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ON CONVERGENCE AND STABILITY OF DIFFERENCE SCHEMES FOR NONLINEAR SCHRÖDINGER TYPE EQUATIONS

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Abstract—The first and the second boundary value problems for a system of nonlinear equations of Schrödinger type

$$\frac{\partial \mathbf{u}}{\partial t} = A\frac{\partial \mathbf{u}}{\partial x} + iB\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{f}(\mathbf{u}, \mathbf{u}^*)$$

are investigated. Here A and B are real and real positive definite, respectively, constant diagonal matrices, $\mathbf{f}$ is a polynomial complex vector function. We do not try to get rid of the addend $A\frac{\partial \mathbf{u}}{\partial x}$. Using a new type of *a priori* estimates, convergence and stability of difference schemes of Crank–Nicolson type for these problems in W_2^1 norm are proved. No restrictions on the ratio of time and space grid steps are assumed.

INTRODUCTION

We consider a class of evolution equations. We prove the convergence and stability of a conservative difference scheme of Crank–Nicolson type for the nonlinear Schrödinger equation system

$$\frac{\partial \mathbf{u}}{\partial t} = A\frac{\partial \mathbf{u}}{\partial x} + iB\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{f}(\mathbf{u}, \mathbf{u}^*).$$

Such equations appear in many models of nonlinear optics [1, 2], in models of energy transfer in molecular systems [3, 4], and they are used in plasma physics, quantum mechanics, and other fields of science.

There are a lot of studies in the field of initial value problems for the Schrödinger equations, but the theory for the initial–boundary problems is less developed. In a majority of the works the first boundary problem [6–8, 11, 12] is considered. The second boundary problem is considered in [9]. Some authors solve problems using finite element methods [11, 12], the others use difference schemes [6–10].

The main difficulties appear due to a nonlinear function $\mathbf{f}(\mathbf{u}, \mathbf{u}^*)$. In this work, as in many models, the nonlinear part is polynomial. It appeared that the existence or blowing up of the continuous or discrete solution of the Schrödinger equation depends on the degree of nonlinearity [5]. In this work we consider the period of time when the solution of continuous problem exists. The convergence and stability of the difference schemes are proved using a new type of *a priori* estimates and technique developed in papers [6–9]. No restrictions on the ratio of the grid steps are assumed.

The present work differs from the works mentioned above because we do not try to get rid of the addend $A\frac{\partial \mathbf{u}}{\partial x}$. Also, due to this addend, we can not use eigenvalue functions to obtain *a priori* estimates. We know that in the case of the first boundary problem we can get rid of the addend $A\frac{\partial \mathbf{u}}{\partial x}$ in our equation using a simple transformation. But in the case of the second boundary problem such transformation does not work. We prove in this paper that one can use the technique [6–9] even in this case.

In Section 1 we find *a priori* estimates for the continuous problem. In Section 2 we prove some properties for the nonlinear part and then find *a priori* estimates for a solution of a difference scheme. In Section 3 we

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prove the convergence of iterative process for a nonlinear difference scheme and get some estimates. In Section 4 we obtain the main result of the paper – the convergence and stability of the scheme in the space W_2^1.

1. STATEMENT OF THE PROBLEM. A PRIORI ESTIMATES

We consider boundary value problems for the nonlinear Schrödinger equation system:

$$\frac{\partial \mathbf{u}}{\partial t} = A\frac{\partial \mathbf{u}}{\partial x} + iB\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{f}(\mathbf{u}, \mathbf{u}^*), \qquad (x, t) \in Q, \tag{1.1}$$

with the initial condition

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \qquad x \in \overline{\Omega}, \tag{1.2}$$

and boundary conditions

$$\mathbf{u}(0, t) = \mathbf{u}(1, t) = 0, \qquad t \in [0; T], \tag{1.3}$$

or

$$\frac{\partial \mathbf{u}}{\partial x}(0, t) = \frac{\partial \mathbf{u}}{\partial x}(1, t) = 0, \qquad t \in [0; T]. \tag{1.4}$$

Here $\Omega = (0; 1)$, $Q = \Omega \times (0; T)$, A, B are real constant diagonal matrices, $B > 0$, $\mathbf{u}(x, t) = (u_1, u_2, \ldots, u_n)$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, where u_i, f_i are complex-valued. We assume that functions $f_i(\mathbf{u}, \mathbf{u}^*)$ are polynomials, that is,

$$f_i(\mathbf{u}, \mathbf{u}^*) = \sum_{k=1}^{s_i} \gamma_{ik}\mathbf{u}^{\beta_{ik}}, \qquad i = 1, \ldots, n, \quad \forall i, \forall k \quad |\beta_{ik}| \geqslant 1, \tag{1.5}$$

here $\mathbf{u}^j = u_1^{j_1} \cdots u_n^{j_n} u_1^{*j_{n+1}} \cdots u_n^{*j_{2n}}$, $|j| = j_1 + \cdots + j_n + j_{n+1} + \cdots + j_{2n}$.

Let $\varphi(y) = \gamma s \beta^2 (y + 1)^{\beta - 1}$, where $\gamma = \max_{ik}\{|\gamma_{ik}|\}$, $\beta = \max_{ik}\{|\beta_{ik}|\}$, $s = \max_i\{s_i\}$. Then

$$|f_i(\mathbf{u}, \mathbf{u}^*)| \leqslant |\mathbf{u}|\varphi(|\mathbf{u}|), \qquad \left|D^j f_i(\mathbf{u}, \mathbf{u}^*)\right| \leqslant \varphi(|\mathbf{u}|) \qquad \forall i, \quad |j| = 1, 2, \tag{1.6}$$

here $|\mathbf{u}| = \max\{|u_i|\}$, $D^j = \partial^{|j|}/\partial u_1^{j_1} \cdots \partial u_n^{j_n} \partial u_1^{*j_{n+1}} \cdots \partial u_n^{*j_{2n}}$, $\varphi(y)$ is a continuous nondecreasing function.

We assume that $\mathbf{u}_0 \in \mathbf{W}_2^2 \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ for the problem (1.1)–(1.3) and there exists a solution $\mathbf{u}(x, t)$ such that

$$\mathbf{u} \in \mathbf{L}_\infty\left(0, T; \mathbf{W}_2^2 \cap \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)\right), \qquad \|\mathbf{u}\|_{C(\overline{Q})} = \max_i\{\|u_i\|_{C(\overline{Q})}\} < \infty. \tag{1.7}$$

Also we assume that $\mathbf{u}_0 \in \mathbf{W}_2^2(\Omega)$ for the problem (1.1), (1.2), (1.4) and there exists a solution $\mathbf{u}(x, t)$ such that

$$\mathbf{u} \in \mathbf{L}_\infty\left(0, T; \mathbf{W}_2^2(\Omega)\right), \qquad \|\mathbf{u}\|_{C(\overline{Q})} = \max_i\{\|u_i\|_{C(\overline{Q})}\} < \infty. \tag{1.8}$$

Here L_2, W_2^1, W_2^2 are Sobolev spaces; $\mathbf{L}_2$, $\mathbf{W}_2^1$, $\mathbf{W}_2^2$ are spaces of n components, that is $\mathbf{B} = \mathrm{B} \times \cdots \times \mathrm{B}$, where B is one of the Sobolev spaces mentioned above; the norms $\|\mathbf{v}\|_{\mathbf{B}}^2 = \sum_{i=1}^n \|v_i\|_{\mathrm{B}}^2$.

We use the well-known imbedding theorem

$$w \in \overset{\circ}{W}{}_2^1(\Omega) \Rightarrow \|w\| \leqslant c_1 \left\|\frac{\partial w}{\partial x}\right\|; \tag{1.9}$$

here $\|w\| = \|w\|_{L_2}$ and $c_1 = c_1(\operatorname{mes}\Omega)$.

Let us denote $b = \min_i\{b_i\}$, $\bar{b} = \max_i\{b_i\}$, $a = \max_i\{|a_i|\}$, where a_i, b_i are the elements of diagonal matrices A, B, respectively.

LEMMA 1.1. *Assume that* (1.6), (1.7), *or* (1.8) *are satisfied, then the following estimates hold: for the solution of* (1.1)–(1.3)

$$\|\mathbf{u}(t)\|_{W_2^1} \leqslant d_1 \|\mathbf{u}(0)\|_{W_2^1} \tag{1.10}$$

and for the solution of (1.1), (1.2), (1.4)

$$\|\mathbf{u}(t)\|_{W_2^1} \leqslant d_2 \|\mathbf{u}(0)\|_{W_2^1}, \tag{1.11}$$

here $d_1 = d_1(a, b, c_1, n, T, \varphi(\|\mathbf{u}\|_{C(\overline{Q})}))$, $d_2 = d_2(a, n, T, \varphi(\|\mathbf{u}\|_{C(\overline{Q})}))$.

Proof. For all j we multiply both sides of (1.1) by u_j^*, then integrate over Ω and take real parts. Integrating by parts we obtain

$$0.5\frac{d}{dt}\|u_j\|^2 = 0.5a_j\left(|u_j|^2\Big|_0^1\right) + \operatorname{Re} ib_j\left(\frac{\partial u_j}{\partial x}u_j^*\Big|_0^1\right) - \operatorname{Re} ib_j\left\|\frac{\partial u_j}{\partial x}\right\|^2 + \operatorname{Re}\int_\Omega f_j u_j^* dx$$
$$= 0.5a_j \operatorname{Re}\left(|u_j|^2\Big|_0^1\right) + \operatorname{Re}\int_\Omega f_j u_j^* dx.$$

If (1.3) is satisfied, we estimate $|\operatorname{Re}\int_\Omega f_j u_j^* dx| \leqslant \int_\Omega |f_j||u_j^*|dx$, then sum these inequalities and then integrate over the interval $[0; t]$, use (1.6), and after that we obtain the estimate (a)

$$\|\mathbf{u}(t)\|^2 \leqslant \|\mathbf{u}(0)\|^2 + 2n\varphi\left(\|\mathbf{u}\|_{C(\overline{Q})}\right)\int_0^t \|\mathbf{u}(\tau)\|^2 d\tau.$$

If (1.4) holds, then we estimate

$$\left|\int_\Omega \frac{\partial}{\partial x}|u_j|^2 dx\right| \leqslant 2\|u_j\|\left\|\frac{\partial u_j}{\partial x}\right\| \leqslant \|u_j\|^2 + \left\|\frac{\partial u_j}{\partial x}\right\|^2 \leqslant \|u_j\|^2_{W_2^1}$$

and obtain the estimate (b)

$$\|\mathbf{u}(t)\|^2 - \|\mathbf{u}(0)\|^2 \leqslant a\int_0^t \|\mathbf{u}(\tau)\|^2_{W_2^1} d\tau + 2n\varphi\left(\|\mathbf{u}\|_{C(\overline{Q})}\right)\int_0^t \|\mathbf{u}(\tau)\|^2 d\tau.$$

Now for all j we multiply both sides of (1.1) by $\frac{\partial u_j^*}{\partial t}$, then integrate over Ω and take imaginary parts. As a result we get

$$\operatorname{Im}\int_\Omega \left|\frac{\partial u_j}{\partial t}\right|^2 dx = a_j \operatorname{Im}\int_\Omega \frac{\partial u_j}{\partial x}\frac{\partial u_j^*}{\partial t}dx + b_j \operatorname{Re}\int_\Omega \frac{\partial^2 u_j}{\partial x^2}\frac{\partial u_j^*}{\partial t}dx + \operatorname{Im}\int_\Omega f_j(\mathbf{u}, \mathbf{u}^*)\frac{\partial u_j^*}{\partial t}dx. \tag{1.12}$$

We take a conjugate equation

$$\frac{\partial u_j^*}{\partial t} = a_j\frac{\partial u_j^*}{\partial x} - ib_j\frac{\partial^2 u_j^*}{\partial x^2} + f_j^*(\mathbf{u}, \mathbf{u}^*)$$

and in the case of problem (1.1)–(1.3) we substitute $\partial u_j^*/\partial t$ by the right-hand side of the conjugate equation only in the third term of the right-hand side of (1.12). In the case of problem (1.1), (1.2), (1.4) we make the same substitution in the first and in the third terms of the same side of (1.12). In the case of problem (1.1)–(1.3) we integrate by parts, then integrate over the interval $[0;t]$, use (1.9) and obtain the inequalities

$$\left\|\frac{\partial u_j}{\partial x}(t)\right\|^2 - \left\|\frac{\partial u_j}{\partial x}(0)\right\|^2 \leqslant a/b\left\|\frac{\partial u_j}{\partial x}(t)\right\|\|u_j(t)\| + a/b\left\|\frac{\partial u_j}{\partial x}(0)\right\|\|u_j(0)\| + 2(1+ac_1/b)\int\limits_0^t \left\|\frac{\partial f_j}{\partial x}(\tau)\right\|\left\|\frac{\partial u_j}{\partial x}(\tau)\right\|d\tau.$$

We use (1.9), ε-inequalities [15] with $\varepsilon = 0.5$, the inequality

$$\left\|\frac{\partial f_j}{\partial x}(\tau)\right\| = \left\|\sum_{k=1}^{n} \frac{\partial f_j}{\partial u_k}\frac{\partial u_k}{\partial x} + \frac{\partial f_j}{\partial u_k^*}\frac{\partial u_k^*}{\partial x}\right\| \leqslant 2\varphi\left(\|\mathbf{u}(\tau)\|_C\right)\sum_{j=1}^{n}\left\|\frac{\partial u_j}{\partial x}(\tau)\right\|,$$

which follows from (1.6), sum the obtained inequalities for $j = 1, \ldots, n$, use (a), and get the estimate

$$\|\mathbf{u}(t)\|^2_{W_2^1} \leqslant \left(a^2/b^2 + 2ac_1/b + 2\right)\|\mathbf{u}(0)\|^2_{W_2^1} + 2n\left(a^2/b^2 + 4ac_1/b + 4\right)\varphi\left(\|\mathbf{u}\|_{C(\overline{Q})}\right)\int\limits_0^t \|\mathbf{u}(\tau)\|^2_{W_2^1}d\tau.$$

Using the Bellman–Gronwall lemma [13] we obtain estimate (1.10) with

$$d_1 = \sqrt{a^2/b^2 + 2ac_1/b + 2}\exp\left(nT\left(4 + 4ac_1/b + a^2/b^2\right)\varphi\left(\|\mathbf{u}\|_{C(\overline{Q})}\right)\right).$$

Similarly, for problem (1.1), (1.2), (1.4) we have the inequality

$$\left\|\frac{\partial \mathbf{u}}{\partial x}(t)\right\|^2 \leqslant \left\|\frac{\partial \mathbf{u}}{\partial x}(0)\right\|^2 + 4n\varphi\left(\|\mathbf{u}\|_{C(\overline{Q})}\right)\int\limits_0^t \left\|\frac{\partial \mathbf{u}}{\partial x}(\tau)\right\|^2 d\tau,$$

then add (b) and get the estimate

$$\|\mathbf{u}(t)\|^2_{W_2^1} \leqslant \|\mathbf{u}(0)\|^2_{W_2^1} + \left(a + 4n\varphi\left(\|\mathbf{u}\|_{C(\overline{Q})}\right)\right)\int\limits_0^t \|\mathbf{u}(\tau)\|^2_{W_2^1}d\tau.$$

From this estimate (1.11) with $d_2 = \exp(T(0.5a + 2n\varphi(\|\mathbf{u}\|_{C(\overline{Q})})))$ follows. The lemma is proved.

2. DISCRETE PROBLEM. A PRIORI ESTIMATES

We introduce the uniform grids with steps τ and h in the domain $\overline{Q}$. $\overline{Q}_{1h} = \overline{\omega}_{1h} * \overline{\omega}_\tau$ and $Q_{1h} = \omega_{1h} * \omega_\tau$ are grids in the case of the first problem, $\overline{Q}_{2h} = \overline{\omega}_{2h} * \overline{\omega}_\tau$ and $Q_{2h} = \omega_{2h} * \omega_\tau$ are grids in the case of the second problem. Here $h = 1/N$, $\tau = T/M$, $t_i = i\tau$, $\overline{\omega}_\tau = \{t_i; i = 0, \ldots, M\}$, $\omega_\tau = \{t_i; i = 0, \ldots, M-1\}$. We denote $x_i = ih$, $\overline{\omega}_{1h} = \{x_i; i = 0, \ldots, N\}$, $\omega_{1h} = \{x_i; i = 1, \ldots, N-1\}$ in the first case, and $x_i = (i-0.5)h$, $\overline{\omega}_{2h} = \{x_i; i = 0, \ldots, N+1\}$, $\omega_{2h} = \{x_i; i = 1, \ldots, N\}$ in the second case. Here, in the second case, we defined the fictitious space grid points $x_0 = -0.5h$ and $x_{N+1} = 1 + 0.5h$.

We will use grid analogues L_{2h}, W^1_{2h}, W^2_{2h}, $\mathbf{L}_{2h}$, $\mathbf{W}^1_{2h}$, $\mathbf{W}^2_{2h}$ of Sobolev spaces and C_h denotes the analogue of the space $C(\overline{Q})$. Let us define scalar products at the grid $\overline{\omega}_{1h}$:

$$(u,v) = \sum_{i=1}^{N-1} u_i v_i^* h, \qquad [u,v] = (u,v) + (h/2)(u_0 v_0^* + u_N v_N^*), \qquad (u,v] = \sum_{i=1}^{N} u_i v_i^* h,$$

similarly at the grid $\overline{\omega}_{2h}$:

$$(u,v)=\sum_{i=2}^{N-1}u_iv_i^*h,\qquad [u,v]=\sum_{i=1}^{N}u_iv_i^*h,\qquad (u,v]=\sum_{i=2}^{N}u_iv_i^*h,\qquad [u,v)=\sum_{i=1}^{N}u_iv_i^*h.$$

The norms in both grids are denoted as follows:

$$|[u]|^2=[u,u],\qquad \|u]|^2=(u,u],\qquad \|u\|^2=(u,u),$$

$$\|u\|^2_{W^1_{2h}}=|[u]|^2+\|u_{\bar{x}}]|^2,\qquad \|u\|^2_{W^2_{2h}}=\|u\|^2_{W^1_{2h}}+\|u_{\bar{x}x}\|^2.$$

The norms in the spaces $\mathbf{L}_{2h}$, $\mathbf{W}^1_{2h}$, $\mathbf{W}^2_{2h}$ are defined in the same way as earlier.

We denote $p=p_i^j=p(x_i,t_j)$, $\hat{p}=p_i^{j+1}$, $\dot{p}=(p+\hat{p})/2$, $p_t=(\hat{p}-p)/\tau$, $p_{\bar{x}x}=(p_x-p_{\bar{x}})/h$, $p_x=(p^j_{i+1}-p^j_i)/h$, $p_{\bar{x}}=(p^j_i-p^j_{i-1})/h$; $p_{\dot{x}}=(p^j_{i+1}-p^j_{i-1})/2h$, $\mathbf{p}=(p_1,\dots,p_n)$.

We relate problem (1.1)–(1.3) with the following Crank–Nicolson type symmetric difference scheme:

$$\mathbf{p}_t=A\dot{\mathbf{p}}_{\dot{x}}+iB\dot{\mathbf{p}}_{\bar{x}x}+\mathbf{f}(\dot{\mathbf{p}},\dot{\mathbf{p}}^*),\qquad (x,t)\in Q_{1h},\tag{2.1}$$

$$\mathbf{p}(x,0)=\mathbf{u}_0(x),\qquad x\in\overline{\omega}_{1h},\tag{2.2}$$

$$\mathbf{p}(x_0,t)=\mathbf{p}(x_N,t)=0,\qquad t\in\overline{\omega}_\tau.\tag{2.3}$$

Also we relate problem (1.1), (1.2), (1.4) with a similar scheme:

$$\mathbf{p}_t=A\dot{\mathbf{p}}_{\dot{x}}+iB\dot{\mathbf{p}}_{\bar{x}x}+\mathbf{f}(\dot{\mathbf{p}},\dot{\mathbf{p}}^*),\qquad (x,t)\in Q_{2h},\tag{2.4}$$

$$\mathbf{p}(x,0)=\mathbf{u}_0(x),\qquad x\in\omega_{2h},\tag{2.5}$$

$$\mathbf{p}(x_0,t)=\mathbf{p}(x_1,t),\qquad \mathbf{p}(x_N,t)=\mathbf{p}(x_{N+1}),\qquad t\in\overline{\omega}_\tau.\tag{2.6}$$

In the case of the first problem, we often deal with functions $u\in\overset{\circ}{W}{}^1_{2h}$, that is, $u_0^j=u_N^j=0$ and $\|u\|=|[u]|$. The following well-known inequalities are valid for such functions [15]

$$\|u\|=|[u]|\leqslant c_2\|u_{\bar{x}}]|,\qquad c_2=c_2(\operatorname{mes}\Omega).\tag{2.7}$$

For functions from W^1_{2h} we have [15]

$$\|u\|_C\leqslant c_3\|u\|_{W^1_{2h}},\qquad c_3=c_3(\operatorname{mes}\Omega)\tag{2.8}$$

Before deriving *a priori* estimates in the discrete case, we prove some properties of the function $\mathbf{f}(\mathbf{u},\mathbf{u}^*)$. For us it is convienent to denote by $|(u)|$ any of the norms in the space L_{2h}, introduced above.

LEMMA 2.1. *Assume that* $\mathbf{f}(\mathbf{u},\mathbf{u}^*)$ *satisfies* (1.5), (1.6) *and* $\mathbf{w},\mathbf{v}\in\mathbf{L}_{2h}$. *Then* $\forall i=1,\dots,n$, *for both problems we have*

$$|(f_i(\mathbf{v},\mathbf{v}^*))|\leqslant\varphi(\|\mathbf{v}\|_C)\,|(\mathbf{v})|,\tag{2.9}$$

$$|(f_i(\mathbf{v},\mathbf{v}^*)-f_i(\mathbf{w},\mathbf{w}^*))|\leqslant 2\sqrt{n}\varphi(\max\{\|\mathbf{v}\|_C,\|\mathbf{w}\|_C\})\,|(\mathbf{v}-\mathbf{w})|.\tag{2.10}$$

Proof. At any point x_k of the grids ω_{1h} or ω_{2h} we have the estimates

$$|f_i(\mathbf{v}_k, \mathbf{v}_k^*)|^2 \leqslant \varphi^2(|\mathbf{v}_k|)\,|\mathbf{v}_k|^2 \leqslant \varphi^2(\|\mathbf{v}\|_C) \sum_{i=1}^{n} |v_{ik}|^2.$$

They lead to the estimates

$$|(f_i(\mathbf{v}, \mathbf{v}^*))|^2 \leqslant \varphi^2(\|\mathbf{v}\|_C) \sum_{k=l_1}^{l_2} \sum_{i=1}^{n} |v_{ik}|^2 h.$$

Hence we get (2.9).

Now at that same space grid point x_k we can write

$$|f_i(\mathbf{v}_k, \mathbf{v}_k^*) - f_i(\mathbf{w}_k, \mathbf{w}_k^*)|^2 \leqslant \left| \sum_{j=1}^{2n} f_i(\xi_{j-1}(\mathbf{v}_k, \mathbf{w}_k)) - f_i(\xi_j(\mathbf{v}_k, \mathbf{w}_k)) \right|^2$$
$$\leqslant 2n \sum_{j=1}^{2n} |f_i(\xi_{j-1}) - f_i(\xi_j)|^2,$$

where $\xi_j(\mathbf{v}_k, \mathbf{w}_k)$ is a $2n$ dimensional vector: $\xi_0 = (v_{1.k}, \ldots, v_{n.k}, v_{1.k}^*, \ldots, v_{n.k}^*)$;

$$\xi_j = \left(w_{1.k}, \ldots, w_{j.k}, v_{j+1.k}, \ldots, v_{n.k}^*\right), \qquad \text{if } j = 1, \ldots, n;$$

$$\xi_j = \left(w_{1.k}, \ldots, w_{j-n.k}^*, v_{j-n+1.k}^*, \ldots, v_{n.k}^*\right), \qquad \text{if } j = n+1, \ldots, 2n.$$

Let $j \leqslant n$, then

$$|f_i(\xi_{j-1}) - f_i(\xi_j)| = \left| \sum_{l=1}^{s_i} \gamma_{il} w_{1.k}^{\beta_{il.1}} \cdots \left(v_{j.k}^{\beta_{il.j}} - w_{j.k}^{\beta_{il.j}}\right) \cdots v_{n.k}^{*\beta_{il.2n}} \right|$$
$$\leqslant |v_{j.k} - w_{j.k}| \left| \sum_{l=1}^{s_i} \gamma_{il} w_{1.k}^{\beta_{il.1}} \cdots \left(\sum_{m=0}^{\beta_{il.j}-1} w_{j.k}^{m} v_{j.k}^{\beta_{il.j}-m-1} \right) \cdots v_{n.k}^{*\beta_{il.2n}} \right|$$
$$\leqslant |v_{j.k} - w_{j.k}|\,\varphi(\max\{|\mathbf{v}_k|, |\mathbf{w}_k|\}).$$

If $j > n$, we can obtain a similar result. Hence, we can write

$$|f_i(\mathbf{v}_k) - f_i(\mathbf{w}_k)|^2 \leqslant 4n\varphi^2(\max\{|\mathbf{v}_k|, |\mathbf{w}_k|\}) \sum_{j=1}^{n} |v_{j.k} - w_{j.k}|^2,$$

and obtain (2.10). The lemma is proved.

We can get some corollaries from this lemma:

COROLLARY 2.1. *Under the conditions of Lemma* 2.1 *for all* $i = 1, \ldots, n$ *and for both problems we have*

$$\|f_{i\bar{x}}(\mathbf{v}, \mathbf{v}^*)]| \leqslant 2\sqrt{n}\varphi(\|\mathbf{v}\|_C)\,\|\mathbf{v}_{\bar{x}}]|, \tag{2.11}$$

$$\|f_i(\mathbf{v}, \mathbf{v}^*)\|_{W_{2h}^1} \leqslant 2\sqrt{n}\varphi(\|\mathbf{v}\|_C)\,\|\mathbf{v}\|_{W_{2h}^1}. \tag{2.12}$$

Proof. We take $\mathbf{v}_k$, $\mathbf{v}_{k-1}$ and the norm $\|\ ]|$ instead of $\mathbf{v}_k$, $\mathbf{w}_k$ and $|(\ )|$, respectively in Lemma 2.1. This leads to (2.11). Formula (2.12) follows from (2.9) and (2.11).

LEMMA 2.2. *Assume that* $\mathbf{f}(\mathbf{u},\mathbf{u}^*)$ *satisfies* (1.5), (1.6) *and* $\mathbf{w},\mathbf{v}\in\mathbf{W}^1_{2h}$. *Then* $\forall i=1,\dots,n$, *for both problems we have*

$$\|(f_i(\mathbf{v},\mathbf{v}^*)-f_i(\mathbf{w},\mathbf{w}^*))_{\bar{x}}]|\leqslant 2\sqrt{2n}\varphi\big(\max\{\|\mathbf{v}\|_C,\|\mathbf{w}\|_C\}\big)\left(\|\mathbf{z}_{\bar{x}}]|+2c_3\|\mathbf{w}_{\bar{x}}]|\|\mathbf{z}\|_{W^1_{2h}}\right), \tag{2.13}$$

here $\mathbf{z}=\mathbf{v}-\mathbf{w}$.

Proof. Similarly as in Lemma 2.1, at every space grid point x_k we have

$$\begin{aligned}&(1/h)|(f_i(\mathbf{v}_k,\mathbf{v}^*_k)-f_i(\mathbf{v}_{k-1},\mathbf{v}^*_{k-1}))-(f_i(\mathbf{w}_k,\mathbf{w}^*_k)-f_i(\mathbf{w}_{k-1},\mathbf{w}_{k-1}))|\\
&\leqslant(1/h)\sum_{j=1}^{2n}\big|(f_i(\xi_{j-1}(\mathbf{v}_k,\mathbf{v}_{k-1}))-f_i(\xi_j(\mathbf{v}_k,\mathbf{v}_{k-1})))-(f_i(\xi_{j-1}(\mathbf{w}_k,\mathbf{w}_{k-1}))-f_i(\xi_j(\mathbf{w}_k,\mathbf{w}_{k-1})))\big|\\
&\leqslant\gamma\sum_{j=1}^{2n}\sum_{l=1}^{s_i}\left|\left(v_{1,k-1}^{\beta_{il,1}}\cdots\sum_{m=0}^{\beta_{il,j}-1}v_{j,k-1}^{\beta_{il,j}-m-1}v_{j,k}^{m}\cdots v_{n,k}^{*\beta_{il,2n}}\right)v_{j,k\bar{x}}\right.\\
&\quad\left.-\left(w_{1,k-1}^{\beta_{il,1}}\cdots\sum_{m=0}^{\beta_{il,j}-1}w_{j,k-1}^{\beta_{il,j}-m-1}w_{j,k}^{m}\cdots w_{n,k}^{*\beta_{il,2n}}\right)w_{j,k\bar{x}}\right|\\
&\leqslant\gamma\sum_{j=1}^{2n}\sum_{l=1}^{s_i}\left(\left|z_{j,k\bar{x}}(v_{1,k-1}^{\beta_{il,1}}\cdots v_{n,k}^{*\beta_{il,2n}})\right|+\left|w_{j,k\bar{x}}(v_{1,k-1}^{\beta_{il,1}}\cdots v_{n,k}^{*\beta_{il,2n}}-w_{1,k-1}^{\beta_{il,1}}\cdots w_{n,k}^{*\beta_{il,2n}})\right|\right).\end{aligned}$$

We estimate the first summand using the expression of function φ, the second one – in the same way as the similar difference in Lemma 2.1. We obtain

$$|(f_i(\mathbf{v},\mathbf{v}^*)-f_i(\mathbf{w},\mathbf{w}^*))_{k\bar{x}}|\leqslant 2\varphi\left(\|\mathbf{v}\|_C\right)\sum_{j=1}^{n}|z_{j,k\bar{x}}|+4\varphi\left(\max\{\|\mathbf{v}\|_C,\|\mathbf{w}\|_C\}\right)\sum_{j=1}^{n}|w_{j,k\bar{x}}|\sum_{r=1}^{n}\|z_r\|_C.$$

Using (2.8), we can obtain the following estimate:

$$\|(f_i(\mathbf{v},\mathbf{v}^*)-f_i(\mathbf{w},\mathbf{w}^*))_{\bar{x}}]|^2\leqslant 8n\varphi^2\left(\|\mathbf{v}\|_C\right)\|\mathbf{z}_{\bar{x}}]|^2+32n^2c_3^2\varphi^2\left(\max\{\|\mathbf{v}\|_C,\|\mathbf{w}\|_C\}\right)\|\mathbf{w}_{\bar{x}}]|\|\mathbf{z}\|^2_{W^1_{2h}}.$$

From this (2.13) follows. Lemma 2.2 is proved.

COROLLARY 2.2. *Under the conditions of Lemma* 2.2 *for all* $i=1,\dots,n$, *for both problems we have*

$$\|(f_i(\mathbf{v},\mathbf{v}^*)-f_i(\mathbf{w},\mathbf{w}^*))\|_{W^1_{2h}}\leqslant 2\sqrt{2n}\varphi\big(\max\{\|\mathbf{v}\|_C,\|\mathbf{w}\|_C\}\big)\big(1+2c_3\|\mathbf{w}\|_{W^1_{2h}}\big)\|\mathbf{v}-\mathbf{w}\|_{W^1_{2h}}. \tag{2.14}$$

Proof. This inequality follows from Lemmas 2.1 and 2.2.

LEMMA 2.3 (Difference Gronwall inequality). *Let functions* $A^{(1)}\geqslant 0$, $A^{(2)}\geqslant 0$, $F\geqslant 0$ *be defined on the grid* ω_τ, *and let the function* $Y\geqslant 0$ *be defined on the grid* $\overline{\omega}_\tau$. *Let* $A=2(A^{(1)}+A^{(2)})$, $\overline{Y}_0=\mathrm{const}\geqslant Y_0$. *If the condition*

$$Y_j\leqslant\overline{Y}_0+\sum_{i=1}^{j}\big(A_i^{(1)}Y_i+A_i^{(2)}Y_{i-1}+F_i\big)\tau_i$$

is satisfied and $\max_i\{\tau_iA_i^{(1)}\}\leqslant 1/2$, *then we have the estimate*

$$\max_j\{Y_j\}\leqslant\left(\overline{Y}_0+2\sum_{j=1}^{M}F_j\tau_j\right)\exp\left(\sum_{i=1}^{M}A_i\tau_i\right).$$

Proof. The proof of this lemma can be found in [16].

COROLLARY 2.3. *Suppose that* $A^{(1)}$, $A^{(2)}$, F, Y, $\overline{Y}_0$ *are the same as in Lemma* 2.3, *and* $A_i^{(1)} = A_i^{(2)} = d$, $F_i = eb_{i-1}$, $\tau_i = \tau = T/M$. *If the conditions*

$$Y_j \leqslant \overline{Y}_0 + \tau d \sum_{i=0}^{j-1} (Y_i + Y_{i+1}) + \tau e \sum_{i=0}^{j-1} b_i$$

and $0 < \tau d \leqslant 1/2$ *are satisfied, then we have the estimate*

$$Y_j \leqslant \left(\overline{Y}_0 + 2et_j \max_{0 \leqslant i < j} \{b_i\}\right) \exp(4dt_j); \tag{2.15}$$

here $t_j = \tau j \leqslant T$.

Proof. The proof of this corollary follows directly from Lemma 2.3.

LEMMA 2.4. *Assume that* (1.6) *is satisfied for problem* (2.1)–(2.3). *Then there exists* $\tau_0 > 0$ *such that* $\forall \tau, 0 < \tau \leqslant \tau_0$ *we have*

$$\|\mathbf{p}(t_j)\|_{W_{2h}^1} \leqslant \bar{d} \|\mathbf{p}(t_0)\|_{W_{2h}^1}, \tag{2.16}$$

here $\bar{d} = \bar{d}(a, b, c_2, n, t_j, \varphi(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}))$, $\tau_0 = \tau_0(a, b, c_2, n, \varphi(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}))$.

Proof. We take a scalar product $(\cdot,\cdot)$ of (2.1) and $\dot{\mathbf{p}}$, use the discrete Green formulas [15], and take the real part. We use (2.3), sum the equations for layers from t_0 up to t_{j-1}, and get the estimate (a)

$$\|\mathbf{p}(t_j)\|^2 \leqslant \|\mathbf{p}(t_0)\|^2 + 2\tau \sum_{k=0}^{j-1} \sum_{i=1}^{n} \|f_i(t_k)\| \, \|\dot{p}_i(t_k)\|.$$

Now we multiply scalarly (using the scalar product $(\cdot,\cdot)$) both sides of (2.1) by τp_{it} and take imaginary part. We take Eq. (2.1) instead of p_{it} in the third summand of the right-hand side of the equation, use the discrete Green formulas and an expression of $\dot{p}$ by p and $\hat{p}$, divide both sides of the equality by $0.5\, b_i$, sum the equations for layers from t_0 up to t_{j-1}, estimate real and imaginary parts of scalar products by their absolute value, estimate $|a_i|/b_i \leqslant a/b$, use inequalities $\|u_{\bar{x}}\| \leqslant \|u_{\bar{x}}]|$, (2.7), and obtain the inequalities

$$\|p_{i\bar{x}}(t_j)]|^2 - \|p_{i\bar{x}}(t_0)]|^2 \leqslant (a/b) \left(\|p_{i\bar{x}}(t_j)]| \, \|p_i(t_j)\| + c_4 \|p_{i\bar{x}}(t_0)]|^2\right) + 2\tau(1 + ac_4/b) \sum_{k=0}^{j-1} \|f_{i\bar{x}}(t_k)]| \, \|\dot{p}_{i\bar{x}}(t_k)]|.$$

We use ε-inequality with $\varepsilon = 0.5$, sum the obtained inequalities, use (a), and obtain

$$\|\mathbf{p}(t_j)\|^2_{W_{2h}^1} \leqslant e_1 \|\mathbf{p}(t_0)\|^2_{W_{2h}^1} + 2\tau(e_1 + 1) \sum_{k=0}^{j-1} \sum_{i=1}^{n} \|f_i(t_k)\|_{W_{2h}^1} \|\dot{p}_i(t_k)\|_{W_{2h}^1}; \tag{2.17}$$

here $e_1 = (2 + 2ac_2/b + a^2/b^2)$.

Now we use (2.12), estimate $\|\dot{p}\|^2 \leqslant 0.5(\|\hat{p}\|^2 + \|p\|^2)$ and obtain

$$\|\mathbf{p}(t_j)\|^2_{W_{2h}^1} \leqslant e_1 \|\mathbf{p}(t_0)\|^2_{W_{2h}^1} + 2\tau(e_1 + 1) n \varphi\left(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\right) \sum_{k=0}^{j-1} \left(\|\mathbf{p}(t_{k+1})\|^2_{W_{2h}^1} + \|\mathbf{p}(t_k)\|^2_{W_{2h}^1}\right).$$

Now (2.16) follows from Corollary 2.3 with

$$\tilde{d} = \left(\sqrt{a^2 + 2abc_2 + 2b^2}/b\right)\exp\Big(4(3 + 2ac_2/b + a^2/b^2)nt_j\varphi\Big(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\Big)\Big),$$

$$\tau_0 = \Big(4(3 + 2ac_2/b + a^2/b^2)n\varphi\Big(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\Big)\Big)^{-1}.$$

The lemma is proved.

LEMMA 2.5. *Assume that* (1.6) *is satisfied for problem* (2.4)–(2.6). *Then there exists* $\tau_0 > 0$ *such that* $\forall\tau$, $0 < \tau < \tau_0$ *we have*

$$\|\mathbf{p}(t_j)\|_{W^1_{2h}} \leqslant \tilde{d}\|\mathbf{p}(t_0)\|_{W^1_{2h}}; \tag{2.18}$$

here $\tilde{d} = \tilde{d}(a, n, t_j, \varphi(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}))$, $\tau_0 = \tau_0(a, n, t_j, \varphi(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}))$.

Proof. We take the scalar product $[\cdot,\cdot]$ of the ith component of Eq. (2.4) and $\dot{p}_i$, use the discrete Green formulas and condition (2.6), and take real part. We get $|[\hat{p}_i]|^2 = |[p_i]|^2 + a_i\tau\big([\dot{p}_{i x}, \dot{p}_i) + (\dot{p}_{i\bar{x}}, \dot{p}_i]\big) + 2\tau\,\mathrm{Re}[f_i, \dot{p}_i]$. We estimate $\big|[\dot{p}_{i x}, \dot{p}_i) + (\dot{p}_{i\bar{x}}, \dot{p}_i]\big| \leqslant \|\dot{p}_{i\bar{x}}]|^2 + |[\dot{p}_i]|^2 = \|\dot{p}_i\|^2_{W^1_{2h}}$, $|a_i| \leqslant a$, then take $\sum_{i=1}^n$, use the Cauchy inequality, and obtain the estimate (a)

$$|[\hat{\mathbf{p}}]|^2 \leqslant |[\mathbf{p}]|^2 + a\tau\|\dot{\mathbf{p}}\|^2_{W^1_{2h}} + 2\tau\sum_{i=1}^n |[f_i]|\,|[\dot{p}_i]|.$$

We find the scalar product $[\cdot,\cdot]$ of the ith component of (2.4) and τp_{it}, take imaginary part, and get

$$0 = a_i\tau\mathrm{Im}[\dot{p}_{i\dot{x}}, p_{it}] + b_i\tau\mathrm{Re}[\dot{p}_{i\bar{x}x}, p_{it}] + \tau\mathrm{Im}[f_i, p_{it}].$$

We take Eqs (2.4) instead of p_{it} in the first and third summands of the right-hand side of the equation, use the discrete Green formulas, condition (2.6), the expression of $\dot{p}$ by p and $\hat{p}$, divide both sides of the equality by $0.5b_i$, use the Cauchy inequality, sum the obtained inequalities, and finally get the estimate

$$\|\hat{\mathbf{p}}_{\bar{x}}]|^2 \leqslant \|\mathbf{p}_{\bar{x}}]|^2 + 2\tau\sum_{i=1}^n \|f_{i\bar{x}}]|\,\|\dot{p}_{i\bar{x}}]|.$$

We add (a), sum inequalities for layers from t_0 up to t_j, and obtain

$$\|\mathbf{p}(t_j)\|^2_{W^1_{2h}} \leqslant \|\mathbf{p}(t_0)\|^2_{W^1_{2h}} + \tau a\sum_{k=0}^{j-1}\|\dot{\mathbf{p}}(t_k)\|^2_{W^1_{2h}} + 4\tau\sum_{k=0}^{j-1}\sum_{i=1}^{n}\|f_i(t_k)\|_{W^1_{2h}}\|\dot{p}_i(t_k)\|_{W^1_{2h}}. \tag{2.19}$$

Hence

$$\|\mathbf{p}(t_j)\|^2_{W^1_{2h}} \leqslant \|\mathbf{p}(t_0)\|^2_{W^1_{2h}} + \tau\Big(0.5a + 4n\varphi\Big(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\Big)\Big)\sum_{k=0}^{j-1}\Big(\|\mathbf{p}(t_{k+1})\|^2_{W^1_{2h}} + \|\mathbf{p}(t_k)\|^2_{W^1_{2h}}\Big)$$

and from here, as in the case of Lemma 2.4, estimate (2.18) follows with

$$\tilde{d} = \exp\Big(t_j\Big(a + 8n\varphi\Big(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\Big)\Big)\Big), \qquad \tau_0 = \Big(a + 8n\varphi\Big(\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\Big)\Big)^{-1}.$$

Lemma 2.5 is proved.

3. THE INVESTIGATION OF THE NEW LAYER

Now we will prove the solvability and uniqueness of problems (2.1)–(2.3) and (2.4)–(2.6). We need the following lemmas.

LEMMA 3.1. *Assume we have the problem* (a) *in the grid* $\overline{Q}_{1h}$: $v_t = \tilde{a}\hat{v}_{\dot{x}} + i\tilde{b}\hat{v}_{\bar{x}x} + g$, *where* $x \in \omega_{1h}$; $v \in W^1_{2h} \cap W^2_{2h}$; $g \in W^1_{2h}$; $\hat{v}_0 = \hat{v}_N = 0$; *then* $\forall\tau\ \exists!\ \hat{v} \in W^1_{2h} \cap W^2_{2h}$ *and we have*

$$\|\hat{v}\|_{W^1_{2h}} \leqslant d_1\|v\|_{W^1_{2h}} + \tau d_2\|g\|_{W^1_{2h}}; \tag{3.1}$$

here $d_j = d_j(a, b, c_2)$, $j = 1, 2$.

If we have the problem (b) *in the grid* $\overline{Q}_{2h}$: $v_t = \tilde{a}\hat{v}_{\dot{x}} + i\tilde{b}\hat{v}_{\bar{x}x} + g$, *where* $x \in \omega_{2h}$; $v_0 = v_1$, $v_N = v_{N+1}$; $v \in W^2_{2h}$; $g \in W^1_{2h}$; $\hat{v}_0 = \hat{v}_1$, $\hat{v}_N = \hat{v}_{N+1}$; *then there exists* $\tau_0 > 0$, $\forall\tau \leqslant \tau_0\ \exists!\ \hat{v} \in W^2_{2h}$ *and we have*

$$\|\hat{v}\|_{W^1_{2h}} \leqslant d_3\|v\|_{W^1_{2h}} + \tau d_4\|g\|_{W^1_{2h}}; \tag{3.2}$$

here $\tau_0 = \tau_0(a)$, $d_j = d_j(a)$, $j = 3, 4$.

In both cases $\tilde{a}, \tilde{b} \in \mathbb{R}$, $|\tilde{a}| \leqslant a$, $0 < b \leqslant \tilde{b}$.

Proof. We gather functions $\hat{v}$ in problems (a) and (b) at the left-hand side of equations. In the case (a) we obtain $\hat{v} - \tilde{a}\tau\hat{v}_{\dot{x}}/2 - i\tilde{b}\tau\hat{v}_{\bar{x}x}/2 = \tilde{g}$ with $x \in \omega_{1h}$, $\hat{v}_0 = \hat{v}_N = 0$. In the case (b) we have $\hat{v} - \tilde{a}\tau\hat{v}_{\dot{x}}/2 - i\tilde{b}\tau\hat{v}_{\bar{x}x}/2 = \tilde{g}$ with $x \in \omega_{2h}$, $v_0 = v_1$, $v_N = v_{N+1}$, $\hat{v}_0 = \hat{v}_1$, $\hat{v}_N = \hat{v}_{N+1}$. In both cases $\tilde{g} = v + \tilde{a}\tau v_{\dot{x}}/2 + i\tilde{b}\tau v_{\bar{x}x}/2 + \tau g$ and $\tilde{g} \in L_{2h}$. We can write $L_1\hat{v} = \tilde{g}$ and $L_2\hat{v} = \tilde{g}$, where L_1 and L_2 are linear operators in a finite-dimensional space. Let $g \equiv 0$, $v \equiv 0$, then we have problems $L_1\hat{v} = 0$ and $L_2\hat{v} = 0$ in this case. Now g satisfies (1.6) and we can use Lemmas 2.4 and 2.5. We obtain the inequality $\|\hat{v}\|_{W^1_{2h}} \leqslant d\|v\|_{W^1_{2h}} = 0$, where d is the constant from the lemmas, mentioned above. Hence, homogeneous linear problems in the finite dimension space have only one solution $\hat{v} \equiv 0$. From here, as in [14], we know, that (a) and (b) problems have unique solutions.

In case (a) we use (2.17) with $n = 1$, $j = 1$, (remember that there $d \geqslant 1$) and obtain the inequality

$$\|\hat{p}\|^2_{W^1_{2h}} - d\|p\|^2_{W^1_{2h}} \leqslant \tau(d+1)\|g\|_{W^1_{2h}}\big(\|\hat{p}\|_{W^1_{2h}} + \sqrt{d}\|p\|_{W^1_{2h}}\big).$$

From here (3.1) follows with $d_1 = \sqrt{(2 + 2ac_2/b + a^2/b^2)}$, $d_2 = d_1^2 + 1$.

In case (b) we use (2.19) with $n = 1$, $j = 1$ and obtain the inequality

$$\|\hat{p}\|^2_{W^1_{2h}} - \|p\|^2_{W^1_{2h}} \leqslant 2\tau\big(\|\hat{p}\|_{W^1_{2h}} + \|p\|_{W^1_{2h}}\big)\big(a/8\big(\|\hat{p}\|_{W^1_{2h}} + \|p\|_{W^1_{2h}}\big) + \|g\|_{W^1_{2h}}\big).$$

From here (3.2) follows with $\tau_0 < 4/a$, $d_3 = 1 + 2a\tau_0/(4 - a\tau_0)$, $d_4 = 8/(4 - a\tau_0)$ and if $\tau_0 \leqslant 2/a$, then $d_3 \leqslant 3$, $d_4 \leqslant 4$.

Using the estimations written above, we can show that $\|\hat{v}\|_{W^1_{2h}}$ is bounded by the norms of functions from W^1_{2h}. Hence, $\hat{v} \in W^1_{2h}$. In both cases we can write

$$\|\hat{v}_{\bar{x}x}\| = \| - 2i\hat{v}/\tau\tilde{b} + \tilde{a}i\hat{v}_{\dot{x}}/\tilde{b} + 2i\tilde{g}/\tau\tilde{b}\| \leqslant (2/\tau b + a/b)\,\|\hat{v}\|_{W^1_{2h}} + 2\|[\tilde{g}]\|/\tau b.$$

If τ is fixed all norms on the right-hand side of this inequality are bounded. Thus, $\hat{v} \in W^2_{2h}$. Lemma 3.1 is proved.

LEMMA 3.2. *Assume we have problem* (c) *in the grid* $\overline{Q}_{1h}$: $v = \tilde{a}\tau v_{\dot{x}}/2 + i\tilde{b}\tau v_{\bar{x}x}/2 + \tau g$, *where* $x \in \omega_{1h}$; $g \in L_{2h}$; $v_0 = v_N = 0$; *then* $\forall\tau\ \exists!\ v \in W^1_{2h} \cap W^2_{2h}$ *and we have*

$$\|[v]\| \leqslant \tau d_1\|[g]\|, \qquad \|v_{\bar{x}}]\| \leqslant d_2\|[g]\|, \qquad \|v_{\bar{x}x}\| \leqslant d_3\|[g]\|. \tag{3.3}$$

If we have problem (d) *in the grid* $\overline{Q}_{2h}$: $v = \tilde{a}\tau v_{\bar{x}}/2 + i\tilde{b}\tau v_{\bar{x}x}/2 + \tau g$, *where* $x \in \omega_{2h}$; $v_0 = v_1$, $v_N = v_{N+1}$; $g \in L_{2h}$; *then there exists* $\tau_0 > 0$, $\forall\tau \leqslant \tau_0$ $\exists!$ $v \in W_{2h}^2$ *and we have*

$$|[v]| \leqslant \tau d_4|[g]|, \qquad \|v_{\bar{x}}]| \leqslant d_4|[g]|, \qquad \|v_{\bar{x}x}\| \leqslant d_6|[g]|, \tag{3.4}$$

here $\tau_0 = \tau_0(a, b)$, $d_j = d_j(a, b)$, $j = 1, \ldots, 6$.
In both cases $\tilde{a}, \tilde{b}$ *are the same as in Lemma* 3.1.

Proof. The existence and uniqueness of the solutions we prove similarly as in Lemma 3.1. We multiply scalarly (using the scalar product $[\cdot,\cdot]$) both sides of equation of our problem (c) (or (d)) by v and take real parts. For (c) we obtain $|[v]| \leqslant \tau|[g]|$ and for (d) we have $|[v]|^2 \leqslant a\tau\|v_{\bar{x}}]|\,|[v]|/2 + \tau|[g]|\,|[v]|$, or $|[v]| \leqslant a\tau\|v_{\bar{x}}]|/2 + \tau|[g]|$. When we take imaginary parts, in case (c) we have $\|v_{\bar{x}}]|^2 \leqslant a\|v_{\bar{x}}]|\,\|v\|/b + 2\|v\|\,\|g\|/b$. Using the estimate, which we have gotten before, from the ε-inequality with $\varepsilon = 0.5$ we obtain the estimate $\|v_{\bar{x}}]| \leqslant d_2|[g]|$. In case (d) we get $\|v_{\bar{x}}]|^2 \leqslant a\|v_{\bar{x}}]|\,|[v]|/b + 2|[v]|\,|[g]|/b$. From here we obtain $\|v_{\bar{x}}]|^2 \leqslant (\tau/2b)\,(a\|v_{\bar{x}}]| + 2|[g]|)^2$ or $\|v_{\bar{x}}]| \leqslant d_5|[g]|$, and then the estimate $|[v]| \leqslant \tau d_4|[g]|$ follows. $\|v\|_{W_{2h}^1}$ is bounded, thus $v \in W_{2h}^1$.

We can obtain the last estimate of (3.3) and (3.4) directly from the equations: $\|v_{\bar{x}x}\| \leqslant 2|[v]|/\tau b + a\|v_{\bar{x}}]|/b + 2|[g]|/b \leqslant d|[g]|$, where $d = d_3$ or $d = d_6$. The right-hand side of the inequality is bounded, thus $v \in W_{2h}^2$.

Here we have $d_1 = 1$, $d_2 = \sqrt{\tau(a^2\tau + 2b)}/b$, $d_3 = (4 + ad_1)/b$, $\forall\tau$. Also $d_4 = \sqrt{2b}/(\sqrt{2b} - a\sqrt{\tau_0})$, $d_5 = 2\sqrt{\tau_0}/(\sqrt{2b} - a\sqrt{\tau_0})$, $d_6 = (2d_3 + ad_4 + 2)/b$ if only $\tau \leqslant \tau_0 < 2b/a^2$; and $d_4 \leqslant 2$, $d_5 \leqslant 2/a$ or $d_5 \leqslant 2\sqrt{2\tau_0}/b$, $d_6 \leqslant 8/b$ if $\tau \leqslant \tau_0 < b/2a^2$.
Lemma 3.2 is proved.

Searching for solutions of problems (2.1)–(2.3) and (2.4)–(2.6) in a new layer, we must solve nonlinear equation systems. We use iterative methods. Now we write iterative processes for both problems and prove their convergence with the exponential rate.

We have the following process for the first problem:

$$\frac{\mathbf{p}^{k+1} - \mathbf{p}}{\tau} = \frac{A}{2}\left(\mathbf{p}_{\dot{x}}^{k+1} + \mathbf{p}_{\dot{x}}\right) + \frac{Bi}{2}\left(\mathbf{p}_{\bar{x}x}^{k+1} + \mathbf{p}_{\bar{x}x}\right) + \mathbf{f}\left(\frac{\mathbf{p}^k + \mathbf{p}}{2}, \frac{\mathbf{p}^{k*} + \mathbf{p}^*}{2}\right), \qquad x \in \omega_{1h},$$

$$\mathbf{p}^0 = \mathbf{p}, \qquad \mathbf{p}_0^{k+1} = \mathbf{p}_N^{k+1} = 0. \tag{3.5}$$

For the second problem the process is given by the following relations:

$$\frac{\mathbf{p}^{k+1} - \mathbf{p}}{\tau} = \frac{A}{2}\left(\mathbf{p}_{\dot{x}}^{k+1} + \mathbf{p}_{\dot{x}}\right) + \frac{Bi}{2}\left(\mathbf{p}_{\bar{x}x}^{k+1} + \mathbf{p}_{\bar{x}x}\right) + \mathbf{f}\left(\frac{\mathbf{p}^k + \mathbf{p}}{2}, \frac{\mathbf{p}^{k*} + \mathbf{p}^*}{2}\right), \qquad x \in \omega_{2h},$$

$$\mathbf{p}^0 = \mathbf{p}, \qquad \mathbf{p}_0^{k+1} = \mathbf{p}_1^{k+1}, \qquad \mathbf{p}_N^{k+1} = \mathbf{p}_{N+1}^{k+1}. \tag{3.6}$$

LEMMA 3.3. *Assume that the following conditions are satisfied*: $\mathbf{p} \in \overset{\circ}{\mathbf{W}}{}_{2h}^1 \cap \mathbf{W}_{2h}^2$, $\mathbf{f}(\mathbf{p}, \mathbf{p}^*) \in \overset{\circ}{\mathbf{W}}{}_{2h}^1$, $\|\mathbf{p}\|_{W_{2h}^1} \leqslant \alpha$. *Then process* (3.5) *produces the unique sequence of the functions* $\{\mathbf{p}^k\}$, $k = 0, 1, \ldots$, *converging to the solution of problem* (2.1)–(2.3) *in the space* $\overset{\circ}{\mathbf{W}}{}_{2h}^1 \cap \mathbf{W}_{2h}^2$. *There is the unique solution* $\hat{\mathbf{p}}$ *of this problem with the condition* $\|\hat{\mathbf{p}}\|_C = O(1)$, *when* $\tau \to 0$. *More over, there exists* $\tau_1 > 0$ *such that* $\forall\tau$ $0 < \tau < \tau_1$, $\forall k$ *we have*

$$\|\mathbf{p}^k\|_{W_{2h}^1} \leqslant d_1\|\mathbf{p}\|_{W_{2h}^1}, \qquad \|\hat{\mathbf{p}}\|_{W_{2h}^1} \leqslant d_1\|\mathbf{p}\|_{W_{2h}^1}, \tag{3.7}$$

here $d_1 = d_1(a, b, c_2)$; $\tau_1 = \tau_1(a, b, c_2, c_3, q, \alpha, \varphi)$, *where* $q < 1$.
If the conditions $\mathbf{p} \in \mathbf{W}_{2h}^2$, $\mathbf{f}(\mathbf{p}, \mathbf{p}^*) \in \mathbf{W}_{2h}^1$, $\|\mathbf{p}\|_{W_{2h}^1} \leqslant \alpha$ *are satisfied, then process* (3.6) *produces the unique sequence* $\{\mathbf{p}^k\}$, $k = 0, 1, \ldots$, *converging to the solution of problem* (2.4)–(2.6) *in the space* $\mathbf{W}_{2h}^2$. $\exists!$ $\hat{\mathbf{p}}$ *satisfying the condition* $\|\hat{\mathbf{p}}\|_C = O(1)$, *when* $\tau \to 0$. *There exists* $\tau_2 > 0$ *such that* $\forall\tau$ $0 < \tau < \tau_2$, $\forall k$ *we have*

$$\|\mathbf{p}^k\|_{W_{2h}^1} \leqslant d_2\|\mathbf{p}\|_{W_{2h}^1}, \qquad \|\hat{\mathbf{p}}\|_{W_{2h}^1} \leqslant d_2\|\mathbf{p}\|_{W_{2h}^1}, \tag{3.8}$$

here $d_2 = d_2(a)$; $\tau_2 = \tau_2(a, b, n, c_3, q, \alpha, \varphi)$, *where* $q < 1$.

Proof. The existence and uniqueness of the sequence in both cases follow from Lemma 3.1. We will prove the estimates using a method of mathematical induction.

a) When $k = 0$, then (3.7) and (3.8) are valid, because $\mathbf{p}^0 = \mathbf{p}$.

b) Suppose these estimates are valid for all $i \leqslant k$. Then, using Lemma 3.1, $\forall j = 1, \dots, n$ we get

$$\|p_j^{k+1}\|_{W_{2h}^1} - e_1\|p_j\|_{W_{2h}^1} \leqslant \tau e_2 \left\| f_j\left((\mathbf{p}^k + \mathbf{p})/2, (\mathbf{p}^{k*} + \mathbf{p}^*)/2\right)\right\|_{W_{2h}^1},$$

where e_1, e_2 are constants in the estimates (3.1) or (3.2). We multiply both sides of the inequalities by $\|p_j^{k+1}\|_{W_{2h}^1} + e_1\|p_j\|_{W_{2h}^1}$, take $\sum_{j=1}^n$, use Corollary 2.1 and estimate $\sum_{j=1}^n \|p_j\| \leqslant \sqrt{n}\|\mathbf{p}\|$, divide both sides by $\|\mathbf{p}^{k+1}\|_{W_{2h}^1} + e_1\|\mathbf{p}\|_{W_{2h}^1}$ and obtain

$$\|\mathbf{p}^{k+1}\|_{W_{2h}^1} - e_1\|p_j\|_{W_{2h}^1} \leqslant \tau e_2 n\varphi\left(\left(\|\mathbf{p}^k\|_C + \|\mathbf{p}\|_C\right)/2\right)\|\mathbf{p}^k + \mathbf{p}\|_{W_{2h}^1}.$$

We use the induction's supposition and get

$$\|\mathbf{p}^{k+1}\|_{W_{2h}^1} \leqslant (e_1 + \tau e_2 n\varphi\left((e_3 + 1)c_3\alpha/2\right)(e_3 + 1))\|\mathbf{p}\|_{W_{2h}^1},$$

here e_3 is one of the constants d_1, d_2 of this lemma. We need the condition $e_1 + \tau e_2 n\varphi\left((e_3 + 1)c_3\alpha/2\right)(e_3 + 1) \leqslant e_3$ to be satisfied. We can take $e_3 = e_1 + 1$, then this condition is valid, when $0 < \tau \leqslant \tau_0$, where $\tau_0 = 1/e_2 n\varphi\left((e_1 + 2)c_3\alpha/2\right)(e_1 + 2)$. The induction step is proved.

Now we subtract the equations for the p_j^k component from the equations for the p_j^{k+1} component. We denote $p_j^{k+1} - p_j^k = v_j^k$. Using Lemma 3.2, we obtain the estimates (a)

$$|[v_j^k]| \leqslant \tau e_4|[g_j^k]|, \qquad \|v_{j\bar{x}}^k]| \leqslant e_5|[g_j^k]|, \qquad \|v_{j\bar{x}x}^k\| \leqslant e_6|[g_j^k]|,$$

here e_4, e_5, e_6 are constants from the inequalities (3.3) or (3.4) and

$$g_j^k = f_j\left(\left(\mathbf{p}^k + \mathbf{p}\right)/2, \left(\mathbf{p}^{k*} + \mathbf{p}^*\right)/2\right) - f_j\left(\left(\mathbf{p}^{k-1} + \mathbf{p}\right)/2, \left(\mathbf{p}^{k-1*} + \mathbf{p}^*\right)/2\right).$$

Using (2.10), we can obtain: $|[g_j^k]| \leqslant \sqrt{n}\varphi\left((e_3 + 1)c_3\alpha/2\right)|[\mathbf{v}^{k-1}]|$. From (a) we easily get (b)

$$|[\mathbf{v}^k]| \leqslant \tau e_4 n\varphi\left((e_3 + 1)c_3\alpha/2\right)|[\mathbf{v}^{k-1}]|$$

and (c)

$$\|\mathbf{v}^k\|_{W_{2h}^2} \leqslant e_7|[\mathbf{v}^{k-1}]|,$$

here $e_7 = n\varphi\left((e_3 + 1)c_3\alpha/2\right)\sqrt{\tau^2 e_4^2 + e_5^2 + e_6^2}$. If $\tau \leqslant q/\left(e_4 n\varphi\left((e_3 + 1)c_3\alpha/2\right)\right)$, where $q < 1$, from (b) we obtain $|[\mathbf{v}^k]| \leqslant q^k|[\mathbf{v}^0]|$. Then from (c) we obtain $\|\mathbf{v}^k\|_{W_{2h}^2} \leqslant e_7 q^{k-1}|[\mathbf{v}^0]|$. It follows that $\forall\, m_1, m_2 \in N$, $m_1 \leqslant m_2$, $\|\mathbf{p}^{m_2} - \mathbf{p}^{m_1}\|_{W_{2h}^2} \leqslant q^{m_1-1}e_7|[\mathbf{v}^0]|/(1-q) \to 0$, when $m_1, m_2 \to \infty$. Thus, the sequence $\{\mathbf{p}^k\}$ is a Cauchy sequence in the complete Banach space $\mathbf{W}_{2h}^2$. It means [14] $\exists!\ \mathbf{w} \in \mathbf{W}_{2h}^2$, such that $\|\mathbf{p}^k - \mathbf{w}\|_{W_{2h}^2} \to 0$, when $k \to \infty$. Due to the inequality $\|\mathbf{p}^k\|_{W_{2h}^1} \leqslant e_3\|\mathbf{p}\|_{W_{2h}^1}$, we obtain $\|\mathbf{w}\|_{W_{2h}^1} \leqslant \|\mathbf{p}^k\|_{W_{2h}^1} + \|\mathbf{w} - \mathbf{p}^k\|_{W_{2h}^1} \leqslant e_3\|\mathbf{p}\|_{W_{2h}^1} + \varepsilon$, where ε is any small positive number. Thus, $\|\mathbf{w}\|_{W_{2h}^1} \leqslant e_3\|\mathbf{p}\|_{W_{2h}^1}$.

We will prove that $\mathbf{w}$ satisfies problems (2.1)–(2.3) or (2.4)–(2.6). We gather all summands at the left-hand side of the equations of corresponding problem, take $\mathbf{w}$ instead of $\hat{\mathbf{p}}$, subtract the equations of iterating process (3.5) or (3.6), take the norm of space $\mathbf{L}_{2h}$ and obtain the inequality

$$\left\| \frac{(\mathbf{w} - \mathbf{p}^k)}{\tau} - \frac{A}{2}\left(\mathbf{w} - \mathbf{p}^k\right)_x - \frac{iB}{2}\left(\mathbf{w} - \mathbf{p}^k\right)_{\bar{x}x} - \left(\mathbf{f}\left(\frac{\mathbf{w} + \mathbf{p}}{2}, \frac{\mathbf{w}^* + \mathbf{p}^*}{2}\right) - \mathbf{f}\left(\frac{\mathbf{w} + \mathbf{p}^{k-1}}{2}, \frac{\mathbf{w}^* + \mathbf{p}^{k-1*}}{2}\right)\right)\right\|$$
$$\leqslant \left(\frac{1}{\tau} + \frac{a + b}{2}\right)\|\mathbf{w} - \mathbf{p}^k\|_{W_{2h}^2} + n\varphi\left(\frac{\max\left\{\|\mathbf{w}\|_C, \|\mathbf{p}^{k-1}\|_C\right\} + \|\mathbf{p}\|_C}{2}\right)\|\mathbf{w} - \mathbf{p}^{k-1}\| \longrightarrow 0,$$

where τ is fixed and $k \to \infty$. From here it follows

$$\left\| \frac{\mathbf{w}-\mathbf{p}}{\tau} - \frac{A}{2}(\mathbf{w}+\mathbf{p})_{\dot{x}} - \frac{iB}{2}(\mathbf{w}+\mathbf{p})_{\bar{x}x} - \mathbf{f}\left(\frac{\mathbf{w}+\mathbf{p}}{2}, \frac{\mathbf{w}^*+\mathbf{p}^*}{2}\right) \right\| = 0.$$

Thus, $\mathbf{w}$ has the same values as the solution of the problem in the grids ω_{1h} or ω_{2h}. Similarly we can show that $\mathbf{w}$ satisfies the equations of the boundary conditions.

Now we will show that if $\mathbf{p} \in \mathbf{W}^2_{2h}$, then $\forall \tau \leqslant \tau_0\ \exists!\ \hat{\mathbf{p}}$, such that $\|\hat{\mathbf{p}}\|_C = O(1)$ when $\tau \to 0$. Suppose we have two such solutions $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$. We denote $\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2 = \mathbf{z}$. Then from (2.1)–(2.3) or (2.4)–(2.6) we obtain the equations in the grids ω_{1h} or ω_{2h}:

$$\mathbf{z} = \frac{A\tau}{2}\mathbf{z}_{\dot{x}} + \frac{iB\tau}{2}\mathbf{z}_{\bar{x}x} + \tau\left(\mathbf{f}\left(\frac{\hat{\mathbf{p}}_1+\mathbf{p}}{2}, \frac{\hat{\mathbf{p}}_1^*+\mathbf{p}^*}{2}\right) - \mathbf{f}\left(\frac{\hat{\mathbf{p}}_2+\mathbf{p}}{2}, \frac{\hat{\mathbf{p}}_2^*+\mathbf{p}^*}{2}\right)\right).$$

Using Lemma 3.2 and (2.10), we can obtain the estimate

$$\|\mathbf{z}\| \leqslant \tau n e_4 \varphi\big((\max\{\|\hat{\mathbf{p}}_1\|_C, \|\hat{\mathbf{p}}_2\|_C\} + \|\mathbf{p}\|_C)/2\big)\|\mathbf{z}\|.$$

We supposed that our solutions are bounded in the norm of the space C_h, thus, $\|\mathbf{z}\| \leqslant O(\tau)\|\mathbf{z}\|$. Hence, $\|\mathbf{z}\| = 0$ and $\hat{\mathbf{p}}_1 = \hat{\mathbf{p}}_2$ in the grids ω_{1h} or ω_{2h}. Similarly we deal with the boundary equations. Hence, the function $\mathbf{w}$, which we can find from the iterative process, is the unique solution of problems (2.1)–(2.3) or (2.4)–(2.6) in the given class of functions. Lemma 3.3 is proved.

4. CONVERGENCE AND STABILITY OF THE DIFFERENCE SCHEME

Let a grid function $\Phi(t_j)$ be an error of approximation of difference schemes (2.1)–(2.3) or (2.4)–(2.6) in a layer t_j, where $t_j = \tau j$ and $\tau = T/M$. In the grids ω_{1h} or ω_{2h} we have

$$\begin{aligned}\Phi(t_j) = {} & \left(\mathbf{u}_t(t_j) - \frac{\partial \mathbf{u}}{\partial t}(\tau(j+0.5))\right) - A\left(\dot{\mathbf{u}}_{\dot{x}}(t_j) - \frac{\partial \mathbf{u}}{\partial x}(\tau(j+0.5))\right) \\ & - \left(iB\left(\dot{\mathbf{u}}_{\bar{x}x}(t_j) - \frac{\partial^2 \mathbf{u}}{\partial x^2}(\tau(j+0.5))\right)\right) - \big(\mathbf{f}(\dot{\mathbf{u}}, \dot{\mathbf{u}}^*(t_j)) - \mathbf{f}(\mathbf{u}, \mathbf{u}^*(\tau(j+0.5)))\big),\end{aligned}$$

where $\mathbf{u}$ is a solution of problem (1.1)–(1.3) or (1.1), (1.2), (1.4). We know that in the case of the first boundary value problem we have $u_0 = u_N = 0$, thus $\Phi(x_0, t_j) = \Phi(x_N, t_j) = 0$. In the case of the second boundary value problem, we define the value of the solution of (1.1), (1.2), (1.4) in the fictitious grid points x_0, x_N similarly as in the difference scheme: $\mathbf{u}_0 = \mathbf{u}_1$, $\mathbf{u}_N = \mathbf{u}_{N+1}$. When $h \to 0$, these conditions and (1.4) are equivalent.

We suppose that $\mathbf{u}(x,t)$ is smooth enough and the following condition is satisfied:

$$\max_{0 \leqslant j \leqslant M-1} \left\{\|\Phi(t_j)\|_{W^1_{2h}}\right\} \longrightarrow 0, \qquad \text{when} \quad \tau, h \to 0. \tag{4.1}$$

Also we introduce the grid function ε as an error of the solution in the grids $\overline{Q}_{1h}$ or $\overline{Q}_{2h}$: $\varepsilon = \mathbf{u} - \mathbf{p}$, where $\mathbf{p}$ is the solution of (2.1)–(2.3) or (2.4)–(2.6).

We subtract a difference problem in a layer t_j from a corresponding differential problem in a time moment $\tau(j+0.5)$. In the case of the first problem we have $u(x_0,t) = u(x_N,t) = p_0 = p_N = 0$, thus, we obtain the following equations:

$$\begin{gathered}\varepsilon_t = A\dot{\varepsilon}_{\dot{x}} + iB\dot{\varepsilon}_{\bar{x}x} + \Psi + \Phi, \qquad (x,t) \in Q_{1h}, \\ \varepsilon(x,0) = 0, \quad x \in \overline{\omega}_{1h}, \qquad \varepsilon(x_0,t) = \varepsilon(x_N,t) = 0, \quad t \in \overline{\omega}_\tau.\end{gathered} \tag{4.2}$$

In the case of the second problem we have defined $\mathbf{u}(x_0,t) = \mathbf{u}(x_1,t)$, $\mathbf{u}(x_N,t) = \mathbf{u}(x_{N+1},t)$, thus

$$\begin{gathered}\varepsilon_t = A\dot{\varepsilon}_{\dot{x}} + iB\dot{\varepsilon}_{\bar{x}x} + \Psi + \Phi, \quad (x,t) \in Q_{2h}, \qquad \varepsilon(x,0) = 0, \quad x \in \overline{\omega}_{2h}, \\ \varepsilon(x_0,t) = \varepsilon(x_1,t), \qquad \varepsilon(x_N,t) = \varepsilon(x_{N+1},t), \quad t \in \overline{\omega}_\tau.\end{gathered} \tag{4.3}$$

In both cases $\Psi = \mathbf{f}(\dot{\mathbf{u}}, \dot{\mathbf{u}}^*) - \mathbf{f}(\dot{\mathbf{p}}, \dot{\mathbf{p}}^*)$.

THEOREM 4.1. *Assume that* (1.5)–(1.7), (4.1) *are satisfied for problem* (1.1)–(1.3). *Then a solution* $\mathbf{p}$ *of problem* (2.1)–(2.3) *converges to a solution* $\mathbf{u}$ *in the norm of space* $\mathbf{L}_\infty(0, T; \mathbf{W}^1_{2h})$ *and there exists* $\tau_0', h_0' > 0$ *such that* $\forall\tau$, $0 < \tau \leqslant \tau_0'$, $\forall h$, $0 < h \leqslant h_0'$ *we have*

$$\max_{0\leqslant j\leqslant M}\left\{\|\varepsilon(t_j)\|_{W^1_{2h}}\right\} \leqslant c_4 \max_{0\leqslant j\leqslant M-1}\left\{\|\Phi(t_j)\|_{W^1_{2h}}\right\}, \tag{4.4}$$

here $M\tau = T$, $c_4 = c_4(a, b, c_2, c_3, \|\mathbf{u}\|_{C(\overline{Q})}, \|\mathbf{u}_0\|_{W^1_{2h}}, \varphi)$.

If (1.5), (1.6), (1.8), (4.1) *are satisfied for problem* (1.1), (1.2), (1.4), *then a solution of* (2.4)–(2.6) *converges to* $\mathbf{u}$ *in the same norm and there exists* $\tau_0'', h_0'' > 0$ *such that* $\forall\tau$, $0 < \tau \leqslant \tau_0''$, $\forall h$, $0 < h \leqslant h_0''$ *we have*

$$\max_{0\leqslant j\leqslant M}\left\{\|\varepsilon(t_j)\|_{W^1_{2h}}\right\} \leqslant c_5 \max_{0\leqslant j\leqslant M-1}\left\{\|\Phi(t_j)\|_{W^1_{2h}}\right\}, \tag{4.5}$$

here $c_5 = c_5(a, b, c_5, \|\mathbf{u}\|_{C(\overline{Q})}, \|\mathbf{u}_0\|_{W^1_{2h}}, \varphi)$.

Proof. In case (4.2) we notice that $\Psi(x_0, t_j) = \Psi(x_N, t_j) = 0$. Then, similarly as in Lemma 2.4, we obtain an inequality similar to (2.17). Using $\|\Psi_i + \Phi_i\| \leqslant \|\Psi_i\| + \|\Phi_i\|$, (2.14), the ε-inequality with $\varepsilon = 0.5$, the $\dot\varepsilon$ expression by $\hat\varepsilon$ and ε, we obtain

$$\begin{aligned}\|\varepsilon(t_j)\|^2_{W^1_{2h}} &\leqslant e_1\|\varepsilon(t_0)\|^2_{W^1_{2h}} + \tau(e_1+1)\sum_{k=0}^{j-1}\|\Phi(t_k)\|^2_{W^1_{2h}}\\ &\quad + \tau(e_1+1)(\tilde e_2 n + 0.5)\sum_{k=0}^{j-1}\left(\|\varepsilon(t_{k+1})\|^2_{W^1_{2h}} + \|\varepsilon(t_k)\|^2_{W^1_{2h}}\right);\end{aligned}$$

here

$$\tilde e_2 = 2\sqrt{2}n\varphi\left(\max\left\{\|\mathbf{u}\|_{C(\overline{Q})}, \|\mathbf{p}\|_{C(\overline{Q}_{t_j})}\right\}\right)\left(1 + 2c_3\max_{0\leqslant k\leqslant j}\left\{\|\mathbf{p}(t_k)\|_{W^1_{2h}}\right\}\right);$$

e_1 is the constant from inequality (2.17).

Let $\tau_1 = 1/((e_1+1)(2\tilde e_2 n + 1))$. Then $\forall\tau$, $0 < \tau \leqslant \tau_1$ we use Corollary 2.3, the equality $\|\varepsilon(t_0)\|_{W^1_{2h}} = 0$, take the square root, and have

$$\|\varepsilon(t_j)\|_{W^1_{2h}} \leqslant \tilde c_4 \max_{0\leqslant k\leqslant j-1}\left\{\|\Phi(t_k)\|_{W^1_{2h}}\right\}, \tag{4.6}$$

here $\tilde c_4 = \sqrt{2(e_1+1)t_j}\exp\big(t_j(e_1+1)(2\tilde e_2 n + 1)\big)$.

In case (4.3) we use (2.19), (2.14) and $\forall\tau$, $0 < \tau \leqslant \tau_2$, where $\tau_2 = 1/(a + 4\tilde e_2 n + 2)$, we get

$$\|\varepsilon(t_j)\|_{W^1_{2h}} \leqslant \tilde c_5 \max_{0\leqslant k\leqslant j-1}\left\{\|\Phi(t_k)\|_{W^1_{2h}}\right\}; \tag{4.7}$$

here $\tilde c_5 = 2\sqrt{t_j}\exp(t_j(a + 4\tilde e_2 n + 2))$.

We notice that if positive parameter $\tilde e_2$ increases the values of parameters $\tilde c_4$ or $\tilde c_5$ increase, too.

For the first problem we will show that $\exists\tau_0', h_0'$ such that $\forall\tau, h$, $0 < \tau \leqslant \tau_0'$, $0 < h \leqslant h_0'$, $\forall j = 0, 1, \dots, M$, $\|\mathbf{p}(t_j)\|_C \leqslant \alpha = 2\|\mathbf{u}\|_{C(\overline{Q})}$. We use mathematical induction:

a) If $j = 0$, then $\|\mathbf{p}(t_0)\|_C \leqslant \|\mathbf{u}(t_0)\|_C \leqslant \|\mathbf{u}\|_{C(\overline{Q})} \leqslant \alpha$.

b) Let $\|\mathbf{p}(t_i)\|_C \leqslant \alpha$ $\forall i = 0, 1, \dots, j-1$. Using Lemma 2.4 we can write the estimates: $\|\mathbf{p}(t_{j-1})\|_{W^1_{2h}} \leqslant \tilde e_3\|\mathbf{p}(t_0)\|_{W^1_{2h}}$. Here $\tilde e_3$ is the parameter from estimate (2.16), it depends on $\|\mathbf{p}\|_{C(\overline{Q}_{t_{j-1}})}$. If that norm increases, $\tilde e_3$ increases, too. Due to the induction's supposition, $\|\mathbf{p}\|_{C(\overline{Q}_{t_{j-1}})} \leqslant \alpha$. Thus, we can write $\|\mathbf{p}(t_i)\|_{W^1_{2h}} \leqslant e_3\|\mathbf{p}(t_0)\|_{W^1_{2h}} = e_3\|\mathbf{u}_0\|_{W^1_{2h}}$ $\forall i = 0, 1, \dots, j-1$, where $\tilde e_3 \leqslant e_3 = \sqrt{e_1}\exp\big(4(e_1+1)nT\varphi(\alpha)\big)$. Using Lemma 3.3, we obtain $\|\mathbf{p}(t_j)\|_{W^1_{2h}} \leqslant e_3 e_4'\|\mathbf{u}_0\|_{W^1_{2h}}$, where e_4' is constant from (3.7). Using (2.8), we obtain the estimate $\|\mathbf{p}(t_j)\|_C \leqslant c_3 e_3 e_4'\|\mathbf{u}_0\|_{W^1_{2h}}$.

In (4.6) parameter $\tilde{c}_4$ depends on $\|\mathbf{p}\|_{C(\overline{Q}_{t_j})}$, $\max_{0\leqslant k\leqslant j}\{\|\mathbf{p}(t_k)\|_{W^1_{2h}}\}$ and increases, when these norms increase. We evaluate these norms by the constants $c_3e_3e_4'\|\mathbf{u}_0\|_{W^1_{2h}}$ and $e_3e_4'\|\mathbf{u}_0\|_{W^1_{2h}}$. We obtain constant $c_4 \geqslant \tilde{c}_4$, where

$$c_4 = \sqrt{2(e_1+1)T}\exp(T(e_1+1)(2e_2'n+1)$$

and

$$e_2' = 2\sqrt{2}n\big(1 + 2c_3e_3e_4'\|\mathbf{u}_0\|_{W^1_{2h}}\big)\varphi\big(c_3e_3e_4'\|\mathbf{u}_0\|_{W^1_{2h}}\big).$$

We can obtain a constant c_4 when $0 < \tau \leqslant \tau_3$, where $\tau_3 = \min\big\{(4(e_1+1)n\varphi(\alpha))^{-1}, ((e_1+1)n(e_4'+1)\varphi((e_4'+1)c_3e_1\|\mathbf{u}_0\|_{W^1_{2h}}/2))^{-1}, ((e_1+1)(2e_2'n+1))^{-1}\big\}$. Now we have $\|\varepsilon(t_j)\|_{W^1_{2h}} \leqslant c_4\max_{0\leqslant k\leqslant j-1}\{\|\Phi(t_k)\|_{W^1_{2h}}\}$, where the right-hand side of the inequality converges to 0 when $\tau, h \to 0$. Then $\exists\tau_0', h_0' > 0$, $\tau_0' \leqslant \tau_3$ such that $\forall\tau$, $0 < \tau \leqslant \tau_0'$, $\forall h$, $0 < h \leqslant h_0'$ we can obtain: $\|\varepsilon(t_j)\|_{W^1_{2h}} \leqslant (1/c_3)\|\mathbf{u}\|_{C(\overline{Q})}$. Using (2.8) and expression of ε by $\mathbf{u}$ and $\mathbf{p}$, we get $\|(\mathbf{u}-\mathbf{p})(t_j)\|_C \leqslant \|\mathbf{u}\|_{C(\overline{Q})}$. Then the inequality follows: $\|\mathbf{p}(t_j)\|_C \leqslant \|\mathbf{u}\|_{C(\overline{Q})} + \|\mathbf{u}(t_j)\|_C \leqslant \alpha$. The induction step is proved.

Similarly we deal with the second problem. As follows from Lemmas 3.3 and 3.1, when τ is small enough, we can estimate $e_4'' \leqslant 4$, where e_4'' is the constant from (3.8). In the same way as before we get $c_5 \geqslant \tilde{c}_5$, where $c_5 = 2\sqrt{T}\exp(T(a+4e_2''n+2))$,

$$e_2'' = 2\sqrt{2}n\big(1 + 8c_3e_5\|\mathbf{u}_0\|_{W^1_{2h}}\big)\varphi\big(4c_3e_5\|\mathbf{u}_0\|_{W^1_{2h}}\big) \geqslant 2\sqrt{2}n\big(1 + 2c_3e_4''e_5\|\mathbf{u}_0\|_{W^1_{2h}}\big)\varphi\big(c_3e_4''e_5\|\mathbf{u}_0\|_{W^1_{2h}}\big),$$

and e_5 is the constant which we obtain from (2.18): $e_5 = \exp(T(a+8n\varphi(\alpha)))$. We can get c_5 when

$$\tau \leqslant \tau_4 = \min\big\{(a+8n\varphi(\alpha))^{-1}, \big(20n\varphi(2.5c_3e_5\|\mathbf{u}_0\|_{W^1_{2h}})\big)^{-1}, (b/2a), (a+4e_2''n+2)^{-1}\big\}.$$

We can find $\tau_0'', h_0'' > 0$, $\tau_0'' \leqslant \tau_4$ such that $\forall\tau$, $0 < \tau \leqslant \tau_0''$, $\forall h$, $0 < h \leqslant h_0''$, the condition of the induction step is satisfied.

When we know such τ_0', h_0' and τ_0'', h_0'', we can write $\|\mathbf{p}(t_j)\|_C \leqslant \alpha$ $\forall j = 0, 1, \ldots, M$. Now in (4.6) and (4.7) we can take constants c_4 and c_5, independent from $\mathbf{p}$, instead of $\tilde{c}_4$ and $\tilde{c}_5$. From this statement and from (4.1) the convergence of schemes in the norm $\mathbf{L}_\infty(0, T; \mathbf{W}^1_{2h})$ follows. Theorem 4.1 is proved.

THEOREM 4.2. *Assume that* $\mathbf{u}_1(x,t)$ *and* $\mathbf{u}_2(x,t)$ *are two solutions of* (1.1)–(1.3) *with the initial conditions* $\mathbf{u}_{10}$ *and* $\mathbf{u}_{20}$. *Let* (1.5)–(1.7), (4.1) *be satisfied in both cases. Then there exists* $\tau_0', h_0' > 0$ *such that* $\forall\tau$, $0 < \tau \leqslant \tau_0'$, $\forall h$, $0 < h \leqslant h_0'$ *the following estimate for the solutions of* (2.1)–(2.3) *is valid*:

$$\max_{0\leqslant j\leqslant M}\big\{\|(\mathbf{p}_1 - \mathbf{p}_2)(t_j)\|_{W^1_{2h}}\big\} \leqslant c_6\|\mathbf{u}_{10} - \mathbf{u}_{20}\|_{W^1_{2h}}, \tag{4.8}$$

here $c_6 = c_6\big(a, b, c_2, c_3, \|\mathbf{u}_1\|_{C(\overline{Q})}, \|\mathbf{u}_2\|_{C(\overline{Q})}, \|\mathbf{u}_{10}\|_{W^1_{2h}}, \varphi\big)$.

Similarly assume that $\mathbf{u}_1(x,t)$ *and* $\mathbf{u}_2(x,t)$ *are two solutions of* (1.1), (1.2), (1.4) *and conditions* (1.5), (1.6), (1.8), (4.1) *are satisfied, then there exists* $\tau_0'', h_0'' > 0$ *such that with the corresponding* τ, h *the following estimate for the solutions of* (2.4)–(2.6) *is valid*:

$$\max_{0\leqslant j\leqslant M}\big\{\|(\mathbf{p}_1 - \mathbf{p}_2)(t_j)\|_{W^1_{2h}}\big\} \leqslant c_7\|\mathbf{u}_{10} - \mathbf{u}_{20}\|_{W^1_{2h}}, \tag{4.9}$$

here $c_7 = c_7(a, b, c_3, \|\mathbf{u}_1\|_{C(\overline{Q})}, \|\mathbf{u}_2\|_{C(\overline{Q})}, \|\mathbf{u}_{10}\|_{W^1_{2h}}, \varphi)$.

Proof. We denote $\mathbf{z} = \mathbf{p}_1 - \mathbf{p}_2$ and $\Upsilon = \mathbf{f}(\dot{\mathbf{p}}_1) - \mathbf{f}(\dot{\mathbf{p}}_2)$. Then, subtracting the equations for $\mathbf{p}_2$ from the equations for $\mathbf{p}_1$, we obtain problems of type (2.1)–(2.3) or (2.4)–(2.6) for a function $\mathbf{z}$ with a function Υ instead of $\mathbf{f}$.

In the first case we use (2.17), estimate $\|\Upsilon_i\|_{W^1_{2h}}$ with the help of (2.14), use Theorem 5.1 and find τ_0', h_0' such that $\forall\tau, h$, $0 < \tau \leqslant \tau_0'$, $0 < h \leqslant h_0'$ the following estimates are valid: $\|\mathbf{p}_i\|_{C(\overline{Q})} \leqslant 2\|\mathbf{u}_i\|_{C(\overline{Q})}$ and

$\max_{0\leqslant k\leqslant M}\left\{\|\mathbf{p}_i(t_k)\|_{W^1_{2h}}\right\} \leqslant e_3e_4'\|\mathbf{u}_{i0}\|_{W^1_{2h}}$, where $i = 1, 2$. Here and in what follows e_1, e_3, e_4', e_5 are constants from Theorem 5.1. From here we obtain the following inequality:

$$\|\mathbf{z}(t_j)\|^2_{W^1_{2h}} \leqslant e_1\|\mathbf{z}(t_0)\|^2_{W^1_{2h}} + \tau(e_1+1)e_6'n\sum_{k=0}^{j-1}\left(\|\mathbf{z}(t_{k+1})\|^2_{W^1_{2h}} + \|\mathbf{z}(t_k)\|^2_{W^1_{2h}}\right),$$

where constant

$$e_6' = 2\sqrt{2}n\left(1+2c_3e_3e_4'\|\mathbf{u}_{10}\|_{W^1_{2h}}\right)\varphi\left(2\max\{\|\mathbf{u}_1\|_{C(\overline{Q})}, \|\mathbf{u}_2\|_{C(\overline{Q})}\}\right).$$

Now we can use Corollary 2.3 and obtain the inequality

$$\|\mathbf{z}(t_j)\|^2_{W^1_{2h}} \leqslant e_1\exp(4T(e_1+1)e_6'n)\|\mathbf{z}(t_0)\|^2_{W^1_{2h}}.$$

This estimate is valid $\forall j = 0, 1, \ldots, M$, hence, we get (4.8) with $c_6 = \sqrt{e_1}\exp(2T(e_1+1)e_6'n)$.

In the second case we prove similarly that there are τ_0'', h_0'' such that (4.9) is valid. Here

$$c_7 = \exp(Tae_6''n), \qquad e_6'' = 2\sqrt{2}n\left(1+8c_3e_5\|\mathbf{u}_{10}\|_{W^1_{2h}}\right)\varphi\left(2\max\{\|\mathbf{u}_0\|_{C(\overline{Q})}, \|\mathbf{u}_2\|_{C(\overline{Q})}\}\right).$$

Theorem 4.2 is proved.

COROLLARY 4.1. *Under the conditions of Theorems* 4.1 *and* 4.2 *we can prove the convergence and stability of difference schemes in the norm* $\|\ \|_{C(\overline{Q})}$.

Proof. This statement follows from (2.8).

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