A new approach to elasto-plasticity using energy and dissipation functionals*

Alexander Mielke
Institut für Analysis, Dynamik und Modellierung
Universität Stuttgart, Germany

30 April 2003

Abstract. We consider models for elasto-plasticity which are based on fully nonlinear elastostatics and rate-independent evolution laws for the plastic deformation and suitable hardening parameters. Accounting for finite strains leads to the multiplicative decomposition of the strain tensor and to a flow rule formulated on a Lie group. Our analysis is based on a recently developed energetic approach to general rate-independent material models which only uses two energy functionals, namely the elastic stored energy and the dissipation distance which plays the role of a metric on the space of internal variables. The evolution law can be reformulated as a static stability condition combined with an energy inequality. This work surveys results on the existence of solutions of an intrinsically associated incremental problem which has the form of a minimization problem. Existence of solutions for the time-continuous problem remains open except for the one dimensional case.

2000 Mathematics Subject Classification: 74 C 15, 49 J 45

Keywords and Phrases: energy functionals, dissipation distance, minimization, incremental problems, finite-strain elasto-plasticity, gradient plasticity.

1 Introduction

The mathematical theory of linearized elasto-plasticity was developed in the 1970s by J.J. Moreau [Mor76] and further developed subsequently up to efficient numerical implementations, see, e.g., [Joh76, HaR95]. This theory relies on the additive

*Research partially supported by SFB 404 "Multifield Problems in Continuum Mechanics"

decomposition

$$\varepsilon = \tfrac{1}{2}(\mathrm{D}u + \mathrm{D}u^{\mathsf{T}}) = \varepsilon_{\mathrm{elast}} + \varepsilon_{\mathrm{plast}}$$

of the linearized strain tensor ε, where $u : \Omega \to \mathbb{R}^d$ denotes the displacement. Moreover, the energy is assumed to be a quadratic functional such that the problem takes the form of a quasi-variational inequality. More general approaches including nonlinear hardening laws and viscoplastic effects can be found in [BeF96, Alb98, ACZ99, Che01a, Che01b]. A first local existence result for smooth solutions in finite-strain elasto-plasticity is obtained in [Nef02].

This work surveys current mathematical developments in the theory of elasto-plasticity which allows for large strains and which is based on the multiplicative decomposition

$$F = \mathrm{D}y = F_{\mathrm{elast}} F_{\mathrm{plast}}. \tag{1.1}$$

The main feature here is that, as in finite strain elasticity also called geometrically nonlinear elasticity, the nonlinearities arise from the multiplicative group of invertible matrices. The main question is to understand the interaction of functional analytical tools, mainly based on linear function spaces, and these algebraic nonlinearities.

Here, $y : \Omega \to \mathbb{R}^d$ is the deformation of the body $\Omega \subset \mathbb{R}^d$. The energy $\mathcal{E}$ stored in a deformed body depends only on the elastic part F_{elast} of the deformation tensor and suitable hardening parameters $p \in \mathbb{R}^m$, but not on the plastic part F_{plast}, which is contained in $\mathrm{SL}(\mathbb{R}^d)$, or another Lie group $\mathfrak{G}$ contained in $\mathrm{GL}_+(\mathbb{R}^d) = \{\, G \in \mathbb{R}^{d\times d} \mid \det G > 0 \,\}$. The energy functional takes the form

$$\mathcal{E}(t, y, (F_{\mathrm{plast}}, p)) = \int_\Omega \widehat{W}(x, \mathrm{D}y(x) F_{\mathrm{plast}}(x)^{-1}, p(x))\,\mathrm{d}x - \langle \ell(t), y\rangle$$

where the external loading $\ell(t)$ is given via

$$\langle \ell(t), y\rangle = \int_\Omega f_{\mathrm{ext}}(t,x)\cdot y(x)\,\mathrm{d}x + \int_\Gamma g_{\mathrm{ext}}(t,x)\cdot y(x)\,\mathrm{d}a.$$

To model the plastic effects one prescribes either a plastic flow law or, equivalently, a dissipation potential $\Delta : \Omega \times \mathrm{T}(\mathfrak{G} \times \mathbb{R}^m) \to [0,\infty]$. We consider $\Delta(x,\cdot,\cdot)$ as an infinitesimal metric which defines the global dissipation distance $D(x,\cdot,\cdot)$ on $\mathfrak{G} \times \mathbb{R}^m$. Thus, the second ingredient of the material model is the dissipation distance between two internal states $P_j = (F^{(j)}_{\mathrm{plast}}, p_j) : \Omega \to \mathrm{SL}(d) \times \mathbb{R}^m$:

$$\mathcal{D}(P_1, P_2) = \int_\Omega D(x, (F^{(1)}_{\mathrm{plast}}(x), p_1(x)),\, (F^{(2)}_{\mathrm{plast}}(x), p_2(x)))\,\mathrm{d}x.$$

Allowing for finite strains we are forced to avoid convexity of the stored-energy density $\widehat{W}$. It rather has to be polyconvex or quasiconvex and frame indifferent, see [Bal77]. These notions work well together with the philosophy that $F = \mathrm{D}y$ is an element of $\mathrm{GL}_+(\mathbb{R}^d)$, i.e., we set $W(F) = \infty$ for $\det F \le 0$. The aim of this work is to show that these assumptions can be connected naturally with the multiplicative decomposition (1.1) and the Lie group structure for $G = F_{\rm plast}$.

We follow the work in [MiT99, MTL02, Mie02a, Mie03a, MiR03] which shows that rate-independent evolution for elastic materials with internal variables ("standard generalized materials") can be formulated by energy principles as follows: A pair $(y, P) : [0, T] \times \Omega \to \mathbb{R}^d \times \mathrm{SL}(\mathbb{R}^d) \times \mathbb{R}^m$ is called a solution of the elasto-plastic problem associated with $\mathcal{E}(t, \cdot, \cdot)$ and $\mathcal{D}$, if **stability (S)** and the **energy inequality (E)** holds:

(S) For all $t \in [0, T]$ we have $\mathcal{E}(t, y(t), P(t)) \leq \mathcal{E}(t, \widetilde{y}, \widetilde{P}) + \mathcal{D}(P(t), \widetilde{P})$.

(E) For all $s, t \in [0, T]$ with $s < t$ we have
$\mathcal{E}(s, y(s), P(s)) + \mathrm{Diss}(P, [s, t]) \leq \mathcal{E}(t, y(t), P(t)) - \int_s^t \langle \dot{\ell}(\tau, y(\tau)\rangle \,\mathrm{d}\tau$.

So far, we are not able to provide existence results for (S) & (E) in the present elasto-plastic setting. However, analogous models in phase transformations [MTL02, MiR03], in delamination [KMR02], in micromagnetism [Kru02, RoK02], and in fracture [FrM98, DMT02] have been treated with mathematical success. In these works two major restrictions had to be made: (i) $\mathcal{E}$ has to be convex in the strains (leading to infinitesimal strains) and (ii) the internal variable P has to lie in a closed convex subset of a Banach space. In finite-strain elasto-plasticity these two assumptions are clearly violated.

Since most of the above-mentioned existence results are based on time-incremental approximations we devote this work to an existence theory for the following incremental problem (IP). We consider this as a first step for finding solutions of (S) & (E) and comment on the problem of treating the time-continuous problem.

(IP) Incremental problem. For given $t_0 = 0 < t_1 < \ldots < t_N = T$ and P_0 find $(y_k, P_k) \in \mathrm{Argmin}\{\, \mathcal{E}(t_k, y, P) + \mathcal{D}(P_{k-1}, P) \mid (y, P)\,\}$, $k = 1, \ldots, N$.

Here "Argmin" denotes the set of all minimizers. Hence, (IP) consists of k minimization problems which are coupled via the dissipation distance. Similar incremental minimization problems are also used in the engineering community, cf. [OrR99, OrS99, MSS99, ORS00, MiL03, MSL02, HaH03]. Hence, it is justified to study the mathematical properties of (IP) even though its relation to (S) & (E) is not clear. In fact, existence and nonexistence for (IP) relates to questions of formation of microstructure, localization or failure, see the discussions in [CHM02, Mie03a].

In Section 6 we introduce the notions of finite-strain elasto-plasticity in detail and establish the relation between the classical flow rules of elasto-plasticity with our energetic formulation (S) & (E). For a more extensive and mechanical treatment we refer to [Mie03a]. In Section 4 study the incremental problem (IP) in specific function spaces. First we establish a rather general result which says that any solution $(y_k, P_k)_{k=1,\ldots,N}$ is stable in the sense of (S) and satisfies a two-sided discretized energy inequality replacing (E).

The key feature of the analysis of (IP), with $\mathcal{E}$ and $\mathcal{D}$ as given above, is to realize that the internal variables $P = (F_{\text{plast}}, p)$ occur under the integral over the body Ω only in a local fashion. Hence, it is possible to minimize in (IP) pointwise in $x \in \Omega$ with respect to $P(x)$. This leads to the **condensed energy density**

$$W^{\text{cond}}(P_{\text{old}}; F) = \min\{\, \widehat{W}(FG^{-1}, p) + D(P_{\text{old}}, (G, p)) \mid (G, p) \in \mathrm{SL}(\mathbb{R}^d) \times \mathbb{R}^m \,\}.$$

The first major assumption for our existence theory is that $W^{\text{cond}}((\mathbf{1}, p_*); \cdot) : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is polyconvex. The second major assumption is that W^{cond} and the dissipation distance D are coercive:

$$W^{\text{cond}}((\mathbf{1}, p_*); F) \geq c|F|^{q_F} - C \quad \text{and} \quad D((\mathbf{1}, p_*), (G, p)) \geq c|G|^{q_G} - C.$$

If the growth exponents satisfy $\frac{1}{q_F} + \frac{1}{q_G} \leq \frac{1}{q} < \frac{1}{d}$, then existence of solutions $(y_k, F^{(k)}_{\text{plast}}, p_k)$ for (IP) is obtained with $y_k \in \mathrm{W}^{1,q}(\Omega, \mathbb{R}^d)$ and $F^{(k)}_{\text{plast}} \in \mathrm{L}^{q_G}(\Omega, \mathbb{R}^{d\times d})$.

In Section 5 we supply a specific two-dimensional example in which all assumptions can be checked explicitly and are fulfilled for suitable parameter values. Thus, we provide a first existence theory for a multi-dimensional elasto-plastic incremental problem in the geometric nonlinear case. Moreover, we discuss a one-dimensional example to highlight the difficulties in proving existence of solutions for the time-continuous problem (S) & (E) by letting the step-size of the time discretization tend to 0. Only by exploiting the very specific properties of the one-dimensional case, we obtain a convergence result for the incremental solution which implies that the time-continuous problem (S) & (E) has a solution as well.

In Section 6 we speculate about the solvability of (IP) and (S) & (E) after regularizing the elastic energy by adding a gradient term in the form

$$\textstyle\int_\Omega \kappa \|\mathrm{D}G\|^r \,\mathrm{d}x \quad \text{or} \quad \int_\Omega \kappa \|(\operatorname{curl} G)G^{\mathsf{T}}\|^r \,\mathrm{d}x.$$

We expect existence results for (IP), but solvability of (S) & (E) remains widely open.

2 Elasto-plasticity at finite strain

Here we describe the framework of multiplicative elasto-plasticity as developed in [Mie02a, Mie03a], where more details on the mechanical modeling are given.

We consider an elastic body $\Omega \subset \mathbb{R}^d$ which is bounded and has a Lipschitz boundary $\partial\Omega$. A deformation is a mapping $y : \Omega \to \mathbb{R}^d$ such that the deformation gradient $F(x) = \mathrm{D}y(x)$ exists for a.e. $x \in \Omega$ and satisfies

$$F(x) \in \mathrm{GL}_+(d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}.$$

The internal plastic state at a material point $x \in \Omega$ is described by the plastic tensor $G = F_{\text{plast}} \in \mathrm{GL}_+(d)$ and a possibly vector-valued hardening variable $p \in \mathbb{R}^m$. We shortly write $P = (G, p)$ to denote the set of all plastic variables. The major assumption in finite-strain elasto-plasticity is the multiplicative decomposition of the deformation gradient F into an elastic and a plastic part

$$F = F_{\text{elast}} F_{\text{plast}} = F_{\text{elast}} G. \tag{2.1}$$

The main feature of this decomposition is that the elastic properties will depend only on F_{elast}, whereas previous plastic transformations through G are completely forgotten. However, the hardening variable p will record changes in G and may influence the elastic properties.

The deformation process is governed by two principles. First we have energy storage which gives rise to the equilibrium equations and second we have dissipation due to plastic transformations which give rise to the plastic flow rule. Energy storage is described by the Gibbs energy

$$\mathcal{E}(t, y, P) = \int_\Omega W(x, \mathrm{D}y(x), P(x))\,\mathrm{d}x - \langle \ell(t), y \rangle,$$

where $\langle \ell(t), y \rangle = \int_\Omega f_{\text{ext}}(t, x) \cdot y(x)\,\mathrm{d}x + \int_{\Gamma_{\text{Neu}}} g_{\text{ext}}(t, x) \cdot y(x)\,\mathrm{d}a(x)$ denotes the process-time dependent loading. The major constitutive assumption is the multiplicative decomposition

$$W(x, F, (G, p)) = \widehat{W}(x, FG^{-1}, p). \tag{2.2}$$

From now on we drop the variable x for notational convenience. However, the whole theory and analysis works in the inhomogeneous case as well.

The dissipational effects are usually modeled by prescribing yield surfaces. For our purpose it is more convenient and mathematically clearer to start on the other side, namely the dissipation metric. In mechanics this metric is called dissipation potential, since the dissipational friction forces are obtained from it via differentiation with respect to the plastic rates. We emphasize that the natural setup for the plastic transformation $G \in \mathrm{GL}_+(d)$ is that of an element of a Lie group $\mathfrak{G} \subset \mathrm{GL}_+(d)$. A usual assumption is plastic incompressibility, which gives $\mathfrak{G} = \mathrm{SL}(d) = \{\, G \mid \det G = 1 \,\}$. However, $\mathfrak{G} = \mathrm{GL}_+(d)$ or a single-slip system $\mathfrak{G} = \{\, \mathbf{1} + \gamma e_1 \otimes e_2 \mid \gamma \in \mathbb{R} \,\}$ may also be possible. A dissipation potential is a mapping

$$\Delta : \Omega \times \mathrm{T}(\mathfrak{G} \times \mathbb{R}^m) \to [0, \infty],$$

which is called a dissipation metric, if it is continuous and if $\Delta(x, (G, p), \cdot)$ is convex and positively homogeneous of degree 1, i.e.,

$$\Delta(x, (G, p), \alpha(\dot{G}, \dot{p})) = \alpha \Delta(x, (G, p), (\dot{G}, \dot{p})) \text{ for } \alpha \geq 0.$$

(Again we drop the variable x for notational convenience.) This condition leads to rate-independent material behavior. Together with the multiplicative decomposition (2.1) one assumes **plastic indifference**

$$\Delta((G\widehat{G}, p), (\dot{G}\widehat{G}, \dot{p})) = \Delta((G, p), (\dot{G}, \dot{p})) \text{ for all } \widehat{G} \in \mathfrak{G}.$$

This amounts to the existence of a function $\widehat{\Delta} : \mathbb{R}^m \times \mathbb{R}^m \times \mathfrak{g} \to [0,\infty]$ such that

$$\Delta((G,p),(\dot G,\dot p)) = \widehat{\Delta}(p,\dot p,\dot G G^{-1}).$$

Here $\mathfrak{g} = \mathrm{T}_{\mathbf{1}}\mathfrak{G}$ is the Lie algebra associated with the Lie group $\mathfrak{G}$, and strictly speaking $\dot G G^{-1}$ is the right translation of $\dot G(t) \in \mathrm{T}_{G(t)}\mathfrak{G}$ to $\mathfrak{g} = \mathrm{T}_{\mathbf{1}}\mathfrak{G}$.

An important feature of our theory is the induced dissipation distance D on $\mathfrak{G}\times\mathbb{R}^m$ defined via (recall $P = (G,p)$)

$$D(P_0,P_1) = \inf\{\, \int_0^1 \Delta(P(s),\dot P(s))\,\mathrm{d}s | P \in \mathrm{C}^1([0,1],\mathfrak{G}\times\mathbb{R}^m), P(0) = P_0, P(1) = P_1 \,\}.$$

It is important to note that we don't assume symmetry (i.e., $\Delta(P,-\dot P) = \Delta(P,\dot P)$) which would contradict hardening. Thus, $D(\cdot,\cdot)$ will not be symmetric either. However, we will often use the triangle inequality

$$D(P_1,P_3) \le D(P_1,P_2) + D(P_2,P_3),$$

eeq which follows immediately from the definition. Plastic difference implies that thedissipation distance satisfies

$$D((G_1,p_1),(G_2,p_2)) = D((\mathbf{1},p_1),(G_2G_1^{-1},p_2)). \tag{2.3}$$

Integration gives the bulk dissipation distance between two internal states $P_j : \Omega \to \mathfrak{G}\times\mathbb{R}^m$ via

$$\mathcal{D}(P_0,P_1) = \int_\Omega D(P_0(x),P_1(x))\,\mathrm{d}x.$$

To make the energetic formulation mathematically rigorous we define the set of kinematically admissible deformations via

$$\mathcal{F} = \{\, y \in \mathrm{W}^{1,q}(\Omega;\mathbb{R}^d) \mid y|_{\Gamma_{\mathrm{Dir}}} = y_{\mathrm{Dir}} \,\}, \tag{2.4}$$

where $\Gamma_{\mathrm{Dir}} = \partial\Omega\backslash\Gamma_{\mathrm{Neu}}$ is a part of the boundary with positive surface measure. Moreover, $y_{\mathrm{Dir}} = Y|_{\Gamma_{\mathrm{Dir}}}$ where $Y \in \mathrm{C}^1(\overline{\Omega};\mathbb{R}^d)$ with $\mathrm{D}Y(x) \in \mathrm{GL}_+(d)$ for all $x \in \overline{\Omega}$. The integrability power q in $\mathrm{W}^{1,q}$ will be chosen larger than d in order to apply the theory of polyconvexity. The loading can then be considered as a function $\ell : [0,T] \to \mathrm{W}^{1,q}(\Omega,\mathbb{R}^d)^*$, where * denotes the dual space (space of all continuous linear forms).

The set of admissible internal states is simply

$$\mathcal{P} = \{\, P : \Omega \to \mathfrak{G}\times\mathbb{R}^m \mid P \text{ measurable} \,\}. \tag{2.5}$$

Because of the image space, which is a manifold, it is not clear whether it is reasonable to consider $\mathcal{P}$ as a subset of a Banach space like $\mathrm{L}^{q_G}(\Omega,\mathbb{R}^{d\times d}) \times \mathrm{L}^{q_p}(\Omega,\mathbb{R}^m)$ or as a general manifold equipped with a metric associated with $\mathcal{D}$. The abstract theory in Section 3 will address the interplay between the topology on $\mathcal{Z} = \mathcal{F}\times\mathcal{P}$ and the metric $\mathcal{D}$.

Definition 2.1. *A process $(y, P) : [0, T] \to \mathcal{F} \times \mathcal{P}$ is called a solution of the elasto-plastic problem defined via $\mathcal{E}(t, \cdot, \cdot)$ and $\mathcal{D}$, if the stability condition (S) and the energy inequality (E) holds:*

$$\begin{array}{ll} \textbf{\textit{(S)}} & \textit{For all } t \in [0,T] \textit{ we have} \\ & \mathcal{E}(t, y(t), P(t)) \leq \mathcal{E}(t, \widetilde{y}, \widetilde{P}) + \mathcal{D}(P(t), \widetilde{P}) \textit{ for all } (\widetilde{y}, \widetilde{P}) \in \mathcal{F} \times \mathcal{P}. \\ \textbf{\textit{(E)}} & \textit{For all } s, t \in [0,T] \textit{ with } s < t \textit{ we have} \\ & \mathcal{E}(t, y(t), P(t)) + \mathrm{Diss}(P, [s,t]) \leq \mathcal{E}(s, y(s), P(s)) - \int_s^t \langle \dot{\ell}(r), y(r) \rangle \,\mathrm{d}r. \end{array} \tag{2.6}$$

Here $-\int_s^t \langle \dot{\ell}, y \rangle \,\mathrm{d}r = \int_s^t \langle \ell, \dot{y} \rangle \,\mathrm{d}r - \langle \ell, y \rangle|_s^t$ is called the reduced work of the external forces, since $\mathcal{E}$ denotes the Gibbs energy instead of the Helmholtz energy. The dissipation reads

$$\mathrm{Diss}(P, [s,t]) = \sup\{ \textstyle\sum_{j=1}^N \mathcal{D}(P(t_{j-1}), P(t_j)) \mid N \in \mathbb{N}, s \leq t_0 < \ldots < t_N \leq t \}$$

for general processes, which equals $\mathrm{Diss}(P, [s,t]) = \int_s^t \int_\Omega \Delta(P(r,x), \dot{P}(r,x)) \,\mathrm{d}x \,\mathrm{d}t$ for differentiable processes.

The major advantage of the energetic formulation via (S) and (E) is that neither derivatives of the constitutive functions W and Δ nor of the solution $(\mathrm{D}y, P)$ are needed. Nevertheless, (S) and (E) are strong enough to determine the physically relevant solutions. We refer to [MiT01] for uniqueness results under additional convexity assumptions. Moreover, it is shown in [Mie03a] that sufficiently smooth solutions (y, P) of (S) and (E) satisfy the classical equations of elasto-plasticity, namely the **equilibrium equation**

$$\left.\begin{array}{rcll} -\operatorname{div} T(t,x) &=& f_{\mathrm{ext}}(t,x) & \text{in } \Omega, \\ y(t,x) &=& y_{\mathrm{Dir}}(x) & \text{on } \Gamma_{\mathrm{Dir}}, \\ T(t,x)\nu(x) &=& g_{\mathrm{ext}}(t,x) & \text{on } \Gamma_{\mathrm{Neu}}, \end{array}\right\}$$

and the **flow rule**

$$0 \in \partial_{\dot{P}}^{\mathrm{sub}} \Delta(P(t,x), \dot{P}(t,x)) - Q(t,x), \tag{2.7}$$

where $T = \frac{\partial}{\partial F} W(\mathrm{D}y, P)) = \frac{\partial}{\partial F_{\mathrm{elast}}} \widehat{W}(\mathrm{D}y G^{-1}, p) G^{-\mathsf{T}}$ is the first Piola-Kirchhoff stress tensor, $\partial_{\dot{P}}^{\mathrm{sub}} \Delta(P, \dot{P})$ denotes the subgradient of the convex function $\Delta(P, \cdot\,) : \mathrm{T}_P(\mathfrak{G} \times \mathbb{R}^m) \to [0, \infty]$ and $Q \in \mathrm{T}_P^*(\mathfrak{G} \times \mathbb{R}^m)$ is the thermodynamically conjugated driving force to $P = (G, p)$, i.e.,

$$Q = -\tfrac{\partial}{\partial (G,p)} W(F, (G,p)) = (G^{-\mathsf{T}} F^{\mathsf{T}} \tfrac{\partial}{\partial F_{\mathrm{elast}}} \widehat{W}(FG^{-1}, p) G^{-\mathsf{T}}, -\tfrac{\partial}{\partial p} \widehat{W}(FG{-}1, p)).$$

Using the elastic domain $\mathbb{Q}(P) = \partial_{\dot{P}}^{\mathrm{sub}} \Delta(P, 0) \subset \mathrm{T}_P^*(\mathfrak{G} \times \mathbb{R}^m)$, the Legendre-Fenchel transform shows that (2.7) is equivalent to the differential inclusion

$$\dot{P} \in \partial \mathcal{X}_{\mathbb{Q}(P)}(Q) = \mathrm{N}_Q \mathbb{Q}(P), \tag{2.8}$$

where $\chi_{\mathbb{Q}(P)}$ is the indicator function and $\mathrm{N}_Q\mathbb{Q}(P)$ denotes the exterior normal cone. If $\mathbb{Q}(P)$ is given by a yield function ϕ in the form $\mathbb{Q}(P) = \{\, Q \mid \phi(P,Q) \leq 0 \,\}$ and $\frac{\partial}{\partial Q}\phi(P,Q) \neq 0$ at $\phi(P,Q) = 0$, then the flow rule (2.7) or (2.8) can be reformulated via the Karush-Kuhn-Tucker conditions:

$$\dot{P} = \lambda \tfrac{\partial}{\partial Q}\phi(P,Q), \quad \lambda \geq 0, \quad \phi(P,Q) \leq 0, \quad \lambda\phi(P,Q) = 0.$$

3 Abstract setup of rate-independent problems

We show here that the energetic formulation derived above is a special case of an abstract formulation for rate-independent processes. This theory was developed in [MiT99, MiT01, MTL02] and takes its most nonlinear form in [MaM03]. Other applications are in the theory of shape-memory alloys [GMH02, MiR03], in ferromagnetism [Kru02, RoK02] or in fracture or delamination [DMT02, FrM98, KMR02]. However, we will see in the subsequent sections that the present state of the theory is not fully applicable in the theory of multiplicative elasto-plasticity.

We start with a topological space $\mathcal{Z}$ and denote convergence in this space by $z_k \xrightarrow{\mathcal{Z}} z$. The rate-independent system consists of two ingredients which both are considered to be energetic quantities. The time-dependent *energy functional* $\mathcal{E} : [0,T] \times \mathcal{Z} \to \mathbb{R}_\infty$ describes the energy-storage mechanism of the system. The *dissipation distance* $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$ describes how the system dissipates energy. The latter is taken to be the minimal amount of dissipated energy when the system changes from one state into another. Hence, $\mathcal{D}$ should satisfy the triangle inequality:

$$\mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3) \quad \text{for all } z_1, z_2, z_3 \in \mathcal{Z},$$

but we allow for the value ∞ and do not enforce symmetry, i.e., we allow for $\mathcal{D}(z_0, z_1) \neq \mathcal{D}(z_1, z_0)$ which is important for elasto-plasticity. For any given curve $z : [0,T] \to \mathcal{Z}$ we define the *dissipation* on $[s,t]$ via

$$\mathrm{Diss}_{\mathcal{D}}(z;[s,t]) = \sup\{\, \textstyle\sum_1^N \mathcal{D}(z(\tau_{j-1}), z(\tau_j)) \mid N{\in}\mathbb{N}, s{=}\tau_0{<}\tau_1{<}\cdots{<}\tau_N{=}t \,\}. \quad (3.1)$$

Definition 3.1. *A curve $z : [0,T] \to X$ is called a* ***solution*** *of the rate-independent model $(\mathcal{E}, \mathcal{D})$, if* ***global stability (S)*** *and* ***energy inequality (E)*** *holds:*

(S) *For all $t \in [0,T]$ and all $\widehat{z} \in \mathcal{Z}$ we have $\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, \widehat{z}) + \mathcal{D}(z(t), \widehat{z})$.*

(E) *For all t_0, t_1 with $0 \leq t_0 < t_1 \leq T$ we have*

$$\mathcal{E}(t_1, z(t_1)) + \mathrm{Diss}_{\mathcal{D}}(z;[t_0,t_1]) \leq \mathcal{E}(t_0, z(t_0)) + \textstyle\int_{t_0}^{t_1} \partial_t \mathcal{I}(t, z(t))\,\mathrm{d}t.$$

The stability condition (S) represents the fact that, while holding the process time t fixed, the system is in a stable state, which means that changing $z(t)$ into $\widetilde{z}$ does not release more stored energy than has to be paid by the dissipation mechanism: $\mathcal{E}(t, z(t)) - \mathcal{E}(t, \widehat{z}) \leq \mathcal{D}(z(t), \widehat{z})$. The energy inequality (E) says that the present stored energy plus the dissipated energy has to be less than the initial energy plus the work of the external forces.

Rate-independency manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that z is a solution of $(\mathcal{E}, \mathcal{D})$, if and only if the reparametrized curve $\widetilde{z} : t \mapsto z(\alpha(t))$, with $\dot{\alpha} > 0$, is a solution of $(\widetilde{\mathcal{E}}, \mathcal{D})$, where $\widetilde{\mathcal{E}}(t, z) = \mathcal{E}(\alpha(t), z)$. In particular, the stability (S) is a static concept and the energy estimate (E) is rate-independent, since the dissipation defined via (3.1) is scale invariant like the length of a curve.

The major importance of the energetic formulation is that neither the given functionals $\mathcal{D}$ and $\mathcal{E}(t, \cdot)$ nor the solutions $z : [0, T] \to \mathcal{Z}$ must be differentiable. In fact, to make sense of such derivatives we would need to impose that $\mathcal{Z}$ is a (Banach) manifold or a suitable subset of a Banach space. Although this will be the case in many applications, it is better to avoid these concepts as long as possible.

A traditional "linear setup" is obtained, if we assume that $\mathcal{Z}$ is a closed convex subset of a Banach space X, that $\mathcal{E}(t, \cdot)$ is strictly convex and that $\mathcal{D}$ has the from $\mathcal{D}(z_0, z_1) = \|z_1 - z_0\|_X$. Then the energetic formulation (S) & (E) is equivalent to the differential inclusion

$$0 \in \partial^{\text{sub}}[\| \cdot \|_X](\dot{z}(t)) + \partial^{\text{sub}}[\mathcal{E}(t, z(t)) + \chi_{\mathcal{Z}}(\cdot)](z(t)) \quad \text{for a.a. } t \in [0, T],$$

where ∂^{sub} denotes the subdifferential for convex functions. We refer to [CoV90, Vis01] for such doubly nonlinear problems and to [MiT01] for exact proofs of the implications between the different formulations.

To develop an existence theory for the problem (S)&(E), one needs to specify conditions on $\mathcal{Z}$, $\mathcal{E}$ and $\mathcal{D}$. We will not discuss the complete existence theory here, mainly because of the fact that in the case of elasto-plasticity the theory is not yet finished. Nevertheless, we give the main flavor of the theory and show how certain major steps work together. The main approach to the energetic formulation is based on time discretization and exploiting the fact that the backward Euler step can be formulated as a minimization problem.

To this end we choose discrete times $0 = t_0 < t_1 < \ldots < t_N = T$. For a given initial datum $z_0 \in \mathcal{Z}$ we formulate the time-incremental problem.

(IP) *For $z_0 \in \mathcal{Z}$ with $\mathcal{E}(0, z_0) < \infty$ find $z_1, \ldots, z_N \in X$ such that*

$$z_k \in \operatorname{Argmin}\{\, \mathcal{E}(t_k, z) + \mathcal{D}(z_{k-1}, z) \mid z \in \mathcal{Z} \,\} \quad \text{for } k = 1, \ldots, N.$$

Here "Argmin" denotes the set of all minimizers. Note that the size of the time step does not enter here, which is due to rate-independence. Of course, the existence of minimizers is nontrivial in general. However, the following result shows that any solution satisfies a discretized version of (S) & (E) and thus justifies (IP) as an approximation to (S) & (E). The stability condition (S) can be reformulated as

$(t, z(t)) \in \mathcal{S}$ for all $t \in [0, T]$, where the *stable set* $\mathcal{S}$ is given via

$$\mathcal{S} := \{ (t, z) \in [0, T] \times X \mid \mathcal{E}(t, z) \le \mathcal{E}(t, \widehat{z}) + \mathcal{D}(z, \widehat{z}) \text{ for all } \widehat{z} \in X \}.$$

Theorem 3.2. *Assume $(0, z_0) \in \mathcal{S}$, then each solution $(z_k)_{k=0,\dots,N}$ of (IP) satisfies the following properties. For $k = 1, \dots, N$*
the state z_k is stable at time t_k, i.e., $(t_k, z_k) \in \mathcal{S}$, and
$\int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, z_k)\, \mathrm{d}s \le \mathcal{E}(t_k, z_k) - \mathcal{E}(t_{k-1}, z_{k-1}) + \mathcal{D}(z_{k-1}, z_k) \le \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, z_{k-1})\, \mathrm{d}s.$

This result shows that (IP) is intrinsically linked with (S) & (E). Its proof is a simple application of the minimization property of z_k together with the triangle inequality for $\mathcal{D}$, cf. [MiT99, MTL02, MaM03].

To obtain existence for (IP) one typically makes the following assumptions: First we assume that the time dependence through the loading is controlled:

$$|\partial_t \mathcal{E}(t, z)| \le C_1 \text{ for all } (t, z) \in [0, T] \times \mathcal{Z}. \tag{3.2}$$

With this we define the set of energetically reachable states

$$\mathcal{R} := \{ (t, z) \in [0, T] \times \mathcal{Z} \mid \mathcal{E}(t, z) + \mathcal{D}(z_0, z) \le \mathcal{E}(0, z_0) + C_1 t + 1 \} \tag{3.3}$$
$$\text{and } \mathcal{R}_{\mathcal{Z}} := \{ z \in \mathcal{Z} \mid \exists\, t \in [0, T] \text{ with } (t, z) \in \mathcal{R} \}. \tag{3.4}$$

Then, the major assumptions are read

$$\begin{array}{ll} \text{(a)} & \mathcal{R} \text{ is a compact subset of } [0, T] \times \mathcal{Z}; \\ \text{(b)} & \mathcal{E} : \mathcal{R} \to \mathbb{R}_\infty \text{ is lower semicontinuous;} \\ \text{(c)} & \mathcal{D} : \mathcal{R}_{\mathcal{Z}} \times \mathcal{R}_{\mathcal{Z}} \to [0, \infty] \text{ is lower semicontinuous.} \end{array} \tag{3.5}$$

If the assumptions (3.2) and (3.5) hold, (IP) has a solution $(z_k)_{k=0,\dots,N}$ which satisfies $(t_k, z_k) \in \mathcal{R}$. To show this one just uses induction over k and employs Theorem 3.2 and the triangle inequality in each step.

Finally we address the question whether the solutions of (IP) can be used to construct a solution of (S) & (E) as a limit of incremental solutions when the temporal step size tends to 0. In [MaM03] the following result was obtained.

Theorem 3.3. *Let the assumptions (3.2) and (3.5) hold and assume that the topology on $\mathcal{Z}$ is compatible with the metric $\mathcal{D}$ in the following way:*

$$\textit{If } (t_k, z_k), (t, z) \in \mathcal{R} \cap \mathcal{S},\ t_k \to t \textit{ and } \min\{\mathcal{D}(z, z_k), \mathcal{D}(z_k, z)\} \to 0, \textit{ then } z_k \stackrel{\mathcal{Z}}{\to} z.$$

Assume further that $\partial_t \mathcal{E} : \mathcal{R} \to [-C_1, C_1]$ is continuous and that the stable set $\mathcal{S}$ is closed (both in the topology of $\mathcal{Z}$).

Then, the problem (S) & (E) has a solution $z : [0, T] \to \mathcal{Z}$ which is obtained as pointwise limit of a subsequence of piecewise constant interpolants $Z^l : [0, T] \to \mathcal{Z}$

of solutions to (IP) for nested partitions with step size tending to 0, that is, for all $t \in [0,T]$ we have $Z^{l_m}(t) \stackrel{\mathcal{Z}}{\rightarrow} z(t)$ for $m \to \infty$. Moreover, the energy inequality (E) is an equality and $\mathcal{E}(t,z(t)) = \lim_{m\to\infty} \mathcal{E}(t,Z^{l_m}(t))$ and $\mathrm{Diss}(z,[s,t]) = \lim_{m\to\infty} \mathrm{Diss}(Z^{l_m},[s,t])$ for all $s,t \in [0,T]$ with $s<t$.

The major difficulty is to establish the closedness of $\mathcal{S}$, since it is defined only implicitly. A typical positive result is obtained when $\mathcal{D}$ and $\mathcal{E}$ are continuous on $\mathcal{R}$. In many applications $\mathcal{Z}$ is a Banach space and it is easy to show that $\mathcal{S}$ is closed in the strong topology, but there $\mathcal{R}$ is not compact. In the weak topology $\mathcal{R}$ would be compact, but since $\mathcal{E}$ and $\mathcal{D}$ are not weakly continuous it is difficult to show that $\mathcal{S}$ is closed. In Section 5 we give an example for elasto-plasticity where this dilemma can be studied explicitly.

4 Incremental problems in elasto-plasticity

Until now, no existence theory for the time continuous problem (S) & (E) is available, except for the case $d=1$ given in the second part of Section 5. However, in computational plasticity [OrR99, MSS99, ORS00, MSL02, HaH03, MiL03] incremental problems are a fundamental tool and hence deserve a mathematical treatment in their right. It was realized in [OrR99, ORS00, CHM02, Mie03a] that existence of solutions for (IP) is not to be expected in general situations. In fact, nonexistence can be connected either with failure of the material due to localization (e.g. in shear bands) or fracture or with formation of microstructure in material domains of positive measure. Here we present constitutive assumptions which allow us to prove existence of solutions for each incremental step, even though the abstract theory of Section 3 is not applicable.

We let $\mathcal{Z} = \mathcal{F} \times \mathcal{P}$, according to (2.4) and (2.5), and choose a time discretization $0 = t_0 < \ldots < t_N = T$. The incremental problem assumes are more specific form as the states $z=(y,P)$ have two components and only one of them appears in the dissipation. For a stable initial state $(y_0,P_0) \in \mathcal{F}\times\mathcal{P}$ we consider the incremental problem

$$\textbf{(IP)}\quad \begin{array}{l} \text{For } k=1,\ldots,N \text{ find } (y_k,P_k) \in \mathcal{F}\times\mathcal{P} \text{ such that} \\ (y_k,P_k) \in \operatorname{Argmin}\{\, \mathcal{E}(t_k,y,P) + \mathcal{D}(P_{k-1},P) \mid (y,P) \in \mathcal{F}\times\mathcal{P}\,\}. \end{array} \tag{4.1}$$

The main point is to show that the set of global minimizers ($\operatorname{Argmin}\{\cdots\}$) is nonempty, i.e. we have to find $z_k = (y_k,P_k) \in \mathcal{F}\times\mathcal{P}$ such that $\mathcal{I}_k(z_k) \le \mathcal{I}_k(z)$ for all $z \in \mathcal{Z}$, where

$$\mathcal{I}_k(y,P) := \textstyle\int_\Omega [W(\mathrm{D}y(x),P(x)) + D(P_{k-1}(x),P(x))]\,\mathrm{d}x - \langle \ell(t_k), y\rangle. \tag{4.2}$$

In fact, there are two ways to attact the problem. The first approach uses that y does not appear in the dissipation. One can define a reduced functional

$$\mathcal{E}^{\rm red} : [0,T] \times \mathcal{P} \to \mathbb{R}_\infty;\ (t,P) \mapsto \min\{\, \mathcal{E}(t,y,P) \mid y \in \mathcal{F}\,\}$$

and then apply the abstract theory with $\mathcal{Z} = \mathcal{P}$ to this new functional. This approach leads to the gneral problem that $\mathcal{E}^{\rm red}$ is defined only implicitly, that the derivative $\partial_t \mathcal{E}^{\rm red}$ may not exist or may not be continuous.

We use a second approach which relies on the special structure that $P \in \mathcal{P}$ occurs under the integral only with its point values and no derivatives appear. Hence, we can minimize with respect to P for each point $x \in \Omega$ separately. We define the **condensed energy density**

$$W^{\rm cond}(P_{\rm old}; F) = \min\{\, W(F,P) + D(P_{\rm old}, P) \mid P \in \mathfrak{G} \times \mathbb{R}^m \,\},$$

the condensed functional

$$\mathcal{I}_k^{\rm cond}(y) = \int_\Omega W^{\rm cond}(P_{k-1}(x); \mathrm{D}y(x))\,\mathrm{d}x - \langle \ell(t_k), y\rangle.$$

and choose a measurable **update function**

$$\begin{aligned}&P^{\rm upd} : (\mathfrak{G} \times \mathbb{R}^m) \times \mathbb{R}^{d\times d} \to \mathfrak{G} \times \mathbb{R}^m \text{ with}\\ &P^{\rm upd}(P_{\rm old}; F) \in \operatorname{Argmin}\{\, W(F,P) + D(P_{\rm old}, P) \mid P \in \mathfrak{G} \times \mathbb{R}^m \,\}.\end{aligned}$$

Lemma 4.1. *If $\widetilde{y} \in \mathcal{F}$ minimizes $\mathcal{I}_k^{\rm cond}$ and if $\widetilde{P}(x) = P^{\rm upd}(P_{k-1}(x); \mathrm{D}\widetilde{y}(x))$, then $(\widetilde{y}, \widetilde{P})$ minimizes $\mathcal{I}_k$. Moreover, $(y,P) \in \mathcal{F}\times\mathcal{P}$ minimizes $\mathcal{I}_k$ if and only if y minimizes $\mathcal{I}_k^{\rm cond}$ and $P(x) \in \operatorname{Argmin}\{\, W(\mathrm{D}y(x),\cdot) + D(P_{k-1}(x),\cdot) \mid P \in \mathfrak{G} \times \mathbb{R}^m \,\}$.*

Hence, each step in (IP) reduces to a classical variational problem of nonlinear elasticity. We now state our main assumptions which are formulated in terms of $W^{\rm cond}$ and D. Thus, the assumptions are quite implicit, since in practice only the stored-energy density W and the dissipation potential Δ are given. In the next section we provide an example where all these conditions are satisfied. Note that the multiplicative decomposition (2.2) and the plastic indifference of the dissipation (2.3) implies $W^{\rm cond}((G_{\rm old}, p_{\rm old}); F) = W^{\rm cond}((\mathbf{1}, p_{\rm old}); FG_{\rm old}^{-1})$.

$$\begin{aligned}
&\text{(a) } W^{\rm cond}((1,\cdot);\cdot) : \mathbb{R}^m \times \mathbb{R}^{d\times d} \to [0,\infty] \text{ and } D(\cdot,\cdot) : (\mathfrak{G}\times\mathbb{R}^m)^2 \to [0,\infty]\\
&\quad\text{are lower semi-continuous.}\\
&\text{(b) For each } p \in \mathbb{R}^m \text{ the function } W^{\rm cond}((1,p),\cdot) : \mathbb{R}^{d\times d} \to [0,\infty]\\
&\quad\text{is polyconvex.}\\
&\text{(c) There exist } C, c > 0, p_* \in \mathbb{R}^m \text{ and exponents } q_F, q_G, q_p \geq 1 \text{ such that}\\
&\quad D((\mathbf{1},p_*),(G,p)) \geq c|G|^{q_D} + c|p|^{q_p} - C \text{ and } W^{\rm cond}((\mathbf{1},p);F) \geq c|F|^{q_F} - C\\
&\quad\text{for all } (F,G,p) \text{ with } D((1,p_*),(G,p)) < \infty.\\
&\text{(d) } P^{\rm upd}((\mathbf{1},\cdot);\cdot) : \mathbb{R}^m \times \mathbb{R}_+^{d\times d} \to \mathfrak{G} \times \mathbb{R}^m \text{ is Borel measurable.}
\end{aligned} \tag{4.3}$$

The following result is established in [Mie03b].

Theorem 4.2. *Let the assumptions in (4.3) be satisfied such that* $\frac{1}{q_F} + \frac{1}{q_G} \leq \frac{1}{q} < \frac{1}{d}$ *holds, where* q *occurs in the definition of* $\mathcal{F} \subset \mathrm{W}^{1,q}(\Omega, \mathbb{R}^d)$ *in (2.4).*

Then, for each $P_0 \in \mathcal{P}$ *with* $\mathcal{D}((1, p_*), P_0) < \infty$ *and* $\ell \in \mathrm{C}^0([0,T], \mathrm{W}^{1,q}(\Omega, \mathbb{R}^d)^*)$ *the incremental problem (IP), see (4.1), has a solution* $((y_k, P_k))_{k=1,\ldots,N}$ *with*

$$y_k \in \mathcal{F} \text{ and } P_k = P^{\mathrm{upd}}(P_{k-1}; \mathrm{D}y_k(\cdot)) \in \mathcal{P} \cap (\mathrm{L}^{q_G}(\Omega, \mathbb{R}^{d\times d}) \times \mathrm{L}^{q_p}(\Omega, \mathbb{R}^m)).$$

Obviously, the result is proved by induction over $k = 1, 2, \ldots, N$. According to Lemma 4.1, the k-th minimization problem for $\mathcal{I}_k$ (cf. (4.2)) reduces to minimization of $\mathcal{I}_k^{\mathrm{cond}} : y \mapsto \int_\Omega W_k(x, \mathrm{D}y(x))\,\mathrm{d}x - \langle \ell(t_k), y\rangle$ with

$$W_k(x, F) = W^{\mathrm{cond}}(P_{k-1}(x); F) = W^{\mathrm{cond}}((\mathbf{1}, p_{k-1}(x)); FG_{k-1}(x)^{-1}).$$

Clearly, $W_k : \Omega \times \mathbb{R}^{d\times d} \to [0, \infty]$ is measurable in x and lower semi-continuous in F, by (4.3c) it satisfies the lower bound

$$W_k(x, F) \geq c|FG_{k-1}(x)^{-1}|^{q_F} - C \geq c|F|^q - C|G_{k-1}(x)|^{q_*} - C.$$

Moreover, (4.3b) gives polyconvexity of $W_k(x, \cdot)$, since W^{cond} is polyconvex and the minors of the product FG_{k-1}^{-1} are linear combinations of products of the minors of F and G_{k-1}^{-1}. Hence, suitable existence results for polyconvex functionals apply. Induction works since $q_* \leq q_G$ holds.

We note that $\mathcal{I}_k : \mathcal{F} \times \mathcal{P} \to \mathbb{R}_\infty$ is not weakly lower semicontinuous because of the geometric nonlinearity coming from the multiplicative decomposition, i.e., $W(F, (G, p)) = \widehat{W}(FG^{-1}, p)$. It is shown in [FKP94, LDR00] that weak lower semicontinuity of $\mathcal{I}_k$ implies that the map $(F, G, p) \mapsto W(F, (G, p)) + D(P_{k-1}(x), (G, p))$ is cross-quasiconvex, which in turn implies convexity in $P = (G, p)$. However, this can only be achieved if $F_{\mathrm{elast}} \mapsto \widehat{W}(F_{\mathrm{elast}})$ is convex, but this contradicts the standard axioms of finite-strain elasto-plasticity, see [CHM02] and below. Of course, lower semi-continuity of $\mathcal{I}_k$ is not necessary and, as shown above, we may obtain minimizers without it.

5 Two examples

The crucial assumption of the above theory is the polyconvexity of W^{cond}. Since polyconvexity is in general hard to check, it is nontrivial to find a multi-dimensional example. Our example only works for the dimension $d = 2$, since it depends on the fact that everything can be calculated explicitly. We consider the isotropic energy density

$$W : \begin{cases} \mathbb{R}^{2\times 2} & \to & \mathbb{R}_\infty, \\ F & \mapsto & \frac{1}{\alpha}(\nu_1^\alpha + \nu_2^\alpha) + V(\det F), \end{cases} \tag{5.1}$$

where $\nu_1, \nu_2 \geq 0$ are the two singular values of F (i.e., the eigenvalues of $(F^\mathsf{T} F)^{1/2}$) and $V : \mathbb{R} \to [0, \infty]$ is convex, continuous and satisfies $V(\delta) = \infty$ for $\delta \leq 0$. For the plastic variables we take $P = (G, p) \in \mathrm{SL}(2) \times \mathbb{R}$ with the dissipation metric

$$\Delta(G, p, \dot{G}, \dot{p}) = \begin{cases} A'(p)\dot{p} & \text{for } \dot{p} \geq \|\dot{G}G^{-1}\|, \\ \infty & \text{else.} \end{cases} \tag{5.2}$$

Here, $A(p) = \mathrm{e}^{\alpha p/\sqrt{2}}$ and $\|\xi\|^2 = \sum \xi_{ij}^2$. The associated dissipation distance satisfies

$$D((\mathbf{1}, p_0), E(s), p_1)) = \begin{cases} \mathrm{e}^{\alpha(p_0/\sqrt{2}+|s|)} - \mathrm{e}^{\alpha p_0/\sqrt{2}} & \text{for } p_1 \geq p_0 + \sqrt{2}|s|, \\ \infty & \text{else,} \end{cases}$$

where $E(s) = \mathrm{diag}(\mathrm{e}^s, \mathrm{e}^{-s})$. Moreover, the condensed stored-energy density takes the explicit form

$$W^{\mathrm{cond}}((1, p); F) = V(\nu_1\nu_2) - \mathrm{e}^{\alpha p/\sqrt{2}} + \begin{cases} \frac{2}{\alpha}\sqrt{\nu_1^\alpha(\nu_2^\alpha + b_p)} & \text{for } \nu_1^\alpha \geq \nu_2^\alpha + b_p, \\ \frac{1}{\alpha}\sqrt{\nu_1^\alpha + \nu_2^\alpha + b_p} & \text{for } |\nu_1^\alpha - \nu_2^\alpha| \leq b_p, \\ \frac{2}{\alpha}\sqrt{\nu_2^\alpha(\nu_1^\alpha + b_p)} & \text{for } \nu_2^\alpha \geq \nu_1^\alpha + b_p, \end{cases}$$

where $b_p = \alpha \mathrm{e}^{\alpha p/\sqrt{2}}$. The update functions can also be given explicitly. For details we refer to [Mie02a, HMM03, Mie03a, Mie03b].

By explicit calculations, it is shown in [Mie02b] that $W^{\mathrm{cond}}((1, p); \cdot)$ is polconvex for $\alpha \geq 2$. Moreover, we have the lower bounds

$$D((\mathbf{1}, p_*), (G, p)) \geq c(\|G\|^\alpha - 1) \quad \text{and} \quad W^{\mathrm{cond}}((\mathbf{1}, p); F) \geq c(\|F\|^{\alpha/2} - b_p)$$

for all $F \in \mathbb{R}^{2\times 2}, p_*, p \in \mathbb{R}$ and $G \in \mathrm{SL}(2)$ with $D((\mathbf{1}, p_*), (G, p)) < \infty$. Thus, this two-dimensional example satisfies the assumptions (4.3) for $\alpha \geq 2$ with exponents $q_F = \alpha/2$ and $q_G = \alpha$. Hence, Theorem 4.2 is applicable if $\frac{1}{2} = \frac{1}{d} > \frac{1}{q} \geq \frac{1}{q_W} + \frac{1}{q_D} = \frac{3}{\alpha}$ holds. Summarizing we obtain the following existence result.

Theorem 5.1. *Let $d = 2$ and $\mathfrak{G} = \mathrm{SL}(2)$. With $\alpha > 6$ let $W : \mathbb{R}^{2\times 2} \to [0, \infty]$ and $\Delta : \mathrm{T}(\mathfrak{G}\times\mathbb{R}) \to [0, \infty]$ be defined via (5.1) and (5.2), respectively. Assume that there exists a $p_* \in \mathbb{R}$, such that the initial condition $P_0 \in \mathcal{P}$ satisfies $\mathcal{D}((\mathbf{1}, p_*), P_0) < \infty$ and let $q = \alpha/3$.*

Then, for each $\ell : [0, T] \to (\mathrm{W}^{1,\alpha/3}(\Omega, \mathbb{R}^2))^*$ the incremental problem (IP) (see (4.1)) has a solution $((y_k, P_k))_{k=1,\ldots,N} \in (\mathcal{F}\times\mathcal{P})^N$. Moreover, there exists a constant C which depends only on α, ℓ, and P_0, but not on the partition $t_1, \ldots, t_N$ nor on the solution such that

$$\|y_k\|_{\mathrm{W}^{1,\alpha/3}} + \|G_k\|_{\mathrm{L}^\alpha} + \|\mathrm{e}^{\alpha p_k/\sqrt{2}}\|_{\mathrm{L}^1} \leq C \text{ for } k = 1, \ldots, N.$$

As a second example we treat the one-dimensional situation, which seems to be trivial regarding the existence of solutions for the incremental problem. Here the major simplification is that polyconvexity is equal to convexity. However, the purpose of this example is to address the question of convergence of the incremental solutions towards a solution of the time-continuous problem (S) & (E). We will see, that general arguments, as given in Theorem 3.3, are not sufficient. Only in using the special one-dimensional structure, we are able to prove convergence (of a subsequence) and obtain finally an existence result for (S) & (E).

Again we treat a special case, but the analysis easily generalizes to far more general constitutive laws W and Δ. We let

$$W(F) = \tfrac{1}{\alpha}(F^{\alpha}+F^{-\alpha}) \text{ for } F>0 \text{ and } \infty \text{ else,}$$

$\mathfrak{G} = \mathrm{GL}_+(1) = (0,\infty)$, $P=(G,p)\in\mathfrak{G}\times\mathbb{R}$, and

$$\Delta((G,p),(\dot G,\dot p)) = \alpha \mathrm{e}^{\alpha p}\dot p \text{ for } \dot p \geq |\dot G/G| \text{ and } \infty \text{ else.}$$

From this we find the condensed stored-energy density

$$W^{\mathrm{cond}}((1,p);F) = \frac{1}{\alpha}\begin{cases} 2\sqrt{1+b_pF^{\alpha}}-b_p & \text{for } F^{\alpha}\geq b_p+F^{-\alpha},\\ F^{\alpha}+F^{-\alpha} & \text{for } |F^{\alpha}-F^{-\alpha}|\leq b_p,\\ 2\sqrt{1+b_pF^{-\alpha}}-b_p & \text{for } F^{-\alpha}\geq b_p+F^{\alpha},\\ \infty, & \text{for } F\leq 0\end{cases}$$

with $b_p=\alpha\mathrm{e}^{\alpha p}$. We see that $W^{\mathrm{cond}}((1,p);\cdot)$ is strictly convex for $\alpha\geq 2$ and that the theory of Section 4 applies for $\alpha>3$, since the exponents in condition (4.3) are $q_F=\alpha/2$ and $q_G=\alpha$. Thus, (IP) has a unique solution for each partition of [0,T].

We now want to discuss the connection to the abstract theory of Section 3. Therefore, we restrict ourselves to the most simple nontrivial case. Let $\Omega=(0,1)\subset\mathbb{R}^1$, $\alpha=6$ and $\mathcal{F}=\{\, y\in \mathrm{W}^{1,2}(\Omega) \mid y(0)=0\,\}$. The loading takes the form

$$\langle \ell(t),y\rangle = \int_0^1 h_{\mathrm{ext}}(t,x)y(x)\,\mathrm{d}x + \sigma_1(t)y(1) = \int_0^1 H_{\mathrm{ext}}(t,x)y'(x)\,\mathrm{d}x$$

where $H_{\mathrm{ext}}(t,x)=\sigma_1(t)+\int_x^1 h_{\mathrm{ext}}(t,\widetilde x)\,\mathrm{d}\widetilde x$ with $H_{\mathrm{ext}}\in \mathrm{C}^1([0,T]\times\overline{\Omega})$. Moreover, we let $\mathcal{P}=\{\,(G,p)\in \mathrm{L}^6(\Omega)^2 \mid G\geq 0 \text{ a.e.}\,\}$ and $(G_0,p_0)\equiv(1,0)$.

With this definition it is clear that $\mathcal{R}\subset[0,T]\times\mathcal{R}_{\mathcal{Z}}$ for some bounded closed set $\mathcal{R}_{\mathcal{Z}}$ in $\mathrm{W}^{1,2}(\Omega)\times L^6(\Omega)^2$. It can be shown that $\mathcal{R}$ is also closed in the strong topology. However, on the one hand compactness of $\mathcal{R}$ in the strong topology fails since $\mathcal{R}$ contains "L^{∞}"-neighborhoods. On the other hand, compactness of $\mathcal{R}$ in the weak topology fails, since this is equivalent to the weak lower semicontinuity of $\mathcal{E}(t,\cdot,\cdot)+\mathcal{D}(P_{k-1},\cdot)$ on $\mathcal{F}\times\mathcal{P}$, which is not satisfied due to lacking cross-quasiconvexity.

As shown in [Mie03a, Mie03b], it is possible to characterize the stable set for this example explicity:

$$\mathcal{S}=\{\,(t,y,G,p)\mid (y'(x),G(x),p(x))\in M(H_{\mathrm{ext}}(t,x)) \text{ for a.a. } x\in\Omega\,\},$$

where $M(H) = \{ (F,G,p) \mid \left|(\frac{F}{G})^\alpha - (\frac{F}{G})^{-\alpha}\right| \le \alpha \mathrm{e}^{\alpha p},\ (\frac{F}{G})^{\alpha-1} - (\frac{F}{G})^{-\alpha-1} = G\,H \}$. Again we see that $\mathcal{S}$ is closed in the strong topology but not in the weak topology, since the sets $M(H) \subset \mathbb{R}^3$ are not convex.

This shows that the abstract theory of Section 3 is not applicable. Nevertheless, it is possible to show that the incremental solutions converge to a solution of the time-continuous problem (S) & (E). Here we use that the compactness of $\mathcal{R}$ is solely used for the purpose of extracting converging subsequences. We can dispense with compactness if convergence can be shown by other means. In the present one-dimensional setting we can use the fact that the problem decouples into "zero-dimensional" plasticity problems for each $x \in \Omega$. Moreover, each of these problems is almost scalar such that monotonicity arguments can be used which imply convergence. As a conclusion the following existence and convergence result is obtained in [Mie03b].

Theorem 5.2. *For the above problem there exists a solution (y, G, p) of (S) & (E) (cf. (2.6)) with $(y, G, p) \in \mathrm{C}^0([0,T], \mathrm{W}^{1,\infty}(\Omega) \times \mathrm{L}^\infty(\Omega)^2)$. Moreover, there exists a constant $C > 0$ such that for each time discretization $0 = t_0 < t_1 < \cdots < t_N = T$ the unique solution $(y_k, G_k, p_k)_{k=0,\dots,N}$ of the incremental problem (4.1) satisfies*

$$\|y(t_k)-y_k\|_{\mathrm{W}^{1,\infty}} + \|G(t_k)-G_k\|_{\mathrm{L}^\infty} + \|p(t_k)-p_k\|_{\mathrm{L}^\infty} \le C \max\{ t_n - t_{n-1} \mid n=1,\dots,k \}$$

for $k = 1, \dots, N$.

The connection with the abstract theory is immediate if we equip $\mathcal{Z} = \mathcal{F} \times \mathcal{P}$ with the strong topology of $\mathrm{W}^{1,\infty}(\Omega) \times \mathrm{L}^\infty(\Omega)$. Then, essentially all the assumptions hold, except for the compactness of $\mathcal{R}$. In particular, $\mathcal{S}$ is closed, $\mathcal{E}$ and $\mathcal{D}$ are lower semicontinuous and $\partial_t \mathcal{E}$ is continuous.

6 Gradient theories

As we have seen in Sections 4 and 5, the above functionals $\mathcal{E}(t,\cdot)$ and $\mathcal{D}(\cdot,\cdot)$ are not weakly lower semi-continuous on spaces of the form $\mathrm{W}^{1,q} \times \mathrm{L}^r$. The major problem is the nonconvexity of the multiplicative term $\mathrm{D}y\, G^{-1}$ and of the Lie group structure underlying $\mathcal{D}$. One possible way out of these problems, which is also often used in engineering, cf. [Gur02, Sve02], consists in adding a regularizing term involving the gradient terms of the plastic tensor $F_{\mathrm{plast}} = G$. (Sometimes these gradient terms are also called "nonlocal terms".) Such terms introduce a length scale into the problem which prevent the formation of arbitrarily fine microstructures which are one of the main obstructions to existence of minimizers.

In this section we just want to indicate how terms involving the gradient $\mathrm{D}G$ (a tensor of third order) may help to establish existence for (IP) and for (S) & (E). Most of the results given here are derived in [MiM03].

We consider the same dissipation distance D and $\mathcal{D}$ as above. However, the energy functional now takes the form

$$\mathcal{E}(t,y,G) = \int_\Omega U(x,\mathrm{D}y,G,\mathrm{D}G)\,\mathrm{d}x - \langle \ell(t),y\rangle,$$

where, for simplicity, we assume that the density U takes one of the two cases

$$U_{\mathrm{grad}}(F,G,A) = \widehat{W}(FG^{-1}){+}\kappa\|A\|^r,$$
$$U_{\mathrm{curl}}(F,G,A) = \widehat{W}(FG^{-1}){+}\kappa\|A_{\mathrm{anti}}G^{\mathsf{T}}\|^r,$$

with $\kappa>0$ and $r>d$. Note that A has the dimension of 1/length and hence κ must have a dimension like $(\text{length})^r$. Here A_{anti} denotes the anti-symmetric part of the tensor A, such that $(\mathrm{D}G)_{\mathrm{anti}} = \operatorname{curl} G$. The density U_{curl} involves the tensor $(\operatorname{curl} G)\,G^{\mathsf{T}}$, which is known to be the physically most relevant tensor for measuring the density of geometrically necessary dislocations [Gur02, Sve02].

The major difference to the previous theory, is that the set $\mathcal{R}$ now provides a priori bounds for $\mathrm{D}G$ or $\operatorname{curl} G$ in $\mathrm{L}^r(\Omega,\mathbb{R}^{d\times d})$ as well. For $U=U_{\mathrm{grad}}$ the set $\mathcal{R}$ is bounded in $[0,T]\times\mathcal{Z}$ with $\mathcal{Z}=\mathrm{W}^{1,q}(\Omega,\mathbb{R}^d)\times\mathrm{W}^{1,r}(\Omega,\mathbb{R}^{d\times d})\times\mathrm{L}^{q_p}(\Omega,\mathbb{R}^m)$. In particular, weak lower semi-continuity of $\mathcal{E}$ on $\mathcal{Z}$ can now be established, since the dangerous product $\mathrm{D}yG^{-1}$ is under control. If

$$y_k \rightharpoonup y \ \text{ in } \ \mathrm{W}^{1,q}(\Omega,\mathbb{R}^d) \qquad \text{and} \qquad G_k \rightharpoonup G \ \text{ in } \ \mathrm{W}^{1,r}(\Omega,\mathbb{R}^{d\times d}),$$

then $\mathbb{M}(\mathrm{D}y_kG_k^{-1}) \rightharpoonup \mathbb{M}(\mathrm{D}yG^{-1})$ in $\mathrm{L}^1(\Omega)^{m_d}$, where $\mathbb{M}(F)$ denotes the vector of all minors of the matrix F. To show this, we use the product rule for minors and that G_k converges strongly to G. Moreover, the dissipation $\mathcal{D}$ which does not involve derivatives of G will be continuous with respect to this convergence.

More precisely, the incremental problem (IP) now consists in minimizing the functional

$$\mathcal{I}_k : (y,G) \mapsto \textstyle\int_\Omega W_k(x,\mathrm{D}y,G,\mathrm{D}g)\,\mathrm{d}x - \langle \ell(t_k),y)\rangle \tag{6.1}$$

with the density $W_k(x,F,G,A) = U(x,F,G,A) + D_k^{\mathrm{cond}}(x,G)$ where the incrementally condensed distance D_k^{cond} is given by

$$D_k^{\mathrm{cond}}(x,G) = \min\{\, D((G_{k-1}(x),p_{k-1}(x)),(G,p) \mid p\in\mathbb{R}^m\,\}.$$

The above arguments indicate that I_k for $U=U_{\mathrm{grad}}$ is weakly lower semi-continuous on $\mathrm{W}^{1,q}(\Omega,\mathbb{R}^d)\times\mathrm{W}^{1,r}(\Omega,\mathbb{R}^{d\times d})$. Coercivity can be obtained with arguments as in Section 4. Thus, the gradient regularization provides an existence theory for (IP) under rather general and easily checkable conditions.

For the case $U=U_{\mathrm{curl}}$ the situation is more difficult. Using $\det G\equiv 1$ we first observe $G^{-1} = \operatorname{ad} G$ and find that the minors $\mathbb{M}_s(FG^{-1})$ of order s are exactly those minors in $\mathbb{M}_d\left(\begin{pmatrix} F \\ G \end{pmatrix}\right)$ which contain s rows from F and $d{-}s$ rows from G. Now, if

$$\begin{array}{rcll} y_k & \rightharpoonup & y & \text{in } \ \mathrm{W}^{1,q}(\Omega,\mathbb{R}^d), \\ G_k & \rightharpoonup & G & \text{in } \ \mathrm{L}^{q_G}(\Omega,\mathbb{R}^{d\times d}), \ \text{and} \\ \operatorname{curl} G_k & \rightharpoonup & \operatorname{curl} G & \text{in } \ \mathrm{L}^{q_c}(\Omega,\mathbb{R}^{d\times d\times d}), \end{array}$$

for suitable exponents q, q_G and $q_c > 1$, then $\mathbb{M}_s(\mathrm{D}_k G_k^{-1}) \rightharpoonup \mathbb{M}_s(\mathrm{D}G^{-1})$ and $\mathbb{M}_s(G_k) \rightharpoonup \mathbb{M}_s(G)$.

Thus, under suitable coercivity assumptions on U and D_k^{cond} the functional I_k in (6.1) can be shown to be weakly lower semi-continuous on a weakly closed subset $\mathcal{Z}$ of the Banach space

$$\mathrm{W}^{1,q}(\Omega,\mathbb{R}^d) \times \{\, G \in \mathrm{L}^{q_G}(\Omega,\mathbb{R}^{d\times d}) \mid \operatorname{curl} G \in \mathrm{L}^{q_c}(\Omega,\mathbb{R}^{d\times d\times d})\,\}$$

for suitable exponents q, q_G and q_c. However, for this we additionally need that $G \mapsto D_k^{\mathrm{cond}}(x,G)$ is polyconvex, since only $\operatorname{curl} G$ is controlled.

Thus, the curl regularization is also strong enough to handle the difficult multiplicative term $\mathrm{D}y\,G^{-1}$. But only under the nontrivial condition that $D_k^{\mathrm{cond}}(x,\cdot)$ is polyconvex we obtain existence for (IP).

Finally, we remark that both regularizations are not strong enough to do the limit from (IP) to (S) & (E). The major problem here is that the dissipation $\mathcal{D}$ only controls distances of the plastic variables $P = (G,p) \in \mathcal{P}$. However, in finite-strain elasto-plasticity there may be more than one global minimizer $y \in \mathcal{F}$ of $\mathcal{E}(t,\cdot,P)$. Thus, the compatibility condition between the topology on $\mathcal{Z} = \mathcal{F}\times\mathcal{P}$ and the dissipation distance $\mathcal{D}$, as stated in Theorem 3.3, does not hold. Of course, temporal oscillations between several global minimizers without any dissipation are unphysical and should be eliminated by adjusting the model in a suitable manner.

Acknowledgments The author is grateful for stimulating discussions with Andreas Mainik, Christian Miehe, Stefan Müller, Patrizio Neff, Tomaš Roubíček and Florian Theil. The research was partially supported by DFG through the SFB 404 (TP C11) *Multifield Problems in Continuum Mechanics.*

References

[ACZ99] J. Alberty, C. Carstensen, and D. Zarrabi. Adaptive numerical analysis in primal elastoplasticity with hardening. *Comput. Methods Appl. Mech. Engrg.*, 171(3-4), 175–204, 1999.

[Alb98] H.-D. Alber. *Materials with Memory*, volume 1682 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. Initial-boundary value problems for constitutive equations with internal variables.

[Bal77] J. Ball. Constitutive inequalities and existence theorems in nonlinear elastostatics. In *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium (Edinburgh, 1976), Vol. I*, pages 187–241. Res. Notes in Math., No. 17. Pitman, London, 1977.

[BeF96] A. Bensoussan and J. Frehse. Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and H^1 regularity. *Comment. Math. Univ. Carolin.*, 37, 285–304, 1996.

[Che01a] K. Chełmiński. Coercive approximation of viscoplasticity and plasticity. *Asymptot. Anal.*, 26, 105–133, 2001.

[Che01b] K. Chełmiński. Perfect plasticity as a zero relaxation limit of plasticity with isotropic hardening. *Math. Methods Appl. Sci.*, 24, 117–136, 2001.

[CHM02] C. Carstensen, K. Hackl, and A. Mielke. Non-convex potentials and microstructures in finite-strain plasticity. *Proc. Royal Soc. London, Ser. A*, 458, 299–317, 2002.

[CoV90] P. Colli and A. Visintin. On a class of doubly nonlinear evolution equations. *Comm. Partial Differential Equations*, 15(5), 737–756, 1990.

[DMT02] G. Dal Maso and R. Toader. A model for quasi-static growth of brittle fractures: existence and approximation results. *Arch. Rat. Mech. Anal.*, 162, 101–135, 2002.

[Efe03] M. Efendiev. On the compactness of the stable set for rate-independent processes. *Comm. Part. Diff. Eqns.*, 2003. Submitted.

[FKP94] I. Fonseca, D. Kinderlehrer, and P. Pedregal. Energy functionals depending on elastic strain and chemical composition. *Calc. Var. Partial Differential Equations*, 2(3), 283–313, 1994.

[FrM98] G. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46, 1319–1342, 1998.

[GMH02] S. Govindjee, A. Mielke, and G. Hall. The free-energy of mixing for n-variant martensitic phase transformations using quasi-convex analysis. *J. Mech. Physics Solids*, 50, 1897–1922, 2002.

[Gur02] M. E. Gurtin. A gradient theory of single–crystal viscoplasticity that accounts for geometrically necessary dislocations. *J. Mech. Phys. Solids*, 50, 5–32, 2002.

[HaH03] K. Hackl and U. Hoppe. On the calculation of microstructures for inelastic materials using relaxed energies. In C. Miehe, editor, *IUTAM Symposium on Computational Mechanics of Solids at Large Strains*, pages 77–86. Kluwer, 2003.

[HaR95] W. Han and B. D. Reddy. Computational plasticity: the variational basis and numerical analysis. *Comput. Mech. Adv.*, 2(4), 283–400, 1995.

[HMM03] K. Hackl, A. Mielke, and D. Mittenhuber. Dissipation distances in multiplicative elastoplasticity. In W. Wendland and M. Efendiev, editors, *Analysis and Simulation of Multifield Problems*. Springer-Verlag, 2003. In press.

[Joh76] C. Johnson. Existence theorems for plasticity problems. *J. Math. Pures Appl. (9)*, 55(4), 431–444, 1976.

[KMR02] M. Kočvara, A. Mielke, and T. Roubíček. A rate-independent approach to the delamination problem. In preparation, 2002.

[Kru02] M. Kružík. Variational models for microstructure in shape memory alloys and in micromagnetics and their numerical treatment. In A. Ruffing and M. Robnik, editors, *Proceedings of the Bexbach Kolloquium on Science 2000 (Bexbach, 2000)*. Shaker Verlag, 2002.

[LDR00] H. Le Dret and A. Raoult. Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results. *Arch. Ration. Mech. Anal.*, 154(2), 101–134, 2000.

[MaM03] A. Mainik and A. Mielke. Existence results for energetic models for rate-independent materials. In preparation, 2003.

[Mie02a] A. Mielke. Finite elastoplasticity, Lie groups and geodesics on SL(d). In P. Newton, A. Weinstein, and P. Holmes, editors, *Geometry, Dynamics, and Mechanics*, pages 61–90. Springer-Verlag, 2002.

[Mie02b] A. Mielke. Necessary and sufficient conditions for polyconvexity of isotropic functions. *J. Convex Analysis*, 2002. Submitted.

[Mie02c] A. Mielke. Relaxation via Young measures of material models for rate-independent inelasticity. *Preprint, February*, 2002.

[Mie03a] A. Mielke. Energetic formulation of multiplicative elasto-plasticity using dissipation distances. *Cont. Mech. Thermodynamics*, 2003. To appear.

[Mie03b] A. Mielke. Existence of minimizers in incremental elasto-plasticity with finite strains. *SIAM J. Math. Analysis*, 2003. Submitted.

[MiL03] C. Miehe and M. Lambrecht. Analysis of microstructure development in shearbands by energy relaxation of incremental stress potentials: large-strain theory for generalized standard solids. *Int. J. Numer. Meth. Eng.*, 2003. In press.

[MiM03] A. Mielke and S. Müller. Lower semi–continuity and existence of minimizers for a functional in elasto-plasticity. In preparation.

[MiR03] A. Mielke and T. Roubíček. A rate-independent model for inelastic behavior of shape-memory alloys. *Multiscale Model. Simul.*, 2003. Submitted.

[MiT99] A. Mielke and F. Theil. A mathematical model for rate-independent phase transformations with hysteresis. In H.-D. Alber, R. Balean, and R. Farwig, editors, *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering"*, pages 117–129. Shaker-Verlag, 1999.

[MiT01] A. Mielke and F. Theil. On rate-independent hysteresis models. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 2001. To appear.

[Mor76] J.-J. Moreau. Application of convex analysis to the treatment of elasto-plastic systems. In P. Germain and B. Nayroles, editors, *Applications of methods of functional analysis to problems in mechanics*, pages 56–89. Springer-Verlag, 1976. Lecture Notes in Mathematics, 503.

[MSL02] C. Miehe, J. Schotte, and M. Lambrecht. Homogenization of inelastic solid materials at finite strain based on incremental minimization princicples. Application to texture analysis of polycrystals. *J. Mech. Physics Solids*, 50, 2123–2167, 2002.

[MSS99] C. Miehe, J. Schröder, and J. Schotte. Computational homogenization analysis in finite plasticity. Simulation of texture development in polycrystalline materials. *Comput. Methods Appl. Mech. Engrg.*, 171, 387–418, 1999.

[MTL02] A. Mielke, F. Theil, and V. Levitas. A variational formulation of rate-independent phase transformations using an extremum principle. *Arch. Rational Mech. Anal.*, 162, 137–177, 2002.

[Nef02] P. Neff. Finite multiplicative plasticity for small elastic strains with linear balance equations and grain boundary relaxation. *Cont. Mech. Thermodynamics*, 2002. To appear.

[OrR99] M. Ortiz and E. Repetto. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids*, 47(2), 397–462, 1999.

[ORS00] M. Ortiz, E. Repetto, and L. Stainier. A theory of subgrain dislocation structures. *J. Mech. Physics Solids*, 48, 2077–2114, 2000.

[OrS99] M. Ortiz and L. Stainier. The variational formulation of viscoplastic constitutive updates. *Comput. Methods Appl. Mech. Engrg.*, 171(3-4), 419–444, 1999.

[RoK02] T. Roubíček and M. Kružík. Mircrostructure evolution model in micromagnetics. *Zeits. angew. math. Physik*, 2002. In print.

[Rou02] T. Roubíček. Evolution model for martensitic phase transformation in shape-memory alloys. *Interfaces Free Bound.*, 4, 111–136, 2002.

[Sve02] B. Svendsen. Continuum thermodynamic models for crystal plasticity including the effects of geometrically necessary dislocations. *J. Mech. Physics Solids*, 50, 1297–1329, 2002.

[Vis01] A. Visintin. A new approach to evolution. *C.R.A.S. Paris*, 332, 233–238, 2001.