.

# Weierstraß-Institut
## für Angewandte Analysis und Stochastik
### Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

# Rate-independent elastoplasticity at finitesrains and its numerical approximation

Alexander Mielke [1,2], Tomáš Roubíček[3,4]

submitted: January 18, 2016

[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

[2] Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin
Germany

[3] Mathematical Institute
Charles University
Sokolovská 83
CZ-186 75 Praha 8
Czech Republic
E-Mail: tomas.roubicek@mff.cuni.cz

[4] Institute of Thermomechanics
Czech Academy of Sciences
Dolejškova 5
CZ-182 00 Praha 8
Czech Republic

No. 2210
Berlin 2016

2010 *Mathematics Subject Classification.* 35K85, 49S05, 65M60, 74A30, 74C15, 74M15, 74S05.

*Key words and phrases.* Plasticity, quasistatic evolution, energetic solutions, dissipation distance, hardening, polyconvexity, Ciarlet-Nečas condition, Signorini contact, finite-element approximation, $\Gamma$-convergence, Lavrentiev phenomenon, 2nd-grade nonsimple materials.

The research of A.M. has been partially supported by DFG through SFB 1114 (subproject B01). T.R. acknowledges partial support from the grants 13-18652S and 14-15264S of the Czech Science Foundation and the institutional support RVO: 61388998 (ČR).

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

**Abstract**

Gradient plasticity at large strains with kinematic hardening is analyzed as quasistatic rate-independent evolution. The energy functional with a frame-indifferent polyconvex energy density and the dissipation are approximated numerically by finite elements and implicit time discretization, such that a computationally implementable scheme is obtained. The non-selfpenetration as well as a possible frictionless unilateral contact is considered and approximated numerically by a suitable penalization method which keeps polyconvexity and simultaneously by-passes the Lavrentiev phenomenon. The main result concerns the convergence of the numerical scheme towards energetic solutions.

In the case of incompressible plasticity and of nonsimple materials, where the energy depends on the second derivative of the deformation, we derive an explicit stability criterion for convergence relating the spatial discretization and the penalizations.

# 1 Introduction

The theory of elastostatics at finite strains has seen a rapid development within the last decades. The fundamental work on polyconvex materials developed in [Bal77] provided a powerful basis for a general theory that allows for the treatment of geometric nonlinearities as well as physically necessary singularities. In particular, a stored-energy density $W : \mathbb{R}^{d\times d}\times\mathbb{R}^m \to \mathbb{R}_\infty \overset{\text{def}}{=} \mathbb{R}\cup\{\infty\}$ as a function of the deformation gradient $\mathsf{F}$ and internal parameters $z$ specified later has to satisfy:

$$\begin{array}{lll} \text{objectivity:} & W(\mathsf{QF},z) = W(\mathsf{F},z) & \text{for } \mathsf{Q}\in\mathrm{SO}(d), \mathsf{F}\in\mathbb{R}^{d\times d}, \quad (1.1\text{a}) \\ \text{local non-selfpenetration:} & W(\mathsf{F},z)\to\infty & \text{for } \det \mathsf{F}\to 0^+, \text{ and} \\ & W(\mathsf{F},z)=\infty & \text{for } \det\mathsf{F}\le 0. \quad (1.1\text{b}) \end{array}$$

Hence, the proper domain of the mapping $\mathsf{F} \mapsto W(\mathsf{F},z)$ is the general linear group $\mathrm{GL}^+(d) \overset{\text{def}}{=} \{\mathsf{F}\in\mathbb{R}^{d\times d};\ \det\mathsf{F} > 0\}$, which already highlights an underlying Lie group structure.

Approximately at the same time the theory of elastoplasticity obtained a sound mathematical basis starting from [Mor74], see also [Alb98,HaR99,Tem85] for surveys on further developments. However, this theory is restricted to the case of small strains and the so-called additive split $e(u) = \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^\mathsf{T} = \mathsf{E}_{\rm el} + \mathsf{E}_{\rm pl}$, as it fundamentally depends on the methods of convex analysis in Hilbert spaces.

A major advance in the mathematical approach to finite-strain elastoplasticity was the observation in [OrR99, OrS99] that the time-incremental problems in rate-independent and in the viscoplastic case can be written as a minimization problem for the sum of the increments in the stored energy and in the dissipated energy. This idea opened up the rich toolbox of the direct methods in the calculus of variations. A general existence theory for the time-continuous problem, which in the classical setting consists of the elastic equilibrium equation and the plastic flow rule, was developed in [MaM09] by using a formulation that allows us to use functional analytical tools that are compatible with the strong nonlinearities generated by the Lie group structures resulting from $\mathrm{GL}^+(d)$ and $\mathrm{SL}(d)$. In particular, it relies on the theory of *energetic solutions* (also called quasistatic evolutions in [DDM06, DaL10]) for rate independent systems as initiated in [MTL02] and further developed in [Mie05, FrM06, MiR15]. In particular, the abstract metric formulation on general topological spaces developed in [MaM05, Mie11, MiR15] is ideally suited for finite-strain elastoplasticity, see [MaM09, DaL10, MiR15]. Hardening mechanisms and regularizing gradient terms for the plastic strain and the possible parameters have to be included to allow for a rigorous analysis. Otherwise the localization phenomena like shearbands may occur (cf. [Mie03, DDM06, BMR12], or the formation of microstructures (cf. [OrR99, CHM02, HHM12]).

Meanwhile, these models are widely used in engineering and are quite successful in predicting macroscopic deformation processes like deep drawing and other forming processes, see e.g. [SiH98, SiO85, MiS92, NeW03]. In particular, also efficient numerical methods have been developed and sucessfully implemented. Sometimes also a global non-penetration or unilateral (self)contact condition is considered and treated by a penalization [GPU04, Stu01].

Yet, to our best knowledge, neither of these numerical schemes has been supported by rigorous convergence analysis. The goal of this article is to use recent advances in abstract numerical approaches in rate-independent processes [MiR09, MiR15] to devise specific finite-element numerical schemes for gradient plasticity at large strains with a guaranteed convergence. Of course, we will use physically relevant models, i.e. with frame-indifferent energies. The particular difficulties are related with the local non-selfpenetration (1.1b). In addition, we also consider the global non-selfpenetration as formulated rigorously in [CiN87] and suggested in this context already in [MaM09, Sect.6]. Rather as a side effect, we will also consider a possible frictionless unilateral contact on the boundary. In contrast to a so-called three-field formulation used often in plasticity, cf. e.g. [SiH98], we do not use any other field beside deformation and plastic strain, which is the minimal scenario and the simplest option to implement computationally.

The model will be formulated in Section 2. For notational simplicity only, we confine ourselves to homogeneous materials (i.e. no explicit $x$-dependence of the energy density $W$ and of the dissipation potential $R$ below) and to kinematic hardening without additional hardening parameters. The case of hardening with additional hardening parameters $z$ (like isotropic hardening) was considered in [MaM09] and can be handled by our numerical considerations, too. Our numerical approximation uses an implicit time discretization via incremental minimization problems, a finite-element method in for space discretization, and suitable regularization of the stored energy as well as a penalization of the non-penetration condition, see Section 3. The convergence is proved in Section 4 provided the spatial discretization converges much faster than the penalization. Section 5 provides a more applicable approximation strategy in the case of 2nd-grade nonsimple materials, i.e. $W$ depends on $\nabla^2 y$. We derive a quite explicit sufficient stability criterion guaranteeing convergence simultaneously in time-space discretization and penalization. All convergence results rely on the general theory of evolutionary $\Gamma$-convergence as developed in [MRS08], where in the present context the main task lies in showing the $\Gamma$-convergence for $(\varepsilon, h) \to (0,0)$ of the penalized and spatially discretrized energies $\mathcal{E}_{\varepsilon h}$ to the limit $\mathcal{E}$ of the continuous model.

In our static $\Gamma$-convergence of $\mathcal{E}_{\varepsilon h}$, we have to avoid the occurrence of a possible Lavrentiev phenomenon [Lav27], which may occur when approximating deformations with determinant constraint by Lipschitz functions. Our approach relies on approximating $W$ monotonously from below by functions $W_\varepsilon$ that have suitable ($\varepsilon$-dependent) upper growth bounds. The relaxation of the global interpenetration condition of Ciarlet-Nečas is done by using a new result concering the weak continuity of the "volume map" $y \mapsto \text{meas}_d\big(y(\Omega)\big)$ on $W^{1,p}(\Omega)$, see Proposition 4.3.

In particular, when plasticity is omitted, as a "side product" our theory devises a regularization and a finite-element approximation for the static elasticity problem (with possibly a unilateral contact) respecting global non-penetrability, showing the $\Gamma$-convergence (or even Mosco convergence) of the regularized and approximate elastic energy, cf. Remark 5.5, thus avoiding effectively the Lavrentiev phenomenon. In comparison with the approaches in literature, which either (i) use a regularization of the discrete problem not having a direct counterpart in the continuous setting (cf. [BaL06, BaL07, Li95, Li96, Neg90, XuH11]) or (ii) introduce auxiliary variables $\eta$ approximating $\nabla y$, cof$\nabla y$, or det $\nabla y$ via a penalization term (cf. [BaK87, CaO10]), we treat the full nonconvex potentials modified in a way that allows for clear physical interpretation on the level of continuous problems.

For the readers' convenience, we summarize the basic notation:

| | |
|---|---|
| $d$ dimension of the problem, | $\widehat{W}_{\rm el} = \widehat{W}_{\rm el}(\mathsf{F}_{\rm el})$ the elastic part of $W$ in invariant form, |
| $\Omega \subset \mathbb{R}^d$ a reference domain, | $R$ dissipation metric, viz. (2.21b), |
| $y : \Omega \to \mathbb{R}^d$ deformation, | $\widehat{R}$ dissipation metric in invariant form, |
| $\mathit{\Pi} : \Omega \to \mathfrak{P} \subset \mathbb{R}^{d\times d}$ plastic strain, | $D$ dissipation distance induced by $R$, |
| $\mathsf{F}$ a placeholder for the deformation gradient $\nabla y$, | $\mathcal{D}$ overall dissipation distance, |
| $\mathsf{P}$ a placeholder for the plastic strain $\mathit{\Pi}$, | $p_{\rm df}$ the exponent of coercivity in $\mathsf{F}$, |
| $\mathsf{A}$ a placeholder for the gradient of plastic strain $\nabla\mathit{\Pi}$, | $p_{\rm el}$ the exponent of coercivity in $\mathsf{F}_{\rm el}$, |
| $\mathsf{F}_{\rm el} = \mathsf{F}\mathsf{P}^{-1}$ elastic strain, | $p_{\rm pl}$ the exponent of coercivity in $\mathsf{P}$, |
| $\mathbb{M}_s(\mathsf{F})$ minors of the order $s$ of a matrix $\mathsf{F}$, | $p_{\rm gr}$ the exponent of coercivity in $\mathsf{A}$, |
| $\mathbb{K}_s(\mathsf{F})$ a generalized cofactor matrix, | $p_{\rm sg}$ the exponent of coercivity in $\nabla^2 y$, |
| $\mathcal{E}$ overall stored energy, | $g_{\rm Dir} : \Gamma_{\rm Dir} \to \mathbb{R}^d$ prescribed time-dependent boundary displacement, |
| $W$ specific stored energy, | $\ell$ outer loading (given by $f_{\rm vol}$ and $f_{\rm surf}$), |
| $W_{\rm el}$ the elastic part of $W$, viz. (2.19), | $\kappa > 0$ hardening coefficient of plasticity, |
| | $\varkappa > 0$ coefficient of the gradient of plasticity, |
| | $\chi > 0$ coefficient of the 2nd-gradient of $y$. |

*Table 1: Main notation used in this paper.*

Moreover, we use the standard notation for the function spaces on $\Omega$: namely, $C(\bar{\Omega})$ denotes the Banach space of continous functions on the closure of $\Omega$, and $L^p(\Omega)$ denotes the Lebesgue space of measurable functions whose $p$-power is integrable, while $W^{k,p}(\Omega)$ is the Sobolev space of functions which are together with all their $k^{\rm th}$-order derivatives in $L^p(\Omega)$. Moreover, we abbreviate $W^{k,2}(\Omega)$ by $H^k(\Omega)$, as usual. For vector or matrix-valued cases, we write e.g. $L^p(\Omega;\mathbb{R}^n)$ or $L^p(\Omega;\mathbb{R}^{n\times m})$ etc. By $p^*$ we will denote the Sobolev exponent $pd/(d{-}p)$ if $1 < p < d$ while $p^* = \infty$ for $p > d$ and $1 < p^* < \infty$ is arbitrary if $p = d$, such that we have the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Moreover, we use the Bochner spaces of Banach-space-valued functions on the time interval $[0,T]$, denoted by $L^p([0,T];X)$. By $\mathrm{B}([0,T];X)$ we denote the space of bounded everywhere defined $X$-valued functions on $[0,T]$, while $C([0,T];X)$ or $C^1([0,T];X)$ are the spaces of $X$-valued continuous or continuously differentiable functions $[0,T] \to X$, respectively. In the proofs, we abbreviate $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_{L^p}$ or $\|\cdot\|_{W^{k,p}(\Omega)}$ by $\|\cdot\|_{W^{k,p}}$ etc.

## 2 The quasistatic gradient plasticity

We introduce some notations. We consider a reference domain $\Omega$ bounded and Lipschitz in $\mathbb{R}^d$, and $\Gamma_{\rm Dir}$ a part of its boundary where (possibly time-dependent) hard-device load is considered, i.e. we presribe the Dirichlet condition $y|_{\Gamma_{\rm Dir}} = g_{\rm Dir} = g_{\rm Dir}(t)$. The state of the body will be prescribed by the deformation $y:\Omega\to\mathbb{R}^d$ and the plastic tensor $\mathit{\Pi}:\Omega\to\mathfrak{P}$ with $\mathfrak{P}\subset\mathbb{R}^{d\times d}$ a closed subset to be specified for a concrete model. We will use also the gradient $\nabla\mathit{\Pi}$, which will be essential to provide compactness and prevent localization and formation of microstructures. Then $z=(\mathsf{P},\mathsf{A})$ in (1.1), where $\mathsf{P}\in\mathbb{R}^{d\times d}$ is the placeholder for $\mathit{\Pi}$ and $\mathsf{A}\in\mathbb{R}^{d\times d\times d}$ is the placeholder for $\nabla\mathit{\Pi}$. We also consider a unilateral frictionless contact that the deformed configuration $y(\bar\Omega)$ is always away of some open set $\mathfrak{C}\subset\mathbb{R}^d$; of course, choosing $\mathfrak{C}=\emptyset$ makes this constraint never active. Then, condering also a global constraint preventing folding of the deformation, we assume that the stored-energy functional takes the form

$$\widetilde{\mathcal{E}}(t,y,\mathit{\Pi}) = \begin{cases} \displaystyle\int_\Omega W(\nabla y(x),\mathit{\Pi}(x),\nabla\mathit{\Pi}(x))\,\mathrm{d}x & \\ \qquad\qquad\qquad -\langle\ell(t),y\rangle & \text{if } \displaystyle\int_\Omega \det(\nabla y)\,\mathrm{d}x \le \mathrm{meas}_d(y(\Omega)), \\ & y(\bar\Omega)\cap\mathfrak{C}=\emptyset, \text{ and } y|_{\Gamma_{\rm Dir}} = g_{\rm Dir}(t), \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that, if $\widetilde{\mathcal{E}}<\infty$, due to (1.1b), it holds simultaneously

$$\det(\nabla y(x))>0 \ \text{ for a.a. } x\in\Omega \qquad\text{and}\qquad \int_\Omega \det(\nabla y)\,\mathrm{d}x \le \mathrm{meas}_d(y(\Omega)), \tag{2.2}$$

which is just what is called the *Ciarlet-Nečas condition* [CiN87] guaranteeing *non-selfpenetration*. An example for an external loading $\ell$ is

$$\big\langle\ell(t),y\big\rangle = \int_\Omega f_{\rm vol}(t,x)\cdot y(x)\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_{\rm Dir}} f_{\rm surf}(t,x)\cdot y(x)\,\mathrm{d}S, \tag{2.3}$$

considering the bulk force $f_{\rm vol}$ and the surface load $f_{\rm surf}$.

Forgetting, for a moment, the global constraints involved in (2.1) related with non-selfpenetration and a possible contact, in our quasistatic setting, we will consider the problem that, in its classical formulation, can be written as

$$\mathrm{div}\big(\partial_{\mathsf{F}}W(\nabla y,\mathit{\Pi},\nabla\mathit{\Pi})\big) + f_{\rm vol} = 0 \quad \text{in } \ \Omega \tag{2.4a}$$

while evolution of $\mathit{\Pi}$ is governed by the thermodynamical-force balance

$$\partial_{\dot{\mathsf{P}}}R(\mathit{\Pi},\dot{\mathit{\Pi}}) + \partial_{\mathsf{P}}W(\nabla y,\mathit{\Pi},\nabla\mathit{\Pi}) - \mathrm{div}\,\big(\partial_{\mathsf{A}}W(\nabla y,\mathit{\Pi},\nabla\mathit{\Pi})\big) \ni 0 \tag{2.4b}$$

where "$\partial$" stands for a partial Fréchet subdifferetial; this might be the standard Gâteaux derivative in the smooth case (as $\partial_{\mathsf{F}}W$, $\partial_{\mathsf{P}}W$, and $\partial_{\mathsf{A}}W$) or the standard convex subdifferential in the convex but nonsmooth case (as $\partial_{\dot{\mathsf{P}}}$). Actually, in the engineering literature, the thermodynamical-force balance (2.4b) is written rather in the explicit form in terms of $\dot{\mathit{\Pi}}$ as the *plastic-flow rule*:

$$\dot{\mathit{\Pi}} \in \big[\partial_{\dot{\mathsf{P}}}R(\mathit{\Pi},\cdot)\big]^{-1}\Big(\mathrm{div}\,\big(\partial_{\mathsf{A}}W(\nabla y,\mathit{\Pi},\nabla\mathit{\Pi})\big) - \partial_{\mathsf{P}}W(\nabla y,\mathit{\Pi},\nabla\mathit{\Pi})\Big),$$

where the inverse to $\partial_{\dot{\mathsf{P}}} R(\varPi,\cdot)$ can further be expressed as the subdifferential of the convex conjugate to $R(\varPi,\cdot)$. To be more specific, let us recall the definition of the subdifferential, which in our case means

$$\partial_{\dot{\mathsf{P}}} R(\varPi,\dot{\varPi}) = \big\{ \mathsf{S}\in\mathbb{R}^{d\times d};\ \forall V\in\mathbb{R}^{d\times d}:\ R(\varPi,V) \geq R(\varPi,\dot{\varPi}) + \mathsf{S}:(V-\dot{\varPi}) \big\}. \tag{2.5}$$

The thermodynamical-force balance (2.4b) involves a potential $R$ of dissipative force such that the functional $\dot{\mathsf{P}} \mapsto R(\mathsf{P},\dot{\mathsf{P}})$ is convex on the tangent space. This thermodynamical-force balance is rate independent if $\dot{\mathsf{P}} \mapsto R(\mathsf{P},\dot{\mathsf{P}})$ is positively homogeneous of degree 1, i.e., $R(\mathsf{P},\lambda\dot{\mathsf{P}}) = \lambda R(\mathsf{P},\dot{\mathsf{P}})$ for any $\lambda \geq 0$. As a consequence the set-valued subdifferential $\dot{\mathsf{P}} \mapsto \partial_{\dot{\mathsf{P}}} R(\mathsf{P},\dot{\mathsf{P}})$ is positively homogeneous of degree 0, i.e., it depends on the direction of the rate $\dot{\mathsf{P}}$ but not on its norm. Thus, if $(y,\varPi)$ solves (2.4) for the loading $\ell$, then for all $\lambda > 0$ the process $t \mapsto (y(\lambda t),\varPi(\lambda t))$ solves (2.4) for the loading $t \mapsto \ell(\lambda t)$.

Of course, (2.4) is to be completed by suitable boundary conditions, namely with Dirichlet boundary conditions on $\Gamma_{\rm Dir}$ and, in accordance with (2.3), the prescribed stress and natural homogeneous conditions for the gradient part on $\partial\Omega\setminus\Gamma_{\rm Dir}$:

$$\begin{aligned} & y|_{\Gamma_{\rm Dir}} = g_{\rm Dir}(t,\cdot) && \text{on } \Gamma_{\rm Dir}, && (2.6a)\\ & \partial_{\mathsf{F}} W(\nabla y,\varPi,\nabla\varPi)\nu = f_{\rm surf} && \text{on } \partial\Omega\backslash\overline{\Gamma}_{\rm Dir}, && (2.6b)\\ & \partial_{\mathsf{A}} W(\nabla y,\varPi,\nabla\varPi)\nu = 0 && \text{on } \partial\Omega. && (2.6c) \end{aligned}$$

In fact, instead of the boundary-value problem (2.6a,b) for the Euler-Lagrange equation (2.4a), we will consider the underlying variational principle,

$$y(t)\ \ \text{minimizes}\ \ \widetilde{y} \mapsto \widetilde{\mathcal{E}}(t,\widetilde{y},\varPi(t))\ \ \text{over all}\ \ \widetilde{y}\ \ \text{with}\ \ \widetilde{y}|_{\Gamma_{\rm Dir}} = g_{\rm Dir}(t). \tag{2.7}$$

This formulation is more natural than (2.4a) as it excludes, e.g., unstable critical points which would never be realized in real systems. Moreover, it allows for giving a rigorous mathematical sense in a simpler way even if one makes transformations as (2.10) below or if one considers global contraints of non-selfpenetration and a possible unilateral contact, which would be very technical in terms of the Euler-Lagrange equation, cf. [Bal02] and also [BaM85, FHM03] and [HaS06, Thm.4.1]. This just documents the advantage of the energetic formulation of the problem we will use later. On the other hand, it should be mentioned that (2.7) excludes local minimizers which real systems may recognized.

As solutions of rate-independent systems are usually not continuous, we need a weak form of the thermodynamical-force balance (2.4b). To this end, the dissipation metric $R$ is replaced by the associated dissipation distance $D$ defined as

$$\begin{aligned} D(\mathsf{P}_0,\mathsf{P}_1) = \inf\Big\{ \int_0^1 & R(\mathsf{P}(s),\dot{\mathsf{P}}(s))\,\mathrm{d}s;\ \mathsf{P}\in W^{1,1}(0,1;\mathbb{R}^{d\times d}),\\ & \mathsf{P}(s) \in \mathfrak{P}\ \forall s\in[0,1],\ \mathsf{P}(0)=\mathsf{P}_0,\ \mathsf{P}(1)=\mathsf{P}_1 \Big\} \end{aligned} \tag{2.8}$$

with $\mathfrak{P} \subset \mathbb{R}^{d\times d}$ as above. The dissipation functional

$$\mathcal{D}(\mathsf{P}_0,\mathsf{P}_1) = \int_\Omega D(\mathsf{P}_0(x),\mathsf{P}_1(x))\,\mathrm{d}x \tag{2.9}$$

measures the minimal amount of energy dissipated when going from the state $\mathsf{P}_0$ to $\mathsf{P}_1$. An important fact is that $\mathcal{D}$ satisfies the (unsymmetric) triangle inequality, i.e. $\mathcal{D}(\mathsf{P}_1,\mathsf{P}_3) \leq \mathcal{D}(\mathsf{P}_1,\mathsf{P}_2) + \mathcal{D}(\mathsf{P}_2,\mathsf{P}_3)$, cf. also (4.5c) below.

Some aspects of the geometric nonlinearities arising through the Lie group structures of finite strains and the multiplicative decomposition were already treated in [FrM06] and [MiM06], respectively. Here we combine and generalize these results in a unified setting. The first nontrivial ingredient is the treatment of time-dependent boundary conditions $g_{\rm Dir}$ as in [MaM05]. For this, we seek for $\widetilde{y}(t,\cdot)$ in the form

$$\widetilde{y}(t,x) = g_{\rm Dir}(t,y(t,x)) \quad \text{with} \quad y(t,x) = x \ \text{ for } \ x\in\Gamma_{\rm Dir}, \tag{2.10}$$

i.e. $\widetilde{y}(t) = g_{\rm Dir}(t,\cdot)\circ y(t) : \Omega \to \mathbb{R}^d$ at each time $t$, and set

$$\mathcal{E}(t,y,\mathit{\Pi}) = \widetilde{\mathcal{E}}(t,g_{\rm Dir}(t,\cdot)\circ y,\mathit{\Pi}). \tag{2.11}$$

Assuming that the Dirichlet loading is realized away from the obstacle $\mathfrak{C}$, one can assume that $g_{\rm Dir}|_{\mathfrak{C}}$ is constant in time, for example the identity, i.e.

$$g_{\rm Dir}(t,x)|_{\mathfrak{C}} = x \qquad \text{for } x\in\mathfrak{C}, \tag{2.12}$$

and the original condition $\widetilde{y}(\Omega)\cap\mathfrak{C} = \emptyset$ transforms simply into $y(\bar{\Omega})\cap\mathfrak{C} = \emptyset$. Also the Ciarlet-Nečas condition keeps its form in term of $y$ equally as in terms of $\widetilde{y}$. Note also that $\nabla\widetilde{y} = \partial_y g_{\rm Dir}(t,y)\nabla y$, which motivates calling (2.10) a multiplicative decomposition. Altogether, this means

$$\mathcal{E}(t,y,\mathit{\Pi}) = \begin{cases} \displaystyle\int_\Omega W(\partial_y g_{\rm Dir}(t,y)\nabla y,\mathit{\Pi},\nabla\mathit{\Pi})\,\mathrm{d}x & \\ \qquad\qquad -\big\langle \ell(t), g_{\rm Dir}(t,y)\big\rangle & \text{if } \displaystyle\int_\Omega \det(\nabla y)\,\mathrm{d}x \leq \mathrm{meas}_d(y(\Omega)) \\ & \text{and } y(\bar{\Omega})\cap\mathfrak{C}=\emptyset, \\ \infty & \text{otherwise,} \end{cases} \tag{2.13}$$

while the (transformed time-constant) Dirichlet conditions are involved in the fixed set of admissible states defined by

$$\mathcal{Q} \stackrel{\text{def}}{=} \big\{(y,\mathit{\Pi}) \in W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d});\ \ y|_{\Gamma_{\rm Dir}} = \text{identity}\big\}. \tag{2.14}$$

We emphasize that the set of all constraints in (2.13) is now constant in time, while time dependence is exclusively moved into the integrand that was time independent originally. This gives a rigorous sense to $\partial_t\mathcal{E}(t,q)$ under suitable data qualifications. Here and subsequently we use $q$ as a placeholder for the state $q = (y,\mathit{\Pi})$.

To allow for finite-strain elasticity complying with (1.1), we assume that $W$ is polyconvex in $\mathsf{F}$. More specifically, denoting the function $\mathbb{M} : \mathbb{R}^{d\times d} \to \mathbb{R}^{\mu_d}$ with $\mu_d = \sum_{s=1}^d \binom{d}{s}^2 = \binom{2d}{d}-1$ maps a matrix to all its *minors* (subdeterminants) and using $\mathsf{A}$ as a placeholder for $\nabla\mathit{\Pi} \in \mathbb{R}^{d\times d\times d}$, we assume

$$\begin{aligned} &\exists\, \mathbb{W} : \mathbb{R}^{\mu_d}\times\mathfrak{P}\times\mathbb{R}^m\times\mathbb{R}^{d\times d\times d} \to \mathbb{R}_\infty : \\ &\quad \text{(i)}\quad \mathbb{W} \text{ is lower semicontinuous,} \\ &\quad \text{(ii)}\quad \forall\,(\mathsf{F},\mathsf{P},\mathsf{A}):\ \ W(\mathsf{F},\mathsf{P},\mathsf{A}) = \mathbb{W}(\mathbb{M}(\mathsf{F}),\mathsf{P},\mathsf{A}), \\ &\quad \text{(iii)}\quad \forall\,\mathsf{P}\in\mathfrak{P}:\ \ (\mathsf{M},\mathsf{A}) \mapsto \mathbb{W}(\mathsf{M},\mathsf{P},\mathsf{A}) : \mathbb{R}^{\mu_d}\times\mathfrak{P}\times\mathbb{R}^{d\times d\times d} \to \mathbb{R}_\infty \text{ is convex.} \end{aligned} \tag{2.15}$$

Thus, (2.15) implies that the function $\mathsf{F} \mapsto W(\mathsf{F}, \mathsf{P}, \mathsf{A})$ is polyconvex in the sense [Bal77] for any $(\mathsf{P}, \mathsf{A})$. For the weak lower semicontinuity, we use that the minors $\mathbb{M}_s(\nabla y)$ of order $s \in \{1, ..., d\}$ of the term in (2.23) are weakly continuous, cf. [Bal77, Res67], and further the strong convergence of $\varPi$, which is obtained by compact-embedding from the coercivity of $W$ in $\mathsf{A}$.

The second nontrivial ingredient that is induced by the multiplicative structure of $\mathrm{GL}^+(d)$ is the control of the power of the external forces $\partial_t \mathcal{E}(t, q)$, which will allow us to replace $\mathcal{P}$ in (4.2) below by $\partial_t \mathcal{E}(t, q(t))$. The crucial assumption is an energy control of the Kirchhoff stress

$$\big|\partial_\mathsf{F} W(\mathsf{F}, \mathsf{P}, \mathsf{A})\mathsf{F}^\mathsf{T}\big| \le c_1^W\big(W(\mathsf{F}, \mathsf{P}, \mathsf{A}) + c_0^W\big), \tag{2.16}$$

which was introduced in [Bal84] and popularized in [Bal02]. The Lie group structure of $\mathrm{GL}^+(d)$ implies that the Kirchhoff stress $\partial_\mathsf{F} W(\mathsf{F})\mathsf{F}^\mathsf{T}$ lies in $\mathrm{gl}(d) = T_\mathbb{I}\mathrm{GL}^+(d)$ and hence is more intrinsic than other stress measures. We obtain the formula

$$\begin{aligned} \partial_t \mathcal{E}(t, q) = \int_\Omega & \partial_\mathsf{F} W(\mathsf{F}, \varPi, \nabla\varPi)\mathsf{F}^\mathsf{T}{:}\mathsf{V}\,\mathrm{d}x \\ & \text{with} \ \ \mathsf{F} = \mathsf{F}(x) = \partial_y g_{\mathrm{Dir}}(t, y(x))\nabla y(x) \\ & \text{and} \ \ \mathsf{V} {=} \mathsf{V}(t, x) {=} \partial_y g_{\mathrm{Dir}}(t, y(x))^{-1}\partial_y \dot g_{\mathrm{Dir}}(t, y(x)), \end{aligned} \tag{2.17}$$

cf. [FrM06, Lemma 5.5]. Under suitable assumptions on $g_{\mathrm{Dir}}$ this allows us to derive an estimate for $\partial_t \mathcal{E}(t, q)$ in terms of $\mathcal{E}(t, q)$ and to deduce further helpful continuity properties of $\partial_t \mathcal{E}$.

Elastoplasticity at finite strains is usually based on the Lee-Liu *multiplicative decomposition*

$$\nabla y = \mathsf{F} = \mathsf{F}_{\mathrm{el}}\mathsf{F}_{\mathrm{pl}}, \tag{2.18}$$

introduced in [LeL67]. This decomposition reflects the Lie group structure of $\mathrm{GL}^+(d)$, where the elastic part $\mathsf{F}_{\mathrm{el}}$ will contribute to the energy storage whereas the plastic tensor $\mathsf{F}_{\mathrm{pl}} = \mathsf{P}$ is simply chosen to evolve according to a plastic flow rule. The plastic tensor maps the material frame (crystallographic lattice) onto itself and is usually assumed to lie in the special linear group $\mathrm{SL}(d) \stackrel{\text{def}}{=} \{\mathsf{P} \in \mathbb{R}^{d\times d};\ \det \mathsf{P} = 1\}$. In this case we choose $\mathfrak{P} = \mathrm{SL}(d)$. Nevertheless, sometimes also volume plastification is involved in the model and then one considers rather $\mathfrak{P} = \mathrm{GL}^+(d)$. Here, we consider more specifically the elastic and the hardening contribution as

$$W(\mathsf{F}, \mathsf{P}, \mathsf{A}) = W_{\mathrm{el}}(\mathsf{F}, \mathsf{P}) + W_{\mathrm{hard}}(\mathsf{P}) + \frac{\varkappa}{p_{\mathrm{gr}}}|\mathsf{A}|^{p_{\mathrm{gr}}}. \tag{2.19}$$

Following [Mie03], beside the objectivity (1.1a), we assume that $W_{\mathrm{el}}$ and $R$ in (2.4b) is invariant under previous plastic strain $\widetilde{\mathsf{P}}$ in the sense

$$\left.\begin{aligned} W_{\mathrm{el}}(\mathsf{F}\widetilde{\mathsf{P}}, \mathsf{P}\widetilde{\mathsf{P}}) = W_{\mathrm{el}}(\mathsf{F}, \mathsf{P}) \\ R(\mathsf{P}\widetilde{\mathsf{P}}, \dot{\mathsf{P}}\widetilde{\mathsf{P}}) = R(\mathsf{P}, \dot{\mathsf{P}}) \end{aligned}\right\} \text{ for all } \widetilde{\mathsf{P}} \in \mathfrak{P}. \tag{2.20}$$

which have to hold for all $\mathsf{F} \in \mathrm{GL}^+(d)$ and $\mathsf{P} \in \mathfrak{P}$. Note that (2.20) postulates this plastic invariance only for $W_{\mathrm{el}}$ and not for the hardening part in (2.19) since hardening is exactly the mechanism that destroys plastic indifference. The conditions (2.20) imply a special form of $W$ and $R$, namely, in view of (2.19), one can consider $W_{\rm el}(\mathsf{F},\mathsf{P}) = \widehat W_{\rm el}(\mathsf{F}\mathsf{P}^{-1})$ and $W_{\rm hard}(\mathsf{P}) = \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}}$ so that altogether one has

$$W(\mathsf{F},\mathsf{P},\mathsf{A}) = \widehat W_{\rm el}(\mathsf{F}\mathsf{P}^{-1}) + \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\mathsf{A}|^{p_{\rm gr}} \tag{2.21a}$$

for some $\widehat W_{\rm el}$ with $\kappa, \varkappa > 0$, and

$$R(\mathsf{P},\dot{\mathsf{P}}) = \widehat R(\dot{\mathsf{P}}\mathsf{P}^{-1}) \tag{2.21b}$$

for some $\widehat R$. Note that the polyconvexity (2.15), i.e. the convexity of $\mathbb{W}(\cdot,\mathsf{P},\cdot)$, here means the conventional polyconvexity of $\widehat W_{\rm el} : \mathbb{R}^{d\times d} \to \mathbb{R}$. Also note that, due to the assumed rate independence, $\widehat R$ is homogeneous of degree 1. Note that (2.21b) implies $\partial_{\dot{\mathsf{P}}} R(\mathsf{P},\dot{\mathsf{P}}) = \partial\widehat R(\dot{\mathsf{P}}\mathsf{P}^{-1})\mathsf{P}^{-\mathsf{T}}$, cf. [MaM09]. In view of the ansatz (2.21), instead of (2.4) we have now:

$$\begin{aligned}
&{\rm div}\big(\partial_{\mathsf{F}}\widehat W_{\rm el}(F_{\rm el})\varPi^{-\mathsf{T}}\big) + f_{\rm vol} = 0 \qquad \text{with} \quad \varPi^{-\mathsf{T}} \stackrel{\rm def}{=} (\varPi^{-1})^{\mathsf{T}}, \quad \text{and} && (2.22a)\\
&\partial\widehat R(\dot\varPi\varPi^{-1})\varPi^{-\mathsf{T}} - F_{\rm el}^{\mathsf{T}}\partial_{\mathsf{F}}\widehat W_{\rm el}(F_{\rm el})\varPi^{-\mathsf{T}}\\
&\qquad\qquad + \kappa|\varPi|^{p_{\rm pl}-2}\varPi - {\rm div}\,\big(\varkappa|\nabla\varPi|^{p_{\rm gr}-2}\nabla\varPi\big) \ni 0. && (2.22b)
\end{aligned}$$

After the transformation (2.10), the integrand $W(F_{\rm el},\mathsf{P},\mathsf{A})$ of $\mathcal{E}$ depends on the product

$$F_{\rm el} = (\nabla\widetilde y)\varPi^{-1} = \partial_y g_{\rm Dir}(t,y(x))\nabla y(x)\varPi(x)^{-1}. \tag{2.23}$$

The Cauchy-Binet relations $\mathbb{M}_s(\mathsf{G}\mathsf{F}) = \mathbb{M}_s(\mathsf{G})\,\mathbb{M}_s(\mathsf{F})$, see e.g. [BrW89, Sect.4.6]), give

$$\mathbb{M}_s(\mathsf{G}\mathsf{F}\mathsf{P}^{-1}) = \mathbb{M}_s(\mathsf{G})\,\mathbb{M}_s(\mathsf{F})\,\mathbb{M}_s(\mathsf{P}^{-1}) = \frac{1}{\det\mathsf{P}}\mathbb{M}_s(\mathsf{G})\,\mathbb{M}_s(\mathsf{F})\,\mathbb{K}_{d-s}^{\mathsf{T}}(\mathsf{P}), \tag{2.24}$$

where the generalized cofactor matrix $\mathbb{K}_s$ is defined as $\mathbb{K}_s(\mathsf{P}) = (\det\mathsf{P})\mathbb{M}_{d-s}^{\mathsf{T}}(\mathsf{P}^{-1})$, and in (2.17), in place of $\mathsf{F}$, one uses $\mathsf{F}_{\rm el}$ from (2.23). For the dissipation density $D$ we choose any left-invariant distance on the Lie group ${\rm SL}(d)$, viz.,

$$D(\mathsf{P}_0,\mathsf{P}_1) = d_{\rm SL}(\mathsf{P}_1\mathsf{P}_0^{-1}) \text{ with } d_{\rm SL} : {\rm SL}(d) \to [0,\infty[\,,$$

where $d_{\rm SL}$ is generated by a norm $\widehat R$ on the Lie algebra ${\rm sl}(d) \stackrel{\rm def}{=} T_{\mathbb{I}}{\rm SL}(d)$ via

$$d_{\rm SL}(\mathsf{P}_1) = \inf\Big\{\int_0^1 \widehat R(\dot{\mathsf{P}}(s)\mathsf{P}(s)^{-1})\,{\rm d}s;\ \mathsf{P}\in C^1([0,1],{\rm SL}(d)),\ \mathsf{P}(0) = \mathbb{I},\ \mathsf{P}(1) = \mathsf{P}_1\Big\}.$$

Clearly this $D$ satisfies the plastic indifference condition $D(\mathsf{P}_0\widetilde{\mathsf{P}},\mathsf{P}_1\widetilde{\mathsf{P}}) = D(\mathsf{P}_0,\mathsf{P}_1)$. According to [Mie02], the mapping $d_{\rm SL}$ is continuous, is strictly positive for $\mathsf{P} \neq \mathbb{I}$, satisfies the multiplicative triangle inequality $d_{\rm SL}(\mathsf{P}_1\mathsf{P}_0) \le d_{\rm SL}(\mathsf{P}_0) + d_{\rm SL}(\mathsf{P}_1)$, and allows for the bounds

$$\delta|\Sigma| \le \widehat R(\Sigma) \le d_{\rm SL}(\mathsf{Q}\,{\rm e}^{\Sigma}) \le C + \widehat R(\Sigma) \quad \text{for} \;\; \Sigma = \Sigma^{\mathsf{T}} \;\; \text{and} \;\; \mathsf{Q}\in{\rm SO}(d), \tag{2.25}$$

with $\delta, C > 0$, see [HMM03, Mie02]. Thus, conditions (4.5) below are fulfilled.

**Example 2.1 (Ogden-type material)** The simplest coercive material that also simultaneously captures polyconvexity property and local non-selfpenetration takes the elastic part in (2.21a) in the form

$$\widehat{W}_{\rm el}(\mathsf{F}_{\rm el}) = \begin{cases} c_1|\mathsf{F}_{\rm el}|^{p_{\rm el}}+c_2/(\det \mathsf{F}_{\rm el})^\gamma & \text{for } \det \mathsf{F}_{\rm el}>0,\\ \infty & \text{otherwise,}\end{cases} \quad \text{with } c_1,c_2,\gamma>0. \tag{2.26}$$

Note that (2.26) is an example of the *Ogden-type material*; in general, Ogden materials may involve still a contribution from $|\operatorname{cof}(\mathsf{F}_{\rm el})|$. The Kirchhoff stress tensor $K(\mathsf{F}) = \partial_\mathsf{F} W(\mathsf{F})\mathsf{F}^\mathsf{T}$ in (2.16) only depends on $\mathsf{F}$ and takes the simple form

$$K(\mathsf{F}) = \begin{cases} c_1 p_{\rm el}|\mathsf{F}|^{p_{\rm el}-2}\mathsf{F}\,\mathsf{F}^\mathsf{T} - \dfrac{c_2\gamma}{(\det\mathsf{F})^\gamma}\mathbb{I} & \text{if } \det\mathsf{F}>0\\ \text{not defined} & \text{otherwise.}\end{cases}$$

Hence, (4.4c) below immediately holds with $c_0^W=0$ and $c_1^W=\max\{p_{\rm el},\gamma\sqrt{d}\}$. Moreover, also condition (4.4d) holds, since $K$ can be differentiated once again giving $|\partial K(\mathsf{F})[H\mathsf{F}]| \le CW(\mathsf{F})|H|$, see [FrM06] for details.

**Remark 2.2 (Isotropic hardening)** The internal variable $\mathsf{P}$ may have a more complicated structure, e.g. from $\mathbb{R}^{d\times d}\times\mathbb{R}$ and that additional scalar variable may act as a hardening parameter to model isotropic hardening, which may or need not be combined with the kinematic hardening considered in this paper.

# 3 Numerical approximation

As said already above, we use a placeholder $q=(y,\Pi)$. We further use the finite-element method (FEM) for space discretization of $q$ with a mesh parameter $h>0$. We assume a polyhedral $\Omega$ for simplicity, and consider its triangulation $\mathfrak{T}_h$ with the simplicial mesh with the mesh-parameter $h$, i.e. $h \overset{\rm def}{=} \max_{\triangle\in\mathfrak{T}_h}\operatorname{diam}(\triangle)$. By $\mathfrak{N}_h$ we denote the set of corresponding nodal points. It gives rise to the finite-dimensional subset $V_h := V_{{\rm df},h}\times V_{{\rm pl},h}$ with

$$V_{{\rm df},h} := \big\{y\in W^{1,\infty}(\Omega;\mathbb{R}^d);\ \forall\triangle\in\mathfrak{T}_h:\ y|_\triangle \text{ affine}\big\}, \tag{3.1a}$$
$$V_{{\rm pl},h} := \big\{\Pi\in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d});\ \forall\triangle\in\mathfrak{T}_h\colon \Pi|_\triangle \text{ affine},\ \forall x\in\mathfrak{N}_h\colon \Pi(x)\in\mathfrak{P}\big\}. \tag{3.1b}$$

It should be emphasized that the constraint $\Pi(\cdot)\in\mathfrak{P}$ is satisfied on $\Omega$ only approximately for $\Pi\in V_{{\rm pl},h}$, cf. Lemma 4.5 below.

In fact, $V_{{\rm df},h}\subset W^{1,\infty}(\Omega;\mathbb{R}^d)$ and also $V_{{\rm pl},h}\subset W^{1,\infty}(\Omega;\mathbb{R}^{d\times d})$. It is well known that the so-called *Lavrentiev phenomenon* [Lav27] may occur in nonlinear elasticity, i.e. mimizing energy on $W^{1,\infty}(\Omega;\mathbb{R}^d)$ may yield a strictly bigger infimum (and thus false solutions or minimizing sequences) than mimizing on a "correct" $W^{1,p_{\rm df}}$-space, as pointed out in [Bal87, Bal02, BaM85, FHM03], following the old observation in [Lav27]. In particular, one cannot hope for convergence of conformal polynomial finite elements as far as discretization of $y$ concerns, which are always only subspaces of $W^{1,\infty}(\Omega;\mathbb{R}^d)$, which was already well recognized for static problems e.g. in [BaL06, CaO10, Li95, Li96]. For quasistatic rate-independent problems as considered here, this difficulty is still present.

We will overcome this problem by making a penalization/regularization of the constraints involved in $\mathcal{E}$. The interaction of discretization with such a penalization needs usually a sufficiently fast refinement of the discretization with respect to penalization, subjected to some (usually rather implicit) convergence criterion, as observed on the abstract level for the static case in [Rou91] and for the quasistatic rate-independent one in [MiR09]. Usually, one cannot hope for unconditional convergence because of the mentioned Lavrentiev phenomenon.

Thus we make a penalization/regularization by a parameter $\varepsilon > 0$ of the stored energy similarly as in [CaO10] and also of the global constraints by considering

$$\mathcal{E}_{\varepsilon h}(t,y,\varPi) \stackrel{\text{def}}{=} \begin{cases} \displaystyle\int_\Omega W_\varepsilon(\partial_y g_{\rm Dir}(t,y)\nabla y, \varPi, \nabla\varPi) + \frac{1}{\varepsilon} d_{\mathfrak{C}}(y)^{r_1}\,\mathrm{d}x - \big\langle \ell(t), g_{\rm Dir}(t,y)\big\rangle \\ \displaystyle\quad + \frac{1}{\varepsilon}\Big(\int_\Omega \det(\nabla y)\,\mathrm{d}x - \mathrm{meas}_d(y(\Omega))\Big)^{r_0} & \text{if } y\in V_{\mathrm{df},h},\ \varPi\in V_{\mathrm{pl},h}, \\ \infty & \text{otherwise} \end{cases} \tag{3.2}$$

with some $r_0, r_1 \geq 1$, where $W_\varepsilon$ denotes a suitable approximation of $W$ and $d_{\mathfrak{C}}(y) =$ distance of $y$ from $\mathbb{R}^d \backslash \mathfrak{C}$. Without restricting substantially the possible applicability, we consider again the ansatz (2.21a) with $\widehat{W}_{\rm el}$ in the additive form

$$\widehat{W}_{\rm el}(\mathsf{F}_{\rm el}) = \mathfrak{W}_1\big(\mathbb{M}_1(\mathsf{F}_{\rm el}), ..., \mathbb{M}_{d-1}(\mathsf{F}_{\rm el})\big) + \mathfrak{W}_0(\det \mathsf{F}_{\rm el}); \tag{3.3}$$

recall that $\mathbb{M}_d = \det$. In other words, we assume that (2.15) has a special structure

$$\begin{aligned} \mathbb{W}(\mathbb{M}(\mathsf{F}), \mathsf{P}, \mathsf{A}) = {}& \mathfrak{W}_1\big(\mathbb{M}_1(\mathsf{F}\mathsf{P}^{-1}), ..., \mathbb{M}_{d-1}(\mathsf{F}\mathsf{P}^{-1})\big) \\ &\qquad + \mathfrak{W}_0\big(\det(\mathsf{F}\mathsf{P}^{-1})\big) + \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\mathsf{A}|^{p_{\rm gr}} \\ = {}& \mathfrak{W}_1\big(\mathbb{M}_1(\mathsf{F})\mathbb{M}_1(\mathsf{P}^{-1}), ..., \mathbb{M}_{d-1}(\mathsf{F})\mathbb{M}_{d-1}(\mathsf{P}^{-1})\big) \\ &\qquad + \mathfrak{W}_0\Big(\frac{\det\mathsf{F}}{\det\mathsf{P}}\Big) + \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\mathsf{A}|^{p_{\rm gr}} \end{aligned} \tag{3.4}$$

for some $\mathfrak{W}_1 : \mathbb{R}^{\mu_d - 1} \to \mathbb{R}$ and $\mathfrak{W}_0 : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ convex. Note that, in (3.4), we used the Cauchy-Binet relation $\mathbb{M}_s(\mathsf{G}\mathsf{F}) = \mathbb{M}_s(\mathsf{G})\,\mathbb{M}_s(\mathsf{F})$ to show that it complies with (2.15). It is important that $\mathfrak{W}_1$ is finite with an upper bound (see (4.22b) below) in contrast to $\mathfrak{W}_0$ which blows up for $\det\mathsf{F}_{\rm el} \searrow 0$ and equals $\infty$ for $\det\mathsf{F}_{\rm el} \leq 0$. Thus, upon replacing $\mathfrak{W}_0$ by a suitable regularization $\mathfrak{W}_\varepsilon$, we will obtain $W_\varepsilon$ with a controlled growth, cf. (4.19) below. The mentioned regularization can consist in replacing $\widehat{W}_{\rm el}$ in (3.3) by

$$\begin{aligned} \widehat{W}_{\mathrm{el},\varepsilon}(\mathsf{F}_{\rm el}) := {}& \mathfrak{W}_1\big(\mathbb{M}_1(\mathsf{F}_{\rm el}), ..., \mathbb{M}_{d-1}(\mathsf{F}_{\rm el})\big) + \mathfrak{W}_\varepsilon(\det\mathsf{F}_{\rm el}) \\ &\qquad \text{with } \ \mathfrak{W}_\varepsilon(\delta) := \min_{\tilde\delta\in\mathbb{R}}\Big(\mathfrak{W}_0(\tilde\delta) + \frac{1}{2\varepsilon}|\tilde\delta - \delta|^2\Big). \end{aligned} \tag{3.5}$$

The convex function $\mathfrak{W}_\varepsilon : \mathbb{R} \to \mathbb{R}$, being called the Yosida approximation of $\mathfrak{W}_0$, is always smooth, has at most quadratic growth, and converges to $\mathfrak{W}_0$ pointwise from below. By replacing $\widehat{W}_{\rm el}$ in (2.21a) with $\widehat{W}_{\mathrm{el},\varepsilon}$, we obtain the following approximation $W_\varepsilon$ of $W$:

$$\begin{aligned} W_\varepsilon(\mathsf{F},\mathsf{P},\mathsf{A}) := {}& \widehat{W}_{\mathrm{el},\varepsilon}(\mathsf{F}\mathsf{P}^{-1}) + \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\mathsf{A}|^{p_{\rm gr}} \\ = {}& \mathfrak{W}_1\big(\mathbb{M}_1(\mathsf{F}\mathsf{P}^{-1}), ..., \mathbb{M}_{d-1}(\mathsf{F}\mathsf{P}^{-1})\big) + \mathfrak{W}_\varepsilon\big(\det(\mathsf{F}\mathsf{P}^{-1})\big) + \frac{\kappa|\mathsf{P}|^{p_{\rm pl}}}{p_{\rm pl}} + \frac{\varkappa|\mathsf{A}|^{p_{\rm gr}}}{p_{\rm gr}}. \end{aligned} \tag{3.6}$$

The important attribute of such an approximation is that $W_\varepsilon$ again satisfies our polyconvex-type structural assumption (2.15). Let us emphasize that this would not be clear if we made the Yosida approximation directly of $W(\cdot,\mathsf{P},\cdot)$, not speaking that such an approximation might destroy the coercivity of $W$ we need later, cf. (4.4a). The mentioned coercivity estimate is also crucial for weak continuity of $\mathbb{M}_s(\mathsf{F})$.

For the *time discretization*, we use an equidistant partition of the time interval $[0,T]$ with the time step $\tau>0$, and use the fully-implicit *backward-Euler formula* leading to *incremental minimization*: For an initial state $q^0_{\varepsilon\tau h}=q_0$, we take $q^k_{\varepsilon\tau h}$ as a solution to the recursive problem

$$\text{Minimize } q\mapsto \mathcal{E}_{\varepsilon h}(t^k_\tau,q)+\mathcal{D}(q^{k-1}_{\varepsilon\tau h},q)\ \text{ subject to }\ q\in V_h:=V_{\mathrm{df},h}\times V_{\mathrm{pl},h} \tag{3.7}$$

to be solved for $k=1,...,T/\tau$; here $\mathcal{D}$ is from (2.9) with $D$ from (2.8). One should note that such $D$ is rather implicit unless the dissipation metric $R=R(\mathsf{P},\dot{\mathsf{P}})$ were independent of $\mathsf{P}$. These minimization problems are close to the ones used in the engineering papers mentioned above, the difference being that we use the (rather implicit) dissipation distance $\mathcal{D}$ whereas most other works approximate this using $R$ and some explicit predictors. In particular, the triangle inequality is essential to derive a priori bounds and to employ a generalized version of Helly's selection principle, cf. [FrM06, MaM05].

It is convenient to introduce the notation of piece-wise constant interpolants $\overline{q}_\tau$ and $\underline{q}_\tau$, defined by

$$\overline{q}_{\varepsilon\tau h}(t):=q^k_{\varepsilon\tau h}\qquad\text{for } t\in\big((k{-}1)\tau,k\tau\big], \tag{3.8a}$$

$$\underline{q}_{\varepsilon\tau h}(t):=q^{k-1}_{\varepsilon\tau h}\qquad\text{for } t\in\big[(k{-}1)\tau,k\tau\big). \tag{3.8b}$$

Beside, we define

$$\overline{\mathcal{E}}_{\varepsilon\tau h}(t,q):=\mathcal{E}_{\varepsilon h}(k\tau,q)\qquad\text{for } t\in\big((k{-}1)\tau,k\tau\big]. \tag{3.8c}$$

The rate independence yields the *discrete stability*:

$$\forall\tilde q\in\mathcal{Q}:\quad \overline{\mathcal{E}}_{\varepsilon\tau h}(t,\overline{q}_{\varepsilon\tau h}(t))\le\overline{\mathcal{E}}_{\varepsilon\tau h}(t,\tilde q)+\mathcal{D}(\overline{q}_{\varepsilon\tau h}(t),\tilde q) \tag{3.9}$$

which holds for any $t\in[0,T]$, and the following *two-sided energy inequality*:

$$\begin{aligned}\int_0^s[\mathcal{E}_{\varepsilon h}]'_t\big(t,\overline{q}_{\varepsilon\tau h}(t)\big)\,\mathrm{d}t\le\overline{\mathcal{E}}_{\varepsilon\tau h}\big(s,\overline{q}_{\varepsilon\tau h}(s)\big)+\mathrm{Diss}_{\mathcal{D}}\big(\overline{q}_{\varepsilon\tau h},[0,s]\big)&\\ -\overline{\mathcal{E}}_{\varepsilon\tau h}(0,u_0,z_0)\le\int_0^s[\mathcal{E}_{\varepsilon h}]'_t\big(t,\underline{q}_{\varepsilon\tau h}(t)\big)\,\mathrm{d}t&\end{aligned} \tag{3.10}$$

which holds for any $s=k\tau\in[0,T]$, $k\in\mathbb{N}$, where (cf. e.g. [Mie05, MiR09])

$$\mathrm{Diss}_{\mathcal{D}}(q;[r,s])=\sup\Big\{\sum_{j=1}^N\mathcal{D}(\mathsf{P}(t_{j-1}),\mathsf{P}(t_j));\ \text{all partitions }\ r\le t_1<t_2<...<t_N\le s\Big\}. \tag{3.11}$$

Unfortunately, in the engineering studies, energetics is usually ignored. Beside its physical importance itself, sometimes it may be advantageously exploited as a certain a-posteriori information to improve or correct the calculations, not speaking about detection of coding mistakes in already seemingly well functioning computer codes. This particularly concerns the evolutionary rate-independent situation at large strains when one has the two-sided estimate (3.10) available and essentially no other mathematically justified concepts of solutions are at disposal, in contrast to small-strains models, cf. the discussion in [MiR15, Chap. 1] or also e.g. [Rou15]. This improvement or correction may be considered if (3.10) has big differences in lower and upper bounds or is even not satisfied e.g. because of a failure of a specific optimization routine used for the global-optimization problem (3.7); cf. [Ben11, MRZ10].

**Example 3.1 (Ogden-type material revisited)** Instead of the general but rather implicit formula (3.6), more explicit formulas can be used in concrete models. E.g. in the particular case (2.26) one can take the explicit regularization

$$\widehat{W}_{\mathrm{el},\varepsilon}(\mathsf{F}_{\mathrm{el}}) = c_1|\mathsf{F}_{\mathrm{el}}|^{p_{\mathrm{el}}} + \begin{cases} \dfrac{c_2}{(\det \mathsf{F}_{\mathrm{el}}+\varepsilon)^\gamma} & \text{if } \det \mathsf{F}_{\mathrm{el}} \geq 0, \\ \dfrac{c_2}{\varepsilon}\Big(|\det \mathsf{F}_{\mathrm{el}}|^2 - \dfrac{\gamma}{\varepsilon^\gamma}\det \mathsf{F}_{\mathrm{el}} + \dfrac{\varepsilon}{\varepsilon^\gamma}\Big) & \text{if } \det \mathsf{F}_{\mathrm{el}} < 0. \end{cases} \tag{3.12}$$

This $\widehat{W}_{\mathrm{el},\varepsilon}$ is a $C^1$-function which is piecewise $C^2$ and polyconvex in the usual sense [Bal77] and has a $\max(p_{\mathrm{el}}, 2d)$ polynomial growth. Note that $W_\varepsilon(\mathsf{F},\mathsf{P},\mathsf{A}) := \widehat{W}_{\mathrm{el},\varepsilon}(\mathsf{F}\mathsf{P}^{-1}) + \frac{\kappa}{p_{\mathrm{pl}}}|\mathsf{P}|^{p_{\mathrm{pl}}} + \frac{\varkappa}{p_{\mathrm{gr}}}|\mathsf{A}|^{p_{\mathrm{gr}}}$, which is a regularized analog of $W$ from of (2.21a), is compatible with the polyconvexity (2.15).

# 4 Convergence towards energetic solutions

The energetic formulation of rate-independent systems provides a certain weak form of system (2.4). For this, in general, we choose a state space $\mathcal{Q}$ for $q = (y, \mathit{\Pi})$ by identifying suitable weakly closed subsets of Sobolev spaces over $\Omega$, here $\mathcal{Q}$ from (2.14). A mapping $q = (y, \mathit{\Pi}) : [0, T] \to \mathcal{Q}$ with $\mathcal{Q}$ from (2.14) is called an *energetic solution* to the rate-independent system determined by the functionals $\mathcal{E}$ and $\mathcal{D}$, abbreviated as RIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, if for all $t \in [0, T]$ the *stability condition* (S) and the *energy balance* (E) hold:

$$\text{(S)} \qquad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \widehat{q}) + \mathcal{D}(q(t), \widehat{q}) \text{ for all } \widehat{q} \in \mathcal{Q}, \tag{4.1a}$$

$$\text{(E)} \qquad \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q; [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \mathcal{P}(s)\, \mathrm{d}s, \tag{4.1b}$$

where $\mathrm{Diss}_{\mathcal{D}}(q; [r, s])$ is as in (3.11) and $\mathcal{P} : [0, T] \to \mathbb{R}$ is the "complementary" power of the external loadings:

$$\mathcal{P}(t) := -\big\langle \dot{\ell}(t), g_{\mathrm{Dir}}(t, y(x))\big\rangle - \int_{\Gamma_{\mathrm{Dir}}} \sigma(t, x) \cdot \dot{g}_{\mathrm{Dir}}(t, y(x))\, \mathrm{d}x, \tag{4.2}$$

with $\ell$ from (2.3) and $\sigma$ being the traction stress $\partial_{\mathsf{F}} W \nu$ on the boundary $\Gamma_{\mathrm{Dir}}$. The adjective "complementary" wants to distinguish (4.2) from the conventional power $\langle \ell(t), \dot{y}(t)\rangle$ related to the Helmholtz-type stored energy. In contrast the latter, $\mathcal{P}$ from (4.2) does not involve the time derivative of the solution, which is more desirable because $\dot{q}$ is not well controlled in rate-independent systems. Actually, in the case of the Dirichlet loading, this last term in (4.1b) should be reformulated in a more complicated form (2.17), since boundary traces $\partial_{\mathsf{F}} W\nu$ are not well defined.

The concept of energetic solutions can be seen as a weak version of the classical plasticity formulation, see [MaM09, Mie03] for discussing the mechanical modeling of elastoplasticity and detailed explanation. The major advantage of (S) and (E) is that it avoids derivatives and is based solely on the functionals $\mathcal{E}$ and $\mathcal{D}$, which need not be smooth or even continuous. This is particularly convenient definition for models at large strains where no reference configuration and no linear structure are apriori given and therefore the notion of a time derivative itself occurring formally in the classical formulation (2.4) is not defined.

Our RIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ is now defined via $\mathcal{Q}$ from (2.14), $\mathcal{E}$ from (2.13), and $\mathcal{D}$ from (2.9), and we will consider an initial-value problem requiring still

$$y(0) = y_0 \quad \text{and} \quad \Pi(0) = \Pi_0. \tag{4.3}$$

We specify rigorous conditions on $D$ and $W$ that guarantee existence of energetic solutions, the conditions on $W$ being quite involved. In particular, we introduce a set $\mathfrak{N} \subset \mathrm{GL}^+(d)$ that contains $\mathfrak{P}$ and will be used for numerical approximations of plastic strains taking values in $\mathfrak{P}$, cf. Lemma 4.5 below. In addition to the polyconvexity (2.15), these conditions include coercivity and continuity, assuming that, for some $\mathfrak{P} \subset \mathfrak{N} \subset \mathrm{GL}^+(d)$, the following qualification of $W$ holds:

$$\begin{aligned}
&\exists\, c > 0,\ p_{\rm el}, p_{\rm pl}, p_{\rm gr}, p_{\rm lp} > 1\ \forall\, (\mathsf{F},\mathsf{P},\mathsf{A}) \in \mathrm{dom}\, W:\\
&\qquad \mathsf{P} \in \mathfrak{N} \ \Rightarrow\ W(\mathsf{F},\mathsf{P},\mathsf{A}) \geq c\left(|\mathsf{F}|^{p_{\rm df}} + |\mathsf{P}|^{p_{\rm pl}} + |\mathsf{A}|^{p_{\rm gr}}\right),
\end{aligned} \tag{4.4a}$$

$$\exists\, c_W > 0,\ \delta > 0,\ \text{and a modulus of continuity } \omega : ]0,\delta[\, \to\, ]0,\infty[$$

$$\forall\, (\mathsf{F},\mathsf{P},\mathsf{A}) \in \mathrm{dom}\, W,\ \mathsf{P} \in \mathfrak{N}\ \ \forall\, N \in \mathbb{R}^{d\times d},\ |N - \mathbb{I}| < \delta:$$

$$W(\cdot,\mathsf{P},\mathsf{A}) \text{ is differentiable on } \mathrm{GL}^+(d) \text{ and} \tag{4.4b}$$

$$\big|\partial_{\mathsf{F}} W(\mathsf{F},\mathsf{P},\mathsf{A})\mathsf{F}^{\mathsf{T}}\big| \leq c_W\big(1 + W(\mathsf{F},\mathsf{P},\mathsf{A})\big) \tag{4.4c}$$

$$\big|\partial_{\mathsf{F}} W(\mathsf{F},\mathsf{P},\mathsf{A})\mathsf{F}^{\mathsf{T}} - \partial_{\mathsf{F}} W(N\mathsf{F},\mathsf{P},\mathsf{A})(N\mathsf{F})^{\mathsf{T}}\big| \leq \omega\big(|N - \mathbb{I}|\big)\big(1 + W(\mathsf{F},\mathsf{P},\mathsf{A})\big). \tag{4.4d}$$

For the extended quasi-distance $D$, we impose the conditions

$$D : \mathfrak{N} \times \mathfrak{N} \to \mathbb{R} \text{ is continuous;} \tag{4.5a}$$

$$\forall\, \mathsf{P}_1, \mathsf{P}_2 \in \mathfrak{N}: \qquad D(\mathsf{P}_1,\mathsf{P}_2) = 0 \iff \mathsf{P}_1 = \mathsf{P}_2; \tag{4.5b}$$

$$\forall\, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3 \in \mathfrak{N}: \ D(\mathsf{P}_1,\mathsf{P}_3) \leq D(\mathsf{P}_1,\mathsf{P}_2) + D(\mathsf{P}_2,\mathsf{P}_3), \tag{4.5c}$$

$$\exists\, C > 0,\ p_1 \in [1, p_{\rm pl}[\,: \qquad |D(\mathsf{P}_1,\mathsf{P}_2)| \leq C\big(1 + |\mathsf{P}_1|^{p_1} + |\mathsf{P}_2|^{p_1}\big). \tag{4.5d}$$

**Proposition 4.1 (Existence of energetic solutions, [MaM09])** *Let (2.15), (4.4) and (4.5) hold for $\mathfrak{N} = \mathfrak{P} \subset \mathrm{GL}^+(d)$ and for*

$$\frac{1}{p_{\rm el}} + \frac{1}{p_{\rm pl}} = \frac{1}{p_{\rm df}} < \frac{1}{d}, \qquad \textit{and } p_{\rm gr} > 1. \tag{4.6}$$

*Let further the Dirichlet loading $g_{\rm Dir} \in C^1([0,T]; W^{1,\infty}(\Omega;\mathbb{R}^d))$ satisfy also (2.12) and be uniformly invertible, satisfying*

$$\exists\, C_{\rm Dir} > 0\ \forall\, (t,y) \in [0,T] \times \mathbb{R}^d: \quad \big|\partial_y g_{\rm Dir}(t,y)\big| + \big|[\partial_y g_{\rm Dir}(t,y)]^{-1}\big| \leq C_{\rm Dir}. \tag{4.7}$$

*Moreover, let the initial condition* $q_0 = (y_0, \varPi_0) \in \mathcal{Q}$ *be stable, i.e.*

$$\forall\, \widehat{q} \in \mathcal{Q}: \quad \mathcal{E}(0, q_0) \le \mathcal{E}(0, \widehat{q}) + \mathcal{D}(q_0, \widehat{q}), \tag{4.8}$$

*and let* $f_{\rm vol} \in C^1([0,T]; W^{1,\infty}(\Omega;\mathbb{R}^d))$ *and* $f_{\rm surf} \in C^1([0,T]; W^{1,\infty}(\partial\Omega\backslash\Gamma_{\rm Dir};\mathbb{R}^d))$. *Then there are energetic solutions* $q = (y, \varPi)$ *according to the definition* (4.1) *with*

$$y \in B([0,T]; W^{1,p_{\rm df}}(\Omega;\mathbb{R}^d)), \tag{4.9a}$$

$$\varPi \in B([0,T]; W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})), \qquad {\rm Diss}_{\mathcal{D}}(q; [0,T]) < \infty. \tag{4.9b}$$

The proof of this assertion can essentially be found in [MaM09]; here we only generalize to a more general $\mathfrak{P}$ with the corresponding coercivity (4.4a), thus admitting also models with $\det \mathsf{P} \ne 1$. We also included the unilateral constraint but the modification of the proof from [MaM09] is self-evident, if the Dirichlet loading is away from $\mathfrak{C}$ so that (2.12) can be assumed. We omit this proof here also because Proposition 4.1 will follow as a by-product of the convergence of our numerical approximation with a bit stronger assumptions or with the same assumptions if only the time (but not space) discretization is considered.

The most essential ingredience for the limit passage from the discrete stability (3.9) to the "continuous" stability (4.1a) is an explicit construction of some, so-called *mutual recovery sequence*. In a full generality, it was designed in [MRS08]. Here, where the dissipation distance is weakly continuous, it suffices to formulate a little easier condition: for all sequences $(t_j, q_j)_{j\in\mathbb{N}}$ which are stable with respect to a sequence of functionals $(\mathcal{E}_{\varepsilon_j h_j}, \mathcal{D})_{j\in\mathbb{N}}$ in the sense

$$\sup_{j\in\mathbb{N}} \mathcal{E}_{\varepsilon_j h_j}(t_j, q_j) < \infty \quad \text{and} \quad \forall q \in \mathcal{Q}: \ \mathcal{E}_{\varepsilon_j h_j}(t_j, q_j) \le \mathcal{E}_{\varepsilon_j h_j}(t_j, q) + \mathcal{D}(q_j, q), \tag{4.10}$$

we require the following:

$$\begin{aligned} &\forall\, (t_j, q_j)_{j\in\mathbb{N}} \text{ from (4.10) with } (t_j, q_j) \rightharpoonup (t_*, q_*) \ \forall\, \widehat{q} \in \mathcal{Q} \ \ \exists\, (\widehat{q}_j)_{j\in\mathbb{N}}: \\ &\qquad\qquad \limsup_{j\to\infty} \big(\mathcal{E}_{\varepsilon_j h_j}(t_j, \widehat{q}_j) + \mathcal{D}(q_j, \widehat{q}_j)\big) \ \le \ \mathcal{E}(t_*, \widehat{q}) + \mathcal{D}(q_*, \widehat{q}). \end{aligned} \tag{4.11}$$

Such a sequence $(\widehat{q}_j)_{j\in\mathbb{N}}$ is called a mutual recovery sequence, since it recovers the stored and the dissipation energies mutually. Even more simply, usually (and, in particular here too), one has weak $\Gamma$-liminf convergence of the sequence of $\{\mathcal{E}_{\varepsilon_j h_j}\}_{j\in\mathbb{N}}$ and one can consider $\widehat{q}_j \rightharpoonup \widehat{q}$ and then, counting that $\mathcal{D}$ is weakly continuous, (4.11) reduces rather to a weak $\Gamma$-convergence of the sequence of $\{\mathcal{E}_{\varepsilon_j h_j}\}_{j\in\mathbb{N}}$, i.e.

$$\forall\, (t,q)\ \forall\, t_j \to t, \ \forall\, q_j \rightharpoonup q: \quad \liminf_{j\to\infty} \mathcal{E}_{\varepsilon_j h_j}(t_j, q_j) \ge \mathcal{E}(t, q), \tag{4.12a}$$

$$\forall\, (t,\widehat{q})\ \forall\, t_j \to t, \ \exists\, \widehat{q}_j \rightharpoonup \widehat{q}: \quad \limsup_{j\to\infty} \mathcal{E}_{\varepsilon_j h_j}(t_j, \widehat{q}_j) \le \mathcal{E}(t, \widehat{q}). \tag{4.12b}$$

Enforcing $\widehat{q}_j \to \widehat{q}$ strongly, one has the stronger concept of the so-called Mosco convergence of $\{\mathcal{E}_{\varepsilon_j h_j}\}_{j\in\mathbb{N}}$, cf. also [MiR15, Sect. 2.1.5].

Without spatial discretization and penalization of the constraints, i.e. $\mathcal{E}_{\varepsilon h} \equiv \mathcal{E}$, the construction of mutual recovery sequences is rather simple in this special problem, and it has been shown in [MaM09] that it suffices to take $\widehat{q}_j = \widehat{q}$. The variant of (4.10)-(4.11) that only deals with penalization without discretization is also relatively simple, and again the constant sequence $\widehat{q}_j = \widehat{q}$ can serve as a mutual recovery sequence.

Involving finite-dimensional numerical approximation, it is important to ensure continuity of $\mathcal{E}_\varepsilon \equiv \mathcal{E}_{\varepsilon 0}$ with $\mathcal{E}_{\varepsilon 0}$ defined again by (3.2) but with $V_{\rm df,0} = W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)$ and $V_{\rm pl,0} = W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$, i.e.

$$\begin{aligned}\mathcal{E}_\varepsilon(t,y,\varPi) = &\int_\Omega W_\varepsilon(\partial_y g_{\rm Dir}(t,y)\nabla y, \varPi, \nabla\varPi) + \frac{d_{\mathfrak{C}}(y)^{r_1}}{\varepsilon}\,\mathrm{d}x\\ &- \big\langle \ell(t), g_{\rm Dir}(t,y)\big\rangle + \frac{1}{\varepsilon}\Big(\int_\Omega \det(\nabla y)\,\mathrm{d}x - \mathrm{meas}_d(y(\Omega))\Big)^{r_0}.\end{aligned}\tag{4.13}$$

For this we need to strengthen (4.4a) by assuming $p_{\rm gr} > d$; note that then formally one has "$p_{\rm pl} = \infty$" so that simply $p_{\rm el} = p_{\rm df}$, cf. also (4.6). In fact, the condition (4.6) from [MaM09] should be read rather as

$$\frac{1}{p_{\rm el}} + \frac{1}{\max(p_{\rm pl}, p_{\rm gr}^*)} = \frac{1}{p_{\rm df}} < \frac{1}{d},\tag{4.14}$$

and, if $p_{\rm gr} > d$, we have $p_{\rm gr}^* = \infty$. Note that if $p_{\rm el} > p_{\rm df}$ were allowed, the growth condition $|W_\varepsilon(\mathsf{F},\mathsf{P},\mathsf{A})| \le C_\varepsilon(|\mathsf{P}|)\big(1 + |\mathsf{F}|^{p_{\rm el}} + |\mathsf{A}|^{p_{\rm gr}}\big)$ naturally corresponding e.g. to the Ogden material (2.26) or (3.12) would not be compatible with the coercivity (4.4a). So, from now on, we will use $p_{\rm el}$ also in place of $p_{\rm df}$.

As we already mentioned in Section 3, the natural choice (3.1b) does not guarantee that $\varPi \in V_{{\rm pl},h}$ is valued in $\mathfrak{P}$. Also here, for approximation purpose, we must consider a neighbourhood of $\mathfrak{P}$ in $\mathbb{R}^{d\times d}$, let us denote it by $\mathfrak{N}$.

Of course, the mentioned formal choice $p_{\rm pl} = \infty$ obviously does not allow us to consider $p_{\rm pl} = \infty$ directly in (4.4a) but one should exploit the embedding $W^{1,p_{\rm gr}}(\Omega) \subset L^\infty(\Omega)$ in the following way:

**Lemma 4.2 (Coercivity of $\mathcal{E}_\varepsilon$)** *Let (4.4a) hold for $W_\varepsilon$ with $p_{\rm gr} > d$ instead of $W$ and with a neighbourhood $\mathfrak{N}$ of $\mathfrak{P}$, and let the Dirichlet loading $\partial_y g_{\rm Dir}(t,y)$ be regular in the sense (4.7). Then $\mathcal{E}_\varepsilon(t,\cdot,\cdot)$ is coercive on the set $\{(y,\varPi)\in\mathcal{Q};\ \varPi(x)\in\mathfrak{N}$ for all $x\in\Omega\}$.*

*Proof.* Considering (2.21a) and using the coercivity $\widehat{W}_{\rm el,\varepsilon}(\mathsf{F}_{\rm el}) \ge c_0|\mathsf{F}_{\rm el}|^{p_{\rm el}}$, we can estimate

$$\begin{aligned}\mathcal{E}_\varepsilon(t,y,\varPi) &\ge \int_\Omega W_\varepsilon(\partial_y g_{\rm Dir}(t,y)\nabla y, \varPi, \nabla\varPi)\,\mathrm{d}x\\
&= \int_\Omega \widehat{W}_{\rm el,\varepsilon}(\partial_y g_{\rm Dir}(t,y)\nabla y\varPi^{-1}) + \frac{\kappa}{p_{\rm pl}}|\varPi|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\nabla\varPi|^{p_{\rm gr}}\,\mathrm{d}x\\
&\ge c_0\|\partial_y g_{\rm Dir}(t,y)\nabla y\varPi^{-1}\|_{L^{p_{\rm el}}}^{p_{\rm el}} + c_1\|\varPi\|_{L^\infty}^{\min(p_{\rm pl},p_{\rm gr})} + \frac{\varkappa}{2p_{\rm gr}}\|\nabla\varPi\|_{L^{p_{\rm gr}}}^{p_{\rm gr}} - \frac{1}{c_1}\\
&\ge c_0\frac{\|\nabla y\|_{L^{p_{\rm el}}}^{p_{\rm el}}}{\|[\partial_y g_{\rm Dir}(t,y)]^{-1}\|_{L^\infty}^{p_{\rm el}}\|\varPi\|_{L^\infty}^{p_{\rm el}}} + c_1\|\varPi\|_{L^\infty}^{\min(p_{\rm pl},p_{\rm gr})} + \frac{\varkappa}{2p_{\rm gr}}\|\nabla\varPi\|_{L^{p_{\rm gr}}}^{p_{\rm gr}} - \frac{1}{c_1}\\
&\ge \frac{c_0}{C_{\rm Dir}^{p_{\rm el}}}\frac{\|\nabla y\|_{L^{p_{\rm el}}}^{p_{\rm el}}}{\|\varPi\|_{L^\infty}^{p_{\rm el}}} + \frac{c_1}{2}\|\varPi\|_{L^\infty}^{\min(p_{\rm pl},p_{\rm gr})} + \frac{\varkappa}{2p_{\rm gr}}\|\nabla\varPi\|_{L^{p_{\rm gr}}}^{p_{\rm gr}} - \frac{1}{c_1}\\
&\ge q\|\nabla y\|_{L^{p_{\rm el}}}^{p_{\rm el}/q} + c_q\|\varPi\|_{L^\infty}^{\min(p_{\rm pl},p_{\rm gr})} + \frac{\varkappa}{2p_{\rm gr}}\|\nabla\varPi\|_{L^{p_{\rm gr}}}^{p_{\rm gr}} - C_q\end{aligned}\tag{4.15}$$

with some $q > 1$ sufficiently large, and with $c_1, c_q > 0$ depending also on the norm of the embedding $W^{1,p_{\rm gr}}(\Omega) \subset L^\infty(\Omega)$ and with some $C_q$ sufficiently large. For the last inequality in (4.15), we used the Young inequality in the form

$$\epsilon^q a^q \geq qab - \frac{(q{-}1)b^{q'}}{\epsilon^{q'}} \quad \text{ with } \ a = \frac{\|\nabla y\|_{L^{p_{\rm el}}}^{p_{\rm el}/q}}{\|\varPi\|_{L^\infty}^{p_{\rm el}/q}} \ \text{ and } \ b = \|\varPi\|_{L^\infty}^{p_{\rm el}/q}$$

so that, for $\epsilon = (c_0/C_{\rm Dir}^{p_{\rm el}})^{1/q}$, we have

$$\frac{c_0}{C_{\rm Dir}^{p_{\rm el}}} \frac{\|\nabla y\|_{L^{p_{\rm el}}}^{p_{\rm el}}}{\|\varPi\|_{L^\infty}^{p_{\rm el}}} \geq q\|\nabla y\|_{L^{p_{\rm el}}}^{p_{\rm el}/q} - \frac{(q{-}1)C_{\rm Dir}^{1/(q-1)}}{c_0^{1/(q-1)}}\|\varPi\|_{L^\infty}^{p_{\rm el}/(q-1)}. \tag{4.16}$$

The strategy is now to choose $q > 1$ so big that $p_{\rm el}/(q{-}1) < \min(p_{\rm pl}, p_{\rm gr})$ so that the term $\frac{c_1}{2}\|\varPi\|_{L^\infty}^{\min(p_{\rm pl},p_{\rm gr})}$ will dominate the last term in (4.16) for large $\varPi$, which eventually yields the last estimate in (4.15) with some $c_q > 0$. This choice of $q$ is always possible. Taking also the Dirichlet conditions for $y$ on $\Gamma_{\rm Dir}$ into account, from (4.15) we obtain the desired coercivity. $\square$

Since the regularized functionals $\mathcal{E}_\varepsilon$ in (4.13) contain the term $\mathrm{meas}_d(y(\Omega))$ we need some continuity result of this term in $y$. To the best of our knowledge the following result is new, where Step 2 in the proof is due to [Mal16]. In the proof of Lemma 5.1 we will derive even Lipschitz continuity of $\mathrm{meas}_d(y(\Omega))$ for maps in $W^{2,p}(\Omega;\mathbb{R}^d)$ with $\det \nabla y$ bounded from below by a positive constant, see (5.11).

**Proposition 4.3** *Assume that $p > d$ and that $\Omega \subset \mathbb{R}^d$ is an bounded, open set with Lipschitz boundary $\partial\Omega$. Then,*

$$y_n \rightharpoonup y \ \textit{in } W^{1,p}(\Omega;\mathbb{R}^d) \quad \Longrightarrow \quad \mathrm{meas}_d(y_n) \to \mathrm{meas}_d(y). \tag{4.17}$$

*Proof.* We will rely on Rellich's embedding theorem giving $\|y_n - y\|_{C^0(\bar\Omega)} \to 0$, where we use $p > d$, the boundedness of $\Omega$ and that $\partial\Omega$ is Lipschitz. Moreover, again using $p > d$, we have the Marcus-Mizel estimate [MaM73, Thm. 1]

$$\mathrm{meas}_d(y(A)) \leq C(p,d)\Big(\int_A |\nabla y(x)|^p \,\mathrm{d}x\Big)^{d/p} \mathrm{meas}_d(A)^{1-d/p} \tag{4.18}$$

for all $y \in W^{1,p}(\Omega)$ and all measurable $A \subset \Omega$. In particular, all $y \in W^{1,p}(\Omega)$ satisfy Lusin's poperty (N), i.e. if $\mathrm{meas}_d(A) = 0$ then also $\mathrm{meas}_d(y(A)) = 0$.

*Step 1: Upper semicontinuity* $\limsup_{n\to\infty} \mathrm{meas}_d(y_n(\Omega)) \leq \mathrm{meas}_d(y(\Omega))$:
Since $\mathrm{meas}_d(\partial\Omega) = 0$ and $\bar\Omega = \Omega \cup \partial\Omega$, Lusin's property (N) gives $\mathrm{meas}_d(y(\Omega)) = \mathrm{meas}_d(y(\bar\Omega))$. Since $y$ is continuous and $\bar\Omega$ compact, so is $K \overset{\text{def}}{=} y(\bar\Omega)$. Hence for all $\varepsilon > 0$ we can find an open set $O_\varepsilon \supset K$ such that $\mathrm{meas}_d(O_\varepsilon \setminus K) < \varepsilon$. With $\|y_n - y\|_{C^0(\bar\Omega)} \to 0$ we find $N_\varepsilon$ such that $y_n(\Omega) \subset O_\varepsilon$ for $n \geq N_\varepsilon$ and conclude

$$\limsup_{n\to\infty} \mathrm{meas}_d(y_n(\Omega)) \leq \mathrm{meas}_d(O_\varepsilon) \leq \mathrm{meas}_d(K) + \varepsilon = \mathrm{meas}_d(y(\Omega)) + \varepsilon,$$

which is the desired result since $\varepsilon > 0$ was arbitrary.

*Step 2: Lower semicontinuity* $\liminf_{n\to\infty} \mathrm{meas}_d(y_n(\Omega)) \geq \mathrm{meas}_d(y(\Omega))$:
Again using Lusin's property (N), [MaZ92, Thm. 3.10] implies that almost all points $z \in$

$y(\Omega)$ are stable. We denote by $S \subset y(\Omega)$ the set of all stable points $z$, i.e. there exists a $\delta_z > 0$ such for all $u \in C(\bar{\Omega})$ with $\|u-y\|_{C^0} < \delta_z$ one also has $z \in u(\Omega)$. Defining the increasing family $B_k = \bigcap_{n=k}^{\infty} y_n(\Omega)$ and using $y_n \to y$ uniformly, we see that for all $z \in S$ there exists $\kappa_z$ such that $z \in B_k$ for $k \geq \kappa_z$. We conclude $S \subset \bigcup_{k=1}^{\infty} B_k$ and obtain

$$\begin{aligned}\mathrm{meas}_d\big(y(\Omega)\big) = \mathrm{meas}_d(S) = \mathrm{meas}_d\big(\textstyle\bigcup_{k=1}^{\infty} B_k\big) &= \lim_{k\to\infty} \mathrm{meas}_d\big(B_k\big)\\ = \mathrm{meas}_d\big(\textstyle\bigcap_{n=k}^{\infty} y_n(\Omega)\big) &\leq \liminf_{n\to\infty} y_n(\Omega),\end{aligned}$$

which is the desired lower semicontinuity result. $\square$

We are now ready to establish the continuity result for $\mathcal{E}_\varepsilon$.

**Lemma 4.4 (Continuity of $\mathcal{E}_\varepsilon$)** *Let the assumptions (4.4b-d) hold for $W_\varepsilon$ with $p_{\rm gr} > d$ instead of $W$ and for a neighbourhood $\mathfrak{N}$ of $\mathfrak{P}$, let $\det(\cdot)$ stay above some positive constant on $\mathfrak{N}$, let the approximation $W_\varepsilon$ satisfy the growth condition*

$$|W_\varepsilon(\mathsf{F},\mathsf{P},\mathsf{A})| \leq C_\varepsilon\big(|\mathsf{P}|\big)\big(1+|\mathsf{F}|^{p_{\rm el}}+|\mathsf{A}|^{p_{\rm gr}}\big) \qquad \text{provided } \ \mathsf{P}\in\mathfrak{N} \tag{4.19}$$

*for some $C_\varepsilon : \mathbb{R} \to \mathbb{R}$ continuous, and assume $p_{\rm el} > d$. Then the functional $\mathcal{E}_\varepsilon$ is continuous on $W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d) \times \{\mathit{\Pi}\in W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d});\ \mathit{\Pi}(x)\in\mathfrak{N}$ for all $x\in\Omega\}$.*

*Proof.* Take converging sequences $y_k \to y$ in $W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)$ and $\mathit{\Pi}_k \to \mathit{\Pi}$ in $W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$. As $p_{\rm gr} > d$, we have also $\mathit{\Pi}_k \to \mathit{\Pi}$ in $C(\bar{\Omega};\mathbb{R}^{d\times d})$. Then also the minors $\mathbb{M}(\mathit{\Pi}_k)$ converge to $\mathbb{M}(\mathit{\Pi})$ in $C(\bar{\Omega};\mathbb{R}^{\mu_d})$. Using the well-known algebraic formula $\mathsf{P}^{-1} = \mathbb{K}_{d-1}^{\mathsf{T}}(\mathsf{P})/\det\mathsf{P}$, cf. (2.24) for $\mathsf{G} = \mathsf{F} = \mathbb{I}$ and $s = d{-}1$, we have

$$\mathit{\Pi}_k^{-1} = \frac{\mathbb{K}_{d-1}^{\mathsf{T}}(\mathit{\Pi}_k)}{\det\mathit{\Pi}_k} \to \frac{\mathbb{K}_{d-1}^{\mathsf{T}}(\mathit{\Pi})}{\det\mathit{\Pi}} = \mathit{\Pi}^{-1} \qquad \text{in } \ C(\bar{\Omega};\mathbb{R}^{d\times d}). \tag{4.20}$$

Here we used continuity of $\mathit{\Pi} \mapsto 1/\det\mathit{\Pi}$ on the set $\mathfrak{N}$. Since $p_{\rm el} > d$, the mapping $y \mapsto \det(\nabla y)$ maps $W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)$ into $L^1(\Omega)$. Altogether, using also (4.17), we can see that the functional $\mathcal{E}_\varepsilon$ is continuous on the set $\{(y,\mathit{\Pi}) \in W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d) \times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d});\ \det\mathit{\Pi}\in\mathfrak{N}\}$. $\square$

Now, together with the implicit time discretization with a time step $\tau > 0$, we consider a conventional P1-finite element approximation both for $y$ and for $\mathit{\Pi}$ with a mesh parameter $h > 0$, cf. (3.1). Recall that $\mathit{\Pi}\in V_{{\rm pl},h}$ satisfy the prescribed constraint $\mathit{\Pi}(x)\in\mathfrak{P}$ at nodal points of the triangulation $\mathcal{T}_h$ but not necessarily at other points of $\Omega$. We will rely on:

**Lemma 4.5 (Approximation of the constraint $\mathit{\Pi}(x)\in\mathfrak{P}$.)** *Let $p_{\rm gr} > d$, and $C \geq 0$ and a neighbourhood $\mathfrak{N}$ of $\mathfrak{P}$ be given. Then, for $V_{{\rm pl},h}$ from (3.1b), the following holds:*

$$\exists\, h^* > 0\ \forall\, h\in(0,h^*]\!:\quad \left.\begin{array}{c}\mathit{\Pi} \in V_{{\rm pl},h} \quad \textit{and}\\ \|\mathit{\Pi}\|_{W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})} \leq C\end{array}\right\} \ \Rightarrow\ \ \forall\, x\in\Omega\!:\ \mathit{\Pi}(x)\in\mathfrak{N}. \tag{4.21}$$

*Proof.* We use the continuous embedding $W^{1,p_{\rm gr}}(\Omega) \subset C^{0,1-d/p_{\rm gr}}(\bar{\Omega})$. Counting that, by the definition of the mesh parameter $h$, it holds $\max_{x\in\bar{\Omega}} \mathrm{dist}(x,\mathcal{N}_h) \leq h$, we have $\min_{\widetilde{x}\in\mathcal{N}_h} |\mathit{\Pi}(x) - \mathit{\Pi}(\widetilde{x})| \leq H_C h^{1-d/p_{\rm gr}}$ with the Hölder-continuity constant $H_C$ dependent on $\|\mathit{\Pi}\|_{W^{1,p_{\rm gr}}}$, i.e. on $C$ from (4.21). Also, we have $|\mathit{\Pi}(x)| \leq CN$ with $N$ the norm of the embedding $W^{1,p_{\rm gr}}(\Omega)\to L^\infty(\Omega)$, which ensures existence of some $\delta>0$ such that the $\delta$-neighbourhood of $\{\mathsf P\in\mathfrak P;\ |\mathsf P|\le CN\}$ is still contained in $\mathfrak N$. Altogether, we get the desired assertion with $h^*=(\delta/H_C)^{p_{\rm gr}/(p_{\rm gr}-d)}$. $\square$

Let us denote by $\overline q_{\varepsilon\tau h}$ the piecewise-constant approximate solution obtained by this way, i.e. using the global-minimization concept to the incremental problems, cf. (3.7). We can expect at least a certain conditional subsequent-limit convergence:

**Proposition 4.6 (Numerical approximation, convergence)** *Let us consider the model with $\mathfrak P$ so that $\det(\cdot)$ stays above some positive constant on $\mathfrak P$ and let the assumptions of Proposition 4.1 be fulfilled with (4.4) using $p_{\rm gr}>d$ and that*

$$\mathfrak W_0:\mathbb R\to[0,\infty]\quad \textit{is convex, lower semicontinuous, and proper,} \tag{4.22a}$$

$$\begin{aligned}\exists p_{\rm el}\ge 2d,\ c>0:\quad & c|\mathsf F_{\rm el}|^{p_{\rm el}}\le\mathfrak W_1(\mathbb M_1(\mathsf F_{\rm el}),...,\mathbb M_{d-1}(\mathsf F_{\rm el}))\le C\big(1+|\mathsf F_{\rm el}|^{p_{\rm el}}\big),\\ & \mathfrak W_1\quad \textit{is convex and continuous}\end{aligned} \tag{4.22b}$$

*hold, and let, for simplicity, the initial value $q_0$ be stable with respect to $\mathcal E_\varepsilon$ and $\mathcal D$ for all $\varepsilon>0$. Then:*

(i) *Fixing $\varepsilon>0$, the numerical approximations $\overline q_{\varepsilon\tau h}=(\overline y_{\varepsilon\tau h},\overline\Pi_{\varepsilon\tau h})$ converge (in terms of subsequences) for $(\tau,h)\to(0,0)$ towards energetic solutions of the regularized RIS $(Q,\mathcal E_\varepsilon,\mathcal D)$ with the initial value $q_0\in Q:=W^{1,p_{\rm el}}(\Omega;\mathbb R^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb R^{d\times d})$ and with $\mathcal E_\varepsilon$ from (4.13) and $\mathcal D$ from (2.9). More specifically, for a subsequence and some $(y_\varepsilon,\Pi_\varepsilon)$, we have*

$$\forall\, t\in[0,T]:\quad \overline\Pi_{\varepsilon\tau h}(t)\rightharpoonup\Pi_\varepsilon(t)\qquad \textit{in}\ \ W^{1,p_{\rm gr}}(\Omega;\mathbb R^{d\times d}), \tag{4.23a}$$

$$\forall\, t\in[0,T]:\quad \mathcal E_\varepsilon\big(t,\overline y_{\varepsilon\tau h}(t),\overline\Pi_{\varepsilon\tau h}(t)\big)\to\mathcal E_\varepsilon\big(t,y_\varepsilon(t),\Pi_\varepsilon(t)\big), \tag{4.23b}$$

$$\forall\, t\in[0,T]:\quad \mathrm{Diss}_{\mathcal D}\big(\overline\Pi_{\varepsilon\tau h},[0,t]\big)\to\mathrm{Diss}_{\mathcal D}\big(\Pi_\varepsilon,[0,t]\big), \tag{4.23c}$$

$$\forall_{\rm a.a.}\, t\in[0,T]:\ \partial_t\mathcal E_\varepsilon\big(t,\overline y_{\varepsilon\tau h}(t),\overline\Pi_{\varepsilon\tau h}(t)\big)\to\partial_t\mathcal E_\varepsilon\big(t,y_\varepsilon(t),\Pi_\varepsilon(t)\big) \tag{4.23d}$$

*and, for all $t\in[0,T]$, there is a further subsequence $\big(q_{\varepsilon\tau_n^th_n^t}(t)\big)_{n\in\mathbb N}$ such that*

$$\forall t\in[0,T]:\quad \overline y_{\varepsilon\tau_n^th_n^t}(t)\rightharpoonup y_\varepsilon(t)\qquad \textit{in}\ \ W^{1,p_{\rm el}}(\Omega;\mathbb R^d), \tag{4.23e}$$

*and any $(y_\varepsilon,\Pi_\varepsilon)$ obtained by this way is an energetic solution to the regularized RIS $(Q,\mathcal E_\varepsilon,\mathcal D)$.*

(ii) *Denoting an energetic solution $(y_\varepsilon,\Pi_\varepsilon)$ obtained in the point (i), there is a converging subsequence for $\varepsilon\to0$ in the mode like (4.23) to some $(y,\Pi)$ and any couple obtained by this way is an energetic solution to the original RIS $(Q,\mathcal E,\mathcal D)$.*

(iii) *For a fixed $(\varepsilon,\tau)$, we can converge with $h\to0$ in the mode like in (4.23) to $(\overline y_{\varepsilon\tau},\overline\Pi_{\varepsilon\tau})$ which solves the time-discrete problem corresponding to the regularized (but not discretized) RIS $(Q,\mathcal E_\varepsilon,\mathcal D)$. Then, converging $(\varepsilon,\tau)\to(0,0)$, one obtains energetic solutions to the original RIS $(Q,\mathcal E,\mathcal D)$.*

*Sketch of the proof.* For fixed $\varepsilon>0$, let us realize the continuity and coercivity of $\mathcal E_\varepsilon$ on $Q=W^{1,p_{\rm el}}(\Omega;\mathbb R^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb R^{d\times d})$ just by using the growth conditions in (4.22) and the continuity of the Nemytskiĭ operators. Here, an important fact is also that we have a uniform estimate of $\Pi_{\varepsilon\tau h}$ in $L^\infty(0,T;W^{1,p_{\rm gr}}(\Omega;\mathbb R^{d\times d}))$ so that, in view of Lemma 4.5, for a sufficiently small $h > 0$, the mapping $\Pi \mapsto \Pi^{-1} : W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d}) \to L^\infty(\Omega;\mathbb{R}^{d\times d})$ is continuous, using $\Pi^{-1} = \mathbb{K}_{d-1}^\top(\Pi)/\det\Pi$. A further ingredient is the standard approximation property of the P1-FE discretization, namely that every $(\widehat{y},\widehat{\Pi})$ is attainable by a sequence $\{(\widehat{y}_h,\widehat{\Pi}_h)\}_{h>0}$ with $(\widehat{y}_h,\widehat{\Pi}_h) \in V_{{\rm df},h} \times V_{{\rm pl},h}$ converging to $(\widehat{y},\widehat{\Pi})$ in the norm topology of $W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$.

We emphasize that $\widehat{W}_\varepsilon$ from (3.6) is polyconvex in the sense of (2.15) because the Yosida approximation $\mathfrak{W}_\varepsilon$ of $\mathfrak{W}_0$ from (3.6) is convex. Hence $\mathcal{E}_\varepsilon$ is also weakly lower semicontinuous on $W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$. Also, it is important that, due to the $\nabla\Pi$-term and the compact embedding $W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d}) \subset L^\infty(\Omega;\mathbb{R}^{d\times d})$, the dissipation distance $\mathcal{D}$ from (2.9) is weakly continuous; hence the mutual-recovery sequences (in the sense of [MRS08]) can simply be taken constant.

As to (ii), the convergence of solutions of the RIS $(Q,\mathcal{E}_\varepsilon,\mathcal{D})$ towards solutions of the RIS $(Q,\mathcal{E},\mathcal{D})$ with the initial condition $q_0$ is relatively standard when exploiting the abstract $\Gamma$-convergence results, cf. [MiR15, MRS08]; here one can rely on the obvious facts that $\mathcal{E}_\varepsilon \le \mathcal{E}$ and $\mathcal{E}_\varepsilon \to \mathcal{E}$ pointwise. One important fact is the uniform coercivity of $\mathcal{E}_\varepsilon$, stated in Lemma 4.2. Another important fact is the weak continuity of $\partial_t\mathcal{E}_\varepsilon(t,\cdot,\cdot)$, which is rather technical and actually needs [MiR15, Prop. 2.1.17] generalized for sequences of energies, exploiting also convergence of energies along stable sequences and uniform approximation of $\partial_t\mathcal{E}_\varepsilon(t,\cdot,\cdot)$ by the time-difference quotients holding as a simple consequence of the $C^1$ time dependence of the loading $\ell(\cdot)$ and $g_{\rm Dir}(\cdot)$, cf. the assumptions in Proposition 4.1.

As to (iii), one can use abstract results about time-discretized problems whose data $\Gamma$-converges in a suitable sense, cf. [MRS08] or also [MiR15, Chap. 2]. Here the standard $\Gamma$-convergence $\mathcal{E}_\varepsilon \to \mathcal{E}$ suffices because $\mathcal{D}$ is continuous. $\square$

**Corollary 4.7 (Implicit convergence criterion)** *There exists $H : \mathbb{R}^+ \to \mathbb{R}^+$ decaying sufficiently fast to zero so that the joint convergence of the numerical approximations $\overline{q}_{\varepsilon\tau h} = (\overline{y}_{\varepsilon\tau h}, \overline{\Pi}_{\varepsilon\tau h})$ converge in terms of subsequences and in the mode as in (4.23) for $(\varepsilon,\tau,h) \to (0,0,0)$ towards energetic solutions of the original RIS $(Q,\mathcal{E},\mathcal{D})$ subject to a "convergence criterion" $h \le H(\varepsilon)$ holds.*

In general, only existence of such $H$ can be proved by rather nonconstructive topological arguments, cf. [KMR05, Prop. 5.6] or [MiR15, Prop. 2.4.6]. Such an unspecified, very implicit criterion has not much practical importance, however, which is a particular motivation for the following section.

# 5 Improved convergence coping with the Lavrentiev phenomenon in case of incompressible plasticity in nonsimple materials

In this section we confine ourselves to the case $\mathbb{M}_d(\Pi) = \det\Pi = 1$, which reflects the phenomenon that plastification is microscopically related with a slip of particular atomic layers due to dislocation movement without substantial change of volume, i.e. incompressiblity of the plastic strain. We thus consider

$$\mathfrak{P} = \big\{\mathsf{P}\in\mathbb{R}^{d\times d};\ \det\mathsf{P} = 1\big\} \quad \text{and} \quad \mathfrak{N} = \big\{\mathsf{P}\in\mathbb{R}^{d\times d};\ \frac{1}{2} \le \det\mathsf{P}\big\}. \tag{5.1}$$

This will facilitate or at least simplify some mathematical aspects; in particular it allows for modification of (3.6) to obtain an approximation of $\mathfrak{W}_0$ that the continuity of $\mathsf{P} \mapsto W_{\mathrm{el},\varepsilon}(\mathsf{F},\mathsf{P})$ is uniform with respect to $\varepsilon$, cf. (5.8b) below.

It should be noted that all variants in Proposition 4.6 concern only the cases that the spatial discretization controlled by $h > 0$ converges first. In reality, we rather refine the discretization together with decreasing $\varepsilon$. Convergence of such a scenario would need the Γ-convergence of the collection $\{\mathcal{E}_{\varepsilon h}\}_{\varepsilon>0,h>0}$, possibly respecting some convergence criterion of the type $h \le H(\varepsilon)$. We however hardly can expect an explicit $H$ because of the mentioned Lavrentiev phenomenon, cf. Corollary 4.7.

Anyhow, to have a chance for some *explicit convergence criterion*, we modify the model in the spirit of the theory of so-called *second-grade nonsimple materials* (also called multipolar solids), cf. e.g. [Pod02, Šil85], alternatively also called the concept of hyper- or couple-stresses [PoV10, Tou62]. It consists in augmenting $\widehat{W}_{\mathrm{el}}$ by a convex higher-order term $\widehat{W}^{\mathrm{SG}}_{\mathrm{el}}(\nabla\mathsf{F}_{\mathrm{el}}) = \frac{\chi}{p_{\mathrm{sg}}}|\nabla\mathsf{F}_{\mathrm{el}}|^{p_{\mathrm{sg}}}$with $\chi > 0$. and $p_{\mathrm{sg}} > d$. This term could easily be replaced by a more physically motivated functional which is convex and satisfies the bounds $c_0|\nabla\mathsf{F}_{\mathrm{el}}|^{p_{\mathrm{sg}}} \le \widehat{W}^{\mathrm{SG}}_{\mathrm{el}}(\nabla\mathsf{F}_{\mathrm{el}}) \le C_0(1+|\nabla\mathsf{F}_{\mathrm{el}}|^{p_{\mathrm{sg}}})$.

Taking into account $\mathsf{F}_{\mathrm{el}} = \partial_y g_{\mathrm{Dir}}(t,y)\nabla y \mathit{\Pi}^{-1}$, cf. (2.23), together with $\mathit{\Pi}^{-1} = \mathbb{K}^{\mathsf{T}}_{d-1}(\mathit{\Pi})/\det\mathit{\Pi}$ with $\det\mathit{\Pi} = 1$, the augmented $\mathcal{E}_\varepsilon$ from (4.13) results to the *second-grade stored energy*

$$\mathcal{E}^{\mathrm{SG}}_{\varepsilon}(t,y,\mathit{\Pi}) := \mathcal{E}_\varepsilon(t,y,\mathit{\Pi}) + \int_\Omega \frac{\chi}{p_{\mathrm{sg}}}\big|\nabla(\partial_y g_{\mathrm{Dir}}(t,y)\nabla y\mathbb{K}^{\mathsf{T}}_{d-1}(\mathit{\Pi}))\big|^{p_{\mathrm{sg}}}\,\mathrm{d}x \tag{5.2}$$

defined on $W^{2,p_{\mathrm{sg}}}(\Omega;\mathbb{R}^d) \times W^{1,p_{\mathrm{gr}}}(\Omega;\mathbb{R}^{d\times d})$. Note that always $W^{2,p_{\mathrm{sg}}}(\Omega;\mathbb{R}^d) \subset W^{1,p_{\mathrm{el}}}(\Omega;\mathbb{R}^d)$ due to $p_{\mathrm{sg}} > d$. We modify analogously also $\mathcal{E}_{\varepsilon h}$ from (3.2), namely

$$\mathcal{E}^{\mathrm{SG}}_{\varepsilon h}(t,y,\mathit{\Pi}) := \mathcal{E}_{\varepsilon h}(t,y,\mathit{\Pi}) + \int_\Omega \frac{\chi}{p_{\mathrm{sg}}}\big|\nabla(\partial_y g_{\mathrm{Dir}}(t,y)\nabla y\mathbb{K}^{\mathsf{T}}_{d-1}(\mathit{\Pi}))\big|^{p_{\mathrm{sg}}}\,\mathrm{d}x \tag{5.3}$$

but, of course, we now cannot use P1-elements for $V_{\mathrm{df},h}$ from (3.1a) and, counting now also the special choice of $\mathfrak{P}$, we choose now:

$$V_{\mathrm{df},h} := \big\{y \in W^{2,\infty}(\Omega;\mathbb{R}^d);\;\; y|_\triangle \;\text{ 2nd-degree polynomal }\; \forall\triangle\in\mathcal{T}_h\big\}, \tag{5.4a}$$

$$V_{\mathrm{pl},h} := \big\{\mathit{\Pi}\in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d});\;\; \mathit{\Pi}|_\triangle \text{ 1st-degree polynomal }\; \forall\triangle\in\mathcal{T}_h,\quad \det\mathit{\Pi}(x) = 1\;\; \forall x\in\mathcal{N}_h\big\}. \tag{5.4b}$$

Note that requiring $\det\mathit{\Pi}(x) = 1$ for all $x$ instead of only for all $x\in\mathcal{N}_h$ used in (5.4b) would make it difficult to construct a projector from $\{\mathit{\Pi}\in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d});\; \det\mathit{\Pi} = 1 \text{ on } \Omega\}$ on such a subspace.

For further purpose, without going into technical details behind the theory of the finite-element method, we denote by $\mathcal{P}^{(k)}_h$ some projector from $W^{k,p}(\Omega)$ into the P$k$-finite element space on the triangulation $\mathcal{T}_h$. We rely on the (standard) approximation properties

$$\forall\, v\in W^{k,p}(\Omega):\qquad \lim_{h\to0}\big\|v-\mathcal{P}^{(k)}_h v\big\|_{W^{k,p}(\Omega)} = 0,\quad\text{and} \tag{5.5a}$$

$$\big\|v-\mathcal{P}^{(k)}_h v\big\|_{W^{l,p}(\Omega)} \le C_{k,l,p}h^{k-l}\big\|v\big\|_{W^{k,p}(\Omega)}\quad\text{for } l = 0,...,k \tag{5.5b}$$

with some $C_{k,l,p}$ independent of $h$ and $v$, cf. e.g. [BrS08, Thm. 4.4.20]. We will use (5.5) component-wise for vector- or matrix-valued functions only for $k = 1$ or 2.

We again assume $W$ in the form (2.21a). As we now use the concept of nonsimple materials, we have "compactness of deformation gradients" and we can relax the polyconvexity (2.15) by requiring convexity of $W(\mathsf{F},\mathsf{P},\mathsf{A})$ only in terms of $\mathsf{A}$. Also the ansatz (3.3) and the assumption (4.22) can then be relaxed to require

$$\widehat{W}_{\rm el}(\mathsf{F}_{\rm el}) = \widehat{W}_{\rm el,1}(\mathsf{F}_{\rm el}) + \mathfrak{W}_0(\det \mathsf{F}_{\rm el}) \quad \text{with } \widehat{W}_{\rm el,1} \text{ continuous, satisfying} \tag{5.6a}$$

$$\exists\, p_{\rm el} \ge 2d,\ c>0: \quad c|\mathsf{F}_{\rm el}|^{p_{\rm el}} \le \widehat{W}_{\rm el,1}(\mathsf{F}_{\rm el}) \le C\big(1+|\mathsf{F}_{\rm el}|^{p_{\rm el}}\big), \quad \text{and} \tag{5.6b}$$

$$\mathfrak{W}_0 : \mathbb{R} \to [0,\infty] \quad \text{lower semicontinous and proper.} \tag{5.6c}$$

If $\mathfrak{W}_0$ is not convex, the construction (3.5) of $\mathfrak{W}_\varepsilon$ is then called the Moreau-Yosida approximation instead of mere Yosida approximation. Altogether, $W_\varepsilon(\mathsf{F},\mathsf{P},\mathsf{A}) = \widehat{W}_{\rm el,1}(\mathsf{F}\mathsf{P}^{-1}) + \mathfrak{W}_\varepsilon(\det(\mathsf{F}\mathsf{P}^{-1})) + \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\mathsf{A}|^{p_{\rm gr}}$. Yet, we now use $\det \mathsf{P} = 1$ so that, using also the relation $\mathsf{P}^{-1} = \mathbb{K}_{d-1}^{\mathsf{T}}(\mathsf{P})/\det \mathsf{P} = \mathbb{K}_{d-1}^{\mathsf{T}}(\mathsf{P})$, we obtain eventually

$$\begin{aligned} W_\varepsilon(\mathsf{F},\mathsf{P},\mathsf{A}) = W_{{\rm el},\varepsilon}(\mathsf{F},\mathsf{P}) + \frac{\kappa}{p_{\rm pl}}|\mathsf{P}|^{p_{\rm pl}} + \frac{\varkappa}{p_{\rm gr}}|\mathsf{A}|^{p_{\rm gr}} \quad \text{with}\\ W_{{\rm el},\varepsilon}(\mathsf{F},\mathsf{P}) = \widehat{W}_{\rm el,1}(\mathsf{F}\mathbb{K}_{d-1}^{\mathsf{T}}(\mathsf{P})) + \mathfrak{W}_\varepsilon(\det \mathsf{F}). \end{aligned} \tag{5.7}$$

It is realistic to assume that $W_{{\rm el},\varepsilon}$ is locally Lipschitz continuous in a specific way respecting not only $\mathfrak{P}$ but even its neighbouhood $\mathfrak{N}$ from (5.1), namely

$$\exists\, L:\mathbb{R}^+ \to \mathbb{R}^+ \text{ increasing } \ \forall\, \mathsf{F},\widetilde{\mathsf{F}} \in \mathbb{R}^{d\times d},\ \ \mathsf{P},\widetilde{\mathsf{P}} \in \mathfrak{N}:$$

$$\big|W_{{\rm el},\varepsilon}(\mathsf{F},\mathsf{P}) - W_{{\rm el},\varepsilon}(\widetilde{\mathsf{F}},\mathsf{P})\big| \le L(|\mathsf{P}|)\varepsilon^{-1}\big|\mathsf{F}-\widetilde{\mathsf{F}}\big|\big(1+|\mathsf{F}|^{p_{\rm el}-1}+|\widetilde{\mathsf{F}}|^{p_{\rm el}-1}\big), \tag{5.8a}$$

$$\big|W_{{\rm el},\varepsilon}(\mathsf{F},\mathsf{P}) - W_{{\rm el},\varepsilon}(\mathsf{F},\widetilde{\mathsf{P}})\big| \le L\big(|\mathsf{P}|+|\widetilde{\mathsf{P}}|\big)\big|\mathsf{P}-\widetilde{\mathsf{P}}\big|\big(1+|\mathsf{F}|^{p_{\rm el}}\big). \tag{5.8b}$$

Let us remark that we could consider a general negative exponent in $\varepsilon^{-1}$ in (5.8b) different from $-1$ but this special choice is fitted to the chosen scaling in (3.2). Recall that in this section, we confine ourselves to $\det \mathit{\Pi} = 1$ and thus $W_\varepsilon$ coincides with the regularization used in (3.6) on $\mathfrak{P}$ but is, in general, different on its neighborhood $\mathfrak{N}$. Further, we strengthen the qualification on $g_{\rm Dir}(t,\cdot)$ by requiring, beside (4.7), also that

$$\exists\, C_{\rm Dir} \in \mathbb{R}\ \forall\, (t,y,\widetilde{y}) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d: \qquad \big|\partial_y^2 g_{\rm Dir}(t,y)\big| \le C_{\rm Dir}. \tag{5.9}$$

Then we can show that the perturbed functional $\mathcal{E}_\varepsilon$ from (5.2) is Lipschitz continuous on bounded sets.

**Lemma 5.1 (Lipschitz continuity of $\mathcal{E}_\varepsilon^{\rm SG}$)** *Let the assumptions (4.4b-d) and (4.19) hold for $\mathfrak{N}$ from (5.1). Moreover, let (5.8) and (5.9) hold. Then, there is $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon \le \varepsilon_0$, the functionals $\mathcal{E}_\varepsilon^{\rm SG} : W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d) \times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d}) \to \mathbb{R}$ are Lipschitz continuous on bounded sets; more specifically, for all $\rho \in [0,\infty)$ there is some*

$\ell_\rho \in [0,\infty)$ *such that it holds*

$$
\begin{aligned}
&\forall\, t\in[0,T]\ \forall\,(y,\varPi),(\widetilde y,\widetilde\varPi)\in W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d}):\\
&\qquad\left.\begin{array}{l}
\mathcal{E}^{\rm SG}_\varepsilon(t,y,\varPi)\le\varrho,\\
\|y\|_{W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d)}+\|\varPi\|_{W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})}\le\rho,\\
\|\widetilde y\|_{W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d)}+\|\widetilde\varPi\|_{W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})}\le\rho,\\
\forall\, x\in\Omega:\quad \varPi(x),\ \widetilde\varPi(x)\in\mathfrak{N}
\end{array}\right\}\ \Rightarrow\\
&\big|\mathcal{E}^{\rm SG}_\varepsilon(t,y,\varPi)-\mathcal{E}^{\rm SG}_\varepsilon(t,\widetilde y,\widetilde\varPi)\big|\le\ell_\rho\Big(\frac{1}{\varepsilon}\big\|y-\widetilde y\big\|_{W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)}\\
&\qquad\qquad\qquad+\big\|y-\widetilde y\big\|_{W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d)}+\big\|\varPi-\widetilde\varPi\big\|_{W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})}\Big).
\end{aligned}\tag{5.10}
$$

*Proof.* We can first estimate the difference in the elastic part by

$$
\begin{aligned}
\bigg|\int_\Omega &W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,y)\nabla y,\varPi)\,{\rm d}x\\
&-\int_\Omega W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\widetilde\varPi)\,{\rm d}x\bigg|\le\int_\Omega I^{(1)}_\varepsilon+I^{(2)}_\varepsilon{\rm d}x
\end{aligned}
$$

with the splitting

$$
\begin{aligned}
&\big|W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,y)\nabla y,\varPi)-W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\widetilde\varPi)\big|\\
&\quad\le\big|W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,y)\nabla y,\varPi)-W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\varPi)\big|\\
&\qquad+\big|W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\varPi)-W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\widetilde\varPi)\big|=:I^{(1)}_\varepsilon+I^{(2)}_\varepsilon.
\end{aligned}
$$

Then, using (5.8a) and the Hölder inequality we obtain

$$
\begin{aligned}
\int_\Omega I^{(1)}_\varepsilon{\rm d}x&\le\int_\Omega L(|\varPi|)\varepsilon^{-1}\big|\partial_y g_{\rm Dir}(t,y)\nabla y-\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y\big|\\
&\qquad\qquad\Big(1+|\partial_y g_{\rm Dir}(t,y)\nabla y|^{p_{\rm el}-1}+|\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y|^{p_{\rm el}-1}\Big)\,{\rm d}x\\
&\le L_1(\|\varPi\|_{L^\infty})\varepsilon^{-1}\Big(\big\|\partial_y g_{\rm Dir}(t,y)-\partial_y g_{\rm Dir}(t,\widetilde y)\big\|_{L^\infty}\big\|\nabla y\big\|_{L^{p_{\rm el}}}\\
&\qquad+\big\|\nabla y-\nabla\widetilde y\big\|_{L^{p_{\rm el}}}\big\|\partial_y g_{\rm Dir}(t,\widetilde y)\big\|_{L^\infty}\Big)\Big(1+\big\|\nabla y\big\|^{p_{\rm el}-1}_{L^{p_{\rm el}}}+\big\|\nabla\widetilde y\big\|^{p_{\rm el}-1}_{L^{p_{\rm el}}}\Big)
\end{aligned}
$$

with $L_1:\mathbb{R}^+\to\mathbb{R}^+$ increasing. We also used that $\partial_y g_{\rm Dir}(t,y)$ is bounded by assumption (4.7). The contribution to (5.10) coming from $I^{(1)}_\varepsilon$ further relies on the Lipschitz continuity $|\partial_y g_{\rm Dir}(t,y)-\partial_y g_{\rm Dir}(t,\widetilde y)|\le C_{\rm Dir}|y-\widetilde y|$ ensured by (5.9). Moreover, using (5.8b), we can eventually estimate

$$
\begin{aligned}
I^{(2)}_\varepsilon&=\big|W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\varPi)-W_{{\rm el},\varepsilon}(\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y,\widetilde\varPi)\big|\\
&\le L\big(|\varPi|+|\widetilde\varPi|\big)\big|\varPi-\widetilde\varPi\big|\big(1+|\partial_y g_{\rm Dir}(t,\widetilde y)\nabla\widetilde y|^{p_{\rm el}}\big),
\end{aligned}
$$

where we used (5.8b). By the Hölder inequality, we can estimate this term as

$$
\int_\Omega I^{(2)}_\varepsilon{\rm d}x\le L_2\big(\|\varPi\|_{L^\infty}+\|\widetilde\varPi\|_{L^\infty}\big)\big\|\varPi-\widetilde\varPi\big\|_{L^\infty}\Big(1+\big\|\nabla\widetilde y\big\|^{p_{\rm el}}_{L^{p_{\rm el}}}\Big)
$$

with some increasing $L_2 : \mathbb{R}^+ \to \mathbb{R}^+$. Thus the terms $I_\varepsilon^{(1)}$ and $I_\varepsilon^{(2)}$ contribute respectively to the first and the third term in the right-hand side of (5.10).

Another contribution to the first right-hand side of (5.10) comes from the penalization terms in $\mathcal{E}_\varepsilon$ in (4.13), using their Lipschitz continuity controlled as $\varepsilon^{-1}$. More in detail, relying on Lipschitz continuity of the distance-functional $d_{\mathfrak{C}}(\cdot)$, we can estimate

$$\Big|\frac{1}{\varepsilon}\int_\Omega d_{\mathfrak{C}}(y)^{r_1}\,\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega d_{\mathfrak{C}}(\widetilde{y})^{r_1}\,\mathrm{d}x\Big| \le \frac{1}{\varepsilon}L_3\big(\|y\|_{L^\infty}^{r_1-1} + \|\widetilde{y}\|_{L^\infty}^{r_1-1}\big)\big\|y-\widetilde{y}\big\|_{L^\infty}$$

with some increasing $L_3 : \mathbb{R}^+ \to \mathbb{R}^+$ (depending on $r_1$). Moreover,

$$\begin{aligned}\Big|\frac{1}{\varepsilon}\Big(\int_\Omega \det(\nabla y)\,\mathrm{d}x - \mathrm{meas}_d(y(\Omega))\Big)^{r_0} - \frac{1}{\varepsilon}\Big(\int_\Omega \det(\nabla\widetilde{y})\,\mathrm{d}x - \mathrm{meas}_d(\widetilde{y}(\Omega))\Big)^{r_0}\Big|&\\ \le \frac{1}{\varepsilon}C\big\|\nabla y-\nabla\widetilde{y}\big\|_{L^{p_{\rm el}}} + \frac{1}{\varepsilon}C\big\|y-\widetilde{y}\big\|_{L^\infty}&\end{aligned}$$

with $C$ dependent on the radius of a ball in $W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)$ where $y$ and $\widetilde{y}$ live; actually, $C = \mathcal{O}(1+\|y\|_{W^{1,p_{\rm el}}}^{r_0 d-1}+\|\widetilde{y}\|_{W^{1,p_{\rm el}}}^{r_0 d-1})$. Here we used also that $y$ and $\widetilde{y}$ live also in a bounded set in $C^{1,1-d/p_{\rm sg}}(\Omega;\mathbb{R}^d)$ and thus $y(\Omega)$ and $\widetilde{y}(\Omega)$ are Lipschitz domains with boundaries $\partial y(\Omega) \subset y(\partial\Omega)$ and $\partial\widetilde{y}(\Omega) \subset \widetilde{y}(\partial\Omega)$ and

$$\begin{aligned}&|\,\mathrm{meas}_d(y(\bar\Omega)) - \mathrm{meas}_d(\widetilde{y}(\bar\Omega))|\\ &\qquad\le C\Big(\mathrm{meas}_{d-1}(\widetilde{y}(\partial\Omega)) + \mathrm{meas}_{d-1}(y(\partial\Omega))\Big)\|y-\widetilde{y}\|_{C(\bar\Omega;\mathbb{R}^d)}\end{aligned}\tag{5.11}$$

with a constant $C$ depending on $d$ and $\varrho$. Here we can use that $y$ and $\tilde{y}$ are locally invertible for any $0 < \varepsilon \le \varepsilon_0$ with $\varepsilon_0$ sufficiently small depending on $\varrho$. This follows from [HeK09, Thm. 3.1] which shows that any level set of $\mathcal{E}^{\rm SG}(t,\cdot,\mathrm{id})$ defined by (5.18) below admits some $\delta > 0$ such that any $y$ from this level set exhibits $\det(\nabla y) \ge \delta$. This holds here possibly with a smaller $\delta$ also for $\mathcal{E}^{\rm SG}(t,\cdot,\Pi)$ with $\Pi$ ranging over bounded sets in $W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$ valued in $\mathfrak{N}$ from (5.1). This implies existence of some $\varepsilon_0 > 0$ such that the level sets $\mathcal{E}_\varepsilon^{\rm SG}(t,\cdot,\cdot)$ considered in (5.10) with any $0 < \varepsilon \le \varepsilon_0$ enjoy that any $y$ from any of these level sets exhibits $\det(\nabla y) \ge \delta/2$ everywhere on $\bar\Omega$. This can be seen by a contradiction argument, assuming that, for any $\delta > 0$ there is some $y_\delta$ and $x_\delta \in \bar\Omega$ for which $\det(\nabla y_\delta(x_\delta)) \le \delta/2$ and using compactness and continuity of $(x,y) \mapsto \det(\nabla y(x))$, for $\delta \to 0$ we would get some $y$ and $x$ such that $\det(\nabla y(x)) \le 0$, which would contradict the mentioned result from [HeK09].

Further we need to estimate

$$\begin{aligned}\Big|\int_\Omega |\nabla(\partial_y g_{\rm Dir}(t,y)\nabla y\mathbb{K}_{d-1}^\top(\Pi))|^2 - |\nabla(\partial_y g_{\rm Dir}(t,\widetilde{y})\nabla\widetilde{y}\mathbb{K}_{d-1}^\top(\widetilde{\Pi}))|^2\,\mathrm{d}x\Big|&\\ \le C_\rho\Big(\|y-\widetilde{y}\|_{W^{2,p_{\rm sg}}} + \|\Pi-\widetilde{\Pi}\|_{W^{1,p}}\Big).&\end{aligned}\tag{5.12}$$

To this goal, we fix $t \in [0,T]$ and use the notation

$$\begin{aligned}&y_{\rm Dir} = g_{\rm Dir}(t,y), \quad \widetilde{y}_{\rm Dir} = g_{\rm Dir}(t,\widetilde{y}), \quad \text{and then}\\ &A = \nabla y_{\rm Dir}, \quad \widetilde{A} = \nabla\widetilde{y}_{\rm Dir}, \quad B = \mathbb{K}_{d-1}^\top(\Pi), \quad \widetilde{B} = \mathbb{K}_{d-1}^\top(\widetilde{\Pi}).\end{aligned}$$

By our assumptions we have $A, \widetilde{A} \in W^{1,p_{\rm sg}(\Omega;\mathbb{R}^{d\times d})}$ (use $p_{\rm el} > d$ here) and $B, \widetilde{B} \in W^{1,p_{\rm el}}(\Omega;\mathbb{R}^{d\times d})$ with $p_{\rm el} > d$. Hence,

$$
\begin{aligned}
&\Big| \int_\Omega \big|\nabla(AB)\big|^{p_{\rm sg}} - \big|\nabla(\widetilde{A}\widetilde{B})\big|^{p_{\rm sg}} \,{\rm d}x \Big| \\
&\qquad \le \Big( \|\nabla(AB)\|_{L^{p_{\rm sg}}}^{p_{\rm sg}-1} + \|\nabla(\widetilde{A}\widetilde{B})\|_{L^{p_{\rm sg}}}^{p_{\rm sg}-1} \Big) \|\nabla(AB{-}\widetilde{A}\widetilde{B})\|_{L^{p_{\rm sg}}} && (5.13)\\
&\qquad \le C_1 \Big( \|A{-}\widetilde{A}\|_{W^{1,p_{\rm sg}}} \|B\|_{W^{1,p_{\rm el}}} + \|\widetilde{A}\|_{W^{1,p_{\rm sg}}} \|B{-}\widetilde{B}\|_{W^{1,p_{\rm el}}} \Big) \\
&\qquad \le \; C_\rho \Big( \|A{-}\widetilde{A}\|_{W^{1,p_{\rm sg}}} + \|B{-}\widetilde{B}\|_{W^{1,p_{\rm el}}} \Big), && (5.14)
\end{aligned}
$$

where $C_1 = (\|A\|_{W^{1,p_{\rm sg}}}^{p_{\rm sg}-1} {+} \|\widetilde{A}\|_{W^{1,p_{\rm sg}}}^{p_{\rm sg}-1})(\|B\|_{W^{1,p_{\rm el}}}^{p_{\rm sg}-1} {+} \|\widetilde{B}\|_{W^{1,p_{\rm el}}}^{p_{\rm sg}-1})$ and $C_\rho$ depends only on $\rho$. Using $p_{\rm el} > d$ (giving $\|\mathit{\Pi}\|_{L^\infty}, \|\widetilde{\mathit{\Pi}}\|_{L^\infty} \le R_\rho$) and the polynomial structure of $\mathbb{K}_{d-1}$ we have $\|B{-}\widetilde{B}\|_{W^{1,p_{\rm el}}} \le C_\rho \|\mathit{\Pi}{-}\widetilde{\mathit{\Pi}}\|_{W^{1,p_{\rm el}}}$. If $g_{\rm Dir} = {\rm id}$, we are already done.

To treat the general case, let us denote

$$
a(x) = \partial_y g_{\rm Dir}(t, y(x)), \quad b = \nabla y, \quad \widetilde{a}(x) = \partial_y g_{\rm Dir}(t, \widetilde{y}(x)), \quad \widetilde{b} = \nabla \widetilde{y},
$$

such that $\nabla y_{\rm Dir} = A = ab$ and $\nabla \widetilde{y}_{\rm Dir} = \widetilde{A} = \widetilde{a}\widetilde{b}$. Obviously, we have

$$
\|A{-}\widetilde{A}\|_{L^{p_{\rm sg}}} = \|ab{-}\widetilde{a}\widetilde{b}\|_{L^{p_{\rm sg}}} \le C \|\partial_y g_{\rm Dir}(t,\cdot)\|_{W^{1,\infty}} \|y{-}\widetilde{y}\|_{W^{1,p_{\rm sg}}} \le C_{\rm Dir} C \|y{-}\widetilde{y}\|_{W^{1,p_{\rm sg}}}
$$

where $C_{\rm Dir}$ is from (5.9). Thus, it remains to estimate $\|\nabla(ab){-}\nabla(\widetilde{a}\widetilde{b})\|_{L^{p_{\rm sg}}}$. Here we use $a, \widetilde{a} \in W^{1,p_{\rm el}}(\Omega)$ (taking into account $p_{\rm el} > d$ and again (4.7) and (5.9)) and $b, \widetilde{b} \in W^{1,p_{\rm sg}}(\Omega;\mathbb{R}^{d\times d})$. Hence, we obtain

$$
\begin{aligned}
\big\|\nabla A{-}\nabla\widetilde{A}\big\|_{L^{p_{\rm sg}}} &\le \big\|\partial_y g_{\rm Dir}(t, y(\cdot)) - \partial_y g_{\rm Dir}(t, \widetilde{y}(\cdot))\big\|_{W^{1,p_{\rm el}}} \big\|\nabla y\big\|_{W^{1,p_{\rm sg}}} \\
&\qquad + \big\|\partial_y g_{\rm Dir}(t, \widetilde{y}(\cdot))\big\|_{W^{1,p_{\rm el}}} \big\|\nabla y - \nabla\widetilde{y}\big\|_{W^{1,p_{\rm sg}}} \\
&\le \big\|\partial_y g_{\rm Dir}(t,\cdot)\big\|_{W^{1,\infty}} \Big( \big\|y{-}\widetilde{y}\big\|_{W^{1,p_{\rm el}}} \big\|y\big\|_{W^{2,p_{\rm sg}}} + \big\|\widetilde{y}\big\|_{W^{1,p_{\rm el}}} \big\|y{-}\widetilde{y}\big\|_{W^{2,p_{\rm sg}}} \Big) \\
&\le \; C_\rho \big\|y{-}\widetilde{y}\big\|_{W^{2,p_{\rm sg}}}. && (5.15)
\end{aligned}
$$

Merging (5.14)–(5.15) shows the desired estimate (5.12).

As to the last two terms in (2.21a), we can estimate

$$
\begin{aligned}
\Big| \int_\Omega \frac{\kappa}{p_{\rm pl}} |\mathit{\Pi}|^{p_{\rm pl}} - \frac{\kappa}{p_{\rm pl}} |\widetilde{\mathit{\Pi}}|^{p_{\rm pl}} {\rm d}x \Big| &\le C_\kappa \int_\Omega |\mathit{\Pi}{-}\widetilde{\mathit{\Pi}}| \big(1{+}|\mathit{\Pi}|^{p_{\rm pl}-1}{+}|\widetilde{\mathit{\Pi}}|^{p_{\rm pl}-1}\big) \,{\rm d}x \\
&\le C_\kappa \big\|\mathit{\Pi}{-}\widetilde{\mathit{\Pi}}\big\|_{L^{p_{\rm pl}}(\Omega;\mathbb{R}^{d\times d})} \Big(1{+}\big\|\mathit{\Pi}\big\|_{L^{p_{\rm pl}}}^{p_{\rm pl}-1}{+}\big\|\widetilde{\mathit{\Pi}}\big\|_{L^{p_{\rm pl}}}^{p_{\rm pl}-1}\Big) && (5.16)
\end{aligned}
$$

and also

$$
\begin{aligned}
\Big| \int_\Omega \frac{\varkappa}{p_{\rm gr}} |\nabla\mathit{\Pi}|^{p_{\rm gr}} - \frac{\varkappa}{p_{\rm gr}} |\nabla\widetilde{\mathit{\Pi}}|^{p_{\rm gr}} {\rm d}x \Big| &\le C_\varkappa \int_\Omega |\nabla\mathit{\Pi}{-}\nabla\widetilde{\mathit{\Pi}}| \big(1{+}|\nabla\mathit{\Pi}|^{p_{\rm gr}-1}{+}|\nabla\widetilde{\mathit{\Pi}}|^{p_{\rm gr}-1}\big) \,{\rm d}x \\
&\le C_\varkappa \big\|\nabla\mathit{\Pi}{-}\nabla\widetilde{\mathit{\Pi}}\big\|_{L^{p_{\rm gr}}} \Big(1{+}\big\|\nabla\mathit{\Pi}\big\|_{L^{p_{\rm gr}}}^{p_{\rm gr}-1}{+}\big\|\nabla\widetilde{\mathit{\Pi}}\big\|_{L^{p_{\rm gr}}}^{p_{\rm gr}-1}\Big). && (5.17)
\end{aligned}
$$

Both (5.16) and (5.17) contribute to the last term in (5.10). $\square$

**Lemma 5.2** **($\Gamma$-convergence of $\mathcal{E}^{\rm SG}_{\varepsilon h}$)** *Let (4.4) and (5.6) and the assumptions of Lemma 5.1 hold. and the loading $(g_{\rm Dir}, f_{\rm vol}, f_{\rm surf})$ satisfy the assumptions of Proposition 4.1. Then for $t_\varepsilon \to t$ the collection $\{\mathcal{E}^{\rm SG}_{\varepsilon h}(t_\varepsilon,\cdot,\cdot)\}_{\varepsilon,h>0}$ $\Gamma$-converges to $\mathcal{E}^{\rm SG}(t,\cdot,\cdot)$ defined as*

$$\mathcal{E}^{\rm SG}(t,y,\Pi) = \begin{cases} \mathcal{E}(t,y,\Pi) \\ \quad + \displaystyle\int_\Omega \frac{\chi}{p_{\rm sg}} \big|\nabla(\partial_y g_{\rm Dir}(t,y)\nabla y \mathbb{K}^\top_{d-1}(\Pi))\big|^{p_{\rm sg}}\,{\rm d}x & \textit{if } \det\Pi = 1 \textit{ on } \Omega, \\ \infty & \textit{otherwise} \end{cases} \tag{5.18}$$

*with $\mathcal{E}$ from (2.13) in the weak topology on bounded sets of $W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d)\times W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$ in the sense (4.12) provided the following stability criterion is satisfied:*

$$\frac{h^{\min(1,p_{\rm sg}/p_{\rm el})}}{\varepsilon} \to 0. \tag{5.19}$$

*Proof.* The liminf-condition (4.12a), i.e. $\liminf \mathcal{E}^{\rm SG}_{\varepsilon h}(t_\varepsilon, y_{\varepsilon h}, \Pi_{\varepsilon h}) \geq \mathcal{E}^{\rm SG}(t,y,\Pi)$ for weakly converging sequences $y_{\varepsilon h} \rightharpoonup y$ and $\Pi_{\varepsilon h} \rightharpoonup \Pi$, holds unconditionally. This follows by showing this convergence for some lower estimate of $\mathcal{E}^{\rm SG}_{\varepsilon h}$, namely for $\mathcal{E}^{\rm SG}_{\varepsilon} \leq \mathcal{E}^{\rm SG}_{\varepsilon h}$ with $\mathcal{E}^{\rm SG}_{\varepsilon}$ defined by (5.2). Actually, $\mathcal{E}^{\rm SG}_{\varepsilon}$ penalizes the constraints occurring in $\mathcal{E}$ defined by (2.13) together with the constraint (1.1b) involved in $W$. The estimate then essentially follows from the weak lower-semicontinuity of $\mathcal{E}^{\rm SG}(t,\cdot,\cdot)$.

The limsup-condition (4.12b), i.e. $\limsup \mathcal{E}^{\rm SG}_{\varepsilon h}(t_\varepsilon, y_{\varepsilon h}, \Pi_{\varepsilon h}) \leq \mathcal{E}^{\rm SG}(t,y,\Pi)$ for any $(y,\Pi)$ and some weakly converging sequences $y_{\varepsilon h} \to y$ and $\Pi_{\varepsilon h} \to \Pi$ needs an explicit construction of such recovery sequences and, in our case, holds only conditionally under the stability criterion (5.19).

For $(y,\Pi)$ given, the recovery sequence can be taken as

$$y_{\varepsilon h} = \mathcal{P}^{(2)}_h y \quad \text{and} \quad \Pi_{\varepsilon h} = \mathcal{P}^{(1)}_h \Pi \qquad \text{(actually independent of } \varepsilon) \tag{5.20}$$

with $\mathcal{P}^{(k)}_h$ from (5.5). In view of (4.12b), we need to prove that

$$\begin{aligned}\limsup\big(\mathcal{E}^{\rm SG}_{\varepsilon h}(t_\varepsilon, y_{\varepsilon h}, \Pi_{\varepsilon h}) - \mathcal{E}^{\rm SG}(t,y,\Pi)\big) &= \limsup\big(\mathcal{E}^{\rm SG}_{\varepsilon}(t_\varepsilon, y, \Pi) - \mathcal{E}^{\rm SG}(t,y,\Pi)\big) \\ &\quad + \lim\big(\mathcal{E}^{\rm SG}_{\varepsilon h}(t_\varepsilon, y_{\varepsilon h}, \Pi_{\varepsilon h}) - \mathcal{E}^{\rm SG}_{\varepsilon}(t_\varepsilon, y, \Pi)\big) \leq 0.\end{aligned} \tag{5.21}$$

If $\mathcal{E}^{\rm SG}(t,y,\Pi) = \infty$, then (5.21) is trivial because its left-hand side is $-\infty$. If $\mathcal{E}^{\rm SG}(t,y,\Pi) < \infty$, then $\limsup(\mathcal{E}^{\rm SG}_{\varepsilon}(t_\varepsilon,y,\Pi) - \mathcal{E}^{\rm SG}(t,y,\Pi)) \leq 0$ follows from $\mathcal{E}^{\rm SG}_{\varepsilon} \leq \mathcal{E}^{\rm SG}$ and from the continuity of $t \mapsto \mathcal{E}^{\rm SG}_{\varepsilon}(t,y,\Pi)$. Therefore, we only need to prove that the last term in (5.21) is zero. More specifically, we can estimate

$$\begin{aligned}&\big|\mathcal{E}^{\rm SG}_{\varepsilon}(t_\varepsilon,y,\Pi) - \mathcal{E}^{\rm SG}_{\varepsilon}(t_\varepsilon,y_{\varepsilon h},\Pi_{\varepsilon h})\big| \\ &\qquad \leq \ell_\rho\Big(\frac{1}{\varepsilon}\big\|y - y_{\varepsilon h}\big\|_{W^{1,p_{\rm el}}} + \big\|y - y_{\varepsilon h}\big\|_{W^{2,p_{\rm sg}}} + \big\|\Pi - \Pi_{\varepsilon h}\big\|_{W^{1,p_{\rm gr}}}\Big) \\ &\qquad \leq \frac{\ell_\rho}{\varepsilon}\big\|y - \mathcal{P}^{(2)}_h y\big\|_{W^{1,p_{\rm el}}} + \ell_\rho\big\|y - \mathcal{P}^{(2)}_h y\big\|_{W^{2,p_{\rm sg}}} + \ell_\rho\big\|\Pi - \mathcal{P}^{(1)}_h \Pi\big\|_{W^{1,p_{\rm gr}}} \\ &\qquad \leq \ell_\rho C \frac{h^{\min(1,p_{\rm sg}/p_{\rm el})}}{\varepsilon}\|y\|_{W^{2,p_{\rm sg}}} + o(1) + o(1) \qquad \text{for } h \to 0,\end{aligned} \tag{5.22}$$

where we used (5.5a) with $k = 2$ and $p = p_{\rm sg}$ for $y$ and with $k = 1$ and $p = p_{\rm gr}$ for $\Pi$, and also we used (5.5b) for $k = 2$, $p = p_{\rm sg}$, and $l = 1$ or $2$ to obtain $\|y - \mathcal{P}^{(2)}_h y\|_{W^{1,p_{\rm sg}}} \leq$

$C_{2,1,p_{\rm sg}}h\|y\|_{W^{2,p_{\rm sg}}}$, which gives (5.22) provided $p_{\rm sg}\geq p_{\rm el}>d$, while for $d<p_{\rm sg}<p_{\rm el}$ we have to use

$$\|y{-}\mathcal{P}_h^{(2)}y\|_{W^{1,\infty}}\leq N\|y{-}\mathcal{P}_h^{(2)}y\|_{W^{2,p_{\rm sg}}}\leq NC_{2,2,p_{\rm sg}}\|y\|_{W^{2,p_{\rm sg}}}$$

with $N$ denoting the norm of the embedding $W^{1,p_{\rm sg}}(\Omega)\subset L^\infty(\Omega)$, relying on $p_{\rm sg}>d$. These estimates can be further interpolated to obtain

$$\big\|y{-}\mathcal{P}_h^{(2)}y\big\|_{W^{1,p_{\rm el}}}\leq Ch^{p_{\rm sg}/p_{\rm el}}\|y\|_{W^{2,p_{\rm sg}}}. \qquad \square$$

Let us note that the recovery sequence (5.20) converges even strongly and therefore the weak Γ-convergence proved in Lemma 5.2 is, in fact, even the so-called Mosco convergence. In the main result of this section, we now do not need to assume the polyconvexity (2.15), which allows for considering more general materials like the St. Venant-Kirchhoff material.

**Proposition 5.3 (Convergence of the approximate evolution)** *Let all the assumptions of Lemma 5.2 together with (4.5) hold and the initial condition $(y_0,\Pi_0)$ be stable and be approximated by $(y_{\varepsilon h,0},\Pi_{h,0})\in V_{{\rm df},h}\times V_{{\rm pl},h}$ so that $(y_{\varepsilon h,0},\Pi_{h,0})\to(y_0,\Pi_0)$ in $\mathcal{Q}$ and $\mathcal{E}_\varepsilon(0,y_{\varepsilon h,0},\Pi_{h,0})\to\mathcal{E}(0,y,\Pi)$. Then, the numerical approximations $\overline{q}_{\varepsilon\tau h}=(\overline{y}_{\varepsilon\tau h},\overline{\Pi}_{\varepsilon\tau h})$ converge (in terms of subsequences) for $(\varepsilon,\tau,h){\to}(0,0,0)$ respecting the stability criterion (5.19) towards energetic solutions of the RIS $(Q^{\rm SG},\mathcal{E}^{\rm SG},\mathcal{D})$ with initial value $q_0\in Q^{\rm SG}\overset{def}{=}W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d){\times}W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d})$ and $(\mathcal{E}^{\rm SG},\mathcal{D})$ from (5.18) and (2.9). Indeed, for a subsequence and some $(y,\Pi)$, we have*

$$\forall\, t\in[0,T]:\quad \overline{\Pi}_{\varepsilon\tau h}(t)\rightharpoonup\Pi(t)\qquad in\ \ W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d}),\tag{5.23a}$$

$$\forall\, t\in[0,T]:\quad \mathcal{E}^{\rm SG}_\varepsilon\big(t,\overline{y}_{\varepsilon\tau h}(t),\overline{\Pi}_{\varepsilon\tau h}(t)\big)\to\mathcal{E}^{\rm SG}\big(t,y(t),\Pi(t)\big),\tag{5.23b}$$

$$\forall\, t\in[0,T]:\quad {\rm Diss}_{\mathcal{D}}\big(\overline{\Pi}_{\varepsilon\tau h},[0,t]\big)\to{\rm Diss}_{\mathcal{D}}\big(\Pi,[0,t]\big),\tag{5.23c}$$

$$\forall_{\rm a.a.}\, t{\in}[0,T]:\ \partial_t\mathcal{E}^{\rm SG}_\varepsilon\big(t,\overline{y}_{\varepsilon\tau h}(t),\overline{\Pi}_{\varepsilon\tau h}(t)\big)\to\partial_t\mathcal{E}^{\rm SG}\big(t,y(t),\Pi(t)\big)\tag{5.23d}$$

*and, for all $t\in[0,T]$, there is a further subsequence $(q_{\varepsilon\tau_n^t h_n^t}(t))_{n\in\mathbb{N}}$ such that*

$$\forall t\in[0,T]:\qquad \overline{y}_{\varepsilon\tau_n^t h_n^t}(t)\rightharpoonup y(t)\qquad in\ \ W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d).\tag{5.23e}$$

*Any $(y,\Pi)$ obtained by this way is an energetic solution to the RIS $(Q^{\rm SG},\mathcal{E}^{\rm SG},\mathcal{D})$.*

*Sketch of the proof.* Using Lemma 4.2, in view of (5.2), we have the estimates

$$\big\|\overline{y}_{\varepsilon\tau h}\big\|_{B([0,T];W^{1,p_{\rm el}}(\Omega;\mathbb{R}^d)}\leq C,\tag{5.24a}$$

$$\big\|\overline{\Pi}_{\varepsilon\tau h}\big\|_{B([0,T];W^{1,p_{\rm gr}}(\Omega;\mathbb{R}^{d\times d}))}\leq C.\tag{5.24b}$$

Then, using the second-gradient term, we obtain also

$$\big\|\overline{y}_{\varepsilon\tau h}\big\|_{B([0,T];W^{2,p_{\rm sg}}(\Omega;\mathbb{R}^d))}\leq C.\tag{5.24c}$$

Indeed (5.24c) follows from $\int_\Omega\frac{\chi}{p_{\rm sg}}\big|\nabla(\partial_y g_{\rm Dir}(t,y_{\varepsilon\tau h})\nabla y_{\varepsilon\tau h}\mathbb{K}_{d-1}^{-\top}(\Pi_{\varepsilon\tau h}))\big|^{p_{\rm sg}}\,{\rm d}x$, cf. the last term in (5.2). To this goal, recall the chain rule

$$\nabla_x\big(g_{\rm Dir}(t,\cdot)\circ y_{\varepsilon\tau h}\big)(x)=\partial_y g_{\rm Dir}(t,y_{\varepsilon\tau h}(x))\nabla y_{\varepsilon\tau h}(x),$$

from which we estimate

$$B_{\varepsilon\tau h} := \partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h})\nabla y_{\varepsilon\tau h}\mathbb{K}_{d-1}^\top(\varPi_{\varepsilon\tau h}) \quad \text{in} \;\; W^{1,p_{\rm sg}}(\Omega;\mathbb{R}^{d\times d}).$$

To estimate $\nabla y_{\varepsilon\tau h} = [\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h})]^{-1}B_{\varepsilon\tau h}\mathbb{K}_{d-1}^{-\top}(\varPi_{\varepsilon\tau h})$ in $W^{1,p_{\rm sg}}(\Omega;\mathbb{R}^{d\times d})$ we use that the gradient of the inverse matrix $x \mapsto [\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(x))]^{-1}$ is given by

$$\begin{aligned}\nabla_x&\big[\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(x))\big]^{-1}\\ &= -\big[\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(x))\big]^{-1}\Big(\partial_y^2 g_{\rm Dir}(t, y_{\varepsilon\tau h})\nabla_x y_{\varepsilon\tau h}\Big)\big[\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(x))\big]^{-1}.\end{aligned}$$

For $[\partial_y g_{\rm Dir}]^{-1}$ and $\partial_y^2 g_{\rm Dir}(t, y_{\varepsilon\tau h})$, we need a-priori bounds, cf. the assumptions (4.7) and (5.9); see already [FrM06, Eqn. (5.7)], and find

$$\begin{aligned}\big\|\, \nabla_x[\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(x))]^{-1}\,\big\|_{L^{p_{\rm el}}} &\le \big\|[\partial_y g_{\rm Dir}]^{-1}\big\|_{L^\infty}^2\big\|\partial_y^2 g_{\rm Dir}\big\|_{L^\infty}^2\big\|y_{\varepsilon\tau h}\big\|_{W^{1,p_{\rm el}}}\\ &\le C_g\big\|y_{\varepsilon\tau h}\big\|_{W^{1,p_{\rm el}}}\end{aligned}$$

with some $C_g$ depending on the qualification of $g_{\rm Dir}$ from (4.7) and (5.9). We find the desired a priori estimate for $\nabla^2 y_{\varepsilon\tau h}$ as follows

$$\begin{aligned}\big\|\nabla^2 y_{\varepsilon\tau h}\big\|_{L^{p_{\rm sg}}} &\le \Big\|\nabla_x\Big([\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(\cdot))]^{-1}B_{\varepsilon\tau h}\mathbb{K}_{d-1}^{-\top}(\varPi_{\varepsilon\tau h})\Big)\Big\|_{L^{p_{\rm sg}}}\\ &\le \big\|[\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(\cdot))]^{-1}B_{\varepsilon\tau h}\mathbb{K}_{d-1}^{-\top}(\varPi_{\varepsilon\tau h})\big\|_{W^{1,p_{\rm sg}}}\\ &\le \big\|[\partial_y g_{\rm Dir}(t, y_{\varepsilon\tau h}(\cdot))]^{-1}\big\|_{W^{1,p_{\rm sg}}}\,\|B_{\varepsilon\tau h}\|_{W^{1,p_{\rm sg}}}\,\big\|\mathbb{K}_{d-1}^{-\top}(\varPi_{\varepsilon\tau h})\big\|_{W^{1,p_{\rm sg}}}\\ &\le C_g\big\|y_{\varepsilon\tau h}\big\|_{W^{1,p_{\rm el}}}C\big\|B_{\varepsilon\tau h}\big\|_{W^{1,p_{\rm sg}}}\big\|\varPi_{\varepsilon\tau h}\big\|_{W^{1,p_{\rm gr}}} \le C_\rho\big\|B_{\varepsilon\tau h}\big\|_{W^{1,p_{\rm sg}}}\,,\end{aligned}$$

where we used already (5.24a) and (5.24b).

The mutual recovery sequence for (4.11) with $\mathcal{E}_{\varepsilon h}^{\rm SG}$ instead of $\mathcal{E}_{\varepsilon h}$ can be taken as in the proof of Lemma 5.2, cf. (5.20), relying on the weak continuity of the dissipation distance due to the assumption (4.5). The rest follows from the abstract arguments, cf. [MRS08] or [MiR15, Sect. 2.4]. $\square$

**Remark 5.4** Regularization (3.6) satisfies (5.8) if $\widehat{W}_{\rm el,1}$ from (5.6) also satisfies

$$\big|\widehat{W}_{\rm el,1}(\mathsf{F}) - \widehat{W}_{\rm el,1}(\widetilde{\mathsf{F}})\big| \le L\big|\mathsf{F}{-}\widetilde{\mathsf{F}}\big|\big(1{+}|\mathsf{F}|^{p_{\rm el}-1}{+}|\widetilde{\mathsf{F}}|^{p_{\rm el}-1}\big) \tag{5.25}$$

Let us note that the Moreau-Yosida approximation $\mathfrak{W}_\varepsilon$ of $\mathfrak{W}_0$ is Lipschitz continuous in the sense $|\mathfrak{W}_\varepsilon(\delta){-}\mathfrak{W}_\varepsilon(\widetilde{\delta})| \le C\varepsilon^{-1}|\delta{-}\widetilde{\delta}|(1+|\delta|+|\widetilde{\delta}|)$ so that $W_\varepsilon(\mathsf{F}) := \mathfrak{W}_\varepsilon(\det \mathsf{F})$ satisfies

$$\big|W_\varepsilon(\mathsf{F}) - W_\varepsilon(\widetilde{\mathsf{F}})\big| \le L\varepsilon^{-1}\big|\mathsf{F}{-}\widetilde{\mathsf{F}}\big|\big(1{+}|\mathsf{F}|^{p_{\rm el}-1}{+}|\widetilde{\mathsf{F}}|^{p_{\rm el}-1}\big). \tag{5.26}$$

In view of (5.25) with (5.26), we obtain (5.8) provided $p_{\rm el} \ge 2d$, as already assumed in (4.22b) and (5.6) anyhow.

**Remark 5.5 (Convergence of static elastic problem)** *For purposes of possible reference, let us pronounce separately the* $\Gamma$- *(or rather Mosco) convergence of the numerical approximation* $\mathcal{E}_{\varepsilon h}^{\rm SG}$ *of the purely elastic problem without any plasticity. More specifically, the functional* $\mathcal{E}_{\varepsilon h}^{\rm SG}(0,\cdot,\mathbb{I})$ *converges towards* $\mathcal{E}(0,\cdot,\mathbb{I})$ *in the Mosco sense provided the stability criterion* (5.19) *holds. This is a simple consequence of Lemma* 5.2 *when fixing* $t$ *and* $\varPi(x) \equiv \mathbb{I}$. *This result does not seem to be reported in the literature (which deals with simple materials only).*

**Remark 5.6 (Some generalizations)** For $\mathfrak{C} = \mathfrak{C}(t)$ time dependent in a reasonable smooth way and remaining always away from $\Gamma_{\rm Dir}$, it is quite straightforward that a suitable definition of $g_{\rm Dir}$ can transform it to the time-independent case we considered up to now. Also, considering $\Omega$ composed from several components possibly mutually in a unilateral contact, each of them being fixed on some part by (possibly time-dependent) Dirichlet condition, represents rather slight notational complication only.

**Acknowledgment.** The authors are grateful to Jiří Plešek for inspiring discussions about computational aspects of large-strain plasticity and to Jan Malý for providing a full and elegant proof of Proposition 4.3. The research of A.M. has been partially supported by DFG through SFB 1114 (subproject B01). T.R. acknowledges partial support from the grants 13-18652S and 14-15264S of the Czech Science Foundation and the institutional support RVO: 61388998 (ČR).

# References

[Alb98] H.-D. Alber. *Materials with Memory*, volume 1682 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[BaK87] J. M. Ball and G. Knowles. A numerical method for detecting singular minimizers. *Numer. Math.*, 51, 181197, 1987.

[Bal77] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4), 337–403, 1977.

[Bal84] J. M. Ball. Minimizers and the Euler-Lagrange equations. In *Trends and applications of pure mathematics to mechanics (Palaiseau, 1983)*, volume 195 of *Lecture Notes in Phys.*, pages 1–4. Springer, Berlin, 1984.

[Bal87] J. M. Ball. Singular mimimizers and their significance in elasticity. In *Directions in partial differential equations*, pages 1–15. Acad. Press, 1987.

[Bal02] J. M. Ball. Some open problems in elasticity. In P. Newton, P. Holmes, and A. Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 3–59. Springer, New York, 2002.

[BaL06] Y. Bai and Z.-P. Li. A truncation method for detecting singular minimizers involving the Lavrentiev phenomenon. *Math. Models Meth. Appl. Sci. ($M^3AS$)*, 16, 847–867, 2006.

[BaL07] Y. Bai and Z. Li. Numerical solution of nonlinear elasticity problems with Lavrentiev phenomenon. *Math. Models Meth. Appl. Sci. ($M^3AS$)*, 17, 1619–1640, 2007.

[BaM85] J. M. Ball and V. J. Mizel. One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation. *Arch. Rational Mech. Anal.*, 90(4), 325–388, 1985.

[Ben11] B. Benešová. Global optimization numerical strategies for rate-independent processes. *J. Global Optim.*, 50, 197–220, 2011.

[BMR12] S. Bartels, A. Mielke, and T. Roubíček. Quasistatic small-strain plasticity in the limit of vanishing hardening and its numerical approximation. *SIAM J. Numer. Anal.*, 50(2), 951–976, 2012.

[BrS08] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, 3rd edition, 2008.

[BrW89] J. G. Broida and S. G. Williamson. *A Comprehensive Introduction to Linear Algebra*. Addison-Wesley, 1989.

[CaO10] C. Carstensen and C. Ortner. Analysis of a class of penalty methods for computing singular minimizers. *Comp. Meth. Appl. Math.*, 10(2), 137–163, 2010.

[CHM02] C. Carstensen, K. Hackl, and A. Mielke. Non–convex potentials and microstructures in finite–strain plasticity. *Proc. Royal Soc. London Ser. A*, 458(2018), 299–317, 2002.

[CiN87] P. G. Ciarlet and J. Nečas. Injectivity and self-contact in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 97(3), 171–188, 1987.

[DaL10] G. Dal Maso and G. Lazzaroni. Quasistatic crack growth in finite elasticity with non-interpenetration. *Ann. Inst. H. Poinc. Anal. Non Lin.*, 27(1), 257–290, 2010.

[DDM06] G. Dal Maso, A. DeSimone, and M. G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Arch. Rational Mech. Anal.*, 180(2), 237–291, 2006.

[FHM03] M. Foss, W. J. Hrusa, and V. J. Mizel. The Lavrentiev gap phenomenon in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 167, 337–365, 2003.

[FrM06] G. Francfort and A. Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. reine angew. Math.*, 595, 55–91, 2006.

[GPU04] D. Gabriel, J. Plešek, and M. Ulbin. Symmetry preserving algorithm for large displacement frictionless contact by the pre-discretisation penalty method. *Int. J. Numer. Meth. Engr.*, 61, 2615–2638, 2004.

[HaR99] W. Han and B. D. Reddy. *Plasticity (Mathematical Theory and Numerical Analysis)*, volume 9 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1999.

[HaS06] D. Habeck and F. Schuricht. Contact between nonlinearly elastic bodies. *Proc. Roy. Soc. Edinburgh Sect. A*, 136(6), 1239–1266, 2006.

[HeK09] T. J. Healey and S. Krömer. Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM Control Optim. Calc. Var.*, 15, 863–871, 2009.

[HHM12] K. Hackl, S. Heinz, and A. Mielke. A model for the evolution of laminates in finite-strain elastoplasticity. *Z. angew. Math. Mech. (ZAMM)*, 92(11-12), 888–909, 2012.

[HMM03] K. Hackl, A. Mielke, and D. Mittenhuber. Dissipation distances in multiplicative elastoplasticity. In W. Wendland and M. Efendiev, editors, *Analysis and Simulation of Multifield Problems*, pages 87–100. Springer-Verlag, 2003.

[KMR05] M. Kružík, A. Mielke, and T. Roubíček. Modelling of microstructure and its evolution in shape-memory-alloy single-crystals, in particular in CuAlNi. *Meccanica*, 40, 389–418, 2005.

[Lav27] M. Lavrentieff. Sur quelques problèmes du calcul des variations. *Annali Matem. Pura Appl.*, 4(Tomo IV), 7–28, 1927.

[LeL67] E. H. LEE and D. T. LIU. Finite-strain elastic-plastic theory with application to plain-wave analysis. *J. Applied Phys.*, 38, 19–27, 1967.

[Li95] Z. LI. Element removal method for singular minimizers in problems of hyperelasticity. *Math. Models Meth. Appl. Sci. ($M^3AS$)*, 5, 387–399, 1995.

[Li96] Z. LI. Numerical methods for mimizers and microstructures in nonlinear elasticity. *Math. Models Meth. Appl. Sci. ($M^3AS$)*, 6, 957–975, 1996.

[Mal16] J. MALÝ. Personal communication, January 2016.

[MaM73] M. MARCUS and V. J. MIZEL. Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. *Bull. Amer. Math. Soc.*, 79(4), 790–795, 1973.

[MaM05] A. MAINIK and A. MIELKE. Existence results for energetic models for rate–independent systems. *Calc. Var. Part. Diff. Eqns.*, 22, 73–99, 2005.

[MaM09] A. MAINIK and A. MIELKE. Global existence for rate-independent gradient plasticity at finite strain. *J. Nonlinear Sci.*, 19(3), 221–248, 2009.

[MaZ92] O. MARTIO and W. P. ZIEMER. Lusin's condition (N) and mappings with nonnegative Jacobians. *Michigan Math. J.*, 39(3), 495–508, 1992.

[Mie02] A. MIELKE. Finite elastoplasticity, Lie groups and geodesics on SL($d$). In P. Newton, A. Weinstein, and P. J. Holmes, editors, *Geometry, Mechanics, and Dynamics*, pages 61–90. Springer–Verlag, New York, 2002.

[Mie03] A. MIELKE. Energetic formulation of multiplicative elasto–plasticity using dissipation distances. *Contin. Mech. Thermodyn.*, 15, 351–382, 2003.

[Mie05] A. MIELKE. Evolution in rate-independent systems (Ch. 6). In C. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, pages 461–559. Elsevier B.V., Amsterdam, 2005.

[Mie11] A. MIELKE. Differential, energetic, and metric formulations for rate-independent processes. In L. Ambrosio and G. Savaré, editors, *Nonlinear PDE's and Applications*, pages 87–170. Springer, 2011. (C.I.M.E. Summer School, Cetraro, Italy 2008, Lect. Notes Math. Vol. 2028).

[MiM06] A. MIELKE and S. MÜLLER. Lower semicontinuity and existence of minimizers for a functional in elastoplasticity. *ZAMM Z. angew. Math. Mech.*, 86(3), 233–250, 2006.

[MiR09] A. MIELKE and T. ROUBÍČEK. Numerical approaches to rate-independent processes and applications in inelasticity. *Math. Model. Numer. Anal. (M2AN)*, 43, 399–428, 2009.

[MiR15] A. MIELKE and T. ROUBÍČEK. *Rate-Independent Systems: Theory and Application.* Applied Mathematical Sciences, Vol. 193. Springer-Verlag New York, 2015.

[MiS92] C. MIEHE and E. STEIN. A canonical model of multiplicative elasto–plasticity. Formulation and aspects of numerical implementation. *Europ. J. Mech. A/Solids*, 11, 25–43, 1992.

[Mor74] J.-J. MOREAU. On unilateral constraints, friction and plasticity. In *New Variational Techniques in Mathematical Physics (Centro Internaz. Mat. Estivo (C.I.M.E.), II Ciclo, Bressanone, 1973)*, pages 171–322. Edizioni Cremonese, Rome, 1974.

[MRS08] A. MIELKE, T. ROUBÍČEK, and U. STEFANELLI. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Part. Diff. Eqns.*, 31, 387–416, 2008.

[MRZ10] A. Mielke, T. Roubíček, and J. Zeman. Complete damage in elastic and viscoelastic media and its energetics. *Comput. Methods Appl. Mech. Engrg.*, 199, 1242–1253, 2010.

[MTL02] A. Mielke, F. Theil, and V. I. Levitas. A variational formulation of rate–independent phase transformations using an extremum principle. *Arch. Rational Mech. Anal.*, 162, 137–177, 2002.

[Neg90] P. V. Negrón-Marrero. A numerical method for detecting singular minimizers of multidimensional problems in nonlinear elasticity. *Numer. Math.*, 58, 135144, 1990.

[NeW03] P. Neff and C. Wieners. Comparison of models for finite plasticity: a numerical study. *Comput. Vis. Sci.*, 6(1), 23–35, 2003.

[OrR99] M. Ortiz and E. Repetto. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids*, 47(2), 397–462, 1999.

[OrS99] M. Ortiz and L. Stainier. The variational formulation of viscoplastic constitutive updates. *Comput. Methods Appl. Mech. Engrg.*, 171(3-4), 419–444, 1999.

[Pod02] P. Podio-Guidugli. Contact interactions, stress, and material symmetry, for nonsimple elastic materials. *Theor. Appl. Mech.*, 28-29, 261–276, 2002.

[PoV10] P. Podio-Guidugli and M. Vianello. Hypertractions and hyperstresses convey the same mechanical information. *Contin. Mech. Thermodyn.*, 22, 163–176, 2010.

[Res67] Y. Reshetnyak. On the stability of conformal maps in multidimensional spaces. *Siberian Math. J.*, 8, 69–85, 1967.

[Rou91] T. Roubíček. A note on an interaction between penalization and discretization. In A. Kurzhanski and I. Lasiecka, editors, *Proceedings of IFIP-IIASA Conference Modelling and Inverse Problems of Control for Distributed Parameter Systems*, pages 145–150, Berlin, 1991. Springer.

[Rou15] T. Roubíček. Maximally-dissipative local solutions to rate-independent systems and application to damage and delamination problems. *Nonlin. Anal., Theory, Meth. Appl.*, 113, 33–50, 2015.

[SiH98] J. Simo and J. Hughes. *Computational Inelasticity.* Springer, Berlin, 1998.

[Šil85] M. Šilhavý. Phase transitions in non-simple bodies. *Arch. Rational Mech. Anal.*, 88, 135–161, 1985.

[SiO85] J. Simo and M. Ortiz. A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive relations. *Comput. Methods Appl. Mech. Engrg.*, 49, 221–245, 1985.

[Stu01] S. Stupkiewicz. Extension of the node-to-segment contact element for surface-expansion-dependent contact laws. *Int. J. Numer. Meth. Engr.*, 50, 739–759, 2001.

[Tem85] R. Temam. *Mathematical Problems in Plasticity.* Gauthier-Villars, Paris, 1985. (french original in 1983).

[Tou62] R. A. Toupin. Elastic materials with couple stresses. *Arch. Rational Mech. Anal.*, 11, 385–414, 1962.

[XuH11] X. Xu and D. Henao. An efficient numerical method for cavitation in nonlinear elasticity. *Math. Models Meth. Appl. Sci. ($M^3AS$)*, 21, 1733–1760, 2011.