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Existence, numerical convergence, and evolutionary relaxation for a rate-independent phase-transformation model

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Abstract

We revisit the two-well model for phase transformation in a linearly elastic body introduced and studied in [MTL02]. This energetic rate-independent model is posed in terms of the elastic displacement and an internal variable that gives the phase portion of the second phase. We use a new approach based on *mutual recovery sequences*, which are adjusted to a suitable energy increment plus the associated dissipated energy and, thus, enable us to pass to the limit in the construction of energetic solutions. We give three distinct constructions of mutual recovery sequences which allow us (i) to generalize the existence result in [MTL02], (ii) to establish the convergence of suitable numerical approximations via space-time discretization, and (iii) to perform the evolutionary relaxation from the pure-state model to the relaxed mixture model. All these results rely on weak converge and involve the H-measure as an essential tool.

1 Introduction

Microstructures occur in many material models and are important for macroscopic effects such as elastoplasticity or the hysteresis in shape-memory materials. On typical macroscopic and mesoscopic length scales, such materials are usually modeled by a strain tensor and some internal variables such as phase indicators, magnetization, plastic tensor, or hardening variables. In most cases, the stored-energy density depends only on the point values of these variables and thus defines a material model without any length scale. Thus, even steady states, which occur as minimizers of the energy, may develop microstructures on arbitrary fine scales. For static problems a rich theory was developed based on the seminal work [BaJ87], which introduced Young measures as an essential tool.

For evolutionary problems the situation is much less developed since the temporal behavior of such microstructures is significantly more difficult. For rate-independent systems, which do not have an intrinsic time-scale and hence are sufficiently close to static problems, a major step forward was done using incremental minimization problems, namely for finite-strain elastoplasticity in [OrR99, ORS00, CHM02], for brittle fracture in [FrM98, DaT02], and for shape-memory materials in [MiT99, MTL02, GMH02].

All these approaches have in common that they are based on incremental minimization problems for an energetic rate-independent systems (ERIS) $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, where $\mathcal{Q}$ is a (possibly nonlinear) state space, $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is the energy potential, and $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ is the dissipation distance, which measures the minimal energy needed to change the state from q to $\widetilde{q}$. Given an initial state $q_0 \in \mathcal{Q}$, the *approximate incremental minimization problem* then reads:

$$
\begin{aligned}
&\text{for } j = 1, ..., J \text{ find } q_j \text{ with} \\
&\mathcal{E}(j\tau, q_j) + \mathcal{D}(q_{j-1}, q_j) \leq \varepsilon\tau + \mathcal{E}(j\tau, \widehat{q}) + \mathcal{D}(q_{j-1}, \widehat{q}) \text{ for all } q \in \mathcal{Q},
\end{aligned}
\tag{1.1}
$$

where $\tau = T/J > 0$ is the timestep. Here the error level $\varepsilon = 0$ is allowed, if there exist minimizers of $\mathcal{E}(t, \cdot) + \mathcal{D}(q_{j-1}, \cdot)$. However, in many cases one has to take $\varepsilon > 0$, since no minimizer exists because of the formation of microstructures. Instead, for every $\varepsilon > 0$ there exists a solution q_j^ε.

Using a fixed initial condition q_0 the static theory can be employed to study the microstructure that arises in q_1^ε for $\varepsilon \to 0$, see e.g. [CDK13]. However, if one wants to study the microstructure in q_2^ε, there will be strong dependence on the microstructure of q_1^ε, and similarly q_j^ε strongly depends on q_{j-1}^ε. This problem gets even more involved, if we define the piecewise constant interpolants $q_{\tau,\varepsilon} : [0,T] \to \mathcal{Q}$ via

$$q_{\tau,\varepsilon}(t) = q_{j-1}^\varepsilon \text{ for } t \in [(j{-}1)\tau, j\tau[\text{ and } q_{\tau,\varepsilon}(T) = q_J^\varepsilon.$$

Then, the major mathematical task in *evolutionary relaxation* is to establish convergence of a suitable subsequence for τ_n, $\varepsilon_n \to 0$ to a limit $q : [0,T] \to \mathcal{Q}$ and to determine an evolution equation for all such limits.

For nonlinear material models without internal length scale this program is largely open. There are particular results for brittle fracture, see e.g. [DFT05, DaL10], in damage modeling [FrG06], and for a very particular plasticity model [CoT05]. The present work is a continuation of the two-phase model introduced in [MiT99, MTL02], where (i) we generalize the existence result for the separately relaxed problem postulated there, (ii) provide a numerical convergence result for space-time discretizations, and (iii) finally show that the above-mentioned evolutionary relaxation holds true, i.e. that all accumulation points of approximations $q_{\tau,\varepsilon}$ are indeed solutions of the separately relaxed model.

All these works lead to so-called *energetic solutions* (also called *quasistatic evolutions* in [DaT02, GaL09, DaL10]) for ERIS, see Definition 2.1. This notion of solutions is formulated in terms of a global stability condition (S) and an energy balance (E). The former simply means that the solution $q : [0,T] \to \mathcal{Q}$ satisfies

$$\forall\, t \in [0,T] : \quad q(t) \in \mathcal{S}(t) := \mathcal{S}^0(t),$$

where the (approximate) stability sets $\mathcal{S}_k^\alpha(t)$ for a ERIS $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ are defined via

$$\mathcal{S}_k^\alpha(t) := \big\{\, q \in \mathcal{Q} \,\big|\, \mathcal{E}_k(t,q) < \infty, \quad \forall\, \widehat{q} \in \mathcal{Q} :\ \mathcal{E}_k(t,q) \le \alpha + \mathcal{E}_k(t,\widehat{q}) + \mathcal{D}_k(q,\widehat{q}) \,\big\}.$$

A crucial step in the existence and Γ-convergence theory for ERIS (see Section 3.2) is the so-called closedness of the stability sets, in the following sense

$$\left.\begin{array}{c} \alpha_k \to 0, \quad t_k \to t_* \\ q_k \in \mathcal{S}_k^{\alpha_k}(t_k), \quad q_k \rightharpoonup q_* \end{array}\right\} \implies q_* \in \mathcal{S}_\infty^0(t),$$

where $\mathcal{S}_\infty^0$ refers to the limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. We say that the latter is the *separately relaxed ERIS* for the family $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ if $\mathcal{E}_k \xrightarrow{\Gamma} \mathcal{E}_\infty$ and $\mathcal{D}_k \xrightarrow{\Gamma} \mathcal{D}_\infty$ in a suitable topology on $\mathcal{Q}$. Yet, in general one cannot conclude that accumulation points q of energetic solutions $q_k : [0,T] \to \mathcal{Q}$ for $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ are energetic solutions for the limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$.

In this work we want to highlight that the method of *mutual recovery sequences* (MRS) (originally called "joint recovery sequences" in [MRS08]) is an ideal tool for existence and convergence theory for ERIS. This is a general abstract version of the jump-transfer or crack-transfer lemmas used in [FrL03, DFT05, DaL10]. It can be seen as an evolutionary counterpart to the classical limsup condition, or condition on the existence of recovery sequences, for static Γ convergence. However, here the condition is for a sequence of ERIS $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$, its supposed limiting system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$, a sequence of states $q_k \rightharpoonup q_*$, and an arbitrary test state $\widehat{q} \in \mathcal{Q}$. Throughout this work we assume that $\mathcal{Q}$ is a weakly or strongly closed subset of a reflexive Banach space $\mathbf{Q}$ and use $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively.

Definition 1.1 (Mutual recovery sequences (MRS)) *Given the ERIS* $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ *for* $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, *a sequence* $(t_k, q_k)_{k\in\mathbb{N}}$ *with* $t_k \to t_*$ *and* $q_k \rightharpoonup q_*$ *in* $\mathcal{Q}$, *and* $\widehat{q} \in \mathcal{Q}$, *a sequence* $(\widehat{q}_k)_{k\in\mathbb{N}}$ *is called a* MRS, *if* $\widehat{q}_k \rightharpoonup \widehat{q}$ *and*

$$\limsup_{k\to\infty} \Big(\mathcal{E}_k(t_k, \widehat{q}_k) + \mathcal{D}_k(q_k, \widehat{q}_k) - \mathcal{E}_k(t_k, q_k)\Big) \leq \mathcal{E}_\infty(t_*, \widehat{q}) + \mathcal{D}_\infty(q_*, \widehat{q}) - \mathcal{E}_\infty(t_*, q_*).$$

The importance here is that we have to recover mutual information on the energy increment $\mathcal{E}_\infty(t_*, \widehat{q}) - \mathcal{E}_\infty(t_*, q_*)$ and the dissipation $\mathcal{D}_\infty(q_*, \widehat{q})$ with the help of *one* sequence $(\widehat{q}_k)_k$. This is clearly distinct from separate relaxation, where there is no interaction between both quantities. In particular, this relates to the obvious fact that for an evolutionary theory we need a recovery condition that couples properties of the energy storage and the dissipation. Another instance of an explicit coupling occurs in EDP-convergence for generalized gradient systems $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ defined in [LM*15].

To highlight the major advantages of MRS, it is sufficient to look at the case $\mathcal{E}_k = \mathcal{E}$ and $\mathcal{D}_k = \mathcal{D}$ for $k \in \mathbb{N}_\infty$, since even for showing existence of energetic solutions for one ERIS, the concept of MRS is relevant and nontrivial. The simplest case occurs if $\mathcal{D}$ is weakly continuous and $\mathcal{E}(t, \cdot)$ is weakly lower semicontinuous; then we can always choose the constant MRS $\widehat{q}_k = \widehat{q}$, since $\mathcal{D}(q_k, \widehat{q}) \to \mathcal{D}(q_*, \widehat{q})$ and $\liminf_{k\to\infty} \mathcal{E}(t, q_k) \geq \mathcal{E}(t, q_*)$. There is a huge literature for nonlocal material models, where the energy is regularized by gradient terms or some nonlocal terms, while the dissipation remains local like $\mathcal{D}((u, z), (\widetilde{u}, \widetilde{z})) = \int_\Omega D(x, z(x), \widetilde{z}(x))\,\mathrm{d}x$, see [Tim09, MPP09, MaM09, MP*10, Han11, HHM12]. Indeed, if $\mathbf{Q} = \mathbf{U} \times \mathrm{W}^{s,q}(\Omega)$ for some $s > 0$ and $q > 1$, then weak continuity of $\mathcal{D}$ holds for Caratheodory functions D (because D has at most linear growth by the triangle inequality). However, in this case the theory of MRS is not really needed.

To see the cancelation effect in the definition of the MRS we consider a Hilbert space $\mathcal{Q} = \mathbf{Q}$, a quadratic energy $\mathcal{E}(t, q) = \frac{1}{2}\langle Aq, q\rangle - \langle \ell(t), q\rangle$, and a translation invariant dissipation distance $\mathcal{D}(q, \widetilde{q}) = \Psi(\widetilde{q} - q)$, which includes the case of classical linearized elasticity. Here, the MRS can be chosen as

$$\widehat{q}_k = q_k - q_* + \widehat{q} \qquad \text{implying } \widehat{q}_k \rightharpoonup \widehat{q} \text{ and } \mathcal{D}(q_k, \widehat{q}_k) = \Psi(\widehat{q} - q_*) = \mathcal{D}(q_*, \widehat{q}).$$

Moreover, using the quadratic structure of $\mathcal{E}(t, \cdot)$ we find

$$\begin{aligned} \mathcal{E}(t, \widehat{q}_k) - \mathcal{E}(t, q_k) &= \frac{1}{2}\big\langle A(\widehat{q} - q_*), \widehat{q}_k + q_k\big\rangle - \langle \ell(t), \widehat{q}_k - q_k\rangle \\ &\to \frac{1}{2}\big\langle A(\widehat{q} - q_*), \widehat{q} + q_*\big\rangle - \langle \ell(t), \widehat{q} - q_*\rangle \ = \ \mathcal{E}(t, \widehat{q}) - \mathcal{E}(t, q_*). \end{aligned} \tag{1.2}$$

Note that $\mathcal{E}(t, \widehat{q}_k) \to \mathcal{E}(t, \widehat{q})$ and $\mathcal{E}(t, q_k) \to \mathcal{E}(t, q_*)$ is false in general. Thus, the appropriate choice of $\widehat{q}_k$ leads to a cancelation, and we conclude that $\widehat{q}_k$ is indeed a MRS.

The full strength of the tool of MRS is seen in material modeling without internal length scale. There we are able to adjust the microstructure in $\widehat{q}_k$ suitably to recover the dissipation as well as the energy increment. Indeed, often (including this work) it is possible to find $\widehat{q}_k$ such that

$$\lim_{k\to\infty} \mathcal{D}_k(q_k, \widehat{q}_k) \ \to \ \mathcal{D}_\infty(q_*, \widehat{q}) \quad \text{and} \tag{1.3a}$$

$$\limsup_{k\to\infty} \big(\mathcal{E}_k(t, \widehat{q}_k) - \mathcal{E}(t, q_k)\big) \ \leq \ \mathcal{E}_\infty(t, \widehat{q}) - \mathcal{E}(t, q_*). \tag{1.3b}$$

After we recall some of the modeling for N-phase materials in Section 2, we concentrate on the special two-phase model of [MTL02], which relies on the relaxed two-well energy derived in [Koh91]. Here $\theta : \Omega \to [0,1]$ denotes the mesoscopic volume fraction of phase 2, and $\widetilde{u} = g_{\mathrm{Dir}}{+}u : \Omega \to \mathbb{R}^d$ is the displacement. Thus the states are $q = (u,\theta) \in \mathcal{Q} = \mathbf{U} \times \mathcal{Z}$ with

$$\mathbf{U} := \big\{\, u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \,\big|\, u|_{\Gamma_{\mathrm{Dir}}} = 0 \,\big\} \text{ and } \mathcal{Z} = \big\{\, \theta \in \mathrm{L}^2(\Omega) \,\big|\, \theta(x) \in [0,1] \text{ a.e.} \,\big\}.$$

The particular case has the special structure that $\mathcal{E}(t,\cdot)$ is quadratic, namely

$$\mathcal{E}(t,u,\theta) = \int_\Omega \frac{1}{2}\Big\langle \mathbb{A}\begin{pmatrix}\mathbf{e}(u)\\ \theta\end{pmatrix}, \begin{pmatrix}\mathbf{e}(u)\\ \theta\end{pmatrix}\Big\rangle \,\mathrm{d}x - \Big\langle l(t), \begin{pmatrix}u\\ \theta\end{pmatrix}\Big\rangle,$$

where $\mathbf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the linearized strain tensor, and $\mathbb{A}$ is a symmetric linear operator. The dissipation distance has the form

$$\mathcal{D}(\theta,\widetilde{\theta}) = \int_\Omega \max\big\{\kappa_{1\to 2}(\widetilde{\theta}{-}\theta)\,,\, \kappa_{2\to 1}(\theta{-}\widetilde{\theta})\big\}\,\mathrm{d}x,$$

where $\kappa_{1\to 2}$ and $\kappa_{2\to 1}$ are positive material constants.

Because of the constraint $\theta \in [0,1]$, the quadratic trick in (1.2) cannot be used to construct MRS. However, it is shown in Proposition 3.3 that

$$\widehat{\theta}_n(x) = \widehat{\theta}(x) + g(x)\big(\theta_n(x){-}\theta_*(x)\big) \tag{1.4}$$

for a suitable $g \in \mathrm{L}^\infty(\Omega;[0,1])$ depending nonlinearly on θ_* and $\widehat{\theta}$ defines a MRS satisfying (1.3). Indeed, the choice of g gives $\mathrm{sign}(\widehat{\theta}_n{-}\theta_n) = \mathrm{sign}(\widehat{\theta}{-}\theta_*)$, and (1.3a) follows by the affine structure in (1.4).

To control the energy difference we exploit the quadratic structure of the energy and the property that the material model is scale invariant. As a consequence the reduced energy

$$\mathcal{I}(t,\theta) := \min\big\{\, \mathcal{E}(t,u,\theta) \,\big|\, u \in \mathbf{U} \,\big\} = \frac{1}{2}\langle \mathbb{L}\theta,\theta\rangle + \langle \beta(t),\theta\rangle + \alpha(t)$$

is defined by a symmetric bounded linear operator $\mathbb{L}$ that is a pseudo-differential operator with non-negative symbol Λ satisfying $\Lambda(r\xi) = \Lambda(\xi)$ for all $r > 0$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. Thus, as was already done in [MTL02, The02] the H-measure theory can be employed. In particular, if θ_n generates the H-measure μ, then $\widehat{\theta}_n$ generates the H-measure $g^2\mu$, and we find

$$\mathcal{I}(t,\widehat{\theta}_n) - \mathcal{I}(t,\theta_n) \;\to\; \mathcal{I}(t,\widehat{\theta}) - \mathcal{I}(t,\theta_*) + \int_\Omega\int_{\mathbb{S}^{d-1}} (g^2{-}1)\,\mathrm{d}\mu(x,\omega).$$

Using $g^2 \leq 1$ and $\mu \geq 0$ gives the desired estimate (1.3b), and $(\widehat{\theta}_n)_n$ is a MRS. This provides the major step in the existence of energetic solutions for the two-phase model, see Theorem 3.1.

In Section 4 we generalize the theory by approximating the spaces $\mathbf{U}$ and $\mathcal{Z}$ by suitable finite-element spaces $\mathbf{U}_k \subset \mathbf{U}_{k+1}$ and $\mathcal{Z}_k \subset \mathcal{Z}_{k+1}$. We provide conditions that all accumulation points of the corresponding approximate minimizers $q_{\tau,k} : [0,T] \to \mathcal{Q}_k \subset \mathcal{Q}$ are indeed solutions for the limiting ERIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$. The MRS is obtained by suitably projecting the sequence defined in (1.4).

The final Section 5 solves the question of *evolutionary relaxation*. We start from the microscopic pure-phase model where θ is restricted to be either 0 or 1, i.e.

$$\theta \in \mathcal{P} := \big\{\, \theta \in \mathrm{L}^2(\Omega) \,\big|\, \theta(x) \in \{0,1\} \text{ for a.a. } x \in \Omega \,\big\} \quad \text{and} \quad \mathcal{Q}^{\mathrm{pure}} = \mathbf{U} \times \mathcal{P}.$$

In terms of the above theory we set $\mathcal{E}_k(t,u,\theta) = \mathcal{E}(t,u,\theta)$ on $\mathcal{Q}^{\rm pure}$ and $\mathcal{E}_k = +\infty$ otherwise. In [The02] it was shown that the "separately relaxed" ERIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is a lower relaxation of $(\mathcal{Q}^{\rm pure},\mathcal{E}_k,\mathcal{D})$ in the sense of [Mie04]. This means that each energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{D})$ can be approximated by solutions of the approximate incremental minimization problem (1.1), but now using the state space $\mathcal{Q}^{\rm pure}$.

Our Theorem 5.1 shows that all accumulation points q of approximate solutions $q_{\tau,\varepsilon}$ are indeed energetic solutions for the ERIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$. Thus, we conclude that the lower relaxation is also an upper relaxation in the sense of [Mie04]. This reveals that the two-phase model under consideration is very special. In general, one should not expect that the separate relaxation is also an upper or a lower relaxation. This can only happen if the macroscopic information kept in the relaxation (here the phase fraction θ) is enough to characterize *all* relevant macroscopic quantities. In [MTL02, The02] it was shown that simple laminates are sufficient to study the separate and the lower relaxation. Interestingly, our method solves the question of upper relaxation even in cases where there are microstructures that are not laminates.

The difficulty in the construction of MRS lies in the fact that $\theta_n(t) \in \mathcal{P}$, while the weak limit $\theta_* \in \mathcal{Z}$ in general. Similarly, for general test functions $\widehat{\theta} \in \mathcal{Z}$ we have to find $\widehat{\theta}_n \in \mathcal{P}$ with $\widehat{\theta}_n \to \widehat{\theta}$. This will be done by constructing hierarchical microstructures based on θ_n and much finer laminates with normal direction ω_* such that $\Lambda(\omega_*) = 0$, see Proposition 5.2 and Figure 1.

2 Pure and relaxed N-phase models

We start with general N-phase models and then restrict to the two-phase model as discussed in [MTL02], where also a detailed physical motivation in terms of separate relaxation is given. We also refer to [CaP01, The02].

2.1 A microscopic model with pure phases

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, where $\Gamma_{\rm Dir} \subset \partial\Omega$ with $\int_{\Gamma_{\rm Dir}} 1\,{\rm d}a > 0$ is the part of the boundary on which displacement (Dirichlet) boundary conditions are applied. The displacement $\widetilde{u}(t)$ will be of the form $g_{\rm Dir}(t) + u(t)$, where u lies in the fixed space

$$\mathbf{U} := \big\{\, u \in {\rm H}^1(\Omega;\mathbb{R}^d) \;\big|\; u|_{\Gamma_{\rm Dir}} = 0 \,\big\}.$$

In the case of N pure phases we consider N different stored-energy densities

$$W_i(E) = \frac{1}{2}(E{-}A_i) : \mathbb{C}_i(E{-}A_i) + \beta_i, \quad i = 1,\ldots,N,$$

where $E = \mathbf{e}(u) = \frac{1}{2}(\nabla u{+}\nabla u^\top)$ denotes the linearized elastic strain, $\mathbb{C}_i$ is the elastic tensor of the ith phase, A_i is the transformation strain, and β_i the height of the ith well. All these quantities may depend on temperature, but we consider an isothermal setting.

For later purposes we associate the ith phase with the ith unit vector $e_i \in \mathbb{R}^N$ and call the functions $z \in \mathcal{P}_N$ a phase-indicator field, where

$$\mathcal{P}_N := \big\{\, z \in {\rm L}^2(\Omega;\mathbb{R}^N) \;\big|\; z(x) \in P_N \ \text{ a.e. in } \Omega \,\big\} \text{ with } P_N := \big\{\, e_i \in \mathbb{R}^N \;\big|\; i = 1,\ldots,N \,\big\}.$$

For characterizing a simple evolutionary model, we add a dissipation distance $d_N : P_N \times P_n \to [0,\infty[$, where $\kappa_{i\to j} := d_N(e_i,e_j)$ denotes the energy per unit volume that is dissipated when a phase transformation from i to j takes place. The induced dissipation distance $\mathcal{D}_N$ on $\mathcal{P}_N$ is defined via

$$\mathcal{D}_N(z_0, z_1) := \int_\Omega d_N(z_0(x), z_1(x))\,\mathrm{d}x.$$

The associated energetic rate-independent system (ERIS) $(\mathcal{Q}_N, \mathcal{E}_N, \mathcal{D}_N)$ for the pure N-phase model is given via the state space $\mathcal{Q}_N := \mathbf{U} \times \mathcal{P}_N$, the dissipation distance $\mathcal{D}_N$ from above, and the energy-storage functional

$$\mathcal{E}_N(t, u, z) := \int_\Omega \mathbb{W}(\mathbf{e}(g_{\mathrm{Dir}}(t)+u)(x), z(x))\,\mathrm{d}x - \langle \ell(t), u\rangle,$$

where $\mathbb{W}(E, e_i) = W_i(E)$ for $i = 1, ..., N$, and $\ell : [0, T] \to \mathbf{U}^*$ includes possible time-dependent volume or surface loadings. In particular, we assume

$$\begin{aligned} &g_{\mathrm{Dir}} \in \mathrm{C}^1\big([0, T]; \mathrm{H}^1(\Omega; \mathbb{R}^d)\big), \quad \ell \in \mathrm{C}^1([0, T]; \mathbf{U}^*), \\ &\text{with } \langle \ell(t), u\rangle = \int_\Omega f_{\mathrm{vol}}(t) \cdot u\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_{\mathrm{Dir}}} f_{\mathrm{surf}} \cdot u\,\mathrm{d}a. \end{aligned} \tag{2.1}$$

2.2 Incremental minimization and energetic solutions

Following the seminal work [OrR99, ORS00] it was suggested in [MiT99, MTL02] to consider incremental minimization problems for ERIS $(\mathcal{Q}_N, \mathcal{E}_N, \mathcal{D}_N)$ for a given time discretization which we take equidistant for simplicity, i.e. $\tau = T/J$ with $J \in \mathbb{N}$. For an initial state $q_0 = (u_0, z_0) \in \mathcal{Q}_N$ we consider approximate minimizers $q_j^{\tau,\varepsilon} \in \mathcal{Q}_N$ satisfying

$$\left.\begin{array}{l} \text{Given } q_0 \in \mathcal{Q}_N, \text{ find iteratively } q_j^{\tau,\varepsilon},\ j = 1, ..., J, \text{ such that} \\ \mathcal{E}_N(j\tau, q_j^{\tau,\varepsilon}) + \mathcal{D}_N(q_{j-1}^{\tau,\varepsilon}, q_j^{\tau,\varepsilon}) \le \varepsilon\tau + \mathcal{E}_N(j\tau, \widehat{q}) + \mathcal{D}_N(q_{j-1}^{\tau,\varepsilon}, \widehat{q}) \text{ for all } \widehat{q} \in \mathcal{Q}_N. \end{array}\right\} \tag{2.2}$$

For positive ε such approximate minimizers always exist, and we can define piecewise constant interpolants $q_{\tau,\varepsilon} : [0, T] \to \mathcal{Q}_N$ via

$$q_{\tau,\varepsilon}(t) = q_{j-1}^{\tau;\varepsilon} \text{ for } t \in [(j{-}1)\tau, j\tau[\text{ with } j = 1, \ldots, J \ \text{ and } \ q_{\tau,\varepsilon}(T) = q_J^{\tau;\varepsilon}. \tag{2.3}$$

For $\varepsilon = 0$ one asks for existence of true minimizers, which in the present, non-relaxed case is not to be expected in general.

The major task is now the characterization of all possible limits, i.e. accumulation points, of $q_{\tau,\varepsilon}$ for $(\tau, \varepsilon) \to (0, 0)$ and to derive a suitable evolutionary model (e.g. in the sense of [Mie04]) having these limits as solutions. In general this task is still much too difficult; however, we will see in Section 5 that it is solvable for the two-phase model (i.e. $N = 2$) with $\mathbb{C}_1 = \mathbb{C}_2$.

The main achievement in [MiT99, MTL02] was the observation of the general fact that all possible limits of the above approximate incremental minimization problem lead to so-called *energetic solutions* for rate-independent systems.

Definition 2.1 (Energetic solutions) *A function $q : [0, T] \to \mathcal{Q}$ is called an energetic solution of the ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, if $t \mapsto \partial_t\mathcal{E}(t, q(t)$ lies in $\mathrm{L}^1([0, T])$ and if for all $t \in [0, T]$ the stability (S) and the energy balance (E) hold:*

$$\begin{aligned} &\textit{(S)} \quad \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \widehat{q}) + \mathcal{D}(q, \widehat{q}) \textit{ for all } \widehat{q} \in \mathcal{Q}; \\ &\textit{(E)} \quad \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_s\mathcal{E}(s, q(s))\,\mathrm{d}s, \end{aligned} \tag{2.4}$$

where the dissipation $\mathrm{Diss}_{\mathcal{D}}(q,[0,t])$ *is defined as the supremum over all partitions* $0 \leq t_0 < t_1 < \cdots < t_{N-1} < t_N \leq t$ *and all* $N \in \mathbb{N}$ *of the sums* $\sum_{i=1}^{N} \mathcal{D}(q(t_{i-1}), q(t_i))$.

We will see in Section 3.2 how under natural conditions the stability and energy balance arise naturally from the incremental minimization problem. However, in the present pure-phase model this does not work, since we have to pass to limits in $q_{\tau_n,\varepsilon_n}(t)$ without any compactness. Thus, we have to work on the weak completion of $\mathcal{Q}_N$. This leads to so-called relaxed models.

2.3 A separately relaxed N-phase model

Instead of treating the microscopic phase indicators z with $z(x) \in P_N$ we may consider a mixture theory on the mesoscopic level, where z is taking values in the Gibbs simplex

$$G_N := \operatorname{conv} P_N = \{\, z = (z_1, .., z_N) \in [0,1]^N \mid \sum_{n=1}^{N} z_n = 1 \,\}.$$

Here $z_i(x) \in [0,1]$ denotes the volume fractions of the ith phase at a mesoscopic material point $x \in \Omega$. With this, we introduce the relaxed state space

$$\mathcal{Q}_N^{\mathrm{rlx}} := \mathbf{U} \times \mathcal{G}_N \text{ with } \mathcal{G}_N := \big\{\, z \in \mathrm{L}^2(\Omega;\mathbb{R}^N) \,\big|\, z(x) \in G_N \text{ a.e. in } \Omega \,\big\}.$$

Extending $\mathcal{E}_N(t,\cdot) : \mathcal{Q}_N \to \mathbb{R}$ to $\mathcal{E}_N(t,\cdot) : \mathcal{Q}_N^{\mathrm{rlx}} \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ by $+\infty$ outside of $\mathcal{Q}_N$, we can define the lower semicontinuous envelope $\mathcal{E}_N^{\mathrm{rlx}}(t,\cdot) : \mathcal{Q}_N^{\mathrm{rlx}} \to \mathbb{R}_\infty$, which is called the (static) relaxation of $\mathcal{E}_N(t,\cdot)$. It has the form

$$\mathcal{E}_N^{\mathrm{rlx}}(t,u,z) = \int_\Omega \mathbb{W}^{\mathrm{rlx}}\big(\mathbf{e}(g_{\mathrm{Dir}}(t){+}u)(x), z(x)\big)\,\mathrm{d}x - \langle \ell(t), u\rangle,$$

where the relaxed stored-energy density is given in terms of the cross-quasiconvexification

$$\begin{aligned} &\mathbb{W}^{\mathrm{rlx}}(E,Z) := \inf\Big\{ \int_{[0,1]^d} \mathbb{W}(E{+}\mathbf{e}(\widetilde{u})(y), \widetilde{z}(y))\,\mathrm{d}y \,\Big|\, \widetilde{u} \in \mathrm{H}^1_{\mathrm{per}}((0,1)^d),\ \widetilde{z} \in \mathfrak{D}(Z) \Big\},\\ &\text{where } \mathfrak{D}(Z) := \big\{\, z : (0,1)^d \to P_N \,\big|\, \textstyle\int_{(0,1)^d} z(y)\,\mathrm{d}y = Z \,\big\}, \end{aligned}$$

see [MTL02, Eqn. (4.5)]. For $\mathbb{C}_i = \mathbb{C}_1$ see also [Mie00, GMH02].

Similarly, by an optimal transport problem based on the weight $d_N : P_N \times P_N \to [0,\infty[$, one can define a dissipation distance $D_N : G_N \times G_N \to [0,\infty[$ which takes the form $D_N(z,\widetilde{z}) = \Psi_N(\widetilde{z}{-}z)$ for a 1-homogeneous function $\Psi_N : \mathbb{R}^n \to [0,\infty[$ (i.e. $\Psi_N(\gamma v) = \gamma\Psi_N(v)$ for all $\gamma \geq 0$ and $v \in \mathbb{R}^N$), see [MTL02, Sect. 4.3]. This leads to the relaxed dissipation distance $\mathcal{D}_N^{\mathrm{rlx}} : \mathcal{G}_N \times \mathcal{G}_n \to [0,\infty[$ defined via

$$\mathcal{D}_N^{\mathrm{rlx}}(z,\widetilde{z}) := \int_\Omega \Psi_N\big(\widetilde{z}(x){-}z(x)\big)\,\mathrm{d}x,$$

and the so-called *separately relaxed ERIS* $(\mathcal{Q}_N^{\mathrm{rlx}}, \mathcal{E}_N^{\mathrm{rlx}}, \mathcal{D}_N^{\mathrm{rlx}})$.

So far, the existence of energetic solutions for such relaxed systems is still open. However, we gained already that the incremental problem (2.2) has solutions for $\varepsilon = 0$. Indeed, since $\mathcal{D}_N(z,\cdot)$ is convex and continuous, it is weakly lower semicontinuous.

Moreover, $\mathcal{E}_N^{\rm rlx}(t,\cdot)$ is weakly lower semicontinuous, because of its construction as a cross-quasiconvexification, cf. [FKP94]. We refer to [BC*04, GHH07, BaJ87] for such static relaxations in the context of material modeling.

However, for passing to the limit of timesteps $\tau \to 0$ it remains open how to show the closedness of the stability sets in the weak topology. Nevertheless there is some hope that energetic solutions for $(\mathcal{Q}_N^{\rm rlx}, \mathcal{E}_N^{\rm rlx}, \mathcal{D}_N^{\rm rlx})$ exist. Unfortunately we are only able to show that this is true for the case $N = 2$ if $\mathbb{C}_1 = \mathbb{C}_2$, see Theorem 3.1. We emphasize that showing existence of energetic solution for the ERIS $(\mathcal{Q}_N^{\rm rlx}, \mathcal{E}_N^{\rm rlx}, \mathcal{D}_N^{\rm rlx})$ is a first step only.

The more important step is to show that accumulation points of approximate solutions $q_{\tau,\varepsilon} : [0,T] \to \mathcal{Q}_N$ of the microscopic pure-state system are indeed solutions of the mesoscopic relaxed model $(\mathcal{Q}_N^{\rm rlx}, \mathcal{E}_N^{\rm rlx}, \mathcal{D}_N^{\rm rlx})$. We expect that this is typically not the case. The point is that the relaxed model only takes into account the mesoscopic volume fractions $z_i(t,x) \in [0,1]$ of the phases $i = 1,...,N$. However, in general situations it is necessary to take into account the type of the microstructures. For instance a rotating laminate may have constant volume fraction, but must dissipate microscopically, see the discussions in [FrG06, GaL09, KoH11, HHM12]. It is surprising that we are able to prove the evolutionary relaxation property in the two-phase case with $\mathbb{C}_1 = \mathbb{C}_2$, see Section 5. The main idea here follows an observation in [The02], where it is shown that looking at suitable laminates with a fixed normal is sufficient, even though other microstructures may occur.

3 Existence for the relaxed two-phase model

In this section we provide the first existence result for the two-phase problem. Moreover, we introduce the general theory of [Mie11, MiR15] for establishing convergence of approximations obtained from incremental minimization procedures. The major step is the proof of the weak closedness of the stability sets, which will be treated afterwards. For this we employ H-measures which are well adapted for the treatment of the quadratic energies occurring in the two-phase problem.

3.1 Setup and existence result

For the rest of this work we restrict to the case of $N = 2$ phases and use the scalar $\theta \in [0,1]$ as the volume fraction of phase $i = 2$, i.e.

$$z = \big(1{-}\theta\,,\theta\big) \in G_2 = \mathrm{conv}\{e_1, e_2\}.$$

Moreover, we assume $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}$, where $\mathbb{C}$ is the symmetric and positive definite elasticity tensor. We also write

$$\mathcal{Q} := \mathbf{U} \times \mathcal{Z} \quad \text{with } \mathbf{U} := \{\, u \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \mid u|_{\Gamma_{\rm Dir}} = 0 \,\} \text{ and } \mathcal{Z} := \mathrm{L}^2(\Omega;[0,1]),$$

equipped with the weak topology. The relaxed energy is defined on $\mathcal{Q}$ and reads

$$\mathcal{E}(t,u,\theta) = \int_\Omega W(\mathbf{e}(g_{\rm Dir}(t){+}u),\theta)\,\mathrm{d}x - \langle \ell(t), u\rangle, \quad \text{where} \tag{3.1}$$
$$W(E,\theta) = \frac{1-\theta}{2}|E{-}A_1|^2_{\mathbb{C}} + \frac{\theta}{2}|E{-}A_2|^2_{\mathbb{C}} - \frac{\gamma}{2}\theta(1{-}\theta) + \beta_1(1{-}\theta) + \beta_2\theta.$$

Here $|E|^2_{\mathbb{C}} := E{:}\mathbb{C}E = \sum_{i,j,k,l=1}^d E_{ij}\mathbb{C}_{ijkl}E_{kl}$, and A_i is the transformation strain of the ith phase. According to [Mie00, MTL02, GMH02] the constant γ is determined by Kohn's relaxation result, see [Koh91], for the elastic double-well problem, that is

$$\gamma := \max\{\, \Sigma(\omega) \mid \omega \in \mathbb{S}^{d-1} \,\} \text{ with } \Sigma(\omega) := \mathbb{C}[A]\omega \cdot \mathbb{A}(\omega)^{-1}\big(\mathbb{C}[A]\omega\big), \tag{3.2}$$

where $A := A_2 - A_1$, and the acoustic tensor $\mathbb{A}(\omega) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is defined via

$$b \cdot \big(\mathbb{A}(\omega)b\big) := \frac{1}{4}\big| b{\otimes}\omega + \omega{\otimes}b \big|^2_{\mathbb{C}} \text{ for all } b \in \mathbb{R}^d.$$

Using the two positive thresholds $\kappa_{1\to2}$ and $\kappa_{2\to1}$ the dissipation distance $\mathcal{D}$ reads

$$\mathcal{D}(\theta,\widetilde{\theta}) = \int_\Omega \psi\big(\widetilde{\theta}(x) - \theta(x)\big)\,\mathrm{d}x \quad \text{with } \psi(a) = \begin{cases} \kappa_{1\to2}\, a & \text{for } a \geq 0, \\ \kappa_{2\to1}|a| & \text{for } a \leq 0. \end{cases} \tag{3.3}$$

Thus, the ERIS $(\mathbf{Q}, \mathcal{E}, \mathcal{D})$ is specified, and we can define the stability sets

$$\mathcal{S}(t) := \big\{\, q \in \mathcal{Q} \,\big|\, \mathcal{E}(t,q) < \infty,\ \forall\, \widehat{q} \in \mathcal{Q}:\ \mathcal{E}(t,q) \leq \mathcal{E}(t,\widehat{q}) + \mathcal{D}(q,\widehat{q}) \,\big\} \text{ for } t \in [0,T].$$

Note that the relaxed energy $\mathcal{E}$ and dissipation $\mathcal{D}$ defined above correspond to $\mathcal{E}^{\mathrm{rlx}}_N$ and $\mathcal{D}^{\mathrm{rlx}}_N$ for $N = 2$ from Section 2.3.

Theorem 3.1 (Energetic solutions) *Under the above assumptions the two-phase model* $(\mathbf{Q}, \mathcal{E}, \mathcal{D})$ *has an energetic solution for all initial conditions* $q(0) \in \mathcal{S}(0)$. *Moreover, every accumulation point* $q : [0,T] \to \mathcal{Q}$ *of the approximations* $q_{\tau,\varepsilon} : [0,T] \to \mathcal{Q}$ *for* $(\tau,\varepsilon) \to (0,0)$ *obtained from the approximate incremental minimization problem* (2.2) *is an energetic solution.*

We mention that the theory in [MiR09] also shows that all energetic solutions are accumulation points of approximations obtained via a slight variant of (2.2), see also [Rin09, Sect. 4.2].

3.2 General strategy for the convergence proof

Here we give the general strategy of constructing energetic solutions that was developed in [MTL02, DFT05]. We follow the six steps as introduced in [FrM06] and [MiR15, Sect. 2.1.6]; but in the present model many features are much simpler, since we can use the quadratic structure of the energy and the weak sequential compactness of the space $\mathcal{Z} = \mathrm{L}^2(\Omega;[0,1])$. Step 3 will rely on the existence of mutual recovery sequences (MRS), which is established in Section 3.4.

Step 0: Construction of approximate solutions. For every timestep $\tau = T/J$ and any $\varepsilon \geq 0$ and the given initial value $q_0 = q(0)$ the approximate incremental problem (2.2) has solutions $q^{\tau,\varepsilon}_j$, $j = 1, \ldots, J$. For $\varepsilon > 0$ this is indeed trivial, while for $\varepsilon = 0$ we can use the weak lower semicontinuity of $q \mapsto \mathcal{E}(j\tau, q) + \mathcal{D}(q^{\tau,\varepsilon}_{j-1}, q)$. Thus, the piecewise constant interpolants $q_{\tau,\varepsilon} : [0,T] \to \mathcal{Q}$ are well defined.

Step 1: A priori estimates. Since $\mathcal{Z}$ lies in a bounded ball of radius $R = |\Omega|^{1/2}$ in $\mathbf{Z} := \mathrm{L}^2(\Omega)$ we always have

$$\forall\, \tau,\varepsilon: \quad \|\theta_{\tau,\varepsilon}(\cdot)\|_{\mathrm{L}^\infty(0,T;\mathbf{Z})} \leq R.$$

Owing to $\mathcal{E}(j\tau, u_j^{\tau,\varepsilon}, \theta_j^{\tau,\varepsilon}) \leq \varepsilon\tau + \mathcal{E}(j\tau, \widehat{u}, \theta_j^{\tau,\varepsilon})$ for $j = 0, \ldots, J$, the quadratic structure of $\mathcal{E}(t, \cdot, \theta)$ together with Korn's inequality show that there is a constant $C_1 > 0$ such that

$$\forall\, \tau, \varepsilon : \quad \|u_{\tau,\varepsilon}(\cdot)\|_{\mathrm{L}^\infty(0,T;\mathbf{U})} \leq C_1.$$

Finally, we may insert $\widehat{q} = q_{j-1}^{\tau,\varepsilon}$ into (2.2) and sum over $j = 1, \ldots, J$ to find

$$\begin{aligned} \mathrm{Diss}_{\mathcal{D}}(q_{\tau,\varepsilon}, [0,T]) \;=\; \sum_{j=1}^{J} \mathcal{D}(q_{j-1}^{\tau,\varepsilon}, q_j^{\tau,\varepsilon}) \;\leq\; \varepsilon T + \sum_{j=1}^{J} \mathcal{E}(j\tau, q_{j-1}^{\tau,\varepsilon}) - \mathcal{E}((j-1)\tau, q_{j-1}^{\tau,\varepsilon}) \\ \leq \varepsilon T + \int_0^T C\big(\|\dot{g}_{\mathrm{Dir}}(t)\|_{\mathrm{H}^1(\Omega)} + \|\dot{\ell}(t)\|_{\mathbf{U}^*}\big)\,\mathrm{d}t \;\leq\; C_2, \end{aligned} \tag{3.4}$$

independently of $\varepsilon \in [0,1]$ and $\tau = T/J$. This estimate doesn't give any information on $u_{\tau;\varepsilon}$, but with $\kappa_* = \min\{\kappa_{1\to2}, \kappa_{2\to1}\} > 0$ we find

$$\mathrm{Diss}_{\|\cdot\|_1}(\theta_{\tau,\varepsilon}, [0,T]) = \mathrm{Var}\big(\theta_{\tau,\varepsilon}; \mathrm{L}^1(\Omega)\big) \leq C_2/\kappa_*.$$

Step 2: Selection of convergent subsequences. Because of the uniform total variation bound for $\theta_{\tau,\varepsilon}$ we can apply the abstract version of Helly's selection principle. Hence, for every sequence $((\tau_k, \varepsilon_k))_{k\in\mathbb{N}}$ with $\tau_k, \varepsilon_k \to 0$ for $k \to \infty$ there exists a subsequence $(\tau_{k_n}, \varepsilon_{k_n})$ with $k_n \to \infty$ and a function $\theta : [0,T] \to \mathcal{Z}$ such that

$$\forall\, t \in [0,T] : \quad \theta_{\tau_{k_n},\varepsilon_{k_n}}(t) \rightharpoonup \theta(t) \text{ and } \mathrm{Diss}_{\mathcal{D}}(\theta, [0,t]) \leq \liminf_{n\to\infty} \mathrm{Diss}_{\mathcal{D}}(\theta_{\tau_{k_n},\varepsilon_{k_n}}, [0,t]). \tag{3.5}$$

Define the function $u : [0,T] \to \mathbf{U}$ to be the unique minimizer of $\mathcal{E}(t, \cdot, \theta(t))$, then it is easy to show that $u_{\tau_{k_n},\varepsilon_{k_n}} \rightharpoonup u(t)$ for all t. Thus, we conclude the convergence along the whole subsequence, namely

$$\forall\, t \in [0,T] : \quad q_{\tau_{k_n},\varepsilon_{k_n}} \rightharpoonup q(t) = (u(t), \theta(t)) \text{ in } \mathbf{U} \times \mathbf{Z}.$$

Step 3: Stability of the limit. The most difficult step in the proof is to show that the accumulation point $q : [0,T] \to \mathcal{Q}$ is stable in the sense of (S) in Definition 2.1, i.e. $q(t) \in \mathcal{S}(t)$. For this we first show that $q_j^{\tau,\varepsilon}$ is approximately stable for time $t = j\tau$, which follows by the triangle inequality for $\mathcal{D}$ as follows. Indeed for all $\widehat{q} \in \mathcal{Q}$ we have

$$\begin{aligned} \mathcal{E}(j\tau, q_j^{\tau,\varepsilon}) &\overset{(2.2)}{\leq} \varepsilon\tau + \mathcal{E}(j\tau, \widehat{q}) + \mathcal{D}(q_{j-1}^{\tau,\varepsilon}, \widehat{q}) - \mathcal{D}(q_{j-1}^{\tau,\varepsilon}, q_j^{\tau,\varepsilon}) \\ &\overset{\text{triangle}}{\leq} \varepsilon\tau + \mathcal{E}(j\tau, \widehat{q}) + \mathcal{D}(q_j^{\tau,\varepsilon}, \widehat{q}), \end{aligned} \tag{3.6}$$

which also will be abbreviated by $q_j^{\tau,\varepsilon} \in \mathcal{S}^{\varepsilon\tau}(j\tau)$.

In order to establish the stability $q(t) \in \mathcal{S}(t)$, we want to pass to the limit along the sequence $(\tau_{k_n}, \varepsilon_{k_n}) \to (0,0)$ by choosing suitable test functions $\widehat{q} = \widehat{q}^{\tau,\varepsilon}$ in the above estimate. The crucial point is to find a MRS $\widehat{q}_n$ such that $\widehat{q}_n \rightharpoonup \widehat{q}$ and

$$\limsup_{n\to\infty} \Big(\mathcal{E}(t, \widehat{q}_n) + \mathcal{D}(q^{\tau_{k_n},\varepsilon_{k_n}}(t), \widehat{q}_n) - \mathcal{E}(t, q^{\tau_{k_n},\varepsilon_{k_n}}(t)) \Big) \leq \mathcal{E}(t, \widehat{q}) + \mathcal{D}(q(t), \widehat{q}) - \mathcal{E}(t, q(t)).$$

This step will be discussed explicitly in the three results of the sections "Mutual recovery sequences I to III" below. Using $j = \widehat{j}_n(t) := \lfloor t/\tau_{k_n} \rfloor \in \mathbb{N}_0$ such that $q_{\tau_{k_n},\varepsilon_{k_n}}(t) = q_{\widehat{j}_n(t)}^{\tau_{k_n},\varepsilon_{k_n}}$ and inserting the MRS into (3.6) yields

$$\begin{aligned} 0 = \lim_{n\to\infty} \big(-\varepsilon_{k_n}\tau_{k_n}\big) \;&\leq\; \limsup_{n\to\infty} \Big(\mathcal{E}(t, \widehat{q}_n) + \mathcal{D}(q_{\tau_{k_n},\varepsilon_{k_n}}(t), \widehat{q}_n) - \mathcal{E}(t, q_{\tau_{k_n},\varepsilon_{k_n}}(t)) \Big) \\ &\leq \mathcal{E}(t, \widehat{q}) + \mathcal{D}(q(t), \widehat{q}) - \mathcal{E}(t, q(t)), \end{aligned}$$

where we used $|t-\tau_{k_n}\widehat{\jmath}_n(t)| \leq \tau_{k_n} \to 0$ and (2.1). This is the desired stability $q(t) \in \mathcal{S}(t)$.

Step 4: Upper energy estimate. We return to the dissipation estimate (3.4) in Step 1, which can be written as

$$\mathcal{E}(T, q_{\tau,\varepsilon}(T)) + \mathrm{Diss}_{\mathcal{D}}(q_{\tau,\varepsilon}, [0,T]) \leq \varepsilon T + \mathcal{E}(0, q(0)) + \int_0^T \partial_t \mathcal{E}(t, q_{\tau,\varepsilon}(t))\,\mathrm{d}t.$$

Since $\partial_t \mathcal{E}(t,q)$ is affine in q, it is weakly continuous, and using $q_{\tau_{k_n},\varepsilon_{k_n}}(t) \rightharpoonup q(t)$ implies the convergence of the last term. Together with the lower semicontinuities $\mathcal{E}(T, q(T)) \leq \liminf_{n\to\infty} \mathcal{E}(T, q_{\tau_{k_n},\varepsilon_{k_n}}(T))$ and (3.5) we find

$$\mathcal{E}(T, q(T)) + \mathrm{Diss}_{\mathcal{D}}(q, [0,T]) \leq \mathcal{E}(0, q(0)) + \int_0^T \partial_t \mathcal{E}(t, q(t))\,\mathrm{d}t,$$

which is the desired upper energy estimate.

Step 5: The lower energy estimate

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q, [s,t]) \geq \mathcal{E}(s, q(s)) + \int_s^t \partial_r \mathcal{E}(r, q(r))\,\mathrm{d}r,$$

holds for all $0 \leq s < t \leq T$ generally for all measurable functions $q : [0,T] \to \mathcal{Q}$ that are stable for all $r \in [s,t]$, which was established in Step 3, see [MTL02, Thm. 2.5] or [Mie11, Prop. 3.11]. Combining this with Step 4 provides the energy balance (E) in Definition 2.1 for energetic solutions, and the proof of Theorem 3.1 is finished, except for the construction of the MRS.

The remaining part in the above proof is the difficult Step 3, where the stability of the accumulation point $q : [0,T] \to \mathcal{Q}$ is established. In [MTL02, Sect. 5] this step was done under the restrictive assumption of convexity of $\mathcal{E}(t,\cdot)$. Here we show that the proof via the construction of MRS is more flexible. Of course, we still need a fine tool from weak-convergence theory, namely H-measure or microlocal defect measures, see [Tar90, Gér91] and [Rin15] for the more general microlocal compactness forms.

3.3 Pseudo-differential operators and H-measures

To understand the set of stable states a little better we can use the fact that $\mathcal{E}(t,\cdot)$ is quadratic, $\mathcal{E}(t,\cdot,\theta)$ is uniformly convex (by Korn's inequality and $\int_{\Gamma_{\mathrm{Dir}}} \mathrm{d}a > 0$), and that $\mathcal{D}$ depends on θ only. Thus,

$$q = (u,\theta) \in \mathcal{S}(t) \implies u \text{ minimizes } \mathcal{E}(t,\cdot,\theta) \implies u = u_{\mathrm{elast}} + \mathbb{B}\theta,$$

where the minimizer $u_{\mathrm{elast}}(t)$ of $\mathcal{E}(t,\cdot,0)$ satisfies $u_{\mathrm{elast}}(\cdot) \in g_{\mathrm{Dir}} + \mathrm{C}^1([0,T];\mathbf{U})$. The linear operator satisfies $\mathbb{B} \in \mathrm{Lin}(\mathrm{L}^2(\Omega);\mathbf{U})$. Defining $\mathcal{J}(t,\theta) := \mathcal{E}(t, u_{\mathrm{elast}}(t) + \mathbb{B}\theta, \theta)$ we arrive at the quadratic functional

$$\mathcal{J}(t,\theta) = \frac{1}{2}\langle \mathbb{L}\theta, \theta\rangle + \langle \beta(t), \theta\rangle + \alpha(t) = \min\big\{\, \mathcal{E}(t,u,\theta) \,\big|\, u \in \mathbf{U} \,\big\},$$

where $\beta \in \mathrm{C}^1([0,T];\mathrm{L}^2(\Omega))$ and $\alpha \in \mathrm{C}^1([0,T])$. While the energetic shift α is irrelevant, the function β can be seen as a time-dependent driving force that depends linearly on $g_{\mathrm{Dir}}(t)$ and $\ell(t)$ via $u_{\mathrm{elast}}(t)$.

The important feature here is that the quadratic functional $\mathcal{J}$ is given in terms of the linear operator $\mathbb{L} \in \mathrm{Lin}(\mathrm{L}^2(\Omega); \mathrm{L}^2(\Omega))$, which is a symmetric pseudo-differential operator of order 0, which means that

$$\mathbb{L}\theta = \mathfrak{F}^{-1}\big(\Lambda(\cdot)\mathfrak{F}(\mathbb{E}\theta)\big)\big|_{\Omega} + \mathbb{K}\theta,$$

where $\mathbb{E} : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\mathbb{R}^d)$ denotes the extension by 0 outside of Ω, and $\mathbb{K}$ is a compact operator in $\mathrm{L}^2(\Omega)$. The more important first part consists of the Fourier transform $\mathfrak{F}$ and the Fourier multiplier Λ, which is also called symbol. The order 0 of the pseudo-differential operator $\mathbb{L}$ relates to the homogeneity of Λ, namely $\Lambda(r\xi) = r^0\Lambda(\xi)$ for $r > 0$ and $\xi \neq 0$. For our two-phase problem Λ takes the specific form

$$\Lambda(\xi) = \gamma - \Sigma(\xi) \quad \text{for } \xi \in \mathbb{R}^d\backslash\{0\},$$

see [MTL02] and (3.2) for the definition of γ and Σ. Thus, the continuous spectrum of $\mathbb{L}$ equals $\{\, \Lambda(\omega) \,|\, \omega \in \mathbb{S}^{d-1} \,\}$, lies in $[0,\infty[$, and contains 0, because of the definition of γ. In particular, a possible negative part of $\mathbb{L}$ must be compact, and $\mathcal{J}(t,\cdot)$ is indeed lower semicontinuous.

For pseudo-differential operators we can use H-measures (cf. [Tar90, Gér91]) to calculate the limits of quadratic functionals under weak convergence in $\mathrm{L}^2(\Omega)$. To formulate our results shortly we simply write $\theta_n \xrightarrow{\mathrm{H}} (\theta_*, \mu) \in \mathrm{L}^2(\Omega) \times \mathrm{M}(\Omega \times \mathbb{S}^{d-1})$, if $\theta_n \rightharpoonup \theta_*$ and the sequence $\theta_n - \theta_*$ generates the H-measure μ. The latter means that for all $\phi \in \mathrm{C_c}(\Omega)$ and $\Psi \in \mathrm{C}(\mathbb{S}^{d-1})$ we have

$$\lim_{n\to\infty} \int_{\xi\in\mathbb{R}^d} \Big|\mathfrak{F}\big[\mathbb{E}\big(\phi(\theta_n{-}\theta_*)\big)\big](\xi)\Big|^2 \Psi(\xi/|\xi|)\,\mathrm{d}\xi = \int_\Omega \int_{\mathbb{S}^{d-1}} |\phi(x)|^2\Psi(\omega)\mu(\mathrm{d}x,\mathrm{d}\omega).$$

The following results will be central for our construction of MRS.

Proposition 3.2 (H-measures) *For $p > 4$ assume that $v_n \rightharpoonup v_*$ in $\mathrm{L}^p(\Omega)$ and $b_n \to b_*$ and $w_m \to 0$ in $\mathrm{L}^p(\Omega)$. Then, we have*

$$v_n \xrightarrow{\mathrm{H}} (v_*, \mu) \implies \mathcal{J}(t, v_n) \to \mathcal{J}(t, v_*) + \int_\Omega \int_{\mathbf{S}^{d-1}} \Lambda(\omega)\mu(\mathrm{d}x, \mathrm{d}\omega), \tag{3.7a}$$

$$v_n \xrightarrow{\mathrm{H}} (v_*, \mu) \implies b_n v_n {+} w_n \xrightarrow{\mathrm{H}} (b_* v_*, b_*^2\mu). \tag{3.7b}$$

Proof. Relation (3.7a) is a well-known standard result, see [Tar90, Cor. 1.2+1.12].

The same reference contains result (3.7b) under the stronger assumption $b_n = b_*$ and $b_* \in \mathrm{C}^0_\mathrm{c}(\Omega)$. Using the a priori bounds $\|v_n\|_{\mathrm{L}^p} + \|b_n\|_{\mathrm{L}^p} \leq C$ we can extend the result since $b_n v_n \rightharpoonup b_* v_*$, and there exists a subsequence $n_k \to \infty$ such that $b_{n_k} v_{n_k} \xrightarrow{\mathrm{H}} (b_* v_*, \widetilde{\mu})$ for $k \to \infty$. We want to show that $\widetilde{\mu} = b_*^2 \mu$.

We approximate $b \in \mathrm{L}^p(\Omega)$ by $B_\delta \in \mathrm{C}^0_\mathrm{c}(\Omega)$ with $B_\delta \to b_*$ in $\mathrm{L}^p(\Omega)$ and write $b_n v_n = z_n + y_n$ with $z_n = B_\delta v_n$ and $y_n = (b_n{-}B_\delta)v_n$. The vector-valued H-measure for the vector $(z_n, y_n)^\top$ has components $(\mu^\delta_{ij})_{i,j=1,2}$ with $\mu^\delta_{11} = B_\delta^2\mu$, where we exploit $B_\delta \in \mathrm{C_c}(\Omega)$. Using $b_n v_n = z_n + y_n$ we have $\widetilde{\mu} = \mu^\delta_{11} + \mu^\delta_{12} + \mu^\delta_{21} + \mu^\delta_{22}$. Moreover, for the total variations of the measures μ_{ij} we have

$$\|\mu^\delta_{12}\|_{\mathrm{TV}} = \|\mu^\delta_{21}\|_{\mathrm{TV}} \leq \limsup_{n\to\infty} \|z_n\|_{\mathrm{L}^2}\|y_n\|_{\mathrm{L}^2}, \quad \|\mu^\delta_{22}\|_{\mathrm{TV}} \leq \limsup_{n\to\infty} \|y_n\|^2_{\mathrm{L}^2}.$$

Using $\|y_n\|_{L^2} \leq \|B_\delta - b_*\|_{L^4} \|v_n\|_{L^4}$, we obtain the estimate

$$\|\widetilde{\mu} - B_\delta^2 \mu\|_{\mathrm{TV}} = \|\widetilde{\mu} - \mu_{11}^\delta\|_{\mathrm{TV}} \leq C\|B_\delta - b_*\|_{L^p} \to 0.$$

Thus, we conclude $\widetilde{\mu} = b_*^2 \mu$ as desired, and $b_n v_n \xrightarrow{\mathrm{H}} (b_* v_*, b_*^2 \mu)$ even without taking a subsequence. ■

More results on H-measures involving fine laminates are given in Proposition 5.2, which is proved in Section 5.3.

3.4 Mutual recovery sequences I

Fix $t \in [0, T]$ and consider a stable sequence $(q_n)_{n\in\mathbb{N}}$, i.e. $q_n \in \mathcal{S}(t)$ with $q_n \rightharpoonup q_*$. To show the stability $q_* \in \mathcal{S}(t)$, we have to find a MRS $(\widehat{q}_n)_{n\in\mathbb{N}}$ for every test function $\widehat{q}$. This will be done with the help of the function

$$F : [0,1]^2 \to [0,1]; \quad F(\theta_0, \theta_1) = \begin{cases} \theta_1/\theta_0 & \text{for } \theta_1 < \theta_0, \\ 1 & \text{for } \theta_0 = \theta_1, \\ (1-\theta_1)/(1-\theta_0) & \text{for } \theta_1 > \theta_0. \end{cases} \tag{3.8}$$

Proposition 3.3 (Mutual recovery sequence I) *Assume that $q_n = (u_n, \theta_n) \in \mathcal{S}(t)$ and $q_n \rightharpoonup q_*$ and that $\widehat{q} = (\widehat{u}, \widehat{\theta})$ is arbitrary. Then, the sequence $\widehat{q}_n = (\widehat{u}_n, \widehat{\theta}_n)$ with*

$$\widehat{u}_n = u_{elast}(t) + \mathbb{B}\widehat{\theta}_n \text{ and } \widehat{\theta}_n = \widehat{\theta} + g\,(\theta_n - \theta_*) \text{ with } g(x) = F(\theta(x), \widehat{\theta}(x))$$

is a recovery sequence satisfying

$$\lim_{n\to\infty} \mathcal{D}(\theta_n, \widehat{\theta}_n) = \mathcal{D}(\theta_*, \widehat{\theta}) \quad \text{and} \quad \limsup_{n\to\infty} \mathcal{E}(t, \widehat{q}_n) - \mathcal{E}(t, q_n) \leq \mathcal{E}(t, \widehat{q}) - \mathcal{E}(t, q_*). \tag{3.9}$$

Proof. We first discuss the dissipation, which only depends on θ. The construction of g via F is such that

$$\mathrm{sign}\big(\widehat{\theta}_n(x) - \theta_n(x)\big) = \mathrm{sign}\big(\widehat{\theta}(x) - \theta_*(x)\big).$$

This follows immediately from the explicit representations

$$\widehat{\theta}_n - \theta_n = \begin{cases} (\widehat{\theta} - \theta_*)(1-\theta_n)/(1-\theta_*) & \text{for } \widehat{\theta} > \theta_*, \\ (\widehat{\theta} - \theta_*) & \text{for } \widehat{\theta} = \theta_*, \\ (\widehat{\theta} - \theta_*)\theta_n/\theta_* & \text{for } \widehat{\theta} < \theta_*. \end{cases}$$

Thus, we can calculate the dissipation by using the domains $\Omega_\pm := \{\, x \in \Omega \mid \pm (\widehat{\theta}(x) - \theta_*(x)) > 0 \,\}$, namely

$$\begin{aligned} \mathcal{D}(\theta_n, \widehat{\theta}_n) &= \kappa_{1\to2} \int_{\Omega_+} \frac{\widehat{\theta} - \theta_*}{1-\theta_*} (1-\theta_n)\,\mathrm{d}x + \kappa_{2\to1} \int_{\Omega_-} \frac{\theta_* - \widehat{\theta}}{\theta_*} \theta_n \,\mathrm{d}x \\ &\to \kappa_{1\to2} \int_{\Omega_+} \frac{\widehat{\theta} - \theta_*}{1-\theta_*} (1-\theta_*)\,\mathrm{d}x + \kappa_{2\to1} \int_{\Omega_-} \frac{\theta_* - \widehat{\theta}}{\theta_*} \theta_* \,\mathrm{d}x \\ &= \kappa_{1\to2} \int_{\Omega_+} (\widehat{\theta} - \theta_*)\,\mathrm{d}x + \kappa_{2\to1} \int_{\Omega_-} (\theta_* - \widehat{\theta})\,\mathrm{d}x \quad = \quad \mathcal{D}(\theta_*, \widehat{\theta}). \end{aligned}$$

Note that the weak convergence $\theta_n \rightharpoonup \theta_*$ and the linearity of the integrals over $\Omega_\pm$ allow us to pass to the limit $n \to \infty$. Thus, the first relation in (3.9) is established.

To establish the second relation we use that for $q = (u,\theta) \in \mathcal{S}(t)$ we have $u = u_{\text{elast}}(t) + \mathbb{B}\theta$, which is equivalent to $\mathcal{E}(t,u,\theta) = \mathcal{I}(t,\theta)$. Thus, it suffices to show

$$\limsup_{n\to\infty} \big(\mathcal{I}(t,\widehat{\theta}_n) - \mathcal{I}(t,\theta_n)\big) \le \mathcal{I}(t,\widehat{\theta}) - \mathcal{I}(t,\theta_*) \le \mathcal{E}(t,\widehat{u},\widehat{\theta}) - \mathcal{E}(t,q_*),$$

where $\widehat{u}$ can be arbitrary.

We now use that $\theta_n \rightharpoonup \theta_*$ in $\mathrm{L}^p(\Omega)$ for all $p > 1$. By the construction of $\widehat{\theta}_*$ we also have $\widehat{\theta}_n \rightharpoonup \widehat{\theta}$ in $\mathrm{L}^p(\Omega)$ for all $p > 1$. Choosing a subsequence (not relabeled) we can assume that θ_n and $\widehat{\theta}_n$ generate H-measures μ and $\widehat{\mu}$ respectively. Applying Proposition 3.2 on H-measures with $b_n = g \in \mathrm{L}^\infty(\Omega)$ we obtain $\widehat{\mu} = g^2\mu$ from (3.7b) and arrive via (3.7a) at

$$\lim_{n\to\infty} \big(\mathcal{I}(t,\widehat{\theta}_n) - \mathcal{I}(t,\theta_n)\big) = \mathcal{I}(t,\widehat{\theta}) - \mathcal{I}(t,\theta_*) + \int_{\Omega\times\mathbb{S}^{d-1}} \big(g(x)^2 - 1\big)\Lambda(\omega)\mu(\mathrm{d}x,\mathrm{d}\omega) \le \mathcal{I}(t,\widehat{\theta}) - \mathcal{I}(t,\theta_*),$$

due to $\Lambda \ge 0$ and $g(x) \in [0,1]$. Because this holds along any subsequence the second relation in (3.9) is established. ∎

4 Numerical approximation

We now exploit the flexibility and robustness of the method of MRS, which allow us to go much further than the theory in [MTL02]. Indeed, we can numerically approximate the problem, e.g. by standard finite-element methods as used in [CaP01].

For this we consider finite-dimensional subspaces $\mathbf{U}_k$ and $\mathbf{Z}_k$ of $\mathbf{U} = \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega;\mathbb{R}^d)$ and $\mathbf{Z} = \mathrm{L}^2(\Omega)$ that are asymptotically dense, i.e.

$$\mathbf{U}_k \subset \mathbf{U}_{k+1} \subset \mathbf{U}, \quad \mathbf{Z}_k \subset \mathbf{Z}_{k+1} \subset \mathbf{Z}, \quad \overline{\bigcup_{k\in\mathbb{N}} \mathbf{U}_k} = \mathbf{U}, \quad \overline{\bigcup_{k\in\mathbb{N}} \mathbf{Z}_k} = \mathbf{Z}. \tag{4.1}$$

Moreover, assume that the discretization of $\theta \in \mathcal{Z}$ is compatible with the constraint $\theta(x) \in [0,1]$. We set $\mathcal{Z}_k = \mathbf{Z}_k \cap \mathcal{Z}$ and assume $0, 1 \in \mathcal{Z}_1$ and that $\cup_{k\in\mathbb{N}}\mathcal{Z}_k$ is dense in $\mathcal{Z}$.

4.1 An abstract convergence result

Based on the above general assumptions we add two major conditions. For each k we need a (maybe nonlinear) mapping $\mathbb{P}_k : \mathcal{Z} \to \mathcal{Z}_k$ such that the following holds

$$\begin{aligned} &\forall\, g,h \in \mathcal{Z} \text{ with } g{+}h \in \mathcal{Z}: \\ &\qquad \alpha_k(g,h) := \sup\{\, \|\mathbb{P}_k(h{+}g\theta) - (h{+}g\theta)\|_{\mathrm{L}^2} \mid \theta \in \mathcal{Z}_k \,\} \;\to\; 0 \text{ for } k \to \infty. \end{aligned} \tag{4.2}$$

To formulate the conditions between the compatibility of the discretization of u through the spaces $\mathbf{U}_k$ and the discretization of θ via $\mathcal{Z}_k$, we again use the quadratic structure of $\mathcal{E}$. For $\theta_k \in \mathcal{Z}_k$ we define the reduced functionals

$$\mathcal{I}_k(t,\theta) := \min\{\, \mathcal{E}(t,u,\theta) \mid u \in \mathbf{U}_k \,\}.$$

By (4.1) we have $\mathcal{I}_k \ge \mathcal{I}_{k+1} \ge \mathcal{I}$ and $\mathcal{I}_k(t,\theta) \to \mathcal{I}(t,\theta)$ for fixed (t,θ). The second major condition is that the convergence is uniform with respect to (t,θ), namely

$$\sigma_k := \sup\{\, \mathcal{I}_k(t,\theta) - \mathcal{I}(t,\theta) \mid \theta \in \mathcal{Z}_k,\ t \in [0,T] \,\} \;\to\; 0 \quad \text{for } k \to \infty. \tag{4.3}$$

To formulate the existence and convergence result, we again use that we are able to restrict to the variable θ. We consider the sequence of ERIS $(\mathcal{Z}, \overline{\mathcal{I}}_k, \mathcal{D}_k)$ given by

$$\overline{\mathcal{I}}_k(t,\theta) := \begin{cases} \mathcal{I}_k(t,\theta) & \text{for } \theta \in \mathcal{Z}_k, \\ \infty & \text{otherwise,} \end{cases}$$

and $\mathcal{D}_k = \mathcal{D}$. We use the discretized stability sets

$$\overline{\mathcal{S}}_k(t) := \{\, \theta \in \mathcal{Z}_k \mid \forall \widehat{\theta} \in \mathcal{Z}_k : \ \overline{\mathcal{I}}_k(t,\theta) \le \overline{\mathcal{I}}_k(t,\widehat{\theta}) + \mathcal{D}_k(\theta,\widehat{\theta}) \,\}.$$

The numerical incremental minimization problem for $\tau = T/J$ with $J \in \mathbb{N}$ reads

$$\theta_0^{k,\tau} = \theta_0^k \in \overline{\mathcal{S}}_k(0), \quad \theta_j^{k,\tau} \text{ minimizes } \mathcal{I}_k(j\tau,\cdot) + \mathcal{D}(\theta_{j-1}^{k,\tau},\cdot)) \text{ for } j = 1, ..., J.$$

As in (2.3) we define the piecewise constant interpolants $\theta_{k,\tau} : [0,T] \to \mathcal{Z}$.

Theorem 4.1 (Convergence of numerical approximation) *Let conditions* (4.1), (4.2)*, and* (4.3) *hold. Moreover, consider stable initial conditions $\theta_0^k \in \overline{\mathcal{S}}_k(0)$ such that*

$$\theta_0^k \rightharpoonup \theta_0 \text{ in } \mathcal{Z} \quad \text{and} \quad \overline{\mathcal{I}}_k(0,\theta_0^k) \to \mathcal{I}(0,\theta_0),$$

then all accumulation points $\theta : [0,T] \to \mathcal{Z}$ for $k \to \infty$ and $\tau \to 0$ (in the sense of (3.5)*) of the numerical approximations $\theta_{k,\tau} : [0,T] \to \mathcal{Z}$ are energetic solutions of $(\mathcal{Z},\mathcal{I},\mathcal{D})$.*

The proof is identical to the one in Section 3.2, where now the crucial construction of MRS for the numerical approximation is given in Section 4.2. We refer to Section 4.3 for possible ways to fulfill the assumptions (4.2) and (4.3) by concrete numerical discretizations.

4.2 Mutual recovery sequences II

The construction follows closely the one for the existence result. However, we have to take care that the MRS lies in the discrete finite-dimensional space $\mathcal{Z}_k = \mathbf{Z}_k \cap \mathcal{Z}$.

Proposition 4.2 (MRS for the discretized system) *Let the conditions* (4.1)*,* (4.2)*, and* (4.3) *be satisfied. Then, for any sequence (θ_k) with $\theta_k \in \overline{\mathcal{S}}(t_k)$, $t_k \to t_*$, and $\theta_k \rightharpoonup \theta_*$ and any $\widehat{\theta} \in \mathcal{Z}$, the sequence*

$$\widehat{\theta}_k = \mathbb{P}_k\big(\widehat{\theta} + g\,(\theta_k - \theta_*)\big) \text{ with } g(x) = F(\theta_*(x), \widehat{\theta}(x))$$

is a MRS satisfying

$$\mathcal{D}(\theta_k,\widehat{\theta}_k) \to \mathcal{D}(\theta_*,\widehat{\theta}) \quad \text{and} \quad \limsup_{n\to\infty} \big(\overline{\mathcal{I}}_k(t_k,\widehat{\theta}_k) - \overline{\mathcal{I}}_k(t_k,\theta_k)\big) \le \mathcal{I}(t_*,\widehat{\theta}) - \mathcal{I}(t_*,\theta_*).$$

In particular, we conclude $\theta_* \in \overline{\mathcal{S}}(t_*)$.

Proof. We first observe that $h := \widehat{\theta} - g\theta_* \in \mathcal{Z}$ and $h + g \in \mathcal{Z}$, which follows from the definition of g via the specific form of F. Setting $\widetilde{\theta}_k = h + g\theta_k$ we have

$$\big|\mathcal{D}(\theta_k,\widehat{\theta}_k) - \mathcal{D}(\theta_*,\widehat{\theta})\big| \le \big|\mathcal{D}(\theta_k,\widetilde{\theta}_k) - \mathcal{D}(\theta_*,\widehat{\theta})\big| + \max\{\kappa_{1\to2},\kappa_{2\to1}\}\|\widetilde{\theta}_k - \widehat{\theta}_k\|_{\mathrm{L}^1} \to 0,$$

where the first term converges to 0 as in the proof of Proposition 3.3. Since $\theta_k \in \mathcal{Z}_k$, the second term is bounded by $C\alpha_k(g,h)$, which converges to 0 by condition (4.2).

For the energy difference we use θ_k, $\widehat{\theta}_k \in \mathcal{Z}_k$ and σ_k as in (4.3) to obtain

$$\overline{\mathcal{I}}_k(t_k,\widehat{\theta}_k)-\overline{\mathcal{I}}_k(t_k,\theta_k) \le \mathcal{I}(t_*,\widehat{\theta}_k) - \mathcal{I}(t_*,\theta_k) + \sigma_k + C|t_k-t_*|.$$

By taking a subsequence we may assume that the limsup is achieved, $\theta_k \xrightarrow{\mathrm{H}} (\theta_*,\mu)$, and $\widehat{\theta}_K \xrightarrow{\mathrm{H}} (\widehat{\theta},\widehat{\mu})$. Using (3.7b) yields $\widehat{\mu} = g^2\mu$, since $\widehat{\theta}_k = \widetilde{\theta}_k + w_n$ with $\widetilde{\theta}_k = h + g\theta_k$ and $\|w_n\|_{\mathrm{L}^2} \le \alpha_k(g,h) \to 0$ by condition (4.2). Thus, using $\sigma_k \to 0$ (i.e. condition (4.3)) and $t_k \to t_*$ we conclude via (3.7a), namely

$$\begin{aligned}&\limsup_{k\to\infty} \big(\overline{\mathcal{I}}_k(t_k,\widehat{\theta}_k)-\overline{\mathcal{I}}_k(t_k,\theta_k)\big) \le \limsup_{k\to\infty} \big(\mathcal{I}_k(t_*,\widehat{\theta}_k)-\mathcal{I}_k(t_*,\theta_k) + 2\sigma_k + C|t_*-t_k|\big)\\ &= \mathcal{I}(t_*,\widehat{\theta}) - \mathcal{I}(t_*,\theta_*) + \int_{\Omega\times\mathbb{S}^{d-1}} (g^2-1)\Lambda\,\mathrm{d}\mu \ \le\ \mathcal{I}(t_*,\widehat{\theta}) - \mathcal{I}(t_*,\theta_*),\end{aligned}$$

since $\Lambda \ge 0$ and $0 \le g \le 1$. This proves the proposition. ∎

4.3 Conditions for numerical approximations

We now show that the two major conditions (4.2) and (4.3) can be easily satisfied by suitable discretizations. For this, we assume that for each $k \in \mathbb{N}$ there is a triangulation $\mathfrak{T}_k$ of Ω, such that Ω decomposes into d-dimensional tetrahedra T (convex hull of $d{+}1$ points) plus some intersections of tetrahedra with Ω along the boundary. By

$$\phi(\mathfrak{T}) := \sup\big\{ \operatorname{diam}(T) \,\big|\, T \in \mathfrak{T} \big\} \text{ with } \operatorname{diam}(T) := \sup\{\, |x_1-x_2| \mid x_1,x_2 \in T \,\}$$

we denote the fineness of the triangulation $\mathfrak{T}$. For any $\mathfrak{T}_k$ we denote by $\mathbf{Z}_k$ the space of functions $\overline{\theta}$ that are constant on each of the subsets $T \subset \mathfrak{T}_k$. To satisfy the condition $\mathbf{Z}_k \subset \mathbf{Z}_{k+1}$ we need to choose a nested triangulation where new tetrahedra are constructed by inserting a point in the interior of T and generating smaller tetrahedra by connecting this point with all the faces of T.

We denote by $\mathbb{P}_k$ the L^2 orthogonal projection from $\mathbf{Z}$ to $\mathbf{Z}_k$ which reads

$$(\mathbb{P}_k\theta)(x) = \frac{1}{|T|}\int_T \theta(y)\,\mathrm{d}y \quad \text{for } x \in T.$$

Given the above construction, the following three conditions are equivalent:

$$\text{(i)}\ \bigcup\nolimits_{k\in\mathbb{N}} \mathbf{Z}_k \text{ is dense in } \mathcal{Z}; \qquad \text{(ii)}\ \ \phi(\mathfrak{T}_k) \to 0 \text{ for } k \to \infty;$$
$$\text{(iii)}\ \ \forall\, \theta \in \mathbf{Z}: \quad \mathbb{P}_k\theta \to \theta \text{ for } k \to \infty.$$

Lemma 4.3 *The operator $\mathbb{P}_k$ constructed above satisfies* (4.2).

Proof. We consider arbitrary $h, g \in \mathcal{Z}$ with $g+h \in \mathcal{Z}$. For $\theta_k \in \mathcal{Z}_k = \mathbf{Z}_k \cap \mathcal{Z}$, we use

$$\begin{aligned}\|\mathbb{P}_k(g\theta_k) - g\theta_k\|^2_{\mathrm{L}^2} &= \sum_{T\in\mathfrak{T}_k} \int_T \Big(\frac{1}{|T|}\int_T g\theta_k\,\mathrm{d}y - g(x)\theta_k(x)\Big)^2 \mathrm{d}x\\ &= \sum_{T\in\mathfrak{T}_k} \int_T (\theta_k|_T)^2 \Big(\frac{1}{|T|}\int_T g\,\mathrm{d}y - g(x)\Big)^2 \mathrm{d}x \le \|\theta_k\|^2_{\mathrm{L}^\infty} \|\mathbb{P}_k g - g\|^2_{\mathrm{L}^2},\end{aligned}$$

where we used that θ_k is constant on each tetrahedron. Using $0 \le \theta_k \le 1$ yields

$\|\mathbb{P}_k(h{+}g\theta_k) - (h{+}g\theta_k)\|_{\mathrm{L}^2} \le \|\mathbb{P}_k h{-}h\|_{\mathrm{L}^2} + \|\mathbb{P}_k(g\theta_k){-}g\theta_k\|_{\mathrm{L}^2} \le \|\mathbb{P}_k h{-}h\|_{\mathrm{L}^2} + \|\mathbb{P}_k g{-}g\|_{\mathrm{L}^2}$.

Thus, we conclude $\alpha_k(g,h) \le \|\mathbb{P}_k h{-}h\|_{\mathrm{L}^2} + \|\mathbb{P}_k g{-}g\|_{\mathrm{L}^2}$ and (iii) from above implies the desired result (4.2). ■

We show that the second condition (4.3) can always be satisfied by choosing a suitably fine discretization for the displacements $u \in \mathbf{U}$. Considering the same family $(\mathfrak{T}_m)_{m\in\mathbb{N}}$ of nested triangulations as above, we set

$$\widetilde{\mathbf{U}}_m := \big\{\, u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \,\big|\, \forall\, T \in \mathfrak{T}_m :\ u|_T \text{ is affine } \big\} \quad \text{and} \quad \mathbf{U}_k = \widetilde{\mathbf{u}}_{m_k}.$$

Here the crucial point is that m_k has to be chosen sufficiently large, i.e. the fineness $\phi(\mathfrak{T}_{m_k})$ of the finite-element space $\mathbf{U}_k$ for the displacements is much higher than that for the phase indicator $\theta_k \in \mathcal{Z}_k$. In particular, this implies that the dimension of $\mathbf{U}_k$ may be much higher that of $\mathbf{Z}_k$. It is well-known (cf. e.g. [BrS08]) that $\cup_{m\in\mathbb{N}}\widetilde{\mathbf{U}}_m$ is dense in $\mathbf{U} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ if and only if $\phi(\mathfrak{T}_m) \to 0$.

Lemma 4.4 *Under the above assumptions there exists a sequence m_k such that condition* (4.3) *holds for $\mathbf{Z}_k$ and $\mathbf{U}_k$ given above.*

Proof. For each $\theta \in \mathcal{Z}$ we set

$$\varsigma_m(\theta) := \sup\big\{\, \mathcal{E}(t,\theta,u) - \mathcal{I}(t,\theta) \,\big|\, u \in \widetilde{\mathbf{U}}_m,\ t\in[0,T] \,\big\},$$

such that σ_k in (4.3) has the form

$$\sigma_k = \sup\big\{\, \varsigma_{m_k}(\theta) \,\big|\, \theta \in \mathcal{Z}_k \,\big\},$$

where it is essential that ς has the larger index m_k while $\theta \in \mathcal{Z}_k$.

If N_k is the number of tetrahedra in $\mathfrak{T}_k$, then $\mathcal{Z}_k$ is the convex hull of the $J_k := 2^{N_k}$ extremal points $(e_j^{(k)})_{j=1,\ldots,J_k}$ in $\mathcal{Z}_k$ which are given by functions taking the values 0 or 1 on each tetrahedron. Because every $\theta \in \mathcal{Z}_k$ has the convex representation $\theta = \sum_{j=1}^{J_k} \lambda_j e_j^{(k)}$ with $\lambda_j \ge 0$ and $\sum_{j=1}^{J_k} \lambda_j = 1$ we can use the convexity of $\mathcal{Q}_m(t,\cdot) := \mathcal{I}_m(t,\cdot){-}\mathcal{I}(t,\cdot)$ (which follows from $\mathbf{U}_k \subset \mathbf{U}$) to obtain

$$\begin{aligned}\varsigma_m(\theta) &= \sup\big\{\, \mathcal{Q}_m(t, \textstyle\sum_j \lambda_j e_j^{(k)}) \,\big|\, t\in[0,T] \,\big\} \le \sup\big\{\, \textstyle\sum_j \lambda_j \mathcal{Q}_m(t, e_j^{(k)}) \,\big|\, t\in[0,T] \,\big\} \\ &\le \textstyle\sum_j \lambda_j \sup\big\{\, \mathcal{Q}_m(t, e_j^{(k)}) \,\big|\, t\in[0,T] \,\big\} = \textstyle\sum_j \lambda_j \varsigma_m(e_j^{(k)}) \le \max\{\, \varsigma_m(e_j^{(k)}) \mid j=1,...,J_k \,\}.\end{aligned}$$

Since $m \mapsto \varsigma_m(\theta)$ decays monotonously to 0 for each k there is a minimal $M(\theta,k)$ such that $\varsigma_m(\theta) \le 1/k$ for $m \ge M(\theta,k)$. We now set

$$m_k := \max\{\, M(e_j^{(k)},k) \mid j=1,\ldots,J_k \,\},$$

then $\varsigma_m(\theta) \le 1/k$ for all $\theta \in \mathcal{Z}_k$ and all $m \ge m_k$. Thus, we conclude $\sigma_k \le 1/k$, which implies the desired condition (4.3). ■

While the above construction shows that it is in principle possible to find converging discretizations, the method is not satisfactory. It would be desirable to show that the discrete spaces $\mathbf{U}_k$ can be formulated on the same triangulation $\mathfrak{T}_k$ instead of the much finer triangulation $\mathfrak{T}_{m_k}$. It is not clear that this can be achieved with some kind of conforming discretization (i.e. $\mathbf{U}_k \subset U$) as used in [CaP01]. However, it might be easier to construct nonconforming scheme like discontinuous Galerkin schemes to satisfy condition (4.3). Moreover, the latter condition turned out to be sufficient for our convergence result in Theorem 4.1, but there might be substantially weaker abstract conditions that would allow for a larger class of discretization schemes.

5 Evolutionary relaxation

The original microscopic problem was described by pure phases with $z(t,x) \in \{e_1, e_2\}$, i.e. the phase indicator θ should only take the values $\theta = 0$ for phase 1 or $\theta = 1$ for phase 2. Thus, we define the pure, or unrelaxed, state space

$$\mathcal{P} := \{\, \theta \in \mathrm{L}^2(\Omega) \mid \theta(x) \in \{0,1\} \text{ for a.a. } x \in \Omega \,\}.$$

Obviously, $\mathcal{P}$ is a subset of $\mathcal{Z}$, but it is not weakly closed. In fact, $\mathcal{Z}$ is the convex hull of $\mathcal{P}$, while $\mathcal{P}$ contains all extremal points of $\mathcal{Z}$.

We may consider the full ERIS $(\mathbf{U}\times\mathcal{P}, \mathcal{E}, \mathcal{D})$ or the equivalent reduced ERIS $(\mathcal{P}, \mathcal{I}, \mathcal{D})$, but it is not clear whether this system has any energetic solutions for general loadings via g_{Dir} and ℓ. However, following the ideas in [MTL02, Mie04, MRS08] (see also [GaL09] for a similar relaxation of a RIS related to fracture) one can define upper and lower incremental relaxations, see [Mie04, Def. 4.1]. Indeed, for a special case of our two-phase problem the lower relaxation was established in [The02].

5.1 The relaxation result

Here we want to address the time-continuous relaxation as introduced in [MRS08, Sect. 4]. For this, we consider approximate incremental minimization problems for $(\mathcal{P}, \mathcal{I}, \mathcal{D})$ defined via (2.2) with a fixed initial state $\theta_0 \in \mathcal{P}$. Now for every $\varepsilon_n > 0$ we choose an approximate solution $\big(\theta_j^{\tau,\varepsilon}\big)_{j=1,\dots,J}$ for the time-discretized problem. As before we denote by $\theta_{\tau,\varepsilon} : [0,T] \to \mathcal{P}$ the piecewise constant interpolants.

Since $\theta_{\tau,\varepsilon}$ satisfies an a priori dissipation bound $\mathrm{Diss}_{\mathcal{D}}(\theta_{\tau,\varepsilon}, [0,T]) \le C$ independently of $\tau = T/N$ and $\varepsilon \in\,]0,1]$, we can extract subsequences $(\tau_k, \varepsilon_k) \to (0,0)$ such that

$$\forall\, t \in [0,T]: \quad \theta_{\tau_k,\varepsilon_k}(t) \rightharpoonup \theta(t) \text{ for } k \to \infty. \tag{5.1}$$

In the spirit of [Mie04, MRS08] we call $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ an (upper) time-continuous relaxation of $(\mathcal{P}, \mathcal{I}, \mathcal{D})$ if all accumulation points θ obtained via (5.1) are energetic solutions for the ERIS $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$.

The following result, which should be seen as a specific nontrivial instance of the general theory in [MRS08, Sect. 4], provides the mathematically rigorous relaxation result that all accumulation points of the pure-phase approximation solutions are indeed solutions of the relaxed model. In particular, it justifies the model derived in [MTL02] via separate relaxation as a true upper relaxation of the evolutionary problem. The property of lower relaxation was already established in [The02].

Theorem 5.1 (Evolutionary relaxation) *Consider the functions $\theta_{\tau,\varepsilon} : [0,T] \to \mathcal{P}$ with $\varepsilon > 0$ and $\tau = T/J$ with $J \in \mathbb{N}$ obtained via the approximate incremental minimization problem* (2.2). *Furthermore assume that $\theta_0 \in \mathcal{P}$ is stable in $\mathcal{P}$, i.e. $\mathcal{I}(0,\theta_0) \le \mathcal{I}(0,\overline{\theta}) + \mathcal{D}(\theta_0,\overline{\theta})$ for all $\overline{\theta} \in \mathcal{P}$. Then, every accumulation point $\theta : [0,T] \to \mathcal{Z}$ satisfying* (5.1) *is an energetic solution of the ERIS $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ as discussed in Section 3.*

As before, the only nontrivial part of the proof is Step 3, where we have to establish the stability of the accumulation points $\theta : [0,T] \to \mathcal{Z}$, i.e. $\theta(t) \in \mathcal{S}(t)$. As before, we will deduce this from the stability of the approximations $\theta_{\tau,\varepsilon}$. However, the nonrelaxed (approximate) stability sets are defined via

$$\mathcal{S}^{\alpha}_{\mathcal{P}}(t) := \{\, \theta \in \mathcal{P} \mid \mathcal{I}(t,\theta) \le \alpha + \mathcal{I}(t,\overline{\theta}) + \mathcal{D}(\theta,\overline{\theta}) \text{ for all } \overline{\theta} \in \mathcal{P} \,\},$$

where the test functions $\overline{\theta}$ are in the much smaller set $\mathcal{P}$ of pure phases only. Thus, the desired closedness condition, which reads

$$\Big(\alpha_k \to 0, \quad \theta_k \in \mathcal{S}_{\mathcal{P}}^{\alpha_k}(t), \quad \theta_k \rightharpoonup \theta_*\Big) \implies \theta_* \in \mathcal{S}(t), \tag{5.2}$$

is more difficult, because we do not only have to pass to the limit, but we also have to enlarge the space of test functions from $\mathcal{P}$ to $\mathcal{Z}$.

Hence, for a construction of MRS we must approximate functions $\widehat{\theta}$ by suitable functions $\overline{\theta}_k \in \mathcal{P}$. In particular, for values $\widehat{\theta}(x) \in \,]0,1[$ we need to introduce new oscillations between the values 0 and 1, which implies that the oscillations captured in the H-measure $\overline{\mu}$ generated by $\overline{\theta}_k$ cannot always be bounded by the H-measure μ, which is generated by θ_k. However, we may introduce the necessary oscillations in such a way that they do not increase the energy $\mathcal{I}$ too much. For this, we essentially use that by the very definition of $\mathcal{I}$ as the relaxation of $\overline{\mathcal{I}}$ there is at least one direction $\omega_* \in \mathbb{S}^{d-1}$ such that $\Lambda(\omega_*) = 0$, i.e. laminates with normal ω_* do only contribute to the energy as much as their weak limit. This is also the essential point in the lower-relaxation result established in [The02].

5.2 Mutal recovery sequences III

We use the following construction for MRS. For $\theta_k \in \mathcal{P}$, $\theta_* \in \mathcal{Z}$, and α_k as in (5.2) and arbitrary test functions $\widehat{\theta} \in \mathcal{Z}$, we have to find a MRS $(\overline{\theta}_k)_k$ with $\overline{\theta}_k \in \mathcal{P}$. We employ the function $H : [0,1] \times \mathbb{R} \to \{0,1\}$ with

$$H(\beta, s) = \begin{cases} 1 & \text{for } \ s \bmod 1 \in [0, \beta[, \\ 0 & \text{for } \ s \bmod 1 \in [\beta, 1[, \end{cases}$$

which satisfies $H\big(\beta, n(\cdot)\big) \rightharpoonup \beta$. For an arbitrary function $\eta \in \mathcal{Z}$ and $m \in \mathbb{N}$ we define the piecewise constant approximations

$$\eta_m(x) = \frac{1}{|\Omega \cap A_m(x)|} \int_{\Omega \cap A_m(x)} \eta(y)\,\mathrm{d}y, \ \text{ where } A_m(x) = \textstyle\times_{i=1}^{d} \big[\frac{1}{m}\lfloor m x_i \rfloor, \frac{1}{m}\lfloor m x_1 {+} 1\rfloor\big[,$$

i.e. $A_m(x) \subset \mathbb{R}^d$ is a semi-open cube of side length $1/m$ containing x. As in [The02, Thm. 3.5] we set

$$S_k^\eta(x) := H(\eta_k(x), k^2 \omega_* \cdot x),$$

which is locally near $x \in \Omega$ a laminate with normal ω_* and volume fraction $\eta_k(x) \approx \eta(x)$. Clearly, $S_k^\eta \in \mathcal{P}$ and $S_k \xrightarrow{\mathrm{H}} (\eta, \widetilde{\mu})$ with $\widetilde{\mu} = \frac{1}{2}\eta(1{-}\eta) \otimes \big(\delta_{\omega_*} {+} \delta_{-\omega_*}\big)$, see e.g. [Rin15, Lem. 12].

The following Theorem 5.3 gives a construction for MRS, which relies on the fact, that we can introduce oscillations via $S_{k_n}^\eta$ which are much faster than oscillations in θ_n. The enforcement of a decoupling of spatial scales via $k_n \gg n$ of the micristructures generated by $S_{k_n}^\eta$ and θ_n, respectively, allows us to calculate the generated H-measure of the maximum function $\zeta_n : x \mapsto \max\{\theta_n(x), S_{k_n}^\eta(x)\}$ explicitly. The proof of this result will be postponed to Section 5.3, and Figure 1 gives a sketch of the construction, where very fine laminates generated by $S_{k_n}^\eta$ are combined with the microstructure of θ_n.

Proposition 5.2 *Assume $\theta_n \in \mathcal{P}$, the convergence $\theta_n \xrightarrow{\mathrm{H}} (\theta_*, \mu)$, and $\eta \in \mathcal{Z}$. Then, there exists a sequence $(\overline{K}_n)_{n\in\mathbb{N}}$ such that for all sequences $(k_n)_{n\in\mathbb{N}}$ with $k_n \geq \overline{K}_n$ for all n, the functions $\zeta_n := \max\{\theta_n, S_{k_n}^\eta\}$ satisfy*

$$\zeta_n \xrightarrow{\mathrm{H}} \big(\theta_* {+} (1{-}\theta_*)\eta, \widehat{\mu}\big) \quad \textit{with } \widehat{\mu} = (1{-}\eta)^2 \mu + (1{-}\theta_*)^2 \eta(1{-}\eta) \otimes \frac{1}{2}\big(\delta_{\omega_*} {+} \delta_{-\omega_*}\big). \tag{5.3}$$

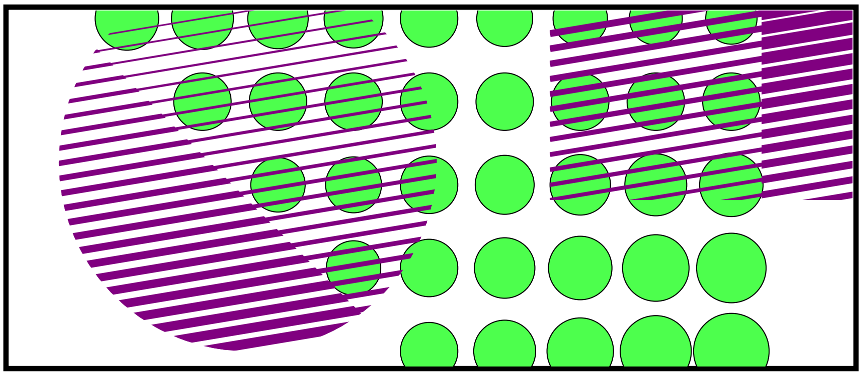

Figure 1: The microstructure of θ_n (green circles) is combined with the much finer laminates (violet) generated by S^η_k for $k \gg n$. The function $\zeta(x) = \max\{\theta_n(x), S^\eta_k(x)\}$ equals 0 in white regions and equals 1 in all colored regions.

We expect that the microlocal compactness forms developed in [Rin15] are the optimal tools to give a clearer proof of the following result and to provide a stronger characterization of the possible limiting objects. Fortunately, for our two-phase problem the H-measure is already sufficient.

The above construction allows us to define suitable MRS $\overline{\theta}_n \in \mathcal{P}$ for a test function $\widehat{\theta} \in \mathcal{Z}$. As before we will be able to guarantee the sign condition

$$\operatorname{sign}(\overline{\theta}_n(x) - \theta_n(x)) = \operatorname{sign}(\widehat{\theta}(x) - \theta_*(x)) \quad \text{a.e. in } \Omega. \tag{5.4}$$

Thus for $\alpha \in \{+, 0, -\}$ we define the indicator functions $\mathbb{1}^\alpha := \mathbb{1}_{\Omega_\alpha}$ for the domains

$$\Omega_\pm := \{\, x \in \Omega \mid \pm(\widehat{\theta}(x) - \theta_*(x)) > 0 \,\} \text{ and } \Omega_0 := \{\, x \in \Omega \mid \widehat{\theta}(x) = \theta_*(x) \,\}.$$

Now we are ready to choose the sequence

$$\begin{aligned} &\overline{\theta}_n := \max\{\theta_n, S^{\eta^+}_{k_n}\}\mathbb{1}^+ + \theta_n \mathbb{1}^0 + \min\{\theta_n, S^{\eta^-}_{k_n}\}\mathbb{1}^-, \quad \text{where} \\ &\eta^+ : \begin{cases} \Omega_+ & \to \quad]0,1[, \\ x & \mapsto \quad (\widehat{\theta}(x) - \theta_*(x))/(1-\theta_*(x)); \end{cases} \quad \text{and } \eta^- : \begin{cases} \Omega_- & \to \quad]0,1[, \\ x & \mapsto \quad \widehat{\theta}(x)/\theta_*(x) \end{cases} \end{aligned} \tag{5.5}$$

and formulate the final result on the MRS for the relaxation problem.

Theorem 5.3 (MRS for evolutionary relaxation) *Let $\theta_k \in \mathcal{P}$, $\theta_* \in \mathcal{Z}$, and α_k be given as in (5.2) and $\widehat{\theta} \in \mathcal{Z}$. Then, there exist $\overline{K}_n \gg n$ such the sequence $(\overline{\theta}_n)_{n\in\mathbb{N}}$ defined in (5.5) with $k_n \geq \overline{K}_n$ is a MRS satisfying the relations*

$$\overline{\theta}_k \rightharpoonup \widehat{\theta}, \quad \mathcal{D}(\theta_k, \overline{\theta}_k) \to \mathcal{D}(\theta_*, \widehat{\theta}), \quad \limsup_{n\to\infty} \big(\mathcal{I}(t, \overline{\theta}_n) - \mathcal{I}(t, \theta_n)\big) \leq \mathcal{I}(t, \widehat{\theta}) - \mathcal{I}(t, \theta). \tag{5.6}$$

Moreover, if $\theta_n \xrightarrow{\mathrm{H}} (\theta_*, \mu)$, then

$$\begin{aligned} &\overline{\theta}_n \xrightarrow{\mathrm{H}} (\widehat{\theta}, \widehat{\mu}) \text{ where } \widehat{\mu} = b^2\mu + a \otimes \frac{1}{2}\big(\delta_{\omega_*} + \delta_{-\omega_*}\big) \\ &\text{with } b = \frac{1-\widehat{\theta}}{1-\theta_*}\mathbb{1}^+ + \mathbb{1}^0 + \frac{\widehat{\theta}}{\theta_*}\mathbb{1}^- \quad \text{and} \quad a = (1-\widehat{\theta})(\widehat{\theta} - \theta_*)\mathbb{1}^+ + \widehat{\theta}(\theta_* - \widehat{\theta})\mathbb{1}^-. \end{aligned} \tag{5.7}$$

Proof. *Step 1:* For the weak convergence we first use Proposition 5.2 to see that $\overline{\theta}_n 1\!\!1^+ \rightharpoonup \big(\theta_*{+}(1{-}\theta_*)\eta^+\big)1\!\!1^+ = \widehat{\theta}1\!\!1^+$. Similarly, $\max\{\theta_n, S^{\eta^-}_{k_n}\}1\!\!1^- = \theta_n S^{\eta^-}_{k_n}1\!\!1^- \rightharpoonup \theta_*\eta^- 1\!\!1^- = \widehat{\theta}1\!\!1^-$, see Step 1 in the proof of Proposition 5.2. Since obviously $\theta_n 1\!\!1^0 \rightharpoonup \theta_* 1\!\!1^0 = \widehat{\theta}1\!\!1^0$, we conclude $\overline{\theta}_n \rightharpoonup \widehat{\theta}$ as desired, if k_n is sufficiently large.

Step 2: The convergence of the dissipation distances $\mathcal{D}(\theta_k, \overline{\theta}_k) \to \mathcal{D}(\theta_*, \widehat{\theta})$ is an easy consequence of Step 1, if we observe the obviously true sign condition (5.4).

Step 3: Next we derive the H-measure relation (5.7) for which we assume $\theta_n \xrightarrow{\mathrm{H}} (\theta_n, \mu)$. By decomposing into $1\!\!1 = 1\!\!1^+ + 1\!\!1^0 + 1\!\!1^-$ we can treat the three parts separately, since $\theta_n 1\!\!1^\alpha \xrightarrow{\mathrm{H}} (\theta_* 1\!\!1^\alpha, 1\!\!1^\alpha \mu)$. Clearly, we obtain $\overline{\theta}_n 1\!\!1^0 \rightharpoonup (\widehat{\theta}1\!\!1^0, 1\!\!1^0\mu)$, i.e. $1\!\!1^0\widehat{\mu} = 1\!\!1^0\mu$ which means $b1\!\!1^0 = 1\!\!1^0$ and $a1\!\!1^0 = 0$.

For the part $\overline{\theta}_n 1\!\!1^+$ we can directly apply Proposition 5.2 with $\eta = \eta^+$, which provides $a1\!\!1^+$ and $b1\!\!1^+$ as given in (5.7). The result on Ω_- follows similarly, e.g. by substituting θ by $1{-}\theta$. Hence, (5.7) is established.

Step 4: To show the limsup estimate in (5.6) we first choose a subsequence realizing the limsup. Choosing a further subsequence (not relabelled), we may assume $\theta_n \xrightarrow{\mathrm{H}} (\theta_*, \mu)$. Thus, owing to Proposition 3.2 and (5.7) we find

$$\mathcal{I}(t, \overline{\theta}_n) - \mathcal{I}(t, \theta_n) = \mathcal{I}(t, \widehat{\theta}) + \int_{\Omega\times\mathbb{S}^{d-1}} \Lambda(\omega)\,\mathrm{d}\widehat{\mu} \ - \ \mathcal{I}(t, \theta_*) - \int_{\Omega\times\mathbb{S}^{d-1}} \Lambda(\omega)\,\mathrm{d}\mu \le \mathcal{I}(t, \widehat{\theta}) - \mathcal{I}(t, \theta_*),$$

where we used $\Lambda\widehat{\mu} \le \Lambda\mu$ because of $\Lambda(\pm\omega_*) = 0$, $\Lambda \ge 0$, and $0 \le b \le 1$ in (5.7). ∎

5.3 Proof of Proposition 5.2

We consider a sequence $\theta_n \in \mathcal{P}$ with $\theta_n \xrightarrow{\mathrm{H}} (\theta_*, \mu)$. For a given $\eta \in \mathcal{Z}$ we have to construct a sequence $(\overline{K}_n)_{n\in\mathbb{N}}$ such that for all $k_n \ge \overline{K}_n$ the functions $\zeta_n := \max\{\theta_n, S^\eta_{k_n}\}$ satisfies

$$\zeta_n \xrightarrow{\mathrm{H}} \big(\theta_*{+}(1{-}\theta_*)\eta, \widehat{\mu}\big) \quad \text{with } \widehat{\mu} = (1{-}\eta)^2\mu + (1{-}\theta_*)^2\eta(1{-}\eta)\otimes\frac{1}{2}\big(\delta_{\omega_*}{+}\delta_{-\omega_*}\big). \tag{5.8}$$

Proof of Proposition 5.2. *Step 1:* As in [The02] we use that for $a, b \in \{0,1\}$ we have the simple relation $\max\{a,b\} = a + (1{-}a)b$. Hence, using $\theta_n, S^\eta_k \in \mathcal{P}$ we have

$$\zeta_n \ = \ Z_{n,k_n} \text{ with } Z_{n,k} := \theta_n + (1{-}\theta_n)S^\eta_k \ = \ (1{-}\theta_n)\,(S^\eta_k{-}\eta) + (1{-}\eta)\theta_n + \eta.$$

We first consider the weak limit, where fixing n and considering $k \to \infty$ we find $Z_{n,k} \rightharpoonup Z_{n,\infty} := (1{-}\eta)\theta_n + \eta$ due to $S^\eta_k \rightharpoonup \eta$. Since the weak L^2-convergence in $\mathcal{Z}$ is metrizable by some metric d_w, we can choose $\widetilde{K}_n$ such that $d_\mathrm{w}(Z_{n,k}, Z_{n,\infty}) \le 1/n$ for all $k \ge \widetilde{K}_n$. Now, $Z_{n,\infty} \rightharpoonup z_\infty := (1{-}\eta)\theta_*{+}\eta$ implies $\varrho(n) := d_\mathrm{w}(Z_{n,\infty}, z_\infty) \to 0$ and

$$d_\mathrm{w}(Z_{n,k_n}, z_\infty) \le d_\mathrm{w}(Z_{n,k_n}, Z_{n,\infty}) + d_\mathrm{w}(Z_{n,\infty}, (1{-}\eta)\theta_*{+}\eta) \le 1/n + \varrho(n) \to 0,$$

whenever $k_n \ge \widetilde{K}_n$, i.e. we have $\zeta_n = Z_{n,k_n} \rightharpoonup z_\infty$ as desired.

Step 2: For the H-measure we use Proposition 3.2 to conclude that the term $(1{-}\eta)\theta_n$ generates the H-measure $\widehat{\mu}^2 := (1{-}\eta)^2\mu$, while the third term η is constant and hence does not contribute to $\widehat{\mu}$.

Step 3: We next show that the first term $\zeta^1_n := (1{-}\theta_n)\,(S^\eta_{k_n}{-}\eta)$ generates the measure $\widehat{\mu}^1 := (1{-}\theta_*)^2\eta(1{-}\eta)\otimes\frac{1}{2}\big(\delta_{\omega_*}{+}\delta_{-\omega_*}\big)$ if k_n grows sufficiently fast. The Fourier transform $\mathfrak{F}$ (where the extension $\mathbb{E}$ by 0 on $\mathbb{R}^d \setminus \Omega$ is suppressed) satisfies the convolution formula

$$\mathfrak{F}\big((1{-}\theta_n)\,(S^\eta_{k_n}{-}\eta)\big) = f_n * g_n \text{ with } f_n(\xi) = \mathfrak{F}\big((1{-}\theta_n)\big)(\xi) \text{ and } g_n(\xi) = \mathfrak{F}\big((S^\eta_{k_n}{-}\eta)\big)(\xi).$$

We choose a radius R_n such

$$\int_{|\xi|>R_n} |f_n(\xi)|^2 \,\mathrm{d}\xi \leq \frac{1}{n}\int_{\mathbb{R}^d} |f_n(\xi)|^2 \,\mathrm{d}\xi = \frac{1}{n}\|f_n\|^2_{\mathrm{L}^2} \leq \frac{C_1}{n}.$$

Since $S^\eta_k{-}\eta \rightharpoonup 0$ the Fourier transform converges to 0 in the balls $\{\, \xi \in \mathbb{R}^d \mid |\xi| \leq R_n \,\}$ for $k \to \infty$ and n fixed. Moreover, the fast oscillations in $x \mapsto H(\eta_k(x), k^2\omega_*{\cdot}x){-}\eta(x)$ lead to a spreading of the Fourier transform in the directions $\xi \approx \pm\lambda\omega_*$ with $\lambda \geq 2k^2\pi$. Indeed, recalling $|\omega_*| = 1$ and setting

$$\Xi_k := \Big\{\, \xi \in \mathbb{R}^d \;\Big|\; |\xi\cdot\omega_*| \geq \max\big\{2k^2\pi{-}k, \frac{k{-}1}{k}\,|\xi|\big\} \,\Big\}$$

we find the relation

$$\rho(k) := \int_{\mathbb{R}^d\setminus\Xi_k} |g_k(\xi)|^2 \,\mathrm{d}\xi \to 0 \text{ for } k \to \infty.$$

Choosing $\overline{K}_n \geq \widetilde{K}_n$ such that $\rho(k) \leq 1/n$ for all $k \geq \overline{K}_n$, we see that the convolution $f_n * g_n$ has most of its mass inside the set $\mathfrak{X}_n := \Xi_{k_n} + \{\, \xi \mid |\xi| \leq R_n \,\}$, i.e.

$$\int_{\mathbb{R}^d\setminus\mathfrak{X}_n} |(f_n * g_n)(\xi)|^2 \,\mathrm{d}\xi \leq C/n.$$

Since the radial projection of $\mathfrak{X}_n$ on $\mathbb{S}^{d-1}$ converges to $\{\omega_*, -\omega_*\}$, the H-measure generated by ζ^1_n is $\alpha(x)\otimes\frac{1}{2}\big(\delta_{\omega_*}{+}\delta_{-\omega_*}\big)$, where α is the weak limit of $\big((1{-}\theta_n)\,(S^\eta_{k_n}{-}\eta)\big)^2$. As in Step 1 we obtain $\alpha = (1{-}\theta_*)^2\eta(1{-}\eta)$, where we may increase $\overline{K}_n$ if necessary.

Step 4: We still have to show that the sum

$$\zeta^1_n + \zeta^2_n := (1{-}\theta_n)\,(S^\eta_{k_n}{-}\eta) + (1{-}\eta)(\theta_n{-}\theta_*)$$

generates the H-measure $\widehat{\mu}^1 + \widehat{\mu}^2$. For this it suffices to show that $\mathfrak{F}(\zeta^1_n)$ and $\mathfrak{F}(\zeta^2_n)$ have their masses well separated. By Step 3 we know that the essential part of the mass of $\mathfrak{F}(\zeta^2_n)$ is contained in $\Xi_{k_n} + \big\{\, \xi \mid |\xi| \leq R_n \,\big\}$, while the essential part of the mass of $\mathfrak{F}\big((1{-}\eta)\theta_n\big)$ is concentrated in $\widetilde{R}_n$. Increasing $\overline{K}_n$ if necessary, for every test function $\varphi \in \mathrm{C_c}(\Omega)$ we find $\|\mathfrak{F}(\varphi\zeta^1_n)\overline{\mathfrak{F}(\varphi\zeta^2_n)}\|_{\mathrm{L}^1(\mathbb{R}^d)} \to 0$. Thus, we conclude

$$\begin{aligned}
&\int_{\Omega\times\mathbb{S}^{d-1}} |\varphi(x)|^2\Psi(\omega)\mu(\mathrm{d}x,\mathrm{d}\omega) = \lim_{n\to\infty}\int_{\mathbb{R}^d} |\mathfrak{F}\big(\varphi(\zeta^1_n{+}\zeta^2_n)\big)(\xi)|^2\Psi(\xi/|\xi|)\,\mathrm{d}\xi \\
&= \lim_{n\to\infty}\Big[\int_{\mathbb{R}^d} |\mathfrak{F}(\varphi\zeta^1_n)|^2\Psi\,\mathrm{d}\xi + 2\,\mathrm{Re}\Big(\int_{\mathbb{R}^d} \mathfrak{F}(\varphi\zeta^1_n)\overline{\mathfrak{F}(\varphi\zeta^2_n)}\;\Psi\,\mathrm{d}\xi\Big) + \int_{\mathbb{R}^d} |\mathfrak{F}(\varphi\zeta^2_n)|^2\Psi\,\mathrm{d}\xi\Big] \\
&= \int_{\Omega\times\mathbb{S}^{d-1}} |\varphi(x)|^2\Psi(\omega)\widehat{\mu}^1(\mathrm{d}x,\mathrm{d}\omega) \;+\; 0 \;+\; \int_{\Omega\times\mathbb{S}^{d-1}} |\varphi(x)|^2\Psi(\omega)\widehat{\mu}^2(\mathrm{d}x,\mathrm{d}\omega).
\end{aligned}$$

Thus, Proposition 5.2 is proved. ∎

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