|^2\big), \tag{4.4}$$

$$\begin{aligned} &|(f(v, v^*)\dot{v} - f(w, w^*)\dot{w}, \dot{v} - \dot{w})| \\ &\quad \leqslant \varphi\big(\max\{\|\check{v}\|_{C_h}, \|v\|_{C_h}, \|\hat{v}\|_{C_h}, \|w\|_{C_h}\}\big)\big(\|\check{v} - \check{w}\|^2 + \|v - w\|^2 + \|\hat{v} - \hat{w}\|^2\big). \end{aligned} \tag{4.5}$$

From (4.5) it follows that

$$|((f(v, v^*)\dot{v})_{\bar{x}}, \dot{v}_{\bar{x}}]| \leqslant \varphi(\max\{\|\check{v}\|_{C_h}, \|v\|_{C_h}, \|\hat{v}\|_{C_h}\})(\|\check{v}_{\bar{x}}]|^2 + \|v_{\bar{x}}]|^2 + \|\hat{v}_{\bar{x}}]|^2). \tag{4.6}$$

The following estimate also holds:

$$\begin{aligned} &|((f(v, v^*)\dot{v}(t_k) - f(w, w^*)\dot{w}(t_k))_{\bar{x}}, \dot{z}_{\bar{x}}(t_k)]| \\ &\quad \leqslant \Big(\max_{i=-1,0,1}\{1, \|v_{\bar{x}}(t_{k+i})]|, \|w_{\bar{x}}(t_{k+i})]|\}\Big) \\ &\qquad \times c\varphi\Big(\max_{i=-1,0,1}\{\|v(t_{k+i})\|_{C_h}, \|w(t_{k+i})\|_{C_h}\}\Big)\Big(\max_{i=-1,0,1}\{\|z_{\bar{x}}(t_{k+i})]|^2\}\Big), \end{aligned} \tag{4.7}$$

where c is a constant, and $z = v - w$.

Estimates (4.4)–(4.7) can be proved as similar estimates in [10].

We prove the convergence and stability in a way slightly different from that used in Secs 2 and 3. Instead of equalities of the form (2.4), (2.7), we will obtain *a priori* estimates of a new type introduced in [8–10].

Let p be a solution of difference scheme (4.3), (2.2), (2.3). Let condition (2.5) be satisfied. As in deducing (2.4), one can get the equality

$$\|\hat{p}\|^2 + \|p\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}, \hat{p}) = \|p\|^2 + \|\check{p}\|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{\check{p}}, p) + 4\tau\operatorname{Re}\sum_{i=1}^{N-1} f(p_i, p_i^*)|\dot{p}_i|^2 h.$$

Hence two estimates follow.

First, estimating summands of the type $\frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}, \hat{p})$ and the nonlinear part, we get the estimate

$$\|\hat{p}\|^2 + \|p\|^2 \leqslant \mu\big(\|p\|^2 + \|\check{p}\|^2\big) + \frac{2\tau}{1 - \nu}\varphi(\|p\|_{C_h})\big(\|\hat{p}\|^2 + \|\check{p}\|^2\big). \tag{4.8}$$

Second, summing the previous equalities over the layers t_k, $k = 1, \dots, j - 1$, and using estimate (4.4), we get the inequality

$$\begin{aligned} \|p(t_j)\|^2 + \|p(t_{j-1})\|^2 &\leqslant \mu\big(\|p(t_1)\|^2 + \|p(t_0)\|^2\big) + \frac{2\tau}{1 - \nu}\varphi(\|p\|_{C(Q_{t_j h})}) \\ &\quad \times \sum_{k=1}^{j-1}\big(\big(\|p(t_{k+1})\|^2 + \|p(t_k)\|^2\big) + \big(\|p(t_k)\|^2 + \|p(t_{k-1})\|^2\big)\big). \end{aligned} \tag{4.9}$$

Here $\|p\|_{C(Q_{t_j h})} = \max_{0\leqslant k\leqslant j}\{\|p(t_k)\|_{C_h}\}$.

Introduce the fictitious grid points of $(-h, \tau j)$ and $(1 + h, \tau j)$, where $j = 0, \dots, M$. Denote the function values at these points by v_{-1} and v_{N+1}. Define the values of the solution of the difference scheme at these points by $p_{-1} = -p_1$ and $p_{N+1} = -p_{N-1}$. This corresponds to zero boundary conditions (1.2) and to the condition $\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0$, where u is a solution of differential problem (4.1), (1.2), (1.3), extended to the boundary of the domain.

Now from (4.3) we get

$$p_{t\bar{x}} = ia\frac{\bar{p}_{\bar{x}} - 2\dot{p}_{\bar{x}}}{h^2} + (f(p,p^*)\dot{p})_{\bar{x}}, \qquad p = p_i^j, \quad j = 1,\dots,M-1, \quad i = 1,\dots,N.$$

Taking the inner product of both sides of the equation with $4\tau\dot{p}_{\bar{x}}$, we obtain the equality

$$\|\hat{p}_{\bar{x}}]|^2 + \|p_{\bar{x}}]|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{p}_{\bar{x}}, \hat{p}_{\bar{x}}] = \|p_{\bar{x}}]|^2 + \|\check{p}_{\bar{x}}]|^2 + \frac{2a\tau}{h^2}\operatorname{Im}(\bar{\check{p}}_{\bar{x}}, p_{\bar{x}}] + 4\tau\operatorname{Re}((f(p,p^*)\dot{p})_{\bar{x}}, \dot{p}_{\bar{x}}].$$

Hence two estimates follow.

First, using (1.8) and the estimate of the nonlinear part

$$|((f(p,p^*)\dot{p})_{\bar{x}}, \dot{p}_{\bar{x}}]| \leqslant (2/h)\|f(p,p^*)\|_{C_h}\|\dot{p}\|\|\dot{p}_{\bar{x}}]| \leqslant (1/h)\varphi(\|p\|_{C_h})\|\dot{p}_{\bar{x}}]|^2,$$

we have

$$\|\hat{p}_{\bar{x}}]|^2 + \|p_{\bar{x}}]|^2 \leqslant \mu\big(\|p_{\bar{x}}]|^2 + \|\check{p}_{\bar{x}}]|^2\big) + \frac{2\tau}{(1-\nu)h}\varphi(\|p\|_{C_h})\big(\|\hat{p}_{\bar{x}}]|^2 + \|\check{p}_{\bar{x}}]|^2\big). \tag{4.10}$$

Second, using estimate (4.6) one can write

$$\begin{aligned}\|p_{\bar{x}}(t_j)]|^2 + \|p_{\bar{x}}(t_{j-1})]|^2 &\leqslant \mu\big(\|p_{\bar{x}}(t_1)]|^2 + \|p_{\bar{x}}(t_0)]|^2\big) + \frac{4\tau}{1-\nu}\varphi(\|p\|_{C(Q_{t_j h})}) \\ &\quad\times\sum_{k=1}^{j-1}\big((\|p_{\bar{x}}(t_{k+1})]|^2 + \|p_{\bar{x}}(t_k)]|^2) + (\|p_{\bar{x}}(t_k)]|^2 + \|p_{\bar{x}}(t_{k-1})]|^2)\big).\end{aligned} \tag{4.11}$$

Summing inequalities (4.8) and (4.10) by condition (2.5), we obtain the estimate

$$\|\hat{p}\|^2_{W^1_{2h}} \leqslant \mu\big(\|p\|^2_{W^1_{2h}} + \|\check{p}\|^2_{W^1_{2h}}\big) + \frac{h\nu}{|a|(1-\nu)}\varphi(\|p\|_{C_h})\big(\|\hat{p}\|^2_{W^1_{2h}} + \|\check{p}\|^2_{W^1_{2h}}\big).$$

For $h \leqslant h_0$, $h_0 = |a|(1-\nu)/2\nu\varphi(\|p\|_{C_h})$ we have

$$\|\hat{p}\|^2_{W^1_{2h}} \leqslant (\mu+1)\max\big\{\|p\|^2_{W^1_{2h}}, \|\check{p}\|^2_{W^1_{2h}}\big\}. \tag{4.12}$$

Summing inequalities (4.9) and (4.11), we get

$$\begin{aligned}\|p(t_j)\|^2_{W^1_{2h}} + \|p(t_{j-1})\|^2_{W^1_{2h}} &\leqslant \mu\big(\|p(t_1)\|^2_{W^1_{2h}} + \|p(t_0)\|^2_{W^1_{2h}}\big) + \frac{4\tau}{1-\nu}\varphi(\|p\|_{C(Q_{t_j h})}) \\ &\quad\times\sum_{k=1}^{j-1}\Big(\big(\|p(t_{k+1})\|^2_{W^1_{2h}} + \|p(t_k)\|^2_{W^1_{2h}}\big) + \big(\|p(t_k)\|^2_{W^1_{2h}} + \|p(t_{k-1})\|^2_{W^1_{2h}}\big)\Big).\end{aligned} \tag{4.13}$$

For the approximation error, we will require that

$$\max_{t_j\in\omega_\tau}\big\{\|\Phi(t_j)\|_{W^1_{2h}}\big\} \longrightarrow 0, \qquad \tau, h \to 0. \tag{4.14}$$

This is a rather natural requirement, since, for $u \in C^4_3(Q)$, from condition (2.5) it follows that

$$\|\Phi_{\bar{x}}(t_j)]| \leqslant c\bigg(\sum_{i=1}^{N}\bigg|\frac{\tau^2 + h^2 + (\tau/h)^2}{h}\bigg|^2 h\bigg)^{1/2} \leqslant c(\nu^2h^3 + h + \nu^2 h) \leqslant ch.$$

For the error of the solution of problem (4.1), (1.2), (1.3), the same equations (3.4), (3.5) are satisfied with

$$\Psi = (f(u, u^*)\dot{u} - f(p, p^*)\dot{p}).$$

Using estimates (4.5) and (4.7), similarly to [10] and almost as in deducing estimate (4.13), one can write the inequality for the error

$$\begin{aligned} &\|\varepsilon(t_j)\|^2_{W^1_{2h}} + \|\varepsilon(t_{j-1})\|^2_{W^1_{2h}} \\ &\leqslant \mu\big(\|\varepsilon(t_1)\|^2_{W^1_{2h}} + \|\varepsilon(t_0)\|^2_{W^1_{2h}}\big) + \frac{2\tau}{1-\nu}\sum_{k=1}^{j-1}\|\Phi(t_k)\|^2_{W^1_{2h}} \\ &+ \frac{\tau}{1-\nu}\Big(1 + 4c\varphi(\max\{\|u\|_{C(\overline{Q})}, \|p\|_{C(Q_{t_j h})}\})\max_{0\leqslant k\leqslant j}\{1, \|u_{\bar{x}}(t_k)]|, \|p_{\bar{x}}(t_k)]|\}\Big) \\ &\times\sum_{k=1}^{j-1}\Big(\big(\|\varepsilon(t_{k+1})\|^2_{W^1_{2h}} + \|\varepsilon(t_k)\|^2_{W^1_{2h}}\big) + \big(\|\varepsilon(t_k)\|^2_{W^1_{2h}} + \|\varepsilon(t_{k-1})\|^2_{W^1_{2h}}\big)\Big). \end{aligned} \tag{4.15}$$

Let us prove a theorem on the convergence of the scheme.

THEOREM 4.1. *Let $u(x,t) \in C^4_3(Q)$, and let conditions (2.5), (3.6) be satisfied. Then the solution of difference problem (4.3), (2.2), (2.3) converges to the solution of problem (4.1), (1.2), (1.3) in the spaces W^1_{2h} and $C(\overline{Q}_h)$. There exist constants τ_0 and h_0 such that, for all $\tau \leqslant \tau_0$ and $h \leqslant h_0$, the following estimates hold:*

$$\max_{t_j\in\bar{\omega}_\tau}\{\|\varepsilon(t_j)\|_{W^1_{2h}}\} \leqslant c_4\|\varepsilon(t_1)\|_{W^1_{2h}} + c_5\max_{t_j\in\omega_\tau}\big\{\|\Phi(t_j)\|_{W^1_{2h}}\big\}, \tag{4.16}$$

$$\|\varepsilon\|_{C(\overline{Q}_h)} \leqslant 0.5c_4\|\varepsilon(t_1)\|_{W^1_{2h}} + 0.5c_5\max_{t_j\in\omega_\tau}\big\{\|\Phi(t_j)\|_{W^1_{2h}}\big\}; \tag{4.17}$$

here $c_i = c_i(a, \varphi, \nu, M_1, T)$, $i = 4, 5$.

Proof. The theorem can be proved almost similarly to the theorems of [8–10].

Let us show by induction the boundedness of the functions $p(t_j)$, $t_j \in \bar{\omega}_\tau$: $\|p(t_j)\|_{W^1_{2h}} \leqslant 2M_1$.

For $j = 0$, by (2.3) we can write $\|p(t_0)\|_{W^1_{2h}} \leqslant 2\|u(t_0)\|_{W^1_{2h}} \leqslant 2M_1$. For $j = 1$, by (3.6) one can assert that, for sufficiently small τ and h, the estimate $\|\varepsilon(t_1)\|_{W^1_{2h}} \leqslant M_1$ holds. Hence we obtain $\|p(t_1)\|_{W^1_{2h}} \leqslant 2M_1$.

Let for all $k = 0, \dots, j-1$, the estimates $\|p(t_k)\|_{W^1_{2h}} \leqslant 2M_1$ hold. Then by (1.8) we have $\|p\|_{C(Q_{t_{j-1}h})} \leqslant M_1$. But then, for all $h \leqslant h_0 = |a|(1-\nu)/(2\nu\varphi(M_1))$, one can apply (4.12), obtaining the estimate $\|p(t_j)\|^2_{W^1_{2h}} \leqslant 2(\mu+1)M_1$. From (1.8) it follows that $\|p\|_{C(Q_{t_j h})} \leqslant (\mu+1)M_1$ and $\max_{0\leqslant k\leqslant j}\{\|p(t_k)\|_{W^1_{2h}}\} \leqslant 2(\mu+1)M_1$. By (3.2), for all sufficiently small $\tau \leqslant 1/2d_1$ and h, one can apply Gronwall inequality (1.9) to (4.15). Here

$$d_1 = (1 + 4c\varphi((\mu+1)M_1)\max\{1, 2(\mu+1)M_1\})/(1-\nu).$$

We obtain the estimate

$$\|\varepsilon(t_j)\|^2_{W^1_{2h}} \leqslant d_2\|\varepsilon(t_1)\|^2_{W^1_{2h}} + d_3\max_{1\leqslant i\leqslant M-1}\big\{\|\Phi(t_i)\|^2_{W^1_{2h}}\big\}, \tag{4.18}$$

where $d_2 = \mu\exp(4d_1T)$, $d_3 = 4T\exp(4d_1T)/(1-\nu)$.

The right side of this estimate converges to zero as $\tau, h \to 0$. Therefore, one can find τ_0 and h_0 such that, for all $\tau \leqslant \tau_0$ and $h \leqslant h_0$, the estimate $\|\varepsilon(t_j)\|^2_{W^1_{2h}} \leqslant M_1$ holds. Hence it follows that $\|p(t_j)\|_{W^1_{2h}} \leqslant 2M_1$, and we are done.

Thus we have proved that $\max_{t_j\in\bar{\omega}_\tau}\{\|p(t_j)\|_{W^1_{2h}}\} \leqslant 2M_1$ and, consequently, estimate (4.18) holds for all t_j, $t_j \in \bar{\omega}_\tau$. In fact, we have estimate (4.16). Applying (1.8), we obtain estimate (4.17) with the constants $c_4 = \sqrt{d_2}$ and $c_5 = \sqrt{d_3}$. The theorem is proved.

As in [8–10], one also proves the stability of the scheme with respect to initial data. Let $u_1(x,t)$, $u_2(x,t)$, and p_1, p_2 be solutions of problems (4.1), (1.2), (1.3) and (4.3), (2.2), (2.3) with initial data $u_{10}(x)$ and $u_{20}(x)$, respectively.

THEOREM 4.2. *Let the hypotheses of Theorem* 4.1 *be satisfied. Then there exist constants* τ_0 *and* h_0 *such that, for* $\tau \leqslant \tau_0$ *and* $h \leqslant h_0$, *the following estimates hold*:

$$\max_{t_j\in\bar{\omega}_\tau}\left\{\|p_1(t_j)-p_2(t_j)\|_{W_{2h}^1}\right\}\leqslant c_6\|u_{10}-u_{20}\|_{W_{2h}^1}, \tag{4.19}$$

$$\|p_1-p_2\|_{C(\overline{Q}_h)}\leqslant 0.5c_6\|u_{10}-u_{20}\|_{W_{2h}^1}. \tag{4.20}$$

Here $c_6=c_6(a,\varphi,\nu,T,\max_{t\in[0:T]}\{\|u_1(t)\|_{W_2^1},\|u_2(t)\|_{W_2^1}\})$.

Proof: As in the proofs of Theorems 3.2 and 4.1, applying estimates (4.5) and (4.7), we obtain the inequality

$$\begin{aligned}\|z(t_j)\|^2_{W^1_{2h}}+\|z(t_{j-1})\|^2_{W^1_{2h}}&\leqslant\mu\big(\|z(t_1)\|^2_{W^1_{2h}}+\|z(t_0)\|^2_{W^1_{2h}}\big)\\&\quad+\tau d_1\sum_{k=1}^{j-1}\Big(\big(\|z(t_{k+1})\|^2_{W^1_{2h}}+\|z(t_k)\|^2_{W^1_{2h}}\big)+\big(\|z(t_k)\|^2_{W^1_{2h}}+\|z(t_{k-1})\|^2_{W^1_{2h}}\big)\Big),\end{aligned}$$

where $d_1=4c\varphi(\max\{\|p_1\|_{C(\overline{Q}_h)},\|p_2\|_{C(\overline{Q}_h)}\})\max_{t_j\in\bar{\omega}_\tau}\{1,\|p_1(t_j)\|_{W^1_{2h}},\|p_2(t_j)\|_{W^1_{2h}}\}$, $z=p_1-p_2$.

In Theorem 4.1 we showed the boundedness of norms $\|p\|_{C(\overline{Q}_h)}\leqslant M_1$ and $\max_{t_j\in\bar{\omega}_\tau}\{\|p(t_j)\|_{W^1_{2h}}\}\leqslant 2M_1$. Therefore, for the inequality obtained, one can apply Gronwall inequality (1.9) for sufficiently small τ and h, and thus get estimates (4.19) and (4.20). The theorem is proved.

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(*Translated by V. Mackevičius*)