Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Variational approaches and methods for dissipative material models with multiple scales

Alexander Mielke

submitted: 19 February, 2015

Weierstraß-Institut
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin-Adlershof
Germany

No. 2084
Berlin 2015

2010 *Mathematics Subject Classification.* 35Q74, 47J35, 49Sxx, 74C15.

Key words and phrases. Generalized gradient systems, energy-dissipation principle, evolutionary Gamma convergence, energeitc solutions, rate-independent systems, balanced-viscosity solutions, finite-strain elastoplasticity, laminate evolution.

Research partially supported by DFG via FOR 797 *MicroPlast*, subproject P5.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

In a first part we consider evolutionary systems given as generalized gradient systems and discuss various variational principles that can be used to construct solutions for a given system or to derive the limit dynamics for multiscale problems. These multiscale limits are formulated in the theory of evolutionary Gamma-convergence. On the one hand we consider the a family of viscous gradient system with quadratic dissipation potentials and a wiggly energy landscape that converge to a rate-independent system. On the other hand we show how the concept of Balanced-Viscosity solution arise as in the vanishing-viscosity limit.

As applications we discuss, first, the evolution of laminate microstructures in finite-strain elastoplasticity and, second, a two-phase model for shape-memory materials, where H-measures are used to construct the mutual recovery sequences needed in the existence theory.

1 Introduction

This work shows how methods from abstract evolutionary systems ba be employed for the study of material models which allow for small or finite-strain elastic deformation y and are characterized by further internal or dissipative variables z which may describe damage, plastic deformations, magnetization, polarization, or phase transformations. The common feature of all models considered is their description in terms of an energy functional $\mathcal{E}$ and a dissipation potential $\mathcal{R}$. Hence the evolution of the state $q = (y, z)$ can be described by a generalized force balance, namely

$$0 \in \partial_{\dot{q}} \mathcal{R}(q(t), \dot{q}(t)) + \mathrm{D}_q \mathcal{E}(t, q(t)). \tag{1}$$

Here $\partial_{\dot{q}} \mathcal{R}(q, \dot{q})$ denotes the convex subdifferential of the dissipation potential $\mathcal{R}$, where for each state q the function $\mathcal{R}(q, \cdot)$ is nonnegative, convex, and lower semicontinuous and satisfies $\mathcal{R}(q, 0) = 0$. Thus, the possibly set-valued subdifferential $\partial_{\dot{q}} \mathcal{R}(q, \dot{q})$ contains the dissipative forces generated by the rate $\dot{q}$ if the system is in the state q. These forces have to be balanced by the potential restoring forces $-\mathrm{D}_q \mathcal{E}(t, q)$.

The formulation of material models in terms of the functionals $\mathcal{E}$ and $\mathcal{R}$ instead of general PDEs shows additional physical structure that can be exploited mathematically. In particular, one can employ the rich theory of the calculus of variations, even for evolutionary systems. As a first case, we see that a very useful time discretization of (1) can be obtained by the time-incremental minimization problem

$$q_{k+1} \in \operatorname*{Arg\,min}_q \Big(\mathcal{E}(t_{k+1}, q) + (t_{k+1} - t_k) \mathcal{R}\big(q_{k+\theta}, \tfrac{1}{t_{k+1}-t_k}(q - q_k)\big) \Big). \tag{2}$$

In the context of abstract evolutionary systems this scheme relates to De Giorgi's theory of minimizing movements, and one way of obtaining solutions is via De Giorgi's $(\mathcal{R}, \mathcal{R}^*)$-principle, also called the energy-dissipation principle (EDP), which is given by the simple variational characterization via

$$\mathcal{E}(T, q(T)) + \int_0^T \mathcal{R}(q, \dot{q}) + \mathcal{R}^*(q, -\mathrm{D}\mathcal{E}(t, q)) \, \mathrm{d}t \le \mathcal{E}(0, q(0)) + \int_0^T \partial_t \mathcal{E}(t, q) \, \mathrm{d}t.$$

This principle and its equivalence to (1) will be discussed in Section 2.1.

The EDP is also extremely useful for studying multiscale problems given in terms of generalized gradient systems $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, where $\varepsilon \in [0,1]$ is a small parameter. The major question is under what conditions the solutions $q_\varepsilon : [0,T] \to \boldsymbol{X}$ for $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converge to a solution $q_0 : [0,T] \to \boldsymbol{X}$ for $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ in the limit $\varepsilon \to 0$. If this holds and additionally the energies converge, i.e. $\mathcal{E}_\varepsilon(t, q_\varepsilon(t)) \to \mathcal{E}_0(t, q_0(t))$ we call this *evolutionary* Γ*-convergence*. In general, the Γ-convergences $\mathcal{E}_\varepsilon \overset{\Gamma}{\to} \mathcal{E}_0$ and $\mathcal{R}_\varepsilon \overset{\Gamma}{\to} \mathcal{R}_0$ are not enough. We discuss some of the results from [Mie14] and give applications to models with wiggly energies, where for $\varepsilon > 0$ the dissipation potentials $\mathcal{R}_\varepsilon(q,v) = \frac{1}{2}\langle v, \mathbb{G}_\varepsilon(q)v\rangle$ are quadratic and satisfy $\mathcal{R}_\varepsilon \to 0$, but the limiting dissipation potential $\mathcal{R}_0$ is 1-homogeneous, such that $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ is a rate-independent system (RIS), such as linearized elastoplasticity, see Section 4.2.

Moreover, the vanishing-viscosity limit $\varepsilon \to 0$ of generalized gradient systems $(\boldsymbol{X}, \mathcal{E}, \mathcal{R}_\varepsilon)$, where the "small-viscosity dissipation potential" has the form $\mathcal{R}_\varepsilon(q,v) = \Psi(q,v) + \frac{\varepsilon}{2}\langle v, \mathbb{G}v\rangle$, can also be studied efficiently using a reparametrized version of the EDP, see Section 4.3. This leads to the notion of *balanced-viscosity solutions* (also called BV solutions) for RIS $(\boldsymbol{X}, \mathcal{E}, \Psi, \mathbb{G})$, where $\mathbb{G}$ indicates the additional viscosity structure which determines the jump behavior.

For purely rate-independent models it is advantageous to replace the infinitesimal dissipation metric Ψ by the dissipation distance $\mathcal{D}(q_0, q_1)$ between two states z_0 and z_1. This leads to the notion of *energetic rate-independent systems* (ERIS). In particular, the time-incremental minimization (2) does not depend on the time step and can be replaced by

$$q^{k+1} \ \in \ \operatorname*{Arg\,min}_{q \in \boldsymbol{X}} \Big(\mathcal{E}(t_{k+1}, q) + \mathcal{D}(q_k, q) \Big). \tag{3}$$

It was observed in [MTL02] that all accumulation points of the piecewise interpolants of the solutions of (3) are so-called *energetic solutions*, see (5) for the purely energetic definition of this solution concept.

A corresponding notion of evolutionary Γ-convergence for ERIS $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ was developed in [MRS08], see also [MiR15] for more details. Using this approach and the general existence theory for finite-strain elastoplasticity from [MaM09, Mie10] it was shown in [MiS13] that linearized elastoplasticity can be derived as the evolutionary Γ-limit of finite-strain elastoplasticity, if the yields stress is tending to 0, see Section 3.2.

In Section 5 we discuss two rate-independent material models that describe the evolution of microstructures. The first one is a mathematical version of the model proposed in [KoH11], where laminates are considered as dissipative internal variables and equipped with a physically motivated dissipation distance, see Section 5.1 and [HHM12]. In Section 5.2 the two-phase model introduced in [MTL02] is reconsidered using a new construction for mutual recovery sequences, which allows us to generalize the original existence proof considerably.

2 Variational formulations for evolution

A main point of looking in different variational principles lies in the fact that theses principles lead to different mathematical formulations. For instance, when looking to global existence results for material models allowing for finite strains and the associated geometric nonlinearities, it is highly desirable to use minimization principles on the energy such that the rich theory of direct methods from the calculus of variations are applicable, such as weak lower semicontinuity, existence of minimizers, Γ-convergence, and relaxation techniques.

2.1 Generalized gradient systems and the energy-dissipation principle

We now convert the formal ideas from the introduction into rigorous mathematical statements. We call a triple $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$ a *generalized gradient system (gGS)*, if $\boldsymbol{X}$ is a Banach space, $\mathcal{E} : [0,T] \times \boldsymbol{X} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is an energy functional, and $\mathcal{R} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty]$ is a *dissipation potential*, which means that for all $q \in \boldsymbol{X}$ the functional $\mathcal{R}(q,\cdot) : \boldsymbol{X} \to \mathbb{R}_\infty$ is lower semicontinuous, nonnegative, convex, and satisfies $\mathcal{R}(q,0) = 0$. We speak of a classical gradient system, or simply a gradient system, if $\mathcal{R}(q,\cdot)$ is quadratic, i.e. there exists a (viscosity) operator $\mathbb{G}(q) = \mathbb{G}(q)^* \geq 0$ such that $\mathcal{R}(q,v) = \frac{1}{2}\langle \mathbb{G}(q)v, v\rangle$. However, plasticity requires non-quadratic dissipation potentials, e.g. of the form $\mathcal{R}(\dot\pi) = \sigma_{\text{yield}}\|\dot\pi\|_{\mathrm{L}^1} + \frac{1}{2}\mu_{\text{visc}}\|\dot\pi\|^2_{\mathrm{L}^2}$. In particular, the rate-independent case requires $\mathcal{R}(q,\lambda v) = \lambda \mathcal{R}(q,v)$ for all $\lambda > 0$, which is incompatible with a quadratic form.

The following proposition from convex analysis shows that there are several completely equivalent formulations of the generalized force balance (1). The equivalences of the points (ii) to (iv) are also called the Fenchel equivalences, cf. [Fen49]. The essential tools is the Fenchel-Legendre transform $\Psi^* : \boldsymbol{X}^* \to \mathbb{R}_\infty$ of a convex function $\Psi : \boldsymbol{X} \to \mathbb{R}_\infty$ defined via

$$\Psi^*(\xi) := \sup\{\, \langle \xi, v\rangle - \Psi(v) \mid v \in \boldsymbol{X} \,\}.$$

Note that in a reflexive Banach space we have $(\Psi^*)^* = \Psi$.

Proposition 2.1 (Equivalent formulations) *Let $\boldsymbol{X}$ be a reflexive Banach space and $\Psi : \boldsymbol{X} \to \mathbb{R}_\infty$ be proper, convex, and lower semicontinuous. Then, for every $\xi \in \boldsymbol{X}^*$ and every $v \in \boldsymbol{X}$ the following five statements are equivalent:*

$$\begin{aligned}
&\textit{(i)} \quad v \in \operatorname*{Arg\,min}_{w \in \boldsymbol{X}} \big(\Psi(w) - \langle \xi, w\rangle\big); \qquad \textit{(ii)}\ \xi \in \partial\Psi(v);\\
&\textit{(iii)} \quad \Psi(v) + \Psi^*(\xi) = \langle \xi, v\rangle;\\
&\textit{(iv)} \quad v \in \partial\Psi^*(\xi); \qquad \textit{(v)} \quad \xi \in \operatorname*{Arg\,min}_{\eta \in \boldsymbol{X}^*} \big(\Psi^*(\eta) - \langle \eta, v\rangle\big).
\end{aligned}$$

Note that the definition of Ψ^* immediately implies the Young-Fenchel inequality $\Psi(w) + \Psi^*(\eta) \geq \langle \eta, w\rangle$ for all w and η. Thus, (iii) expresses an optimality as well.

Defining the dual dissipation potential $\mathcal{R}^*$ via $\mathcal{R}^*(q,\cdot) := (\mathcal{R}(q,\cdot))^*$ we can apply these equivalences to reformulate (1) in the following ways:

(I) Helmholtz-Rayleigh principle [Hel69, Ray71]

(HRP) $\quad \dot q \in \operatorname{Arg\,min}\Big(\mathcal{R}(q,v) - \langle \mathrm{D}\mathcal{E}(t,q), v\rangle\Big);$

(II) Force balance in $\boldsymbol{X}^*$ Rayleigh-Biot equation [Ray71, Bio55]

(FB) $\quad 0 \in \partial_{\dot q}\mathcal{R}(q,\dot q) + \mathrm{D}\mathcal{E}(t,q) \ \in \boldsymbol{X}^*;$

(III) Power balance in $\mathbb{R}$ De Giorgi's $(\mathcal{R},\mathcal{R}^*)$ formulation [DMT80]

(PB) $\quad \mathcal{R}(q,\dot q) + \mathcal{R}^*(q,-\mathrm{D}\mathcal{E}(t,q)) = -\langle \mathrm{D}\mathcal{E}(t,q),\dot q\rangle;$

(IV) Rate equation in $\boldsymbol{X}$ Onsager equation [Ons31]

(RE) $\quad \dot q \in \partial_\xi \mathcal{R}^*(q,-\mathrm{D}\mathcal{E}(t,q)) \ \in \boldsymbol{X};$

(V) Maximum dissipation principle cf. e.g. [HaF08]

(MDP) $\quad \mathrm{D}\mathcal{E}(t,q) \in \operatorname{Arg\,max}\Big(\langle \xi,\dot q\rangle - \mathcal{R}^*(q,\xi)\Big).$

Note that we have changed the sign in (V) to justify the name of (MDP). The reason for this will become apparent in the rate-independent setting where $\mathcal{R}^*$ only takes the two values 0 and ∞, see (4) and [HaF08].

Before returning to the general situation, we highlight the three different cases (II)–(IV) for the classical viscous dissipation, i.e. $\mathcal{R}(u,v) = \frac12\langle \mathbb{G}v, v\rangle$ and $\mathcal{R}^*(u,\xi) = \frac12\langle \xi, \mathbb{K}\xi\rangle$ with $\mathbb{K} = \mathbb{G}^{-1}$. Then, we have

$$\begin{aligned} &\text{(FB)} \quad \mathbb{G}\dot u = -\mathrm{D}\mathcal{E}(u) \qquad\qquad \text{(RE)} \quad \dot u = -\mathbb{K}\mathrm{D}\mathcal{E}(u) = -\nabla_{\mathbb{G}}\mathcal{E}(u)\\ &\text{(PB)} \quad \frac12\langle \mathbb{G}\dot u,\dot u\rangle + \frac12\langle \mathrm{D}\mathcal{E}(u), \mathbb{K}\mathrm{D}\mathcal{E}(u)\rangle = -\langle \mathrm{D}\mathcal{E}(u),\dot u\rangle, \end{aligned}$$

where (RE) can be seen as a "gradient evolution", as $\nabla_{\mathbb{G}}$ is the gradient operator.

The above forms can already be understood as variational formulations, since the evolution is expressed by extremizing a functional or by variations or derivatives of the two functionals $\mathcal{E}$ and $\mathcal{R}$. However, for mathematical purposes it is desirable to have variational formulations for the whole solution trajectories $q : [0,T] \to \boldsymbol{X}$. One such principle can be derived on the basis of the power balance (PB) by integration in time and using the chain rule and finally employing the Young-Fenchel inequality $\Psi(w) + \Psi^*(\eta) \geq \langle \eta, w\rangle$, cf. [DMT80] or the survey [Mie14].

Theorem 2.2 (De Giorgi's *energy-dissipation principle*) *Under suitable technical conditions on $(\boldsymbol{X},\mathcal{E},\mathcal{R})$ a function $q : [0,T] \to \boldsymbol{X}$ satisfies (I)–(V) from above for almost all $t \in [0,T]$ if and only if the* Energy-Dissipation Principle *(EDP) holds:*

$$\text{(EDP)} \qquad \begin{cases} \mathcal{E}(T,q(T)) + \displaystyle\int_0^T \mathcal{R}(q,\dot q) + \mathcal{R}^*(q,-\mathrm{D}\mathcal{E}(t,q))\,\mathrm{d}t \\ \qquad\qquad \leq \mathcal{E}(0,q(0)) + \displaystyle\int_0^T \partial_t\mathcal{E}(t,q(t))\,\mathrm{d}t. \end{cases}$$

Then, the EDP is equivalent to the energy-dissipation balance *(EDB), where "$\leq$" in (EDP) is replaced by "$=$".*

It is obvious how to obtain (EDB) (and hence (EDP) from (I)–(V). For this one simply integrates the power balance (III) in time and uses a *abstract chain rule*

$$\mathcal{E}(t,q(t)) = \mathcal{E}(r,q(r)) + \int_r^t \langle \mathrm{D}\mathcal{E}(s,q(s)), \dot{q}(s)\rangle + \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s.$$

Starting from (EDP) and using the chain rule one easily obtains the power balance (III) as an estimate, namely $\int_0^T \mathcal{R}+\mathcal{R}^*\,\mathrm{d}t \le \int_0^T -\langle \mathrm{D}\mathcal{E},\dot{q}\rangle\,\mathrm{d}t$. However, the Young-Fenchel inequality gives $\mathcal{R}+\mathcal{R}^* \ge -\langle \mathrm{D}\mathcal{E},\dot{q}\rangle$ for almost all $t\in[0,T]$, so that the power balance (III) has to hold.

The importance of the EDP is that a discrete counterpart can be derived based on the incremental minimization problem (2) and De Giorgi's variational interpolants $\widetilde{q}_\tau$. In a classical Banach-space setting on can use the piecewise constant right and left-continuous interpolants $\underline{q}_\tau$ and $\overline{q}_\tau$ as well as the piecewise affine interpolant $\widehat{q}_\tau$ (all satisfying $q_\tau(t_k)=q_k$) and obtains the discrete version of EDP in the form

$$\mathcal{E}(t_k,\widehat{q}_\tau(t_k))+\int_{t_l}^{t_k}\mathcal{R}(\underline{q}_\tau,\dot{\widehat{q}})+\mathcal{R}^*(\underline{q}_\tau,-\mathrm{D}\mathcal{E}(t,\widetilde{q}_\tau))\,\mathrm{d}t \le \mathcal{E}(t_l,\widehat{q}_\tau(t_l)) + \int_{t_l}^{t_k}\partial_t\mathcal{E}(t,\overline{q}_\tau)\,\mathrm{d}t.$$

Under suitable assumptions it is possible to take the time-step limit $\tau\to 0$ and arrive at the notion of *weak energy-dissipation solutions*, defined by the condition that

$$\mathcal{E}(t,q(t))+\int_r^t\mathcal{R}(q,\dot{q})+\mathcal{R}^*(q,-\mathrm{D}\mathcal{E}(s,q))\,\mathrm{d}s \le \mathcal{E}(r,q(r)) + \int_r^t\partial_s\mathcal{E}(s,q)\,\mathrm{d}s$$

holds for all $t\in[0,T]$, $s=0$, and almost all $s\in[0,T]$. An existence proof for weak energy-dissipation solutions for a model of finite-strain *viscoplasticity* using the multiplicative decomposition is given in [MRS15]. There it is not possible to derive the missing chain-rule estimate to return back to the differential inclusions (I)–(V).

Another very useful variational principle is only valid for classical gradient systems, where it is possible to define a dissipation distance $\mathcal{D}$. If the energy functionals $\mathcal{E}(t,\cdot)$ are geodesically λ-convex, then one reformulate the evolutionary problem via a so-called evolutionary variational inequality (EVI), see [AGS05, Mie14]. For an application of this theory of geodesically λ-convex gradient systems in one-dimensional viscoelasticity we refer to [MOŞ14]. This one-dimensional existence theory, where $q=y$, relies on time-incremental minimization problems

$$y^{k+1} = \operatorname{Arg\,min}\Big(\frac{1}{2(t_{k+1}-t_k)}\mathcal{D}(w,y^k)^2+\mathcal{E}(w)\Big)$$

and establishes strong convergence of the solution even in the case of nonconvex $\mathcal{E}$.

An approximative variational characterization of whole trajectories can be obtained by the *weighted energy-dissipation functional* (WED functional), which is defined via

$$\mathcal{W}_\varepsilon(q) = \int_0^T \mathrm{e}^{-t/\varepsilon}\big(\mathcal{R}(q(t),\dot{q}(t))+\frac{1}{\varepsilon}\mathcal{E}(t,q(t))\big)\,\mathrm{d}t, \quad q(0)=q_0.$$

and which was introduced in [MiO08]. Under sufficient smoothness of $\mathcal{E}$ and $\mathcal{R}$ we see that the Euler-Lagrange equation takes the form

$$\mathrm{D}_{\dot q}\mathcal{R}(q,\dot q)+\mathrm{D}_q\mathcal{E}(t,q)=\varepsilon\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{D}_{\dot q}\mathcal{R}(q,\dot q)\big)-\mathrm{D}_q\mathcal{R}(q,\dot q)\Big),\quad \mathrm{D}_{\dot q}\mathcal{R}(q(T);\dot q(T))=0.$$

Thus, we obtain an "elliptic regularization" of the original evolutionary problem. The advantage is that showing the existence of minimizers $\widehat{q}_\varepsilon:[0,T]\to\boldsymbol{X}$ for $\mathcal{W}_\varepsilon$ is usually much easier than establishing the existence of solutions for the gGS. Yet, the major problem then is to pass to the limit $\varepsilon\to 0$ to find a limit q of the approximations $\widehat{q}_\varepsilon$. For the rate-independent case $\mathcal{R}(q,v)=\Psi(v)$ this was done in [MiO08] obtaining energetic solutions q. For classical gradient system $\mathcal{R}(q,v)=\frac12\langle\mathbb{G}v,v\rangle$ with $\mathbb{G}$ independent of q the convergence $\widehat{q}_\varepsilon\to q$ was established in [MiS11].

The general aim of introducing the WED functional in [MiO08] was the possibility of using relaxation techniques that are invented originally only for stationary problems also in the context of evolutionary problems. First results on such relaxations are presented in [MiO08, Sec. 4.4+5], mainly in the context of RIS. For a proper relaxation of a viscous PDE we refer to [CoO08, Sec. 4], where the case

$$\boldsymbol{X}=\mathrm{L}^2(\Omega),\quad \mathcal{E}(q)=\int_\Omega F(\nabla q(x))-f(t,)q\,\mathrm{d}x,\quad \mathcal{R}(\dot q)=\frac12\int_\Omega \dot q^2\,\mathrm{d}x$$

was considered, with $\Omega\subset\mathbb{R}^2$ and $F(A)=0$ for $A\in K:=\{\pm(1,0),\pm(0,1)\}$ and ∞ else. It is proved that quasiminimizers $\widetilde{q}_\varepsilon$ of $\mathcal{W}_\varepsilon$ converge to solutions of the relaxed evolution defined via the differential inclusion

$$\dot q=\frac12\operatorname{div}\sigma+\frac12 f,\quad \text{where } \sigma(t,x)\in\partial\chi_S(\nabla u(t,x)),$$

where $S=\operatorname{conv}K=\{\,(A_1,A_2)\in\mathbb{R}^2\mid |A_1|+|A_2|\le 1\,\}$ and χ_S is indicator function of convex analysis, i.e. $\chi_S(A)=0$ for $A\in S$ and ∞ otherwise.

2.2 Rate-independent systems and energetic solutions

The case of purely rate-independent dissipation is distinct from the general dissipation potentials. It is characterized by the condition on $\mathcal{R}(q,\lambda v)=\lambda\mathcal{R}(q,v)$ for all $\lambda>0$. In that case we call $(\boldsymbol{X},\mathcal{E},\mathcal{R})$ a *rate-independent system (RIS)*. Then, the force-velocity relation $v\mapsto\partial_v\mathcal{R}(q,v)$ is meant in the sense of subdifferentials of convex functions, which is set-valued:

$$\partial\Psi(v)=\{\,\eta\in\boldsymbol{X}^*\mid\forall\, w\in\boldsymbol{X}:\ \Psi(w)\ge\Psi(v)+\langle\eta,w-v\rangle\,\}.$$

For rate-independent cases we have

$$\partial_v\mathcal{R}(q,\lambda v)=\partial_v\mathcal{R}(q,v)=\{\,\eta\in K(q)\mid\mathcal{R}(q,v)=\langle\eta,v\rangle\,\},$$

where $K(q):=\partial_v\mathcal{R}(q,0)$ is called the elastic domain. Moreover, for the dual dissipation potential we find the simple form

$$\mathcal{R}^*(q,\xi)=\chi_{K(q)}(\xi)=\begin{cases}0 & \text{for } \xi\in K(q),\\ \infty & \text{for } \xi\notin K(q),\end{cases}$$

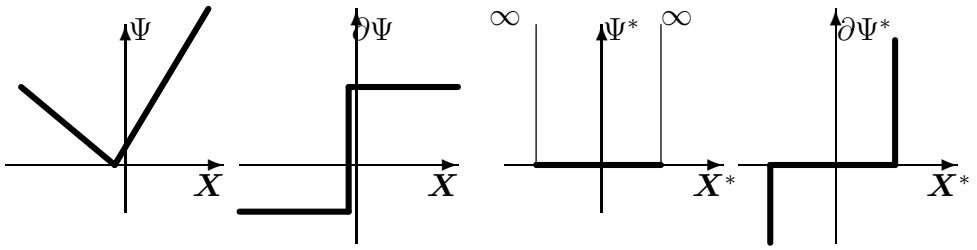

Figure 1: Primal and dual dissipation potential for RIS.

see Figure 1.

In principle the five formulations I to V of the previous subsection are still valid for RIS. However, one can use the special structure of $\partial_v \mathcal{R}$ and $\mathcal{R}^*$ to simplify the presentation. For instance, the maximum-dissipation principle reduces to the simpler form

$$\text{rate-independent MDP:} \qquad \mathrm{D}_q \mathcal{E}(t,q) = \operatorname*{Arg\,max}_{\xi \in K(q)} \langle \xi, \dot{q} \rangle. \tag{4}$$

Second the energy-dissipation principle in the rate-independent case takes a simpler form as $\mathcal{R}^*$ is either 0 or ∞. A differentiable function $q : [0,T] \to \boldsymbol{X}$ solves I to V if and only if

$$\begin{aligned} &(\mathrm{S})_{\mathrm{loc}} \quad -\mathrm{D}_q \mathcal{E}(t,q) \in K(q) := \partial_v \mathcal{R}(q,0), \\ &(\mathrm{E}) \qquad \mathcal{E}(T;q(T)) + \int_0^T \mathcal{R}(q,\dot{q})\,\mathrm{d}t = \mathcal{E}(0,q(0)) + \int_0^T \partial_t \mathcal{E}(t,q)\,\mathrm{d}t. \end{aligned}$$

We call the first condition a *local stability condition*, since the system stays in a state $q(t)$ in which the driving force $\xi(t) = \mathrm{D}_q \mathcal{E}(t,q(t))$ is not big enough to overcome the possible dissipative forces $\eta \in K(q)$.

The major problem for RIS is that the solutions will in general develop jumps, i.e. the three values $q(t{-}0) := \lim_{s \nearrow t} q(s)$, $q(t)$, and $q(t{+}0) := \lim_{s \searrow t} q(s)$ may be different. In such a discontinuous situation the differential formulations are not really useful. Of course, if there is enough convexity in the system the solution will not develop jumps and the above formulations are optimal.

In general cases, the notion of energetic solutions can be used to characterize solutions with jump in a variational way. Instead of the infinitesimal dissipation potential $\mathcal{R}$, which in mathematical terms plays the role of a infinitesimal Finsler metric, is not suitable but can be replaced by a *dissipation distance* $\mathcal{D} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty]$ which is assumed to satisfy the triangle inequality $\mathcal{D}(q_1,q_3) \le \mathcal{D}(q_1,q_2) + \mathcal{D}(q_2,q_3)$, but the symmetry $\mathcal{D}(q_1,q_2) = \mathcal{D}(q_2,q_1)$ is not needed. The triple $(\boldsymbol{X},\mathcal{E},\mathcal{D})$ is called an *energetic rate-independent systems* (ERIS), and a function $q : [0,T] \to \boldsymbol{X}$ is called an *energetic solution* if for all $t \in [0,T]$ the *global stability (S)* and the *energy balance (E)* hold:

$$\begin{aligned} &(\mathrm{S}) \quad \mathcal{E}(t,q(t)) \le \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(q(t),\widetilde{q}) \text{ for all } \widetilde{q} \in \boldsymbol{X}; \\ &(\mathrm{E}) \quad \mathcal{E}(T,q(T)) + \mathrm{Diss}_{\mathcal{D}}(q;[0,T]) = \mathcal{E}(0,q(0)) + \int_0^T \partial_s \mathcal{E}(s,q)\,\mathrm{d}s, \end{aligned} \tag{5}$$

where the total dissipation along a possibly discontinuous solutions is defined via

$$\mathrm{Diss}_{\mathcal{D}}(q;[r,s]) := \{ \sum_{j=1}^{N} \mathcal{D}(q(t_{j-1}),q(t_j)) \mid N \in \mathbb{N},\ r \le t_0 < t_1 < \cdots < t_N \le s \}. \tag{6}$$

For energetic solutions, possible jumps can be given a natural physical interpretation. First, the *energy balance (E)* implies the exact energy conservation $\mathcal{E}(t, q(t+0)) = \mathcal{E}(t, q(t-0)) - \mathcal{D}(q(t-0), q(t+0))$. Second, (S) implies that a jump immediately occurs if it is possible, which is called the principle of realizability in [MTL02].

The notion of energetic solutions was first introduced in [MTL02], and under suitable technical assumptions it was shown that all limits of the piecewise constant interpolants of the solutions of the time-incremental minimization problems

$$q^{k+1} \in \operatorname*{Arg\,min}_{\widetilde{q} \in \boldsymbol{X}} \Big(\mathcal{D}(q^k, \widetilde{q}) + \mathcal{E}(t_{k+1}, \widetilde{q}) \Big) \tag{7}$$

converge to energetic solutions. We refer to [Mie11b, MiR15] for a detailed account of this theory.

Note that in the incremental problems (7) one is doing a global minimization, which is reflected in the global stability condition (S). This leads to a jump behavior which is sometimes unrealistic, since potential barriers are not seen. To define a notion of solutions that do not show the problem of too early jump, one can treat RIS as limits of rate-dependent systems, i.e. systems with a small viscosity proportional to ε and then consider the vanishing-viscosity limit $\varepsilon \to 0$. The corresponding notion of solutions is called *Balanced-Viscosity solutions*, which will be discussed in Section 4.3.

The two major stimuli in the development of the theory of energetic solutions for RIS were the theory of crack evolution in brittle materials, see [DFT05] for linearized elasticity and [DaL10] for finite-strain elasticity, and the theory of *finite-strain elastoplasticity*, see [MaM09, Mie10]. In the former case the name *irreversible quasistatic evolution* is used for what is called energetic solutions here. In both cases, there is not a useful underlying linear structure in a function space $\boldsymbol{X}$, and the full strength of the abstract definition of energetic solutions is needed.

3 Evolutionary Γ-convergence

Following the notions in the survey article [Mie14] we consider families of gGS $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)_{\varepsilon \in]0,1[}$ and ask the questions whether the solutions q_ε for these system have a limit q for $\varepsilon \to 0$ and whether the limit q is again a solution to a gGS $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$. Ideally, one might hope that it is sufficient that $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ convergence in a suitable topology to $\mathcal{E}_0$ and $\mathcal{R}_0$, respectively. We will show that such results exist, but we will also discuss situations where we start with quadratic $\mathcal{R}_\varepsilon$ and end up with a limiting dissipation $\mathcal{R}_0$ that is rate independent.

We first give the general definition of *pE-convergence*, which is a short name of *evolutionary* Γ*-convergence* with wellprepared initial conditions. Hence, the letter"E" stands for both, 'E'volutionary convergence and 'E'nergy convergence. while the letter "p" stands for well'P'reparedness of the initial conditions, in contrast to E-convergence, where the latter is not needed.

Definition 3.1 (pE-convergence of $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$) *We say that the generalized gradient*

systems $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ *pE-converge to* $(\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0)$*, and write* $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)\overset{pE}{\to}(\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0)$*, if*

$$\left.\begin{array}{c} q_\varepsilon:[0,T]\to\boldsymbol{X} \\ \text{is sol. of } (\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon), \\ q_\varepsilon(0)\to q^0, \text{ and} \\ \mathcal{E}_\varepsilon(0,q_\varepsilon(0))\to\mathcal{E}_0(0,q^0)<\infty \end{array}\right\} \implies \left\{\begin{array}{c} \exists\, q \text{ sol. of } (\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0) \text{ with } q(0)=q^0 \\ \text{and a subsequence } \varepsilon_k\to 0: \\ \forall\, t\in\,]0,T]\text{: } q_{\varepsilon_k}(t)\to q(t) \text{ and} \\ \mathcal{E}_{\varepsilon_k}(q_{\varepsilon_k}(t))\to\mathcal{E}_0(q(t)). \end{array}\right. \tag{8}$$

Similarly, we define the pE-convergence for ERIS $(\mathcal{Q},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)\overset{pE}{\to}(\mathcal{Q},\mathcal{E}_0,\mathcal{D}_0)$*, if "solution" is understood in the sense of energetic solutions.*

In the following subsection we discuss some abstract results for pE-convergence.

3.1 pE-convergence for generalized gradient systems

The first general approach to the evolutionary Γ-convergence for classical gradient systems, where the variational structure was exploited systematically, goes back to [SaS04], see also [Ser11, Mie14]. This approach is based on the energy-dissipation principle for the gGS $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ presented in Theorem 2.2, which transforms the evolutionary system $0\in\partial_{\dot q}\mathcal{R}_\varepsilon(q_\varepsilon,\dot q_\varepsilon)+\mathrm{D}_q\mathcal{E}_\varepsilon(t,q_\varepsilon)$ into the upper energy-dissipation estimate

$$\mathcal{E}_\varepsilon(t,q_\varepsilon(T))+\mathcal{J}_\varepsilon(q_\varepsilon(\cdot))\le\mathcal{E}_\varepsilon(0,q_\varepsilon(0))+\int_0^T\partial_s\mathcal{E}_\varepsilon(s,q_\varepsilon(s))\,\mathrm{d}s,$$
$$\text{where } \mathcal{J}_\varepsilon(q):=\int_0^T\mathcal{R}_\varepsilon(q(t),\dot q(t))+\mathcal{R}_\varepsilon^*(q(t),-\mathrm{D}_q\mathcal{E}_\varepsilon(t,q(t)))\,\mathrm{d}t$$

Having a variational principle for the whole trajectory, one can now use variational techniques to pass to the limit $\varepsilon\to 0$. First we observe that the first term on the right-hand converges to the desired limit by the assumption of the wellpreparedness of the initial conditions. For the second term on the right-hand side we may assume that it is lower order and can be handled by compactness. In fact, often one has $\mathcal{E}_\varepsilon(t,q)=\mathcal{U}_\varepsilon(q)-\langle\ell_\varepsilon(t),q\rangle$, then $\partial_t\mathcal{E}(t,q)=-\langle\dot\ell_\varepsilon(t),q\rangle$ is linear in q and strong convergence of $\dot\ell_\varepsilon(t)\to\dot\ell(t)$ is $\boldsymbol{X}^*$ is sufficient.

Hence, it remains to estimate the two terms on the left-hand side. Here we can take advantage that we only need an estimate from above, i.e. the liminf estimates

$$\mathcal{E}_0(T,q(T))\le\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(T,q_\varepsilon(T))\quad\text{and}\quad\mathcal{J}_0(q(\cdot))\le\liminf_{\varepsilon\to0}\mathcal{J}_\varepsilon(q_\varepsilon(\cdot))$$

are sufficient. For this, one has to derive suitable a priori estimates on the solutions q_ε such that one is able to extract a subsequence q_{ε_k} which converges in a sufficiently strong topology to establish the desired liminf estimates.

The famous Sandier-Serfaty approach [SaS04, Ser11] relies on the two liminf estimates

$$\int_0^T\mathcal{R}_0(q_0(t),\dot q_0(t))\,\mathrm{d}t\le\liminf_{\varepsilon\to0}\int_0^T\mathcal{R}_\varepsilon(q_\varepsilon(t),\dot q_\varepsilon(t))\,\mathrm{d}t\quad\text{and}$$
$$\mathcal{R}_0^*(q_0,-\mathrm{D}_q\mathcal{E}_0(t,q_0))\le\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon^*(q_\varepsilon,-\mathrm{D}_q\mathcal{E}_\varepsilon(t,q_\varepsilon)).$$

However, the energy-dissipation principle (EDP) is even more flexible, since we do not need these two separate lower bounds. In passing to the liminf for the total dissipation $\int_0^T\mathcal{R}_\varepsilon+\mathcal{R}_\varepsilon^*\,\mathrm{d}t$ we may even give up the special dual form $\mathcal{R}+\mathcal{R}^*$ of the integrand.

This idea, which was applied successfully in [AM*12, Mie12, MPR14, LM*15], can be summarized as follows.

Defining the functional $\mathcal{J}_\varepsilon : \mathrm{W}^{1,1}([0,T];\boldsymbol{X}) \to [0,\infty]$ via

$$\mathcal{J}_\varepsilon(u) := \int_0^T \mathcal{R}_\varepsilon(u,\dot u) + \mathcal{R}_\varepsilon^*(u,-\mathrm{D}\mathcal{E}_\varepsilon(u))\,\mathrm{d}t,$$

we have to find a sufficiently good lower bound for the Γ-liminf, namely

$$\text{(i)}\quad u_\varepsilon(\cdot) \overset{*}{\rightharpoonup} u(\cdot) \text{ in } \mathrm{L}^\infty([0,T];\boldsymbol{X}) \implies \int_0^T \mathcal{M}_0(u(t),\dot u(t))\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon),$$

where the integrand $\mathcal{M}_0$ does not need to be of the form $\mathcal{R}_0 + \mathcal{R}_0^*$. Hence, finding the best (i.e. largest) $\mathcal{M}_0$ is nothing else than finding the (static) Γ-limit of the functionals $\mathcal{J}_\varepsilon$. It suffices to find $(\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0)$ and $\mathcal{M}_0$ such that

(ii) $\mathcal{E}_\varepsilon \overset{\Gamma}{\rightharpoonup} \mathcal{E}_0$;
(iii) $\mathcal{M}_0(u,v) \ge -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$;
(iv) the chain rule holds for$(\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0)$;
(v) $\mathcal{M}_0(u,v) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle \implies \mathcal{R}_0(u,v)+\mathcal{R}_0^*(u,-\mathrm{D}\mathcal{E}_0(u)) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$.

As before, we can start from the EDP $\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{E}_\varepsilon(u_\varepsilon(0))$. Using the wellpreparedness of the initial datum, (i), and (ii) we pass to the limit and obtain the EDP

$$\mathcal{E}_0(u(T)) + \int_0^T \mathcal{M}_0(u(t),\dot u(t))\,\mathrm{d}t \le \mathcal{E}_0(u(0)).$$

Now using the (iii) and the chain rule (iv) we find

$$\begin{aligned}\mathcal{E}_0(u(0)) &\overset{\text{(iv)}}{=} \mathcal{E}_0(u(T)) - \int_0^T \langle \mathrm{D}\mathcal{E}(u(t)),\dot u(t)\rangle\,\mathrm{d}t\\ &\overset{\text{(iii)}}{\le} \mathcal{E}_0(u(T)) + \int_0^T \mathcal{M}_0(u(t),\dot u(t))\,\mathrm{d}t \le \mathcal{E}_0(u(0)).\end{aligned}$$

Thus, we conclude that we must have equality in (iii) for almost all $t\in[0,T]$, such that we can use (v) to conclude that u is a solution for $(\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0)$. Hence, the pE-convergence $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon) \overset{\mathrm{pE}}{\to} (\boldsymbol{X},\mathcal{E}_0,\mathcal{R}_0)$ is established.

Section 4.1 summarizes the results of [Mie12, MiT12], which show that the above strategy can even be applied to justify the passage from small viscous dissipation (i.e. $\mathcal{R}_\varepsilon(u,\cdot)$ is quadratic) to a limit problem with large rate-independent dissipation (i.e. $\mathcal{R}_0(u,\cdot)$ is positively homogeneous of degree 1, see Section 2.2).

In fact, under a slight and natural strengthening of the conditions (i) to (v), it is possible to construct $\mathcal{R}_0$ directly from $\mathcal{M}_0$. Indeed, assume that $\mathcal{M}_0(u,\cdot)$ is additionally even, convex, $\mathbb{R}$-valued, and lower semicontinuous, then $\mathcal{R}_\mathcal{M}$ defined via

$$\mathcal{R}_\mathcal{M}(u,v) := \mathcal{M}_0(u,v) - \mathcal{M}_0(u,0)$$

is a dissipation potential. Moreover, using property (iii) we find the estimate

$$\mathcal{R}^*_{\mathcal{M}}(u,-\mathrm{D}\mathcal{E}_0(u)) = \sup_{v\in\boldsymbol{X}} \Big(\langle -\mathrm{D}\mathcal{E}_0(u),v\rangle - \mathcal{M}_0(u,v) + \mathcal{M}_0(u,0)\Big) \le \mathcal{M}_0(u,0).$$

Thus, we find the desired EDP $\mathcal{E}_0(u(T)) + \int_0^T \mathcal{R}_{\mathcal{M}} + \mathcal{R}^*_{\mathcal{M}}\,\mathrm{d}t \le \mathcal{E}_0(u(0))$. We emphasize that the choice $\mathcal{R}_0 = \mathcal{R}_{\mathcal{M}}$ in (iv) and (v) is admissible, but not unique. In particular, it may be possible to find simpler $\mathcal{R}_0$ as is the case in the application discussed in Section 4.1.

3.2 pE-convergence for rate-independent systems

A quite general theory of evolutionary Γ-convergence for ERIS $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$ was already developed in [MRS08], see also [MiR15] for more details and applications. For simplicity, here we restrict to the case that the energies have the form

$$\mathcal{E}_\varepsilon(t,q) = \mathcal{F}_\varepsilon(q) - \langle \ell_\varepsilon(t),q\rangle, \tag{9a}$$

where $\boldsymbol{X}$ is a reflexive Banach space. We allow for the case that $\mathcal{F}$ is not convex and that the dissipation distances $\mathcal{D}_\varepsilon$ are not translation invariant. A typical set of assumptions reads as follows:

$$\exists\, c,C>0\ \forall\,\varepsilon\in[0,1],\ q\in\boldsymbol{X}:\ \mathcal{F}_\varepsilon(q)\ge c\|q\|^2 - C; \tag{9b}$$

$$\forall\,\varepsilon\in[0,1]:\quad \mathcal{F}_\varepsilon:\boldsymbol{X}\to\mathbb{R}_\infty \text{ is weakly lower semicontinuous;} \tag{9c}$$

$$\exists\, C>0\ \forall\,\varepsilon\in[0,1]:\ \|\ell_\varepsilon\|_{\mathrm{C}^1([0,T])}\le C; \tag{9d}$$

$$\forall\, t\in[0,T]:\quad \dot\ell_\varepsilon(t)\to\dot\ell_0(t) \text{ in } \boldsymbol{X}^* \text{ as } \varepsilon\to 0; \tag{9e}$$

$$\forall\,\varepsilon\in[0,1]\ \forall\, q_j\in\boldsymbol{X}:\ \begin{cases} \mathcal{D}_\varepsilon(q_1,q_3)\le \mathcal{D}_\varepsilon(q_1,q_2)+\mathcal{D}_\varepsilon(q_2,q_3),\\ \mathcal{D}_\varepsilon(q_1,q_2)=0 \implies q_1=q_2. \end{cases} \tag{9f}$$

In general, these conditions together with Γ convergence of the energies and the dissipation are not strong enough to show pE-convergence. Even for existence for a fixed ε we need additional conditions, e.g. weak continuity of $\mathcal{D}_\varepsilon$ is sufficient.

Our first result on pE-convergence for ERIS assumes that the dissipation distances $\mathcal{D}_\varepsilon$ weakly continuously converge to $\mathcal{D}_0$, viz.

$$\mathcal{D}_\varepsilon \overset{\mathrm{C}}{\rightharpoonup} \mathcal{D}_0, \quad \text{which means that } \ q_\varepsilon \rightharpoonup q_0,\ \widehat{q}_\varepsilon\rightharpoonup\widehat{q}_0 \implies \mathcal{D}_\varepsilon(q_\varepsilon,\widehat{q}_\varepsilon)\to\mathcal{D}_0(q_0,\widehat{q}_0).$$

Theorem 3.2 (pE-convergence for ERIS) *Assume that the ERIS $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$ satisfy (9), $\mathcal{E}_\varepsilon \overset{\Gamma}{\rightharpoonup} \mathcal{E}_0$, and $\mathcal{D}_\varepsilon \overset{\mathrm{C}}{\rightharpoonup} \mathcal{D}_0$ in $\boldsymbol{X}$; then $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon) \overset{\mathrm{pE}}{\rightharpoonup} (\boldsymbol{X},\mathcal{E}_0,\mathcal{D}_0)$.*

We refer to [MRS08] for the first proof and to [Mie14, Thm. 5.4] for a shorter proof. In fact, it is rather straightforward to establish the EDP, i.e. (E) in (5) where "=" is replaced by "$\le$". The major difficulty lies in showing that the global stability condition (S) holds for the limit $\varepsilon = 0$. This stability then implies a "chain-rule estimate", which show that (E) holds even with equality "=".

The major tool for passing to the limit in the stability condition is the existence of so-called *mutual recovery sequences*. (A very similar condition is already very useful in

showing existence of energetic solutions.) Given a family $(q_\varepsilon)_{\varepsilon\in[0,1]}$ with $q_\varepsilon \rightharpoonup q_0$ and a test state $\widehat{q}_0$, we say that the family $(\widehat{q}_\varepsilon)_{\varepsilon\in]0,1[}$ is a mutual recovery sequences at time t, if

$$\limsup_{\varepsilon\to 0} \big(\mathcal{E}_\varepsilon(t,\widehat{q}_\varepsilon)-\mathcal{E}_\varepsilon(t,q_\varepsilon)+\mathcal{D}_\varepsilon(q_\varepsilon,\widehat{q}_\varepsilon)\big) \ \le\ \mathcal{E}_0(t,\widehat{q})-\mathcal{E}_0(t,q_0)+\mathcal{D}_0(q_0,\widehat{q}_0). \tag{10}$$

Clearly, if all q_ε satisfy the stability condition at time t, then all term in the limsup are nonnegative; hence we conclude that the right-hand side is nonnegative, which is the stability of q_0 if the test state $\widehat{q}_0$ can be chosen arbitrary. Under the conditions of the above Theorem 3.2 we see that the existence of mutual recovery sequence easily holds, since it suffices to choose recovery sequences for the energy $\mathcal{F}_\varepsilon$ and use the weak continuity of $\mathcal{D}_\varepsilon$ and $\langle \ell_\varepsilon(t),\cdot\rangle$.

In the case that $\boldsymbol{X}$ is a Hilbert space $\boldsymbol{H}$, the energies are quadratic, and the dissipation distances are translationally invariant, viz.

$$\mathcal{F}_\varepsilon(q)=\frac{1}{2}\langle A_\varepsilon q,q\rangle \ge c\|q\|_{\boldsymbol{H}}^2 \quad\text{and}\quad \mathcal{D}_\varepsilon(q_1,q_2)=\Psi_\varepsilon(q_2-q_1), \tag{11}$$

one can construct mutual recovery sequences in the form $\widehat{q}_\varepsilon = q_\varepsilon + w_\varepsilon$ with $w_\varepsilon \rightharpoonup \widehat{q}_0 - q_0$ and exploit the better convergence $\widehat{q}_\varepsilon - q_\varepsilon = w_\varepsilon \rightharpoonup \widehat{q}_0 - q_0$ (strong convergence in $\boldsymbol{H}$!) in the following terms:

$$\mathcal{F}_\varepsilon(\widehat{q}_\varepsilon)-\mathcal{F}_\varepsilon(q_\varepsilon)=\frac{1}{2}\langle A_\varepsilon w_\varepsilon,\widehat{q}_\varepsilon+q_\varepsilon\rangle \quad\text{and}\quad \mathcal{D}_\varepsilon(q_\varepsilon,\widehat{q}_\varepsilon)=\Psi_\varepsilon(w_\varepsilon). \tag{12}$$

Using this, the following result was derived in [LiM11] and [MiR15, Ch. 3.5.4]. Here the Mosco convergence $\mathcal{E}_\varepsilon \overset{\mathrm{M}}{\to} \mathcal{E}_0$ means $\mathcal{E}_\varepsilon(t,\cdot) \overset{\Gamma}{\to} \mathcal{E}_0(t,\cdot)$ and $\mathcal{E}_\varepsilon(t,\cdot) \overset{\Gamma}{\rightharpoonup} \mathcal{E}_0(t,\cdot)$ for all $t\in[0,T]$.

Theorem 3.3 (pE-convergence for quadratic ERIS) *Let $(\boldsymbol{H},\mathcal{E}_\varepsilon,\Psi_\varepsilon)_{\varepsilon\in[0,1]}$ satisfy* (9) *and* (11). *If $\mathcal{E}_\varepsilon \overset{\mathrm{M}}{\to} \mathcal{E}_0$, $\Psi_\varepsilon \overset{\mathrm{C}}{\to} \Psi_0$, and $\Psi_\varepsilon \overset{\Gamma}{\rightharpoonup} \Psi_0$, then $(\boldsymbol{H},\mathcal{E}_\varepsilon,\Psi_\varepsilon) \overset{\mathrm{pE}}{\rightharpoonup} (\boldsymbol{H},\mathcal{E}_0,\Psi_0)$.*

In contrast to Theorem 3.2 we need the continuous convergence $\Psi_\varepsilon \overset{\mathrm{C}}{\to} \Psi_0$ here only in the strong topology of $\boldsymbol{H}$. Applications of this theory occur in linearized elastoplasticity in the context of homogenization in [MiT07, GiM11, Han11] and in the derivation of elastoplastic plate models.

A highly non-trivial application of pE-convergence is treated in [MiS13], where the ERIS $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$ for $\varepsilon>0$ describe models for *finite-strain elastoplasticity* for which existence of energetic solutions was established in [MaM09, Mie10]. In [MiS13], the energy, the dissipation distance, and the loadings are scaled by $\varepsilon>0$ in such a way that the system converges to *linearized elastoplasticity* in the sense of pE-convergence. The major assumption is that the yield stress (contained in $\mathcal{D}_\varepsilon$) scales in the same way as the displacement. Thus, *linearized elastoplasticity* is a justifiable model only under the condition that the yield stress is so small that even small strains can generate plastic effects.

4 Justification of rate-independent models

In this section we discuss two distinct cases in which RIS arise as limits of rate-dependent systems. The typical situation we are interested in is a system with slow loading, where

we always assume that the loading time $t \in [0,T]$ is our relevant time scale. In fact, in mechanics this time scale is often called process time, since it may be significantly larger than the intrinsic time scales inside the material.

In Section 4.1 we consider purely viscous systems, i.e. with a quadratic dissipation potential $\mathcal{R}_\varepsilon(q,v) = \frac{\varepsilon^\alpha}{2}\langle \mathbb{G}(q)v, v\rangle$, where the small parameter ε indicates that the relaxation times due to viscous effects are much smaller, namely of order $O(\varepsilon^\alpha)$. However, to prevent the system to relax into a global minimum for each macroscopic time we consider an energy that has microscopic wiggles that keeps the system outside macroscopic minimizers.

In Section 4.3 we consider gGS with a dissipation potential consisting of a fixed rate-independent and a small rate-dependent part, e.g. $\mathcal{R}_\varepsilon(q,v) = \mathcal{R}_{\mathrm{ri}}(q,v) + \frac{\varepsilon}{2}\langle \mathbb{G}(q)v, v\rangle$. For $\varepsilon > 0$ the solutions q_ε will be absolutely continuous with respect to $t \in [0,T]$ and the task is to characterize the jumps that develop in the vanishing-viscosity limit $\varepsilon \to 0$.

We also refer to [LOR07] for a derivation of macroscopic rate-independent behavior in the case of crack propagation.

4.1 Wiggly energies give rise to rate-independent friction

This section deals with the question how macroscopic RIS can arise from purely viscous systems in the limit of vanishing viscosity $\varepsilon \to 0$. We refer to [PuT02, MiT12, Mie12] for the full details. We stay in the framework of evolutionary Γ-convergence of gGS $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. In particular, we will start with the cases $\mathcal{R}_\varepsilon(q,v) = \frac{\varepsilon^\alpha}{2}\langle \mathbb{G}v, v\rangle$, where obviously $\mathcal{R}_\varepsilon \to 0$, and end up with a limit system $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$, where $\mathcal{R}_0$ is rate-independent. The first example will show very clearly that $\mathcal{R}_0$ is determined not by $\mathcal{R}_\varepsilon$, but by microscopic variations in the energies $\mathcal{E}_\varepsilon$, hence one uses the name *wiggly energies*.

In [Mie12] the following slight generalization of the wiggly-energy model of [Jam96] was studied. The latter was analyzed already in [PuT02, PuT05], but the gradient structure was first exploited in [Mie12]. As viscous gradient system $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ it takes the form

$$\boldsymbol{X} = \mathbb{R}, \qquad \mathcal{E}_\varepsilon(t,q) = \mathcal{F}(q) + \varepsilon W(q, \tfrac{1}{\varepsilon}q) - \ell(t)q, \qquad \mathcal{R}_\varepsilon(v) = \frac{\varepsilon^\alpha}{2}v^2.$$

Here $\mathcal{F} \in \mathrm{C}^2(\mathbb{R})$ denotes the macroscopic part of the energy, $W \in \mathrm{C}^2(\mathbb{R}\times\mathbb{S}^1)$ denotes the wiggly part, and $\ell \in \mathrm{C}^1([0,T])$ is the given time-dependent loading. Here $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ indicated that W is nontrivially periodic with period 1 in the second variable. In particular, writing $W = W(q,p)$, we assume

$$\rho_+(q) := \max\{\, \mathrm{D}_p W(q,p) \,|\, p \in \mathbb{S}^1 \,\} > 0 \quad \text{and} \tag{13a}$$

$$\rho_-(q) := \min\{\, \mathrm{D}_p W(q,p) \,|\, p \in \mathbb{S}^1 \,\} < 0. \tag{13b}$$

Defining $\mathcal{E}_0(t,q) = \mathcal{F}(q) - \ell(t)q$, we see that the energies $\mathcal{E}_\varepsilon$ uniformly converge to the macroscopic limit $\mathcal{E}_0$ via $|\mathcal{E}_\varepsilon(t,q) - \mathcal{E}_0(t,q)| \le C\varepsilon$, i.e. the wiggles are not seen on the energetic level. However, for the restoring force $\mathrm{D}_q\mathcal{E}_\varepsilon(t,q)$ we see a strong deviation from $\mathrm{D}_q\mathcal{E}_0(t,q)$. In particular, the functions $q \mapsto \mathrm{D}_q\mathcal{E}_\varepsilon(t,q)$ has many zeros (local equilibria of $\mathcal{E}_\varepsilon$).

The ODE $0 = \mathrm{D}_{\dot q}\mathcal{R}_\varepsilon(\dot q) + \mathrm{D}_q\mathcal{E}_\varepsilon(t,q)$ generated by $(\mathbb{R}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ reads

$$0 = \varepsilon^\alpha \dot q + \mathcal{F}'(q) + \mathrm{D}_p W(q, \tfrac{1}{\varepsilon}q) - \varepsilon \mathrm{D}_q W(q, \tfrac{1}{\varepsilon}q) - \ell(t). \tag{14}$$

The aim of evolutionary Γ-convergence is to show that the solutions q_ε of the viscous gradient system $(\mathbb{R}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converge to a solutions of the RIS $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_0)$, where the macroscopic energy $\mathcal{E}_0$ is given above and the rate-independent dissipation potential $\mathcal{R}_0$ is defined via

$$\mathcal{R}_0(z,v) := \begin{cases} \rho_+(z)v & \text{for } v \geq 0, \\ \rho_-(z)v & \text{for } v \leq 0. \end{cases} \tag{15}$$

Hence the solutions q of the limiting RIS $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_0)$ are given by the differential inclusion

$$0 \in \partial_{\dot q}\mathcal{R}_0(q,\dot q) + \mathrm{D}_q\mathcal{E}_0(t,q). \tag{16}$$

We emphasize that the definition of $\mathcal{R}_0$ does only involve characteristics of the wiggly microscopic energy landscape of $\mathcal{E}_\varepsilon$, namely the p-derivate of the wiggle function $W(q,p)$.

The main convergence result states that the solutions q_ε of (14) converge to solutions of the RIS $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_0)$.

Theorem 4.1 ([PuT02, Mie12]) *Let $\mathcal{F}$, W, ℓ, $\mathcal{E}_\varepsilon$, and $\mathcal{R}_\varepsilon$ be as described above, $\alpha > 0$, and assume that the mutual-convexity condition*

$$\inf\{\, \mathcal{E}''(q) \mid q \in \mathbb{R} \,\} > \sup\{\, |\mathrm{D}_q\mathrm{D}_pW(q,p)| \mid q \in \mathbb{R},\ p \in \mathbb{S}^1 \,\} \tag{17}$$

holds. Then $(\mathbb{R}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \overset{\mathrm{E}}{\to} (\mathbb{R}, \mathcal{E}_0, \mathcal{R}_0)$.

The proof in [Mie12] relies on three major pillars, namely (a) suitable a priori estimates, (b) a liminf-estimate for the energy-dissipation principle, and (c) uniqueness of the limiting systems. For (a) and (c) the standard energy estimates and the mutual-convexity condition (17) are used. The major difficulty lies in the limit passage (b) for the energy-dissipation principle as described in Section 3.1. For this we define the total dissipation functional

$$\mathcal{J}_\varepsilon(q) = \int_0^T \mathcal{M}_\varepsilon(t, q_\varepsilon(t), \dot q_\varepsilon(t))\,\mathrm{d}t \text{ with } \mathcal{M}_\varepsilon(t,q,v) = \mathcal{R}_\varepsilon(q,v) + \mathcal{R}_\varepsilon^*(q, -\mathrm{D}_q\mathcal{E}_\varepsilon(t,q)).$$

Inserting the specific forms of $\mathcal{R}_\varepsilon$, $\mathcal{R}_\varepsilon^*$, and $\mathcal{E}_\varepsilon$ we find

$$\mathcal{M}_\varepsilon(t,q,v) = \frac{\varepsilon^\alpha}{2}v^2 + \frac{1}{2\varepsilon^\alpha}\big|\mathcal{F}'(q) - \ell(t) + \mathrm{D}_pW(q,q/\varepsilon) + \varepsilon\mathrm{D}_qW(q,q/\varepsilon)\big|^2.$$

Homogenization arguments from [Bra02, Sect. 3] yield the liminf estimate

$$\liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(q_\varepsilon) \geq \mathcal{J}_0(q) := \int_0^T \mathcal{M}_0(t,q,\dot q)\,\mathrm{d}t \text{ with } \mathcal{M}_0(t,q,v) = \mathfrak{P}(v, \mathcal{F}'(q) - \ell(t)),$$

$$\mathfrak{P}(q,\xi) := |v|K(q,\xi) + \chi_{[\rho_-(q),\rho_+(q)]}(\xi), \text{ and } K(q,\xi) = \int_{\mathbb{S}^1} |\xi + \mathrm{D}_pW(q,p)|\,\mathrm{d}p.$$

It is easy to check the conditions (ii)–(v) in Section 3.1 for $\mathcal{E}_0$ and $\mathcal{R}_0$ given above. First note that (ii) and (iv) are trivial. Next observe $K(q,\xi) \geq |\xi|$, which implies (iii). For the crucial condition (v) we use that $\mathcal{M}_0(t,q,v) = -v\mathrm{D}_q\mathcal{E}_0(t,q)$ means $\xi = \mathrm{D}_q\mathcal{E}_0(t,q) \in [\rho_-(q), \rho_+(q)]$ and $|v|K(q,\xi) = -v\xi$. However, $K(q,\xi) = |\xi|$ holds if and only if $\xi \notin]\rho_-(q), \rho_+(q)[$. Thus, the equivalence to $0 \in \partial_v\mathcal{R}_0(q,v) + \xi$ (or any other of the five equivalent formulations in Proposition 2.1) follows easily.

4.2 1D elastoplasticity as limit of a chain of bistable springs

A second evolutionary Γ-limit with wiggly energies is established in [MiT12]. The system models a chain of N *bistable springs* with small viscous damping. Denoting by e_j the strain in the jth spring, the system reads

$$\left.\begin{aligned} &\nu\dot{e}_j = -\mathcal{F}'_{\rm biq}(e_j) + \mu_j^N + G(t,j/N) + \sigma(t) \quad \text{for } j=1,...,N;\\ &\mathcal{C}_N((e_j)) := \tfrac{1}{N}\textstyle\sum_{j=1}^N e_j = \ell(t), \end{aligned}\right\} \tag{18}$$

where the biquadratic double-well potential $\mathcal{F}_{\rm biq}(e) := \frac{k}{2}\min\{(e+a)^2,(e-a)^2\}$ generates the bistability. The coefficients μ_j^N are biases that act as quenched disorder (time-independent) and are chosen randomly, namely independently and identically distributed according to a probability density $f\in \mathrm{L}^1([-\mu_*,\mu_*])$ with average 0.

The system is driven by the volume loading $G\in \mathrm{C}^1([0,T]\times[0,1])$ and the constraint $\mathcal{C}_N$ corresponding to a Dirichlet loading $\ell\in \mathrm{C}^1([0,T])$ prescribing the total length of the chain, where σ is the Lagrange parameter for this constraint.

Using $\boldsymbol{e}=(e_1,...,e_N)$ as a state vector, the system has the energy functional $\mathcal{E}_N$ and the viscous dissipation potential $\mathcal{R}_N$:

$$\mathcal{E}_N(t,\boldsymbol{e}) = \frac{1}{N}\sum_{j=1}^N \Big(\mathcal{F}_{\rm biq}(e_j) - \mu_j^N e_j + G(t,j/N)e_j\Big) \text{ and } \mathcal{R}_N(\boldsymbol{e},\dot{\boldsymbol{e}}) = \frac{\nu}{2N}\sum_{j=1}^N \dot{e}_j^2.$$

The total system can now be written abstractly as a viscous gradient flow via

$$0 = \mathrm{D}_{\dot{\boldsymbol{e}}}\mathcal{R}_N(\boldsymbol{e},\dot{\boldsymbol{e}}) + \mathrm{D}_{\boldsymbol{e}}\mathcal{E}_N(t,\boldsymbol{e}) + \sigma(t)\mathrm{D}\mathcal{C}_N(\boldsymbol{e}) \quad \text{with} \quad \mathcal{C}_N(\boldsymbol{e}) = \ell(t).$$

Our small parameter is now $\varepsilon = 1/N$, which is the ratio between the length of the springs and the total length. Clearly, the energy $\mathcal{E}_N$ is wiggly in the sense that there are many local minimizers for a given constraint $\mathcal{C}_N(\boldsymbol{e})=\ell$, namely up to 2^N.

The limit of particle number $N\to\infty$ and viscosity $\nu\to 0$ can be studied by embedding the system into a spatially continuous setting on the physical domain $\Omega =]0,1[$. The potential $\mathcal{F}_{\rm biq}$ has two wells and hence two phases for each spring, which we characterize by the phase indicators $z_j = \mathrm{sign}(e_j)\in\{-1,0,1\}$. With the indicator functions

$$\varphi_j^N(x) := \begin{cases} 1 & \text{for } x\in\big((j-1)/N, j/N\big),\\ 0 & \text{otherwise.}\end{cases} \tag{19}$$

we define elastic and plastic strains via $(\overline{e}^N(t),\overline{p}^N(t)) := \mathcal{P}_N(e^N(t))$, where

$$\mathcal{P}_N : \begin{cases} \mathbb{R}^N & \to & \mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Omega),\\ \boldsymbol{e}=(e_j)_{j=1,...,N} & \mapsto & \Big(\sum_{j=1}^N e_j^N\varphi_j^N\ ,\ a\sum_{j=1}^N z_j^N\varphi_j^N\Big)\end{cases} \tag{20}$$

The definition of $(\overline{e}^N,\overline{p}^N)$ is such that we obtain a linear stress-strain relation

$$\mathcal{F}'_{\rm biq}(\overline{e}^N(t,x)) = k\big(\overline{e}^N(t,x) - \overline{p}^N(t,x)\big),$$

since the nonlinearity is moved into the definition of $\overline{p}$ via $z_j = \mathrm{sign}(e_j)$.

The limiting gGS $(\boldsymbol{H}, \mathcal{E}_0, \mathcal{R}_0)$ describes *linearized elastoplasticity* with hardening and is defined via

$$\boldsymbol{H} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega), \qquad \mathcal{R}_0(\dot{\overline{p}}) = \int_\Omega ka\big|\dot{\overline{p}}(x)\big| \,\mathrm{d}x,$$
$$\mathcal{E}_0(\overline{e}, \overline{p}) = \int_\Omega \frac{k}{2}\big(\overline{e}(x) - \overline{p}(x)\big)^2 + H_f(\overline{p}(x)\big) + G(t,x)\overline{e}(x) \,\mathrm{d}x,$$

where the hardening potential H_f is a convex function that is uniquely determined by the distribution function f for the random biases μ_j^N. Indeed, defining L_f such that $L_f'' = f$ one obtains H_f as Legendre transform of L_f, see [MiT12].

Together with the constraint $\mathcal{C}_0(\overline{e}) := \int_\Omega \overline{e}(x) \,\mathrm{d}x = \ell(t)$, we obtain the RIS $(\boldsymbol{H}, \mathcal{E}_0, \mathcal{R}_0, \mathcal{C}_0)$ with a 1-homogeneous dissipation potential $\mathcal{R}_0$ given in terms of the "yield stress ka". The associated differential inclusion

$$\begin{aligned} 0 &= \mathrm{D}_{\overline{e}}\mathcal{E}(\overline{e}, \overline{p}) + \sigma(t)\mathrm{D}\mathcal{C}(\overline{e}) = k(\overline{e} - \overline{p}) + \sigma, \qquad \mathcal{C}(\overline{e}) = \ell(t), \\ 0 &\in \partial\mathcal{R}(\dot{\overline{p}}) + \mathrm{D}_{\overline{p}}\mathcal{E}(\overline{e}, \overline{p}) = ka\mathrm{Sign}(\dot{\overline{p}}) + k(\overline{p} - \overline{e}) + \partial H_f(\overline{p}). \end{aligned} \tag{21}$$

describes one-dimensional elastoplasticity with Dirichlet loading $u(t,0) = 0$ and $u(t,1) = \ell(t)$, if the displacement is defined by $u(t,x) = \int_0^x \overline{e}(t,y) \,\mathrm{d}y$.

The following convergence result shows that the rate-independent evolution (21) is indeed the evolutionary Γ-limit of the finite-dimensional viscous systems (18).

Theorem 4.2 ([MiT12, Thm. 5.2]) *Assume $\nu_N = 1/N^\alpha$ for a fixed $\alpha > 1$. Consider the solutions $\boldsymbol{e}^N : [0,T] \to \mathbb{R}^N$ of the gradient system $(\mathbb{R}^N, \mathcal{E}_N, \mathcal{R}_N)$, where the biases μ_j^N are chosen randomly (and independently and identically distributed) according to the distribution f. Then, with probability 1 with respect to the random biases μ_j^N we have $(\mathbb{R}^N, \mathcal{E}_N, \mathcal{R}_N) \overset{\mathrm{pE}}{\rightharpoonup} (\boldsymbol{H}, \mathcal{E}_0, \mathcal{R}_0)$ in the sense of the embedding $\mathcal{P}_N$: If the initial conditions $\boldsymbol{e}^N(0)$ satisfy $e_j^N(0) < 0$ for all j,*

$$\mathcal{P}_N(\boldsymbol{e}^N(0)) \rightharpoonup (\overline{e}_0, \overline{p}_0) \text{ in } \boldsymbol{H}, \quad \text{and} \quad \mathcal{E}^N(0, \boldsymbol{e}^N(0)) \to \mathcal{E}(0, \overline{e}_0, \overline{p}_0);$$

then, for all $t \in [0,T]$ we have

$$\mathcal{P}_N(\boldsymbol{e}^N(t)) \rightharpoonup (\overline{e}(t), \overline{p}(t)) \text{ in } \boldsymbol{H} \quad \text{and} \quad \mathcal{E}^N(t, \boldsymbol{e}^N(t)) \to \mathcal{E}(t, \overline{e}(t), \overline{p}(t)),$$

where $(\overline{e}, \overline{p})$ is the unique solution of (21).

We again emphasize that the limiting dissipation potential $\mathcal{R}_0$ is not related to the original quadratic potentials $\mathcal{R}_N$. In the definition of $\mathcal{R}_0$ the constants k and a appear, which are part of the definition of the double-well potential $\mathcal{F}_{\mathrm{biq}}$.

4.3 Balanced-viscosity solutions as vanishing-viscosity limits

Assuming rate independence for an evolutionary system is always an approximation: the loading time-scale is taken to be much slower than all the internal relaxation processes. Moreover, in most material models there are two kinds of variables, i.e. we write the state variable q as a couple $q = (y, z)$, where y denotes the elastic or fast variables, usually

containing the elastic deformation $\phi : \Omega \to \mathbb{R}^d$ or the small displacement $u : \Omega \to \mathbb{R}^d$. The variable z are taken to be internal variables which are slower and may be modeled by rate-independent friction such as plastic yields or activated phase transformation. Hence, a typical quasistatic material model (where we still neglect inertial terms) will have the form of a coupled system

$$0 = \varepsilon^\alpha \mathbb{G}_1(y,z)\dot{y} + \mathrm{D}_y\mathcal{E}(t,y,z), \qquad 0 \in \partial\Psi(y,z,\dot{z}) + \varepsilon\mathbb{G}_2(y,z)\dot{z} + \mathrm{D}_z\mathcal{E}(t,y,z),$$

where we again assume that the loading rate is scaled to be of order one, such that the viscous relaxation times for the variable y are $O(\varepsilon^\alpha)$ while the variable z has rate-independent terms (instantaneous relaxation is possible) as well as additional viscous relaxation on the time scale $O(\varepsilon)$. Clearly, we have a generalized gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R}_\varepsilon)$ with

$$\boldsymbol{X} = \boldsymbol{Y}\times\boldsymbol{Z} \ \text{ and } \ \mathcal{R}_\varepsilon(y,z,\dot{y},\dot{z}) = \Psi(y,z,\dot{z}) + \frac{\varepsilon^\alpha}{2}\langle\mathbb{G}_1(y,z)\dot{y},\dot{y}\rangle_{\boldsymbol{Y}} + \frac{\varepsilon}{2}\langle\mathbb{G}_2(y,z)\dot{z},\dot{z}\rangle_{\boldsymbol{Z}}.$$

Again, we can ask the question of evolutionary Γ-convergence of $(\boldsymbol{X}, \mathcal{E}, \mathcal{R}_\varepsilon)$ towards a limit system $(\boldsymbol{X}, \mathcal{E}, \Psi, \Xi)$, in the sense that solutions q_ε of the former converge to the solutions q_0 of the latter system. Here the additional structure "Ξ" indicates that the simple RIS $(\boldsymbol{X}, \mathcal{E}, \Psi)$ needs to be enhanced by some information characterizing the jumps.

To obtain a rate-independent limit, one is again interested in the case $\varepsilon \to 0$, which is called the *vanishing-viscosity limit*. Formally, it is expected that the limits $q_0 = (y_0, z_0)$ of solutions $q_\varepsilon = (y_\varepsilon, z_\varepsilon)$ will satisfy the different inclusion

$$0 = \mathrm{D}_y\mathcal{E}(t,q_0(t)) \quad \text{and} \quad 0 \in \partial_{\dot{z}}\Psi(q_0(t),\dot{z}_0(t)) + \mathrm{D}_z\mathcal{E}(t,q_0(t)) \tag{22}$$

for almost all $t \in [0,T]$. However, in general the limits $q_0 : [0,T] \to \boldsymbol{X}$ will develop jumps with $q_0(t{-}0) \neq q_0(t{+}0)$ and (22) will not be enough to characterize these jumps. Moreover, the jumps arising in the vanishing-viscosity limit will depend on the different viscosity choices $\varepsilon^\alpha\mathbb{G}_1(q)$ and $\varepsilon\mathbb{G}_2(q)$.

Indeed, in [MRS14b] the dependence of the exponent $\alpha > 0$ was investigated in a situation where $q = (y,z) \in \mathbb{R}^n \times \mathbb{R}^m$ and where $\mathcal{E}(t,\cdot,z)$ is strictly convex. It turns out that the jump behavior is quite different for the three cases $\alpha \in]0,1[$, $\alpha = 1$, and $\alpha > 1$. For $\alpha > 1$ the component y can relax into the unique minimizer of $\mathcal{E}(t,\cdot,z(t))$ much faster than any changes in z. Hence, it is possible to reduce the situation by eliminating the variable y by defining $y = Y(t,z) = \operatorname{Arg\,min}_{\widetilde{y}\in\boldsymbol{Y}}\mathcal{E}(t,\widetilde{y},z)$ and $\widehat{\mathcal{E}}(t,z) = \mathcal{E}(t,Y(t,z),z)$.

For $\alpha \leq 1$ the situation is much more difficult and new jump phenomena occur, which are not yet understood, see [MRS14b] for some first results.

In light of the above discussion for $\alpha > 1$ we restrict ourself to the case $\boldsymbol{X} = \boldsymbol{Z}$ and consider gGS $(\boldsymbol{Z}, \mathcal{E}, \mathcal{R}_\varepsilon)$ with the simplest "vanishing-viscosity dissipation potential"

$$\mathcal{R}_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\langle\mathbb{G}v, v\rangle, \tag{23}$$

where Ψ is positively homogeneous of degree 1 and $\mathbb{G} = \mathbb{G}^* > 0$. The important observation is that $\mathbb{G}$ generates a Hilbert-space norm $\|v\|_{\boldsymbol{V}} := \big(\langle\mathbb{G}v, v\rangle\big)^{1/2}$, which is defines the Hilbert space $\boldsymbol{V}$. Throughout, we assume that $\boldsymbol{Z}$ is continuously embedded into $\boldsymbol{V}$,

which is certainly the case for the model system studied in [Mie11b, MiZ14]:

$$\text{(MS)} \quad \begin{cases} \boldsymbol{Z} = \mathrm{L}^1(\Omega), \quad \boldsymbol{V} = \mathrm{L}^2(\Omega), \quad \mathcal{R}_\varepsilon(v) = \displaystyle\int_\Omega |v| + \frac{\varepsilon}{2}|v|^2 \,\mathrm{d}x, \\ \text{and} \quad \mathcal{E}(t,z) = \displaystyle\int_\Omega \frac{\kappa}{2}|\nabla z|^2 + W(z) - \ell(t) z \,\mathrm{d}x \text{ for } z \in \mathrm{H}^1_0(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, W is the double-well potential $W(z) = (z^2-1)^2/4$, and ℓ is a smooth loading. The evolutionary equation is

$$\begin{aligned} 0 \in \mathrm{Sign}(\dot z) + \varepsilon \dot z - \kappa \Delta z + W'(z) - \ell(t) \quad &\text{for } (t,x) \in [0,T] \times \Omega, \\ z(t,x) = 0 \quad &\text{for } (t,x) \in [0,T] \times \partial\Omega, \end{aligned} \tag{24}$$

which is extensively studied in [MiZ14] by direct PDE methods.

For passing to the limit $\varepsilon \to 0$ and still controlling the jump behavior it is useful to reparametrize the solutions $t \mapsto (t, z_\varepsilon(t)) \in [0,T] \times \boldsymbol{Z}$ in the extended state space and study the convergence there. This idea was introduced in for RIS in [EfM06] and turned into an energetic framework in the series of papers [MRS09, MRS12, MRS14a, MRS14b].

For the reparametrization we let $t = \mathfrak{t}(s)$ and $z(t) = \mathfrak{z}(s)$, where $s \in [0,S]$ is now an arclength-like parameter. We write $\mathfrak{z}'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{z}(s)$ and note $\dot z(\mathfrak{t}(s))\mathfrak{t}'(s) = \mathfrak{z}'(s)$.

Definition 4.3 (Parametrized solutions) *Let the RIS $(\boldsymbol{Z}, \mathcal{E}, \Psi, \mathbb{G})$ and $\boldsymbol{V}$ be given as above. Then, a pair $(\mathfrak{t}, \mathfrak{z}) : [0,S] \to [0,T] \times \boldsymbol{Z}$ is called a* $\mathbb{G}$-parametrized solution, *if $(\mathfrak{t}, \mathfrak{z}) \in \mathrm{W}^{1,1}(0,T; \mathbb{R} \times \boldsymbol{V})$ and there exists $\lambda : [0,S] \to [0,\infty[$ such that*

$$\left.\begin{aligned} &\mathfrak{t}(0) = 0, \quad \mathfrak{t}(S) = T, \quad \mathfrak{t}'(s) \geq 0, \quad \lambda(s) \geq 0, \quad \lambda(s)\mathfrak{t}'(s) = 0, \\ &0 \in \partial\Psi(\mathfrak{z}'(s)) + \lambda(s)\mathbb{G}\mathfrak{z}'(s) + \mathrm{D}_z\mathcal{E}(\mathfrak{t}(s), \mathfrak{z}(s)), \end{aligned}\right\} \text{ a.e. on } [0,S]. \tag{25}$$

The definition clearly displays the rate independence of the notion of $\mathbb{G}$-parametrized solutions, since $\mathfrak{z}'$ only occurs in the rate-independent term $\partial\Psi$ or together with λ which can be scaled freely.

For a variational approach we transform the EDP, cf. Theorem 2.2, by time rescaling and obtain for $(\mathfrak{t}, \mathfrak{z})$ the following identity:

$$\begin{aligned} \mathcal{E}(\mathfrak{t}(S), \mathfrak{z}(S)) + \int_{s=0}^{S} \mathfrak{P}_\varepsilon\big(\mathfrak{t}'(s), \mathfrak{z}'(s), -\mathrm{D}_z\mathcal{E}(\mathfrak{t}(s), \mathfrak{z}(s))\big) \,\mathrm{d}s \\ = \mathcal{E}(mft(0), \mathfrak{z}(0)) + \int_0^S \partial_t\mathcal{E}(\mathfrak{t}(s), \mathfrak{z}(s))\mathfrak{t}'(s) \,\mathrm{d}s, \end{aligned} \tag{26}$$

$$\text{where} \quad \mathfrak{P}_\varepsilon(\tau, V, \xi) = \tau\mathcal{R}_\varepsilon(\tfrac{1}{\tau}V) + \tau\mathcal{R}^*_\varepsilon(\xi). \tag{27}$$

Using the special form of $\mathcal{R}_\varepsilon$ we obtain a quite explicit form for $\mathfrak{P}_\varepsilon$, namely

$$\mathfrak{P}_\varepsilon(\tau, V, \xi) = \Psi(V) + \frac{\varepsilon}{2\tau}\langle \mathbb{G}V, V\rangle + \frac{\tau}{2\varepsilon} M_{\boldsymbol{V}}(\xi)^2 \quad \text{with } M_{\boldsymbol{V}}(\xi) := \inf_{\eta \in \partial\Psi(0)} \|\xi - \eta\|_{\boldsymbol{V}^*}.$$

It is now easy to see that the Γ-limit of $\mathfrak{P}_\varepsilon : [0,\infty[\times \boldsymbol{Z} \times \boldsymbol{V}^* \to [0,\infty]$ for $\varepsilon \to 0$ takes the form

$$\mathfrak{P}_0(\tau, V, \xi) := \begin{cases} \Psi(V) + \Psi^*(\xi) & \text{for } \tau > 0, \\ \Psi(V) + \|V\|_{\boldsymbol{V}} M_{\boldsymbol{V}}(\xi) & \text{for } \tau = 0. \end{cases}$$

Clearly, $\mathfrak{P}_0(\tau,V,\xi) \geq -\langle \xi, V\rangle$ for all (τ,V,ξ). Moreover, equality holds if and only if $0 \in \partial\Psi(V)+\xi$ in the case $\tau>0$ and $0 \in \partial\Psi(V)+\lambda\mathbb{G}V+\xi$ in the case $\tau=0$ see [MRS12, Sec. 3.2]. Thus, all parametrized solutions satisfy the limiting EDP

$$\mathcal{E}(\mathfrak{t}(S),\mathfrak{z}(S)) + \int_{s=0}^{S} \mathfrak{P}_0\big(\mathfrak{t}'(s),\mathfrak{z}'(s),-\mathrm{D}_z\mathcal{E}(\mathfrak{t}(s),\mathfrak{z}(s))\big)\,\mathrm{d}s \tag{28}$$

$$= \mathcal{E}(\mathfrak{t}(0),\mathfrak{z}(0)) + \int_0^S \partial_t\mathcal{E}(\mathfrak{t}(s),\mathfrak{z}(s))\mathfrak{t}'(s)\,\mathrm{d}s, \tag{29}$$

and vice versa, sufficiently smooth solutions of the EDP are parametrized solutions. The advantage of (29) is that we do not need to assume $\mathfrak{z} \in \mathrm{W}^{1,1}([0,T];\boldsymbol{V})$. All solutions $(\mathfrak{t},\mathfrak{z})$ with $\mathfrak{t} \in \mathrm{W}^{1,1}([0,T])$ and $\mathfrak{z} \in \mathrm{BV}([0,T];\boldsymbol{Z}) \cap \mathrm{C}^0([0,T];\boldsymbol{V})$ of (29) are called *parametrized balanced-viscosity* solutions of $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$. Here the term "balanced viscosity" relates to the subtle balance of rate-independent and viscous dissipations along jumps, that is seen in $\mathfrak{P}_0$ for $\tau=0$ in the term $\Psi(V)+\|V\|_{\boldsymbol{V}}M_{\boldsymbol{V}}(\xi)$.

The advantage of reformulating subdifferential equations like (24) and (25) in terms of the reparametrized EDP (27) is that we can control the limit $\varepsilon \to 0$ easily. In particular, if the define the solutions of $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$ to be parametrized balanced-viscosity solution, then we have evolutionary Γ-convergence of $(\boldsymbol{Z},\mathcal{E},\mathcal{R}_\varepsilon)$ (with $\mathcal{R}_\varepsilon$ from (23)) to $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$.

However, the introduction of the parametrization may appear ad hoc and disturbing. So one can define the notion of *Balanced-Viscosity solutions* as follows: $z:[0,T]\to\boldsymbol{Z}$ is called a BV solutions for $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$ if there exists a parametrized balanced-viscosity solutions $(\mathfrak{t},\mathfrak{z}):[0,S]\to[0,T]\times\boldsymbol{Z}$ such that for all $t\in[0,T]$ there exists an $s\in[0,S]$ with $t=\mathfrak{t}(s)$ and $z(t)=\mathfrak{z}(s)$. This simply means that the image of $(\mathfrak{t},\mathfrak{z})$ in $[0,T]\times\boldsymbol{X}$ contains the graph of $z:[0,T]\to\boldsymbol{Z}$.

One major achievement in [MRS12, MRS14a] is a proper intrinsic definition of BV solutions without referring to parametrizations. For this one defines a new (time-dependent) dissipation distance $\boldsymbol{\Delta}(t,\cdot,\cdot)$ that measures the minimal dissipation according to $\mathfrak{P}_0$ along all curves connecting to states z_0 and z_1:

$$\boldsymbol{\Delta}(t,z_1,z_2) := \inf\Big\{ \int_0^1 \mathfrak{P}_0\big(0,\dot{\mathfrak{y}}(r),-\mathrm{D}_z\mathcal{E}(t,\mathfrak{y}(r))\big)\,\mathrm{d}r \;\Big|\; \mathfrak{y}\in\mathrm{C}^1([0,1];\boldsymbol{V}),\ \mathfrak{y}(0)=z_1,\ \mathfrak{y}(1)=z_2\Big\}. \tag{30}$$

Note that $\boldsymbol{\Delta}$ is defined with time t as a frozen parameter, i.e. $\mathfrak{t}'(r)=\tau=0$. Clearly, we have the triangle inequality $\boldsymbol{\Delta}(t,z_0,z_2)\le\boldsymbol{\Delta}(t,z_0,z_1)+\boldsymbol{\Delta}(t,z_1,z_2)$ and the lower estimate $\boldsymbol{\Delta}(t,z_1,z_2)\ge\Psi(z_2-z_1)$. For the definition of BV solutions we use a supplemented dissipation functional $\mathrm{Diss}_{\mathfrak{p},\mathcal{E}}$ defined on functions $z\in\mathrm{BV}([0,T];\boldsymbol{X})$. Here $J(z)\subset[0,T]$ is the jump set of z, i.e. all the times t where the three values $z(t-0)$, $z(t)$, and $z(t+0)$ are not equal. The new dissipation functional $\mathrm{Diss}_{\mathfrak{M},\mathcal{E}}(z;[t_1,t_2])$ is bigger than the purely rate-independent functional Diss_Ψ defined in (6), because it properly accounts for the additional dissipation through the viscous terms during jumps:

$$\begin{aligned}\mathrm{Diss}_{\mathfrak{p},\mathcal{E}}(z;[t_1,t_2]) := \mathrm{Diss}_\Psi(z;[t_1,t_2]) &+ \widehat{\boldsymbol{\Delta}}(t_1,z(t_1),z(t_1^+))+\widehat{\boldsymbol{\Delta}}(t_2,z(t_2^-),z(t_2))\\ &+\sum\nolimits_{t\in J(z)}\big(\widehat{\boldsymbol{\Delta}}(t,z(t^-),z(t))+\widehat{\boldsymbol{\Delta}}(t,z(t),z(t^+))\big),\end{aligned}$$

where $\widehat{\boldsymbol{\Delta}}(t,z_0,z_1) := \boldsymbol{\Delta}(t,z_0,z_1)-\Psi(z_1-z_0)\ge 0$.

Definition 4.4 (Balanced-Viscosity solutions) *A function $z \in \mathrm{BV}([0,T];\boldsymbol{Z})$ is called a Balanced-Viscosity solution, in short* BV solution*, for $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$, if*

$$\forall\, t\in[0,T]\setminus J(z):\quad z(t)\in\mathcal{S}_{loc}(t):=\{\,z\in\boldsymbol{Z}\mid 0\in\partial\Psi(0)+\mathrm{D}_z\mathcal{E}(t,z)\,\}\ \textit{and} \tag{31a}$$

$$\forall\, t\in[0,T]:\quad \mathcal{E}(t,z(t))+\mathrm{Diss}_{\mathfrak{M},\mathcal{E}}(z;[0,t])=\mathcal{E}(0,z(0))+\int_0^t\partial_t\mathcal{E}(t,z(t))\,\mathrm{d}t. \tag{31b}$$

It is interesting to see that the definition of BV solutions again consists of a static stability condition and an energy balance as in the case of energetic solutions, see (5). However, no the stability is local instead of global and it is only valid at continuity points of the solution. To compensate for this the dissipation is enhanced at jumps deriving from the additional dissipation through balanced viscosity.

We now use the advantage that BV solutions are defined as functions from the time interval $[0,T]$ into the state space $\boldsymbol{Z}$ like the viscous approximations. Thus, the natural question is how the solutions z_ε converge to BV solutions. This question was first answered in [MRS12] for the finite-dimensional setting and in [MRS14a, Thm. 3.9] for a general infinite-dimensional setting.

Theorem 4.5 (Vanishing-viscosity limit gives BV solutions) *Under suitable technical conditions on $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$ and the initial condition $z^0\in\boldsymbol{Z}$, the solutions $z_\varepsilon:[0,T]\to\boldsymbol{Z}$ of $(\boldsymbol{X},\mathcal{E},\mathcal{R}_\varepsilon)$ with $z_\varepsilon(0)=z^0$ and $\mathcal{R}_\varepsilon$ from (23) exist and there exist a subsequence $\varepsilon_k\to0$ and a BV solution $z:[0,T]\to\boldsymbol{Z}$ for $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$ such that*

$$\forall\, t\in[0,T]:\quad z_{\varepsilon_k}(t)\rightharpoonup z(t)\ \ \textit{in}\ \boldsymbol{Z}\ \ \textit{and}\ \ \mathcal{E}(t,z_{\varepsilon_k}(t))\to\mathcal{E}(t,z(t))\ \ \textit{for}\ k\to\infty.$$

Moreover, any pointwise limit z of a subsequence of $(z_\varepsilon)_{\varepsilon>0}$ is a BV solution.

Our final result concerns the vanishing-viscosity limit jointly with time discretizations, which provides an easy way of numerically calculating BV solutions. We discretize the time interval by partitions $\Pi=(t_0,t_1,....,t_{N_\Pi})$ with fineness $\phi(\Pi)=\max\{\,t_k{-}t_{k-1}\mid k=1,...,N_\Pi\,\}$. The incremental minimization problem for the viscous problem reads

$$z_k^\varepsilon\in\mathrm{Arg\,min}_{z\in\boldsymbol{Z}}\mathcal{E}(t_k,z)+\Psi(z{-}z_{k-1}^\varepsilon)+\frac{\varepsilon}{2(t_k{-}t_{k-1})}\big\|z{-}z_{k-1}^\varepsilon\big\|_{\boldsymbol{V}}^2,\quad z_0^\varepsilon=z^0.$$

We denote by $z^{\Pi,\varepsilon}:[0,T]\to\boldsymbol{Z}$ the piecewise constant interpolant. The following result was first proved in [EfM06, MRS12] for the finite-dimensional setting. For a quite general infinite-dimensional version we refer to [MRS14a, Thm. 3.10].

Theorem 4.6 (Convergence of viscous time discretizations) *Assume suitable technical conditions on $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$ and $z_0\in\boldsymbol{Z}$ (see [Mie11b, MRS14a]) and consider a sequences $(\Pi_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ such that*

$$\varepsilon_n\to0\ \ \textit{and}\ \ \phi(\Pi_n)/\varepsilon_n\to0. \tag{32}$$

Then, there exists a subsequence $n_l\to\infty$ and a BV solution z for $(\boldsymbol{Z},\mathcal{E},\Psi,\mathbb{G})$ such that the piecewise constant interpolants z^{Π_n,ε_n} satisfy

$$\forall\, t\in[0,T]:\quad z^{\Pi_{n_l},\varepsilon_{n_l}}(t)\rightharpoonup z(t)\ \textit{in}\ \boldsymbol{Z}\ \textit{and}\ \mathcal{E}(t,z^{\Pi_{n_l},\varepsilon_{n_l}}(t))\to\mathcal{E}(t,z(t))\ \ \textit{for}\ l\to\infty.$$

Moreover, any such pointwise limit of a subsequence of $(z^{\Pi_n,\varepsilon_n})_{n\in\mathbb{N}}$ is a BV solution.

5 Rate-independent evolution of microstructures

The theory of RIS provides an ideal framework for studying microstructures in the sense of the calculus of variations, namely those given by laminates or more general Young measures. The starting point of most of these works was the seminal paper [OrR99] on microstructures in finite-strain plasticity. In the sequel a lot of work was done for the relaxation of a single elastoplastic time step, see [CHM02, CDK13b, CDK13a]. We also refer to [HeK14, Hei15, Hei14] for the characterization and numerical calculation of *quasiconvex hulls*.

In contrast, the evolution of microstructures in plasticity is mathematically much less developed, see e.g. [Mie04, CoT05]. However, the same theory was soon transferred to easier dissipative material models such as damage (cf. e.g. [FrG06, GaL09, Mie11a]) and phase transformations in elastomers (cf. e.g. [DeD02]) or shape-memory materials (cf. e.g. [BC*04, BaH09, KoH11, CLR15]).

In the following we discuss two applications of the evolutionary theory, both based on energetic solutions for RIS, see Section 2.2. The first application is treated in [HHM12] and deals with the evolution of microstructure in the form of laminates, where laminates are explicitly takes as an allowed microstructure with an appropriate dissipation distance as proposed in [KoH11]. The second application reconsiders the evolutionary model from [MTL02], where the microstructure is captured by a macroscopic phase fraction $z(t,x) \in [0,1]$.

5.1 Laminate evolution in finite-strain plasticity

We summarize the results in [HHM12], which analyze a rate-independent model for finite-strain elastoplasticity with microstructure. The state of the system is described by the deformation $\phi : \Omega \to \mathbb{R}^d$ and by a Young measure $\Lambda : \Omega \to \mathcal{L} \subset \mathrm{Prob}(K)$, where $K := \mathbb{R}^{d\times d} \times \mathrm{SL}(\mathbb{R}^d))$, and $\mathrm{SL}(\mathbb{R}^d) = \{ P \in \mathbb{R}^{d\times d} \mid \det P = 1 \}$ is the special linear group containing the plastic strains, whereas $\mathbb{R}^{d\times d}$ will contain microfluctuations of the deformation gradient.

The main idea is to specify a physically relevant subset $\mathcal{L}$ of admissible Young measures, like laminates of a fixed order as in [OrR99], to define a suitable dissipation distance between these measures, and to prevent formation of different microstructures by a suitable regularization. Following [KoH11] the simplest set of admissible probability measures are laminates of first order:

$$\mathcal{L} := \{ \alpha\delta_{((1-\alpha)b\otimes n,Q)} + (1-\alpha)\delta_{(-\alpha b\otimes n,R)} \mid \alpha \in [0,1],\ b,n \in \mathbb{R}^d,\ R,Q \in \mathrm{SL}(\mathbb{R}^d) \}.$$

Of course, more complicated lamination trees on the sense of [OrR99] would be possible. The point is now to define a dissipation distance $D_{\mathrm{lam}} : \mathcal{L} \times \mathcal{L} \to [0,\infty]$ between such laminates, which properly accounts for changes in the microstructure. In particular, one wants to model the fact that it is very difficult to rotate the normal vector n in such microstructures. When keeping n fixed, then the deformation fluctuation $b \in \mathbb{R}^d$ may change without dissipation, while changes of the volume fraction α dissipate according to the distance $D_{\mathrm{SL}}(Q_0,Q_1)$ or $D_{\mathrm{SL}}(R_0,R_1)$.

The ERIS is now constructed via the state space $\mathcal{Q} = \boldsymbol{Y} \times \mathcal{Z}$ with $\boldsymbol{Y} = \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$

and $\mathcal{Z} = \{\,\Lambda \in \mathrm{YM}(\Omega;K) \,|\, \Lambda(x) \in \mathcal{L} \text{ a.e. }\}$ and the energy functional

$$\mathcal{E}(t,\phi,\Lambda) = \int_\Omega \int_{\mathcal{L}} \big(W(\nabla\phi(I_d{+}A)P^{-1}) + H(P)\big)\Lambda(\mathrm{d}A,\mathrm{d}P)\,\mathrm{d}x \\ + \sigma\mathcal{G}(\Lambda) - \langle \ell(t),\phi\rangle \quad \text{with } \mathcal{G}(\Lambda) := \int_\Omega\int_\Omega \frac{d_{\mathrm{W}}(\Lambda(x),\Lambda(y))^p}{|x{-}y|^{d+\theta p}}\,\mathrm{d}x\,\mathrm{d}y,$$

where d_{W} defines a 1-Wasserstein like norm on $\mathcal{L}$, namely

$$d_{\mathrm{W}}(\Lambda_0,\Lambda_1) := \sup\{\int_K g(A,P)\Lambda_1(\mathrm{d}A,\mathrm{d}P) - \int_K g(B,Q)\Lambda_0(\mathrm{d}B,\mathrm{d}Q) \;\; |\,\mathrm{Lip}_K(g)\le 1\,\}.$$

Thus, $\mathcal{G}(\Lambda)$ serves as a spatial regularization for the laminate field $\Lambda : \Omega \to \mathcal{L}$ which prevents the formation of further more complicated microstructures.

The dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ is defined as

$$\mathcal{D}(\Lambda_0,\Lambda_1) = \int_\Omega D_{\mathrm{lam}}(\Lambda_0(x),\Lambda_1(x))\,\mathrm{d}x$$

Under suitable assumptions on the polyconvex energy density W and the hardening energy H it is shown in [HHM12, Thm. 2.4] that the ERIS $\mathcal{Q},\mathcal{E},\mathcal{D})$ has for each stable initial condition (ϕ^0,Λ^0) an *energetic solution* describing the *laminate evolution*. Indeed, using the regularizing term $\mathcal{G}$ one has a compactness for the laminate fields, which allows to establish suitable lower semicontinuity results for $\mathcal{E}$ and $\mathcal{D}$ as well as mutual recovery sequences in the sense of (10).

5.2 A two-phase shape-memory model for small strains

Finally we present some new results for the *two-phase model* for introduced in [MTL02]. In fact, this model was the origin for the development of energetic solutions.

The two elastic phases are described by linearized elasticity with the same elastic tensor $\mathbb{C}$, but have different transformation strains A_j. On the microscopic level one may use the stored energy density

$$\widehat{W}(e) = \min\{\frac{1}{2}(e{-}A_1){:}\mathbb{C}(e{-}A_1) + c_1, \frac{1}{2}(e{-}A_2){:}\mathbb{C}(e{-}A_2) + c_2\},$$

where $e = e(u) := \frac{1}{2}(\nabla u{+}\nabla u^\top)$ is the infinitesimal strain tensor. The *relaxation* of $\widehat{W}$ with given volume fraction $z \in [0,1]$ for phase 2 was derived in [Koh91]:

$$W(e,z) = (1{-}z)\big(\tfrac{1}{2}(e{-}A_1){:}\mathbb{C}(e{-}A_1) + c_1\big) + z\big(\tfrac{1}{2}(e{-}A_2){:}\mathbb{C}(e{-}A_2) + c_2\big) - \rho z(1{-}z),$$

where the relaxation coefficient $\rho > 0$ can be calculated explicitly.

The ERIS studied in [MTL02] is given by $\boldsymbol{Q} = \mathrm{H}^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)\times \mathrm{L}^1(\Omega;[0,1])$,

$$\mathcal{E}(t,u,z) = \int_\Omega W(e(u),z) - \ell(t)\cdot u\,\mathrm{d}x, \quad \text{and } \mathcal{D}(z_0,z_1) = \delta\|z_1{-}z_0\|_{\mathrm{L}^1} \tag{33}$$

for some smooth loading and some dissipation coefficient $\delta > 0$. A first existence result for energetic solutions was obtained in [MTL02, Thm. 5.1] under the unnatural assumption

that the energy $\mathcal{E}(t,\cdot)$ is convex. A corresponding numerical algorithm using space-time discretization and incremental minimizations (cf. (7)) were developed in [CaP01]. Using the abstract theory for ERIS in [Mie11b, MiR15], the existence theory was recently improved, see [HeM15], by a new construction of mutual recovery sequences, see (10).

Theorem 5.1 ([HeM15]) *The ERIS* (33) *with* $\ell \in \mathrm{W}^{1,1}([0,T];\mathrm{H}^1_D(\Omega)^*)$ *has, for each stable initial state* $q_0=(u_0,z_0)$, *an* energetic solution $(u,z):[0,T]\to \boldsymbol{Q}$.

The proof relies in reducing the system to a problem in z alone. For this note that the equation $\mathrm{D}_u\mathcal{E}(t,u,z)=0$ is a linear elliptic PDE for u with a right-hand side that is linear in z and ℓ. Hence, the unique solution $u=U(z,\ell)\in \mathrm{H}^1_D(\Omega;\mathbb{R}^d)$ can be inserted into $\mathcal{E}$ to obtain the reduced ERIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$ with

$$\mathcal{Z}:=\mathrm{L}^1(\Omega;[0,1]) \quad \text{and} \quad \mathcal{I}(t,\theta)=\mathcal{E}(t,U(z,\ell(t)),z)=\frac{1}{2}\big\langle Lz+\gamma(t),z\big\rangle+\alpha(t).$$

Here L is a pseudo-differential operator of order 0, and the symbol, which can be calculated explicitly, is non-negative by the explicit formula for ρ from [Koh91]. The symbol attains the value 0 along the optimal laminates and ρ is the largest number such that the symbol remains non-negative.

Because of the constraint $z\in[0,1]$ the quadratic trick indicated in (12) cannot be used for showing the closedness of the set of stable states. Indeed, from the incremental minimization problem (7) we obtain piecewise constant interpolants $z^\tau:[0,T]\to\mathcal{Z}$ that are globally stable, i.e. (S) in (5) holds at $t=k\tau$ for $k\in\mathbb{N}_0$. For a subsequence $\tau_k\to 0$ we have $z^{\tau_n}(t)\rightharpoonup z(t)$ and we have to show that $z(t)$ is stable as well.

Since stability is a static concept we can fix t and drop it for notational convenience. To establish stability of z we start from the stability of z_n in the form

$$\mathcal{I}(t,\widehat{z}_n)+\mathcal{D}(z_n,\widehat{z}_n)-\mathcal{I}(t,z_n)\ge 0 \text{ for all } \widehat{z}_n\in\mathcal{Z}.$$

To pass to the limit we can only use $z_n\rightharpoonup z$, but may choose a suitable mutual recovery sequence $\widehat{z}_n\rightharpoonup\widehat{z}$ for a given test state $\widehat{z}$. In [HeM15] the following choice was introduced:

$$\widehat{z}_n(x)=\widehat{z}(x)+g(x)\,(z_n(x)-z(x)),\ \text{where } g(x)=\begin{cases}\frac{\widehat{z}(x)}{z(x)} & \text{for } \widehat{z}(x)<z,\\ 1 & \text{for } \widehat{z}(x)=z,\\ \frac{1-\widehat{z}(x)}{1-z(x)} & \text{for } \widehat{z}(x)>z.\end{cases}$$

Clearly we have $\widehat{z}_n\in\mathcal{Z}$, $\widehat{z}_n\rightharpoonup\widehat{z}$ and $\operatorname{sign}(\widehat{z}_n-z_n)=\operatorname{sign}(\widehat{z}-z)$. Decomposing Ω into Ω_+ and Ω_- such that $\widehat{z}\ge z$ and $\widehat{z}<z$, respectively, we obtain

$$\begin{aligned}&\tfrac{1}{r}\mathcal{D}(z_n,\widehat{z}_n)=\|z_n-\widehat{z}_n\|_{\mathrm{L}^1}=\textstyle\int_{\Omega_+}\widehat{z}_n-z_n\,\mathrm{d}x+\int_{\Omega_-}z_n-\widehat{z}_n\,\mathrm{d}x\\ &=\textstyle\int_{\Omega_+}\frac{\widehat{z}-z}{1-z}(1-z_n)\,\mathrm{d}x+\int_{\Omega_-}\frac{z-\widehat{z}}{z}\,z_n\,\mathrm{d}x\ \to\ \int_{\Omega_+}\widehat{z}-z\,\mathrm{d}x+\int_{\Omega_-}z-\widehat{z}\,\mathrm{d}x=\tfrac{1}{r}\mathcal{D}(z,\widehat{z}).\end{aligned}$$

To control the energy differences $\mathcal{I}(t,\widehat{z}_n)-\mathcal{I}(t,z_n)$ we exploit the quadratic form of the energy. In fact, the sequence $v_n:=z_n-z\rightharpoonup 0$ generates an *H-measure* $\mu\ge 0$ which exactly characterizes the limit of the quadratic energy, namely

$$\lim_{n\to\infty}\mathcal{I}(t,z_n)=\mathcal{I}(t,z)+\int_\Omega\int_{\omega\in\mathbb{S}^{d-1}}\Sigma_L(\omega)\mu(x,\mathrm{d}\omega)\,\mathrm{d}x,$$

where $\Sigma_L(\omega) \geq 0$ is the symbol of L. The construction of $\widehat{z}_n$ gives $\widehat{v}_n := \widehat{z}_n - \widehat{z} = gv_n \rightharpoonup 0$, such that $\widehat{v}_n$ generates the H-measure $g^2\mu$. Thus, we obtain

$$\begin{aligned}&\lim_{n\to\infty} \big(\mathcal{I}(t,\widehat{z}_n) - \mathcal{I}(t,z_n)\big)\\ &= \mathcal{I}(t,\widehat{z}) - \mathcal{I}(t,z) + \int_\Omega \int_{\omega\in\mathbb{S}^{d-1}} (g(x)^2-1)\Sigma_L(\omega)\mu(x,\mathrm{d}\omega)\,\mathrm{d}x.\end{aligned}$$

Now, using $g^2 \leq 1$ we conclude the desired limsup estimate

$$0 \leq \limsup_{n\to\infty} \Big(\mathcal{I}(t,\widehat{z}_n) + \mathcal{D}(z_n,\widehat{z}_n) - \mathcal{I}(t,z_n)\Big) \leq \mathcal{I}(t,\widehat{z}) + \mathcal{D}(z,\widehat{z}) - \mathcal{I}(t,z).$$

Since $\widehat{z}$ was arbitrary, the global stability (S) of z is established.

We refer to [HeM15] for a detailed analysis, which includes the convergence of space-time discretizations in suitable finite-element spaces as well as the strong convergence of certain Riesz projections related to the directions of the microstructures between the two phases.

Acknowledgments. This paper surveys the research done in the subproject P5 *Microstructures in evolutionary material models* in the years 2007 to 2014 which was largely supported by Deutsche Forschungsgemeinschaft within the Research Unit FOR 797 *Analysis and computation of microstructure in finite plasticity.* The author is grateful to his collaborators Sebastian Heinz, Klaus Hackl, Michael Ortiz, Christoph Ortner, Riccarda Rossi, Tomáš Roubíček, Guiseppe Savaré, Yasemin Şengül, Ulisse Stefanelli, Marita Thomas, and Sergey Zelik.

References

[AGS05] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[AM*12] S. Arnrich, A. Mielke, M. A. Peletier, G. Savaré, and M. Veneroni. Passing to the limit in a Wasserstein gradient flow: from diffusion to reaction. *Calc. Var. Part. Diff. Eqns.*, 44, 419–454, 2012.

[BaH09] T. Bartel and K. Hackl. A micromechanical model for martensitic phase-transformations in shape-memory alloys based on energy-relaxation. *Z. angew. Math. Mech. (ZAMM)*, 89(10), 792–809, 2009.

[BC*04] S. Bartels, C. Carstensen, K. Hackl, and U. Hoppe. Effective relaxation for microstructure simulations: algorithms and applications. *Comput. Methods Appl. Mech. Engrg.*, 193, 5143–5175, 2004.

[Bio55] M. A. Biot. Variational principles in irreversible thermodynamics with applications to viscoelasticity. *Phys. Review*, 97(6), 1463–1469, 1955.

[Bra02] A. Braides. Γ-*Convergence for Beginners.* Oxford University Press, 2002.

[CaP01] C. Carstensen and P. Plecháč. Numerical analysis of a relaxed variational model of hysteresis in two-phase solids. *M2AN Math. Model. Numer. Anal.*, 35(5), 865–878, 2001.

[CDK13a] S. Conti, G. Dolzmann, and C. Kreisbeck. Relaxation and microstructure in a model for finite crystal plasticity with one slip system in three dimensions. *Discr. Cont. Dynam. Systems Ser. S*, 6, 1–16, 2013.

[CDK13b] S. Conti, G. Dolzmann, and C. Kreisbeck. Relaxation of a model in finite plasticity with two slip systems. *Math. Models Meth. Appl. Sci. (M^3AS)*, 23, 2111–2128, 2013.

[CHM02] C. Carstensen, K. Hackl, and A. Mielke. Non–convex potentials and microstructures in finite–strain plasticity. *Proc. Royal Soc. London Ser. A*, 458(2018), 299–317, 2002.

[CLR15] S. Conti, M. Lenz, and M. Rumpf. Hysteresis in magnetic shape memory composites: modeling and limulation. *J. Mech. Physics Solids*, 2015. Submitted.

[CoO08] S. Conti and M. Ortiz. Minimum principles for the trajectories of systems governed by rate problems. *J. Mech. Physics Solids*, 56(5), 1885–1904, 2008.

[CoT05] S. Conti and F. Theil. Single-slip elastoplastic microstructures. *Arch. Rational Mech. Anal.*, 178, 125–148, 2005.

[DaL10] G. Dal Maso and G. Lazzaroni. Quasistatic crack growth in finite elasticity with non-interpenetration. *Ann. Inst. H. Poinc. Anal. Non Lin.*, 27(1), 257–290, 2010.

[DeD02] A. DeSimone and G. Dolzmann. Macroscopic response of nematic elastomers via relaxation of a class of SO(3)-invariant energies. *Arch. Ration. Mech. Anal.*, 161(3), 181–204, 2002.

[DFT05] G. Dal Maso, G. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 176, 165–225, 2005.

[DMT80] E. De Giorgi, A. Marino, and M. Tosques. Problems of evolution in metric spaces and maximal decreasing curve. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 68(3), 180–187, 1980.

[EfM06] M. Efendiev and A. Mielke. On the rate–independent limit of systems with dry friction and small viscosity. *J. Convex Anal.*, 13(1), 151–167, 2006.

[Fen49] W. Fenchel. On conjugate convex functions. *Canadian J. Math.*, 1, 73–77, 1949.

[FrG06] G. Francfort and A. Garroni. A variational view of partial brittle damage evolution. *Arch. Rational Mech. Anal.*, 182, 125–152, 2006.

[GaL09] A. Garroni and C. J. Larsen. Threshold-based quasi-static brittle damage evolution. *Arch. Rational Mech. Anal.*, 194(2), 585–609, 2009.

[GiM11] A. Giacomini and A. Musesti. Two-scale homogenization for a model in strain gradient plasticity. *ESAIM Control Optim. Calc. Var.*, 17(4), 1035–1065, 2011. Published online. DOI 10.1051/cocv/2010036.

[HaF08] K. Hackl and F. D. Fischer. On the relation between the principle of maximum dissipation and inelastic evolution given by dissipation potentials. *Proc. R. Soc. A*, 464, 117–132, 2008.

[Han11] H. Hanke. Homogenization in gradient plasticity. *Math. Models Meth. Appl. Sci. (M^3AS)*, 21(8), 1651–1684, 2011.

[Hei14] S. Heinz. On the structure of the quasiconvex hull in planar elasticity. *Calc. Var. Part. Diff. Eqns.*, 50, 481–489, 2014.

[Hei15] S. Heinz. Quasiconvexity equals lamination convexity for isotropic sets of 2x2 matrices. *Adv. Calc. Var.*, 8, 43–53, 2015.

[HeK14] S. Heinz and M. Kružík. Computations of quasiconvex hulls of isotropic sets. *WIAS preprint no. 2049*, 2014.

[Hel69] H. v. Helmholtz. Zur Theorie der stationären Ströme in reibenden Flüssigkeiten. *Verhandlungen des naturhistorisch-medizinischen Vereins zu Heidelberg*, pages 1–7, 1869. Band V.

[HeM15] S. Heinz and A. Mielke. Analysis and numerics of a phase-transformation model. *In preparation*, 2015.

[HHM12] K. Hackl, S. Heinz, and A. Mielke. A model for the evolution of laminates in finite-strain elastoplasticity. *Z. angew. Math. Mech. (ZAMM)*, 92(11-12), 888–909, 2012.

[Jam96] R. D. James. Hysteresis in phase transformations. In *ICIAM 95 (Hamburg, 1995)*, volume 87 of *Math. Res.*, pages 135–154. Akademie Verlag, Berlin, 1996.

[Koh91] R. V. Kohn. The relaxation of a double-well energy. *Contin. Mech. Thermodyn.*, 3(3), 193–236, 1991.

[KoH11] D. M. Kochmann and K. Hackl. The evolution of laminates in finite crystal plasticity: a variational approach. *Contin. Mech. Thermodyn.*, 23(1), 63–85, 2011.

[LiM11] M. Liero and A. Mielke. An evolutionary elastoplastic plate model derived via Γ-convergence. *Math. Models Meth. Appl. Sci. (M^3AS)*, 21(9), 1961–1986, 2011.

[LM*15] M. Liero, A. Mielke, M. A. Peletier, and D. R. M. Renger. On the microscopic origin of generalized gradient structures. *In preparation*, 2015.

[LOR07] C. Larsen, M. Ortiz, and C. Richardson. Fracture paths from front kinetics: relaxation and rate-independence. *Arch. Rational Mech. Anal.*, 193(3), 539–583, 2007.

[MaM09] A. Mainik and A. Mielke. Global existence for rate-independent gradient plasticity at finite strain. *J. Nonlinear Sci.*, 19(3), 221–248, 2009.

[Mie04] A. Mielke. Deriving new evolution equations for microstructures via relaxation of variational incremental problems. *Comput. Methods Appl. Mech. Engrg.*, 193(48-51), 5095–5127, 2004.

[Mie10] A. MIELKE. Existence theory for finite-strain crystal plasticity with gradient regularization. In K. Hackl, editor, *IUTAM Symposium on Variational Concepts with Applications to the Mechanics of Materials*, pages 171–183. Springer, 2010. Proceedings of the IUTAM Symposium on Variational Concepts, Bochum, Germany, Sept. 22–26, 2008.

[Mie11a] A. MIELKE. Complete-damage evolution based on energies and stresses. *Discr. Cont. Dynam. Systems Ser. S*, 4(2), 423–439, 2011.

[Mie11b] A. MIELKE. Differential, energetic, and metric formulations for rate-independent processes. In L. Ambrosio and G. Savaré, editors, *Nonlinear PDE's and Applications*, pages 87–170. Springer, 2011. (C.I.M.E. Summer School, Cetraro, Italy 2008, Lect. Notes Math. Vol. 2028).

[Mie12] A. MIELKE. Emergence of rate-independent dissipation from viscous systems with wiggly energies. *Contin. Mech. Thermodyn.*, 24(4), 591–606, 2012.

[Mie14] A. MIELKE. On evolutionary Γ-convergence for gradient systems. *WIAS Preprint 1915*, 2014. To appear in Proc. Summer School in Twente University June 2012.

[MiO08] A. MIELKE and M. ORTIZ. A class of minimum principles for characterizing the trajectories of dissipative systems. *ESAIM Control Optim. Calc. Var.*, 14, 494–516, 2008.

[MiR15] A. MIELKE and T. ROUBÍČEK. *Rate-Independent Systems: Theory and Application*. In preparation, 2015.

[MiS11] A. MIELKE and U. STEFANELLI. Weighted energy-dissipation functionals for gradient flows. *ESAIM Control Optim. Calc. Var.*, 17, 52–85, 2011.

[MiS13] A. MIELKE and U. STEFANELLI. Linearized plasticity is the evolutionary Γ-limit of finite plasticity. *J. Europ. Math. Soc.*, 15(3), 923–948, 2013.

[MiT07] A. MIELKE and A. M. TIMOFTE. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM J. Math. Analysis*, 39(2), 642–668, 2007.

[MiT12] A. MIELKE and L. TRUSKINOVSKY. From discrete visco-elasticity to continuum rate-independent plasticity: rigorous results. *Arch. Rational Mech. Anal.*, 203(2), 577–619, 2012.

[MiZ14] A. MIELKE and S. ZELIK. On the vanishing viscosity limit in parabolic systems with rate-independent dissipation terms. *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5)*, XIII, 67–135, 2014.

[MOŞ14] A. MIELKE, C. ORTNER, and Y. ŞENGÜL. An approach to nonlinear viscoelasticity via metric gradient flows. *SIAM J. Math. Analysis*, 46(2), 1317–1347, 2014.

[MPR14] A. MIELKE, M. PELETIER, and M. RENGER. On the relation between gradient flows and the large-deviation principle, with applications to Markov chains and diffusion. *Potential Analysis*, 41(4), 1293–1327, 2014.

[MRS08] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Part. Diff. Eqns.*, 31, 387–416, 2008.

[MRS09] A. Mielke, R. Rossi, and G. Savaré. Modeling solutions with jumps for rate-independent systems on metric spaces. *Discr. Cont. Dynam. Systems Ser. A*, 25(2), 585–615, 2009.

[MRS12] A. Mielke, R. Rossi, and G. Savaré. BV solutions and viscosity approximations of rate-independent systems. *ESAIM Control Optim. Calc. Var.*, 18(1), 36–80, 2012.

[MRS14a] A. Mielke, R. Rossi, and G. Savaré. Balanced-viscosity (BV) solutions to infinite-dimensional rate-independent systems. *J. Europ. Math. Soc.*, 2014. To appear. (WIAS preprint 1845. http://arxiv.org/abs/1309.6291).

[MRS14b] A. Mielke, R. Rossi, and G. Savaré. Balanced-Viscosity solutions for multi-rate systems. *J. Physics, Conf. Series*, 2014. WIAS Preprint 2001. Submitted.

[MRS15] A. Mielke, R. Rossi, and G. Savaré. Existence results for viscoplasticity at finite strain. *In preparation*, 2015.

[MTL02] A. Mielke, F. Theil, and V. I. Levitas. A variational formulation of rate–independent phase transformations using an extremum principle. *Arch. Rational Mech. Anal.*, 162, 137–177, 2002.

[Ons31] L. Onsager. Reciprocal relations in irreversible processes, I+II. *Physical Review*, 37, 405–426, 1931. (part II, 38:2265-2279).

[OrR99] M. Ortiz and E. Repetto. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids*, 47(2), 397–462, 1999.

[PuT02] G. Puglisi and L. Truskinovsky. Rate independent hysteresis in a bi-stable chain. *J. Mech. Phys. Solids*, 50(2), 165–187, 2002.

[PuT05] G. Puglisi and L. Truskinovsky. Thermodynamics of rate-independent plasticity. *J. Mech. Phys. Solids*, 53, 655–679, 2005.

[Ray71] L. Rayleigh (Hon. J. W. Strutt). Some general theorems relating to vibrations. *Proc. London Math. Soc.*, s1-4, 357–368, 1871.

[SaS04] E. Sandier and S. Serfaty. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, LVII, 1627–1672, 2004.

[Ser11] S. Serfaty. Gamma-convergence of gradient flows on Hilbert spaces and metric spaces and applications. *Discr. Cont. Dynam. Systems Ser. A*, 31(4), 1427–1451, 2011.