Lipschitz Lecture held in Bonn

Modeling and analysis of rate-independent processes

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These lectures are mainly based on the survey [Mie05]. For some of the applications we refer to [Mie06a]. These notes were taken from the six 2-hours lectures and typed afterwards by Simone Hock, Dorothee Knees, Carsten Patz, Adrien Petrov, Florian Schmid, and Marita Thomas.

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1 Motivating examples

1.1 Rate-independent evolutionary problems

In this Lipschitz Lecture we consider evolutionary problems, hence problems that depend on time. A solution of such a problem will be denoted by q and maps from some time interval $[0,T]$ into a state-space $\mathcal{Q}$, i.e. $q:[0,T]\to\mathcal{Q}$. Thus, we want to know how the solution q **evolves** in time.

We are interested in the subclass of **rate-independent** evolutionary problems. A typical evolutionary problem consists of an initial value $q_0\in\mathcal{Q}$ which determines the initial position of the solution, i.e. $q(0)=q_0$ and of course on some time-dependent data. Let us for simplicity assume that this time-dependent data can explicity be expressed by some function $l\in \mathrm{C}^0\big([0,T],\mathcal{Q}^*\big)$. Then we are able to characterize the rate-independence of such a problem with the help of the

Input-output operator

$$\mathcal{Q}:\text{state-space}$$
$$\mathcal{H}:\begin{cases}\mathcal{Q}\times \mathrm{C}^0\big([0,T],\mathcal{Q}^*\big) & \to & \mathrm{C}^0\big([0,T],\mathcal{Q}\big),\\ q_0\times l & \mapsto & q(\cdot).\end{cases}$$

The Input-output operator $\mathcal{H}$ maps the given data (i.e. the initial value q_0 and the (external) loading $l\in \mathrm{C}^0\big([0,T],\mathcal{Q}^*\big)$) to a solution of the problem.

Definition 1.1 (Abstract definition of rate-independence) *An evolutionary problem, which can be expressed by the above input-output operator is called* **rate-independent** *if the solution (output) depends on a change of the rate of the loading as follows:*

$$\mathcal{H}(q_0,l\circ\alpha)=\mathcal{H}(q_0,l)\circ\alpha \qquad \textit{for every } \alpha\in \mathrm{C}^1\big([0,T]\big) \textit{ with } \dot\alpha>0.$$

If for example the external loading acts twice as fast ($\dot\alpha\equiv 2$) then the solution of a rate-independent problem responds just twice as fast, too. To give the reader a first intuition we present some examples.

1.2 First example of a rate-independent problem: sliding ball

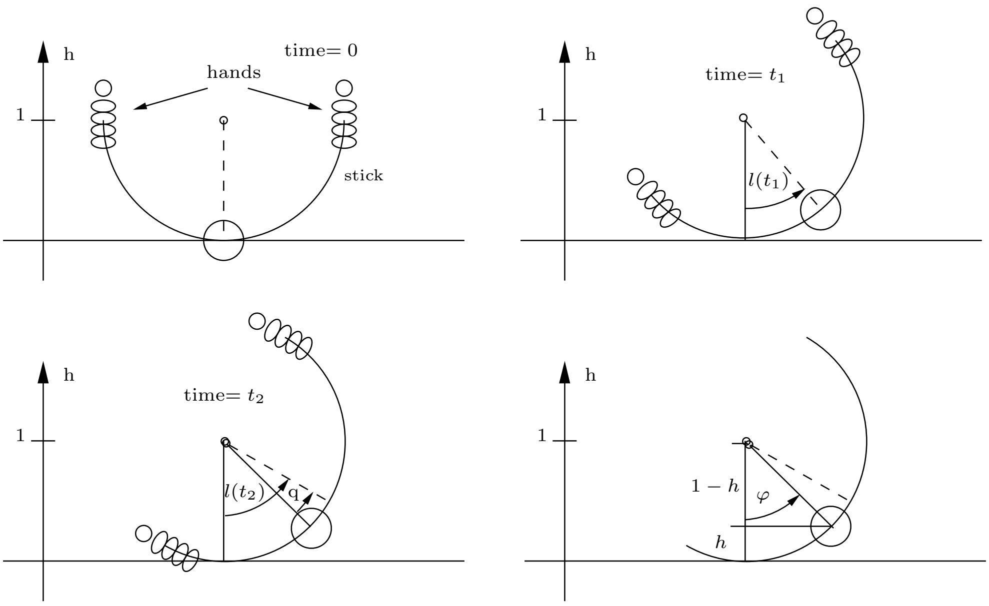

Figure 1: sliding ball on a stick

We consider a one dimensional problem $\mathcal{Q} = \mathbb{R}^1$ of a sliding ball or a sliding pearl on a bended stick, (see figure 1). Somebody is holding the stick with his hands and as time evolves the person inclines the stick by the angle $l(t)$. The question is: What will the ball do? Our solution $q(t)$ is the angle between the position of the ball and the dotted line corresponding to the initial position of the ball. This angle describes how far the ball slides along the stick. The problem turns out to be nontrivial if we take dry or Coulomb friction into account.

We now explain the four pictures of figure 1. The first picture describes the initial position of the system at time $t = 0$ and we assume $0 = l(0) = q(0)$. In the second picture the person inclined the stick by the angle $l(t_1)$. We assume that due to friction the ball does not slide and we still have $q(t_1) = 0$. In the third picture the person inclined the stick even more and we have $l(t_2) > l(t_1)$. Now, the ball did slide by the angle $-q$. The last picture will be used to motivate the following mathematical model.

To model the above problem we describe the energy inside the system. For this we introduce the energy storage functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R},\ (t,q) \mapsto \mathcal{E}(t,q)$. Note first, that we do not take into account kinetic energy, hence our energy functional does not depend on the velocity of the solution $\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}q$. This is physically only allowed if we can assume that the physical velocity of the ball remains small.

Let us now consider the fourth picture and assume that the mass of the ball is 1, then its potential energy is described by $\mathcal{E} = h$ with h being the height of the position of the ball. The height h depends on the prescribed angle $l(t)$ and the excursion q of the ball as follows

$$h = 1 - (1-h) = 1 - \cos(\varphi) = 1 - \cos(l(t) - q).$$

To keep the model simple we assume $l(t) - q$ to be small and use $\cos(\varphi) \approx 1 - \frac{\varphi^2}{2}$ and we deduce $h \approx \frac{(l(t)-q)^2}{2}$. This motivates the definition

$$\mathcal{E}(t,q) := \frac{1}{2}q^2 - l(t)q.$$

Note, that we let drop the term $\frac{1}{2}l^2(t)$. We will motivate later on why this does not have any impact on our solution q. The negative energy restoring force (the force, that is needed to keep the particle in its position) is

$$\sigma(t,q) = \mathrm{D}_q\mathcal{E}(t,q) = q - l(t).$$

During its sliding friction occurs and we denote by $\kappa > 0$ the coefficient of friction which models the roughness of the stick. For simplicity we assume that the ball is pressed with a constant force to the stick. Then the following implications are the Coulomb friction law and describe the behavior of the system.

$$\begin{cases} |\sigma| & < \kappa & \Rightarrow & \dot{q} = 0 \\ \sigma & = \kappa & \Rightarrow & \dot{q} \geq 0 \\ \sigma & = -\kappa & \Rightarrow & \dot{q} \leq 0 \\ |\sigma| & > \kappa & & \text{forbidden} \end{cases} \tag{1}$$

Hence, if the energy restoring force is less than the maximal frictional force (described by $\kappa\cdot$"*normal force*" $= \kappa \cdot 1$) then the particle does not move.

This case study can be expressed in a more convenient form using the multivalued function

$$\operatorname{Sign}(v) = \begin{cases} [-1,1] & \text{if} & v = 0, \\ \{+1\} & \text{if} & v > 0, \\ \{-1\} & \text{if} & v < 0. \end{cases}$$

See also figure 2.

It is now easy to see that (1) is equivalent to the differential inclusion

$$0 \in \underbrace{\kappa\operatorname{Sign}(\dot{q})}_{\text{frictional forces}} + \underbrace{q - l}_{\text{energy restoring force}} . \tag{DI}$$

This is a typical force balance law.

Problem 1.2 (Sliding ball) *For given $q_0 \in (-\pi, +\pi)$ find a solution $q \in \mathrm{W}^{1,1}([0,T],\mathbb{R})$ such that for almost all $t \in [0,T]$ we have*

$$0 \in \kappa\operatorname{Sign}(\dot{q}(t)) + q(t) - l(t).$$

We check that the model (DI) is rate-independent. Assume that $q \in \mathrm{W}^{1,1}([0,T],\mathbb{R})$ is a solution for the above problem. We now replace the input $l(t)$ by $l(\alpha(t))$ with $\alpha \in$

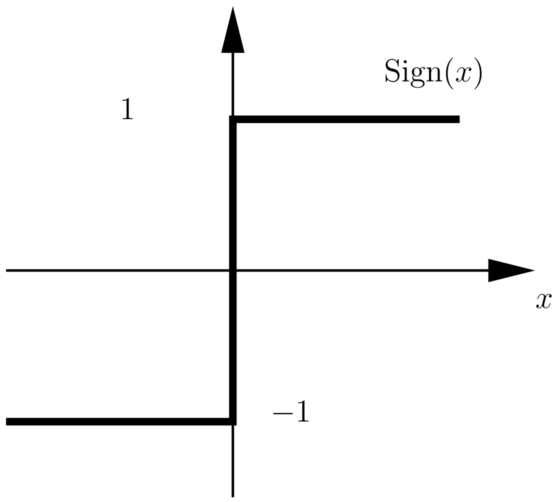

Figure 2: multivalued sign function

$\mathrm{C}^1([0,T],\mathbb{R}), \alpha(0) = 0$ and $\dot{\alpha} > 0$. We claim that $q(\alpha(t))$ is the new solution. Inserting it into the differential inclusion leads us to

$$0 \in \kappa\, \mathrm{Sign}\left(\dot{\alpha}(t)\dot{q}(\alpha(t))\right) + q(\alpha(t)) - l(\alpha(t)).$$

The model is now rate-independent due to the fact that the multivalued Sign is homogenous of degree 0, i.e.

$$\mathrm{Sign}(\gamma\dot{q}) = \mathrm{Sign}(\dot{q}) \qquad \text{for } \gamma > 0. \tag{2}$$

We next present a solution curve for a periodic loading $l(t) = \sin(t)$ in figure 3. Note that we have $l \in \mathrm{C}^\infty([0,T],\mathbb{R})$ for the input while the output satisfies $q \in \mathrm{C}^{\mathrm{lip}}([0,T],\mathbb{R})$. Such a behavior is typical for non-smooth dynamics.

1.3 General remarks on our rate-independent modeling

The above model already contains some important features which will be present during the whole lecture. First we will mainly follow an energetic approach and formulate our problems using an energy storage functional $\mathcal{E}$ that only depends on time t and the state q. This is a first and severe limitation since any dependence on the velocity $\dot{q}$ is neglected. $\mathcal{E}$ is only the potential energy. Physically such models are only meaningful if the time scale of the external loading is quite small compared with the intrinsic time scale of the problem, e.g. if the stick in the above example is inclined slowly within hours or days.

Apart of the energy we will have dissipation in all models. In the sliding ball example dissipation was due to friction. In general the dissipation describes the loss of energy of the system, due to changes in the system $\int_{t_0}^{t_1} \mathcal{R}(\dot{q})\mathrm{d}t$ with $\mathcal{R}(v) = \kappa|v|$. Hence, the dissipation always depends on the velocity of the solution. As for the modeling of the energy functional we also have to restrict ourself for the modeling of the dissipation to special cases. Namely,

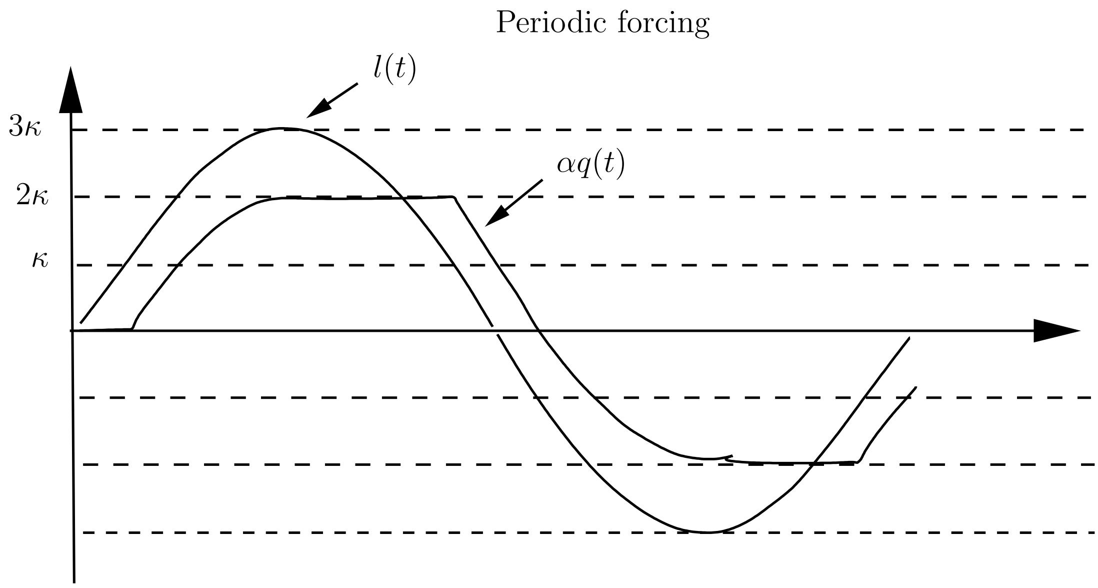

Figure 3: Sliding ball with periodic forcing

the dissipation potential $\mathcal{R}$ has to be homogenous of degree 1 with respect to the velocity or respectively the dissipational force $\mathcal{R}'(v)$ has to be homogenous of degree 0 with respect to the velocity v, see (2). This 1-homogeneity gives the rate independence. Classical viscous dissipation is 2-homogenous, i.e. $\mathcal{R}(v) = \frac{a}{2}v^2$.

As a last remark we want to mention that in all problem formulations we are going to consider the solution will never depend on a purely time dependent term of the energy, i.e. we get the same solutions for the energies $\mathcal{E}(t,q)$ and $\mathcal{E}(t,q) + e(t)$ for all smooth functions e, since energies are usually only defined up to a constant.

1.4 Second example of a rate-independent problem: folding ruler

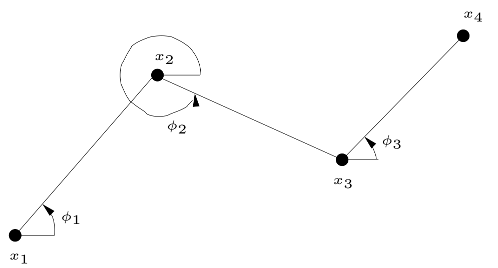

Figure 4: folding ruler

The folding ruler (see figure (4)) is a typical example for a rate-independent model as a geometric evolutionary system. The problem consists in determining the positions of the four joints $(x_1, \ldots, x_4) \in \mathbb{R}^8$. In fact, x_1 and x_4 are the both ends while x_2 and x_3 are the positions of the joints in which dry friction occurs when the angle between the two legs of the joint changes. Since the distance of the joints is pair-wise fixed we have to consider the manifold $\mathcal{M} := \{ (x_1, \ldots, x_4) \in \mathbb{R}^8 \mid \|x_j - x_{j+1}\| = d_j, j = 1, 2, 3 \}$ which has dimension five. Equivalently we can follow the approach of modeling the ruler by $x_1, \phi_1, \phi_2, \phi_3$ and put $\mathcal{M} = \{ (x_1, \phi_1, \phi_2, \phi_3) \in \mathbb{R}^5 \mid \phi_j \in \mathbb{S}^1,\ j = 1, 2, 3 \}$. Here x_1 denotes the position of the first joint while ϕ_j denote the angle at the corresponding joint. In both cases the manifold $\mathcal{M}$ has no linear structure since we cannot add to elements $q_1, q_2 \in \mathcal{M}$ and assure that $q = q_1 + q_2 \in \mathcal{M}$ holds.

Using coordinates $(x_1, \phi_1, \phi_2, \phi_3)$ on $\mathcal{M}$ and assuming gravity acting in the direction $(0, -1)^{\mathrm{T}}$ we have the energy functional $\mathcal{E}(t, x_1, \phi_1, \phi_2, \phi_3) = (m_1 + m_2 + m_3)[x_1 \cdot (0, 1)^T + d_1 \cos \phi_1] + (m_1 + m_3) d_2 \cos \phi_2 + m_3 d_3 \cos \phi_3$ and the dissipational functional $\mathcal{R}(\dot{\phi}_1, \dot{\phi}_2) = \kappa(|\dot{\phi}_1| + |\dot{\phi}_2|)$.

1.5 Generalized formulation

Let be given an energy potential

$$\mathcal{E} : [0, T] \times \mathcal{M} \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$$

and a dissipation metric

$$\mathcal{R} : T\mathcal{M} = \cup_{q \in \mathcal{M}} (q, T_q \mathcal{M}) \to [0, \infty]$$

Here $T\mathcal{M}$ denotes the tangent bundle of the manifold $\mathcal{M}$. From a physical point of view the dissipation metric $\mathcal{R}$ is a power and its derivative with respect to its second variable (i.e. $\mathrm{D}_v \mathcal{R}(q, \dot{q})$) corresponds to the dissipational forces. As in the sliding ball example we formulate the problem to solve as a

Force balance

$$0 = \mathrm{D}_v \mathcal{R}(q, \dot{q}) + \mathrm{D}_q \mathcal{E}(t, q) \in T_q^* \mathcal{M}. \tag{3}$$

This problem turns out to be a gradient flow if the dissipative metric $\mathcal{R}$ is 2-homogeneous, e.g. $\mathcal{R}(q, v) = \frac{1}{2} \langle G(q) v, v \rangle$. Then (3) reads $G(q) \dot{q} = -\mathrm{D}_q E(t, q)$ or using the gradient (with respect to the metric G) we have $\dot{q} = -\nabla_q \mathcal{E}(t, q)$.
The problem is rate-independent if the dissipative metric $\mathcal{R}$ is 1-homogeneous with respect to its second variable, i.e. $\mathcal{R}(q, \gamma v) = \gamma \mathcal{R}(q, v)$ for all $\gamma \geq 0$. Note that in such a situation a nontrivial metric $\mathcal{R}$ is not differentiable in $v = 0$. This leads us to the definition of subdifferentials.

Definition 1.3 (Subdifferential) *Let X be a Banach space. The subdifferential $\partial \varphi(v)$ of a functional $\varphi : X \to (-\infty, \infty]$ in a point $v \in X$ is defined by*

$$\partial \varphi(v) = \{ \sigma \in X^* \mid \varphi(w) - \varphi(v) \geq \langle \sigma, w - v \rangle \text{ for every } w \in X \}.$$

We rewrite the force balance law (3) in the rate-independent situation of $\mathcal{R}(q, \cdot)$ being 1-homogeneous as a

Subdifferential formulation:

$$0 \in \partial_v \mathcal{R}(q, \dot{q}) + \mathrm{D}_q \mathcal{E}(t, q). \tag{SDF}$$

Note that

$$\mathcal{R}(q, \cdot) \text{ 1-homogeneous} \quad \Leftrightarrow \quad \partial_v \mathcal{R}(q, \cdot) \text{ 0-homogeneous.}$$

2 Evolutionary variational inequality - the quadratic case

2.1 Equivalent problem formulations

The classical theory was developed by Moreau in 1974 (see [Mor74]) for models in elasto-plasticity.
After having seen how to formulate rate-independent problems as subdifferential formulations (see (SDF)) we now introduce the formulation as an evolutionary variational inequality.

In the following $\mathcal{Q}$ denotes a Hilbert space and $\mathcal{Q}^*$ its dual. The dual pairing is denoted by $\langle \cdot, \cdot \rangle$. We assume that the potential energy functional has the form

$$\mathcal{E}(t, q) = \frac{1}{2} \langle \mathcal{A} q, q \rangle - \langle l(t), q \rangle$$

with $\mathcal{A} \in \mathrm{Lin}\big(\mathcal{Q}, \mathcal{Q}^*\big)$, $\mathcal{A} = \mathcal{A}^* \geq \alpha > 0$. This assumption is classical to model elastic material properties. As usual we denote by $l \in \mathrm{C}^1\big([0, T], \mathcal{Q}^*\big)$ the external loading. For the dissipative metric we assume $\mathcal{R}(q, v) = \mathcal{R}(v)$ and

$$\mathcal{R} : \mathcal{Q} \to [0, \infty] \quad \text{convex, lower semi-continuous and 1-homogeneous.} \tag{4}$$

From this assumption we deduce the triangle inequality $\mathcal{R}(v_1 + v_2) \leq \mathcal{R}(v_1) + \mathcal{R}(v_2)$.
We want to emphasize that we do not need assumptions of the following type $\mathcal{R}(-v) = \mathcal{R}(v)$ i.e., symmetry $\mathcal{R}(v) \geq c\|v\|$ i.e., coercivity or $\mathcal{R}(v) \leq C\|v\|$ i.e., boundedness. The subdifferential formulation now reads

$$0 \in \partial \mathcal{R}(\dot{q}) + \underbrace{\mathcal{A} q - l(t)}_{= \mathrm{D}_q \mathcal{E}(t, q)}. \tag{SDF}$$

By the definition of subdifferentials, (SDF) is equivalent to the

evolutionary variational inequality:

$$\forall v \in \mathcal{Q} \quad \langle \mathcal{A} q - l(t), v - \dot{q} \rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \geq 0. \tag{EVI}$$

Definition 2.1 *We call a function q a solution of* (SDF)*, if $q \in \mathrm{W}^{1,1}\big((0, T), \mathcal{Q}\big)$ and* (SDF) *holds a.e. in* $[0, T]$.

We next introduce a third equivalent formulation which will be used most in this lecture.

Definition 2.2 (Energetic solution, quadratic case) *We call a function q an energetic solution if $q \in W^{1,1}((0,T),\mathcal{Q})$ and* (S) *and* (E) *are valid for every $t \in [0,T]$,:*
Stability:

$$\forall \tilde{q} \in \mathcal{Q}: \quad \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\tilde{q}) + \mathcal{R}(\tilde{q}-q), \tag{S}$$

Energy balance:

$$\mathcal{E}(t,q(t)) + \int_0^t \mathcal{R}(\dot{q}(s))\,\mathrm{d}s = \mathcal{E}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(s,q(s))\,\mathrm{d}s. \tag{E}$$

Proposition 2.3

$$\text{(SDF)} \Leftrightarrow \text{(EVI)} \Leftrightarrow \text{(S)\&(E)}$$

Proof. By convexity of $\mathcal{E}(t,\cdot)$ the stability condition (S) is equivalent to

$$\forall v \in \mathcal{Q}: \quad \langle \mathrm{D}\mathcal{E}(t,q), v\rangle + \mathcal{R}(v) \geq 0. \tag{S$_{\text{loc}}$}$$

Applying $\frac{\mathrm{d}}{\mathrm{d}t}$ to (E) and using $q \in \mathrm{W}^{1,1}$ we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) + \mathcal{R}(\dot{q}(t)) = 0 + \partial_t\mathcal{E}(t,q(t))$ holds for almost all $t \in [0,T]$. Using the chain rule this is in turn equivalent to

$$\langle \mathrm{D}\mathcal{E}(t,q(t)), \dot{q}(t)\rangle + \mathcal{R}(\dot{q}(t)) = 0. \tag{E$_{\text{loc}}$}$$

Subtracting (E$_{\text{loc}}$) from (S$_{\text{loc}}$) leads us to (EVI). Hence we have prove up to now (E)&(S) $\Rightarrow$ (EVI).

We next show the opposite direction, i.e. (EVI) $\Rightarrow$ (S)&(E). We multiply (EVI) with ε and replace v by $\frac{1}{\varepsilon}v$. From $\varepsilon \to 0$ we derive then (S$_{\text{loc}}$). Plugging $v = 0$ in (EVI) and $v = \dot{q}$ in (S$_{\text{loc}}$) we conclude (E$_{\text{loc}}$). Hence, for all t where (EVI) holds we have (S$_{\text{loc}}$)&(E$_{\text{loc}}$). Integrating $\int_0^t$ (E$_{\text{loc}}$) $\mathrm{d}s$ gives (E) for all $t \in [0,T]$ whereas (S$_{\text{loc}}$) gives (S) only for almost all t. To resolve this we show the closedness of the set of stable states, i.e.

$$\text{if } \left\{ \begin{array}{ccc} q(t_j) & \text{satisfies} & \text{(S)} \\ t_j & \to & t \\ q(t_j) & \to & q(t) \end{array} \right\} \text{ then } q(t) \text{ satisfies (S), too.}$$

We have to show

$$\forall \tilde{q} : \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\tilde{q}) + \mathcal{R}(\tilde{q}-q(t)).$$

We know $\forall j \in \mathbb{N} \forall \tilde{q}$:

$$\begin{array}{ccccc} \mathcal{E}(t_j,q(t_j)) & \leq & \mathcal{E}(t_j,\tilde{q}) & + & \mathcal{R}(\tilde{q}-q(t_j)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}(t,q(t)) & & \mathcal{E}(t,\tilde{q}) & & ?? \end{array}.$$

Hence, our task is to find a sequence $\tilde{q}_j \to \tilde{q}$ with $\mathcal{E}(t_j,\tilde{q}_j) \to \mathcal{E}(t,\tilde{q})$ and $\mathcal{R}(\tilde{q}_j - q(t_j)) \to \mathcal{R}(\tilde{q}-q(t))$. A good choice is $\tilde{q}_j(t) := q(t_j) + \tilde{q} - q(t)$ since $\tilde{q}_j(t) \to \tilde{q}$. It follows $\mathcal{E}(t_j,\tilde{q}_j) \to \mathcal{E}(t,\tilde{q})$ and $\mathcal{R}$ is constant along the sequence. Consequently, we derive that (S) holds for all $t \in [0,T]$ which is our desired result.

Note that the validity of (S) for all $t \in [0,T]$ implies that any solution to (SDF), (EVI) or (S)&(E) must have an initial value $q_0 = q(0)$ that is stable. The evolutionary problem is not solvable for all initial data! $\square$

2.2 Classical theory for (EVI)

1. Existence for the initial value problem holds under the assumption $q(0) = q_0 \in \mathcal{S}(0)$. The set of stable states at time t is defined by

$$\mathcal{S}(t) := \{\, q \in \mathcal{Q} \mid \forall \tilde{q} \in \mathcal{Q} : \mathcal{E}(t, q) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{R}(\tilde{q} - q) \,\}.$$

   The existence proof is a corollary of the general existence result which we formulate in section 3.2.

2. Uniqueness. The uniqueness is proven using the idea of contractive semigroups. Assume that q_1 and q_2 are solutions, i.e.

$$\begin{aligned} \forall v \in \mathcal{Q} : \quad & \langle \mathcal{A} q_1 - l, v - \dot{q}_1 \rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}_1) \geq 0, \\ \forall \tilde{v} \in \mathcal{Q} : \quad & \langle \mathcal{A} q_2 - l, \tilde{v} - \dot{q}_2 \rangle + \mathcal{R}(\tilde{v}) - \mathcal{R}(\dot{q}_2) \geq 0. \end{aligned}$$

   Choosing $v = \dot{q}_2$ and $\tilde{v} = \dot{q}_1$ and adding makes all $\mathcal{R}$ disappear and we end with

$$\langle \mathcal{A}(q_1 - q_2), \dot{q}_2 - \dot{q}_1 \rangle \geq 0.$$

   Due to $\mathcal{A} = \mathcal{A}^*$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \langle \mathcal{A}(q_1 - q_2), q_1 - q_2 \rangle \leq 0.$$

   Because of $\mathcal{A} \geq \alpha > 0$, i.e. $\langle \mathcal{A} v, v \rangle \geq \alpha \|v\|^2$ for all $v \in \mathcal{Q}$, this implies $q_1 \equiv q_2$ provided that $q_1(0) = q_2(0)$.

2.3 Linearized elastoplasticity

As example we study a model of linearized elastoplasticity. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain describing the body under consideration. The state space $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ is partitioned into the set of admissible deformations

$$\mathcal{F} = \big(\mathrm{H}^1_{\Gamma_{Dir}}(\Omega)\big)^d = \{\, u : \Omega \to \mathbb{R}^d \mid u \in (\mathrm{H}^1(\Omega))^d,\ u|_{\Gamma_{Dir}} = 0 \,\}$$

and the set of plastic tensors $e_{pl} \in \mathcal{Z} = \mathrm{L}^2(\Omega, \mathbb{R}^{d\times d}_{sym,0})$. Thereby $\mathbb{R}^{d\times d}_{sym,0} = \{\, A \in \mathbb{R}^{d\times d} \mid A = A^\top,\ \mathrm{tr} A = 0 \,\}$. The linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top) \in \mathbb{R}^{d\times d}_{sym}$ satisfies **Korn's inequality** on $\mathcal{F}$, i.e. there exists a constant $c_{Korn} > 0$, such that

$$\int_\Omega |e(u)|^2 \mathrm{d}x \geq c_{Korn} \|u\|^2_{\mathrm{H}^1(\Omega)} .$$

The energy is given as the sum of the elastic energy, the hardening energy and the potential energy:

$$\mathcal{E}(t, u, e_{pl}) = \int_\Omega \frac{1}{2}(e(u) - e_{pl}) : \mathbb{C} : (e(u) - e_{pl}) + \frac{1}{2} e_{pl} : \mathbb{A} : e_{pl} \mathrm{d}x - \langle l(t), u \rangle$$

with symmetric and positive definite fourth order tensors $\mathbb{C}$ and $\mathbb{A}$. The following estimate holds:

$$\begin{aligned}
\frac{1}{2}\left\langle \mathcal{A}\begin{pmatrix} u \\ e_{pl}\end{pmatrix}, \begin{pmatrix} u \\ e_{pl}\end{pmatrix}\right\rangle &= \int_\Omega \frac{1}{2}(e(u)-e_{pl}):\mathbb{C}:(e(u)-e_{pl}) + \frac{1}{2}e_{pl}:\mathbb{A}:e_{pl}\mathrm{d}x \\
&\geq c_{\mathbb{C}}\|e(u)-e_{pl}\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{sym})} + c_{\mathbb{A}}\|e_{pl}\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{sym})} \\
&\geq c_0\left(\|e(u)\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{sym})} + \|e_{pl}\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}_{sym})}\right) \\
&\geq c_0 c_{\mathrm{Korn}}\|u\|^2_{\mathrm{H}^1(\Omega)} + c_0\|e_{pl}\|^2_{\mathrm{L}^2}.
\end{aligned}$$

The dissipation function is defined independently of $\dot{u}$:

$$\mathcal{R}(\dot{u},\dot{e}_{pl}) = \tilde{\mathcal{R}}(\dot{e}_{pl}) = \int_\Omega \tilde{R}(\dot{e}_{pl}(x))\mathrm{d}x\ . \tag{5}$$

As a first case, we put

$$\tilde{R}:\begin{cases} \mathbb{R}^{d\times d}_{sym,0} & \to [0,\infty) \\ \dot{e}_{pl} & \mapsto \rho|\dot{e}_{pl}| \end{cases}\ , \tag{6}$$

where $|\cdot|$ denotes a matrix norm that depends on the applied engineering model, i.e.

$$\begin{aligned}
|A| &= \max\{\,|\lambda(A)|\mid \lambda(A)\text{ eigenvalue of } A\,\} && \text{for Tresca plasticity,} \\
|A| &= (A:A)^{\frac{1}{2}} && \text{for von Mises plasticity .}
\end{aligned}$$

Under consideration of definition equation (6) the dissipation function in equation (5) reads as follows: $\tilde{\mathcal{R}}(\dot{e}_{pl}) = \rho\|\dot{e}_{pl}\|_{\mathrm{L}^1(\Omega)}$.

As a second model we consider

$$\tilde{R}:\begin{cases} \mathbb{R}^{d\times d}_{sym,0} & \to [0,\infty] \\ A & \mapsto \begin{cases} \rho(A:A)^{\frac{1}{2}} & \text{if } A \text{ is positive semi-definite} \\ \infty & \text{else.} \end{cases} \end{cases}$$

Here $\tilde{\mathcal{R}}$ is neither continuous in the L^2- topology nor in the L^1-topology, however it satisfies all assumptions of the abstract theory, cf, (4).

Since $\mathcal{R}$ is independent of $\dot{u}$, the subdifferential of $\mathcal{R}$ with respect to $\dot{q} = (\dot{u},\dot{e}_{pl})^\top$ can be expressed as follows:

$$\partial\mathcal{R}(\dot{u},\dot{e}_{pl}) = \{0\}\times\partial\tilde{\mathcal{R}}(\dot{e}_{pl})\ .$$

Thereby $\partial\mathcal{R}(\dot{u},\dot{e}_{pl})\subset\mathcal{F}^\star\times\mathcal{Z}^\star$ with $\{0\}\subset\mathcal{F}^\star$ and $\partial\tilde{\mathcal{R}}(\dot{e}_{pl})\subset\mathcal{Z}^\star$. Thus, the **subdifferential formulation** for linearized elastoplasticity is given by:

$$\begin{pmatrix}0\\0\end{pmatrix} \in \partial\mathcal{R}(\dot{u},\dot{e}_{pl}) + \begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^\star & \mathcal{A}_{22}\end{pmatrix}\begin{pmatrix}u\\e_{pl}\end{pmatrix} - \begin{pmatrix}l\\0\end{pmatrix} \quad \subset \begin{pmatrix}\mathcal{F}^*\\\mathcal{Z}^*\end{pmatrix}, \tag{SDF}$$

which can be split up in the elastic equilibrium equation

$$0 = \mathcal{A}_{11}u + \mathcal{A}_{12}e_{pl} - l(t) = \mathrm{D}_u\mathcal{E}(t,u,e_{pl}) \quad \in\mathcal{F}^*$$

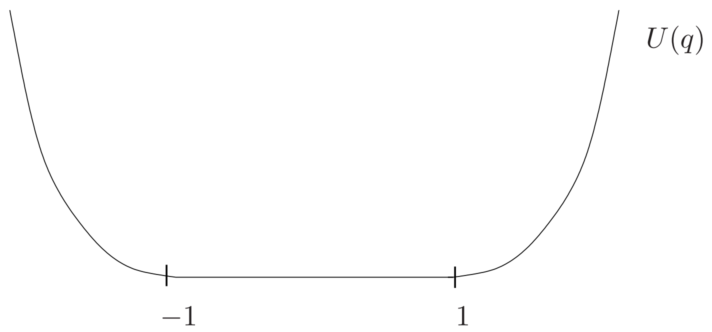

Figure 5: Example for non-uniqueness. The potential U not being strictly convex.

and the plastic flow rule

$$0 \in \partial\tilde{\mathcal{R}}(\dot{e}_{pl}) + \mathcal{A}_{12}^{\star}u + \mathcal{A}_{22}e_{pl} = \partial\tilde{\mathcal{R}}(\dot{e}_{pl}) + \mathrm{D}_{e_{pl}}\mathcal{E}(t,u,e_{pl}) \quad \subset \mathcal{Z}^* .$$

The above problem (SDF) has a unique solution q for stable initial conditions. Existence follows from the main theorem 3.4, see Section 3.2. Uniqueness was established above in Section 2.2.

2.4 Remarks regarding uniqueness

Let us consider the general case:

$$0 \in \partial\mathcal{R}(q,\dot{q}) + \mathrm{D}_q\mathcal{E}(t,q) , \tag{SDF}$$

where

$$\mathcal{E}(t,q) = U(q) - q$$

with the potential U as in figure 5. The dissipation metric $\mathcal{R}$ is assumed to be defined by

$$\mathcal{R}(\dot{q}) = |\dot{q}| .$$

In this case $q(t)$ solves (SDF) if

$$\dot{q} \geq 0 \quad \text{and} \quad -1 \leq q(t) \leq 1 . \tag{7}$$

Remark 2.4 *Uniqueness can not be expected without strict convexity.*

We want to call attention to the literature given below, where some modified energy functionals have been analyzed and uniqueness could be achieved:

- [MT04]: Here, no state dependence is assumed:

$$\mathcal{R}(q,\dot{q}) = \mathcal{R}(\dot{q}) .$$

For $\mathcal{E} \in \mathrm{C}^{2,Lip}([0,T]\times\mathcal{Q})$ with $\mathrm{D}^2\mathcal{E} \geq \alpha > 0$ on the whole space, uniqueness is achieved.

- [BKS04]: In this case the energy is given by

$$\mathcal{E}(t,q) = \frac{1}{2}|q|^2 - \langle l(t), q\rangle \ .$$

  On the dissipation metric $\mathcal{R}$ the following assumptions are made:

$$\begin{aligned} &\mathrm{D}_q\mathcal{R}(q,\dot q)^2 \quad \text{small}\,, && \frac{1}{c}\|\dot q\| \le \mathcal{R}(q,\dot q) \le C\|\dot q\|\,,\\ &\mathcal{R}^2 \in \mathrm{C}^2(\mathcal{Q}\times\mathcal{Q})\,, && \mathrm{D}^2_{\dot q}\mathcal{R}(q,\cdot) \ge \alpha > 0\,. \end{aligned}$$

- [MR07] combines the results of the two papers above and slightly generalizes their union.

We finally mention a seemingly simple example where uniqueness is still an open problem. Let $\mathcal{Q} = \mathbb{R}^2$, $|q| = \sqrt{q_1^2 + q_2^2}$, and

$$\mathcal{E}(t,q) = \frac{1}{2}|q|^2 - l(t)\cdot q + \chi_{B_1(0)}(q)\,,$$

where $\chi_{B_1(0)}(q) = \begin{cases} 0 & \text{for } |q| \le 1 \\ \infty & \text{for } |q| \ge 1 \end{cases}$ and $\mathcal{R}(\dot q) = |\dot q|$. We point out that the existence of a solution $q \in \mathrm{C}^{\mathrm{lip}}([0,T],\mathcal{Q})$ follows easily by the theory, but its uniqueness is still an **open problem**.

3 General existence theory

3.1 Energetic formulation in the general case

Up to now, the state space $\mathcal{Q}$ was assumed to be a Banach space. In this section the existence theory is generalized on $\mathcal{Q}$ being the product of arbitrary Hausdorff topological spaces $\mathcal{F}$ and $\mathcal{Z}$. Thereby $\mathcal{F}$ is the space of the "elastic", dissipation-free variables and $\mathcal{Z}$ denotes the space of the internal, dissipational variables.
In order to motivate a generalized definition of the dissipation distance, we firstly consider, as in the sections before, the subdifferential formulation referring to $\mathcal{Q}$ being a Banach space:

$$0 \in \partial\mathcal{R}(q,\dot q) + \mathrm{D}_q\mathcal{E}(t,q)\,, \tag{SDF1}$$

where the dissipation metric $\mathcal{R}$ has to be understood as an infinitesimal metric in the sense of differential geometry, mapping from the tangential bundle $T\mathcal{Q}$ into $[0,\infty]$. Via the dissipation metric we introduce the (global) dissipation distance

$$\begin{aligned} &\mathcal{D}: \mathcal{Q}\times\mathcal{Q} \to [0,\infty],\\ &\mathcal{D}(q_0,q_1) = \inf\{\int_0^1 \mathcal{R}(q(s),\dot q(s))\mathrm{d}s \mid q \in W^{1,1}([0,1],\mathcal{Q}),\ q(0) = q_0, q(1) = q_1\}\,. \end{aligned}$$

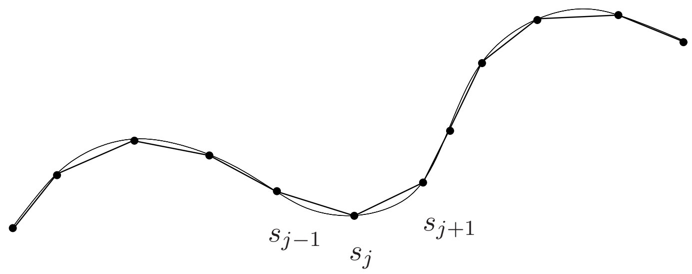

Figure 6: Approximation of the dissipation along a path q using partitions, see $\mathrm{Diss}_{\mathcal{D}}(q,[r,t])$.

Since $\mathcal{R}$ is a metric, $\mathcal{D}$ satisfies the triangle inequality by definition

$$\mathcal{D}(q_0,q_2) \leq \mathcal{D}(q_0,q_1) + \mathcal{D}(q_1,q_2) \quad \text{and } \mathcal{D}(q,q) = 0.$$

As we didn't assume $\mathcal{R}(q,-\dot{q}) = \mathcal{R}(q,\dot{q})$ the dissipation distance may allow for $\mathcal{D}(q_0,q_1) \neq \mathcal{D}(q_1,q_0)$ is allowed.

In the general case of a Hausdorff topological space $\mathcal{Q}$, we consider an energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ and an even more general dissipation distance $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0,\infty]$, for which the properties will be given below. For this general setting we now state the definition of an energetic solution:

Definition 3.1 (Energetic solution for Hausdorff topological spaces) *A function $q : [0,T] \to \mathcal{Q}$ is called an* **energetic solution** *associated with $\mathcal{E}$ and $\mathcal{D}$ if*

$$t \mapsto \partial_t \mathcal{E}(t,q(t)) \in \mathrm{L}^1((0,T))$$

if for every $t \in [0,T]$ we have

$$\mathcal{E}(t,q(t)) < \infty,$$

and if for every t stability (S) *and energy balance* (E) *hold:*

$$\forall \tilde{q} \in \mathcal{Q} : \ \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\tilde{q}) + \mathcal{D}(q(t),\tilde{q}) \ , \tag{S}$$

$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_t \mathcal{E}(s,q(s)) \mathrm{d}s \, , \tag{E}$$

where

$$\mathrm{Diss}_{\mathcal{D}}(q,[r,t]) = \sup_{\textit{all partitions of } [r,t]} \sum_{j=1}^{N} \mathcal{D}(q(s_{j-1}),q(s_j)) \ .$$

Remark 3.2 *The energetic formulation is totally independent of a differentiable structure for $\mathcal{Q}$. We do not need $\mathrm{D}\mathcal{E} = \partial_z \mathcal{E}$. Even if $\mathcal{Q}$ is a convex subset of a Banach space, we may consider $\mathcal{D}$ and $\mathcal{E}(t,\cdot)$ that are not even continuous or convex (on suitably defined dense subspaces). Moreover, $\dot{q}$ is not needed.*

3.2 The existence theorem

Here we follow the general theory developed in [MM05, FM06] and surveyed in [Mie05]. The essential advance between [MM05] and [FM06] is due to abstract versions of the ideas in [DFT05]. However, the theory presented here is slightly generalized due to the further development in [MRS06].

The following assumptions are made on the energy potential and on the dissipation distance:

General assumptions on the energy potential $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$:

Compactness of energy sublevels:
$\forall\, t \in [0,T]\ \forall\, E \in \mathbb{R} :\ L_{t,E} := \{\, q \in \mathcal{Q} \mid \mathcal{E}(t,q) \le E \,\}$ is s-compact. (E1)

Uniform control of the power $\partial_t \mathcal{E}$:
$\exists\, c_0^E \in \mathbb{R}\ \exists\, c_1^E > 0\ \forall\, q \in \mathcal{Q}$ with $\mathcal{E}(0,q) < \infty$: (E2)
$\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T])$ and $|\partial_t \mathcal{E}(t,q)| \le c_1^E (c_0^E + \mathcal{E}(t,q))$ for all $t \in [0,T]$.

Uniform time-continuity of the power $\partial_t \mathcal{E}$:
$\forall\, \varepsilon > 0\ \forall\, E \in \mathbb{R}\ \exists\, \delta > 0$: (E3)
$\mathcal{E}(0,q) \le E$ and $|t_1 - t_2| < \delta \implies |\partial_t \mathcal{E}(t_1,q) - \partial_t \mathcal{E}(t_2,q)| < \varepsilon$.

Conditions of the dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$

Pseudo distance:
$\forall\, z_1, z_2, z_3 \in \mathcal{Z} :\ \mathcal{D}(z_1,z_1) = 0$ and $\mathcal{D}(z_1,z_3) \le \mathcal{D}(z_1,z_2) + \mathcal{D}(z_2,z_3)$. (D1)

Lower semi-continuity: $\mathcal{D} :\ \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$ is s-lower semi-continuous. (D2)

Positivity of $\mathcal{D}$: For all s-compact sets $\mathcal{K} \subset \mathcal{Z}$:
If $z_k \in \mathcal{K}$ and $\min\{\mathcal{D}(z_k,z), \mathcal{D}(z,z_k)\} \to 0$, then $z_k \xrightarrow{\mathcal{Z}} z$. (D3)

By s-compact and s-continuous we mean sequentially compact and sequentially continuous. Furthermore we introduce the **set of stable states** associated with the time t:

$$\mathcal{S}(t) = \{\, q \in \mathcal{Q} \mid \mathcal{E}(t,q) < \infty,\ \forall \tilde{q} \in \mathcal{Q} : \mathcal{E}(t,q) \le \mathcal{E}(t,\tilde{q}) + \mathcal{D}(q,\tilde{q}) \,\}\ .$$

This leads to the definition of stable sequences:

Definition 3.3 *A sequence $(t_j, q_j)_{j\in\mathbb{N}} \subset [0,T] \times \mathcal{Q}$ is called a stable sequence, if*

(i) $\sup\{\, \mathcal{E}(t_j,q_j) \mid j \in \mathbb{N} \,\} < \infty\,$,

(ii) $q_j \in \mathcal{S}(t_j)\,$.

We are now ready to state the main theorem on the existence of energetic solutions for the energetic formulation (S) and (E).

Theorem 3.4 (Main Existence Theorem) *Let $\mathcal{E}$ and $\mathcal{D}$ satisfy conditions* (E1)–(E3) *and* (D1)–(D3)*. Moreover, let the following* **compatibility condition** (CC) *hold:*

For every stable sequence $(t_k, q_k)_{k\in\mathbb{N}}$ with $(t_k, q_k) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t_*, q_*)$ we have

$$\partial_t\mathcal{E}(t_k, q_k) \to \partial_t\mathcal{E}(t_*, q_*), \tag{CCa}$$

$$q_* \in \mathcal{S}(t_*). \tag{CCb}$$

Then, for each $q_0 \in \mathcal{S}(0)$ there exists an energetic solution $q : [0, T] \to \mathcal{Q}$ associated with $\mathcal{E}$ and $\mathcal{D}$ with initial datum $q(0) = q_0$.

A useful sufficient criterion for the compatibility condition (CC) is the following, see [MM05].

Proposition 3.5 *If (E1)-(D3) hold and $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ is s-continuous, then* (CCa) *and* (CCb) *hold.*

This proposition will be reformulated and proved on page 21.

3.3 Example: Linearized elastoplasticity

We check now the validity of conditions (E1)-(D3), (CCa) and (CCb) for the model of linearized elastoplasticity introduced in section 2.3. Let u and z be respectively the elastic displacements and the plastic variables. We assume that u and z belong respectively to $\mathcal{F} = \mathrm{H}^1_{\Gamma_D}(\Omega, \mathbb{R}^d)$ and $\mathcal{Z} = \mathrm{L}^2(\Omega; \mathbb{R}^m)$. We introduce also the following notations: $q = (u, z)^\top$, $f(t) = (l(t), 0)^\top$ and $\mathcal{Q} = \mathrm{H}^1_{\Gamma_D}(\Omega, \mathbb{R}^d) \times \mathrm{L}^2(\Omega; \mathbb{R}^m)$. The Hilbert space $\mathcal{Q}$ is equipped with the weak topology. The potential energy at time t is defined as

$$\mathcal{E}(t, q) := \tfrac{1}{2}\langle \mathcal{A}q, q\rangle - \langle f(t), q\rangle,$$

where $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}^*$ is linear and bounded and satisfies $\mathcal{A} = \mathcal{A}^* \geq \alpha > 0$. Furthermore, $f \in C^1([0, T]; \mathcal{Q}^*)$. The dissipation distance is given as

$$\mathcal{D}(z_0, z_1) := \int_\Omega \tilde{R}(z_1(x) - z_0(x))\, dx \text{ with } \tilde{R} \text{ as in Subsection 2.3.}$$

(E1) Sublevels of $\mathcal{E}$. Since $\mathcal{E}$ is strongly continuous, coercive and convex, the sets $L_{t,E}$ are convex, bounded and strongly closed and thus weakly compact.
(E2) Observe that $\partial_t\mathcal{E}(t, q) = \lim_{h\to 0}(\mathcal{E}(t+h, q)-\mathcal{E}(t, q))$ and therefore $\partial_t\mathcal{E}(t, q) = -\langle \dot{l}(t), u\rangle$. We deduce that

$$|\partial_t\mathcal{E}(t, q)| \leq \underbrace{\|\dot{l}(t)\|_{\mathcal{Q}^*}}_{\leq c_0}\|q\|_{\mathcal{Q}} \leq \frac{1}{2}c_0^2 + \frac{1}{2}\|q\|^2_{\mathcal{Q}} \leq \frac{1}{2}c_0^2 + c_1(c_2 + \mathcal{E}(t, q)).$$

(E3) Observe that $\dot{f}(t) = (\dot{l}(t), 0)^\top \in C^0([0, T]; \mathcal{Q}^*)$ is uniformly continuous.

(D1) The triangle inequality follows immediately from the definition of $\mathcal{D}$.

(D2) holds since $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ is convex and strongly lower semi continuous and thus weakly lower semi-continuous.

(D3) Sequential compactness of $\{z_k,\, k \in \mathbb{N}\}$ implies boundedness in $\mathrm{L}^2(\Omega; \mathbb{R}^m)$. Moreover $\min\{\mathcal{D}(z, z_k), \mathcal{D}(z_k, z)\} \to 0$ shows that $\|z_k - z\|_{\mathrm{L}^1(\Omega)}$ tends to zero. Hence, we have $z_k \rightharpoonup z$ weakly in $\mathrm{L}^2(\Omega)$ as desired.

(CCa) Let $(t_k, q_k) \to (t_*, q_*)$. Then $\dot{l}(t_k) \to \dot{l}(t_*)$ strongly in $\mathcal{Q}^*$ and $q_k \rightharpoonup q_*$ weakly in $\mathcal{Q}$ and we deduce that $\partial\mathcal{E}(t_k, q_k) = -\langle \dot{l}(t_k), q_k\rangle \to \partial\mathcal{E}(t_*, q_*) = -\langle \dot{l}(t_*), q_*\rangle$.

(CCb) Already proved at the end of the proof of Proposition 2.3. Note that each $\mathcal{S}(t)$ is strongly closed and convex.

3.4 Proof of the main theorem 3.4

The existence proof relies on the following incremental minimization problem. We denote by $\mathrm{Argmin}_{\tilde{q}\in\mathcal{Q}}\{\varphi(\tilde{q})\}$ the set of all minimizers of a function $\varphi : \mathcal{Q} \to \mathbb{R}_\infty$. For a given partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_n = T$. We define the following *incremental problem*:

$$\text{For } k = 1, \ldots, N, \text{ find } q_k \in \underset{\tilde{q}\in\mathcal{Q}}{\mathrm{Argmin}}\{\mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(q_{k-1}, \tilde{q})\}. \tag{9}$$

Note that q_0 is the given initial condition.

Example 3.6 *Incremental minimization problems occur in many contexts of time discretizations for PDEs. Rate independent systems are special since the incremental problem does not depend on the length of the time step.*

Consider the heat equation:

$$\frac{\partial\theta}{\partial t} = \Delta\theta \text{ in } \Omega, \tag{10}$$

with Dirichlet boundary conditions $\theta_{|\partial\Omega} = 0$ and initial conditions $\theta(0, \cdot\,) = \theta_0$. Observe that (10) *can be discretized in time, via the backward Euler scheme:*

$$\frac{1}{h}(\theta_k - \theta_{k-1}) = \Delta\theta_k.$$

Hence in each time step we have to solve an elliptic PDE for θ_k, namely $-\Delta\theta_k + \frac{1}{h}\theta_k = \frac{1}{h}\theta_{k-1}$. This PDE is easily identified as the Euler-Lagrange equations of the incremental problem

$$\theta_k \in \underset{\tilde{\theta}\in H^1_0(\Omega)}{\mathrm{Argmin}}\Big\{\frac{1}{2}\|\nabla\tilde{\theta}\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2h}\|\tilde{\theta} - \theta_{k-1}\|^2_{\mathrm{L}^2(\Omega)}\Big\}.$$

Remark 3.7 *Rate independence makes the incremental problem independent of time step.*

The proof of existence theorem 3.4 consists of five steps.

Let $\mathcal{J}_k : \mathcal{Q} \to \mathbb{R}_\infty$ be defined by $\mathcal{J}_k(\tilde{q}) = \mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(q_{k-1}, \tilde{q})$. We deduce from (E1) and (D2) that $\mathcal{J}_k$ is *s-lower semi-continuous* and has *s-compact sublevels*. Then minimizers $\mathcal{J}_k$ exist for all $k \in \{1, \ldots, N\}$. More precisely, for q_0 given, there exist $q_1, \ldots, q_N$.

Step 1: A priori estimates. Since (q_k) are minimizers, we may deduce the following result which shows that the fully implicit *incremental problem* is a very convenient discretization from the analytical standpoint.

Theorem 3.8 *Assume that $q_0 \in \mathcal{S}(0)$, then every solution $(q_k)_{k=0,1,\ldots N}$ of the incremental problem* (9) *satisfies the discrete version of stability (S) and energy equality (E), namely for all $k \in \{1, \ldots, N\}$ we have*

$$q_k \in \mathcal{S}(t_k), \tag{11a}$$

$$\int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\, ds \le e_k + \delta_k - e_{k-1} \le \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\, ds, \tag{11b}$$

where $e_k := \mathcal{E}(t_k, q_k)$ and $\delta_k := \mathcal{D}(q_{k-1}, q_k)$. Moreover, we have the following inequalities:

$$\mathcal{E}(t_k, q_k) \le (e_0 + c_0^E) \exp(c_1^E t_k) - c_0^E, \tag{12a}$$

$$\sum_{j=1}^{N} \mathcal{D}(z_{j-1}, z_j) \le (e_0 + c_0^E) \exp(c_1^E T). \tag{12b}$$

Proof. Since q_k is a minimizer, we have $e_k + \delta_k \le \mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(q_{k-1}, \tilde{q})$ for all $\tilde{q}$ belonging to $\mathcal{Q}$. We prove (11a) using the triangle estimate $\mathcal{D}(q_{k-1}, \tilde{q}) \le \delta_k + \mathcal{D}(q_k, \tilde{q})$.

The lower estimate in (11b) comes from $q_{k-1} \in \mathcal{S}(t_{k-1})$. We test by $\tilde{q} = q_k$ and obtain:

$$e_{k-1} \le \mathcal{E}(t_{k-1}, q_k) + \delta_k = e_k + \delta_k - \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(s, q_k)\, \mathrm{d}s.$$

The upper estimate in (11b) follows from the minimality of q_k, i.e. we may deduce from (9) that

$$e_k + \delta_k \le \mathcal{E}(t_k, q_{k-1}) + \mathcal{D}(q_{k-1}, q_{k-1}) = e_{k-1} + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\, \mathrm{d}s.$$

Observe that the upper estimate in (11b) gives

$$e_k + \delta_k \le e_{k-1} + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\, \mathrm{d}s. \tag{13}$$

On the other hand, we estimate $\mathcal{E}(s, q_{k-1}) = \mathcal{E}(t_{k-1}, q_{k-1}) + \int_{t_{k-1}}^{s} \partial_s \mathcal{E}(r, q_{k-1})\, \mathrm{d}r$ making use of $|\partial_t \mathcal{E}(s, q_{k-1})| \le c_1^E(\mathcal{E}(s, q_{k-1}) + c_0^E)$, i.e. (E2). By the classical Gronwall lemma we infer that

$$\mathcal{E}(s, q_{k-1}) \le (\mathcal{E}(t_{k-1}, q_{k-1}) + c_0^E) \exp(c_1^E |s - t_{k-1}|) - c_0^E. \tag{14}$$

Introducing (14) in (13), we arrive at

$$\begin{aligned} e_k + \delta_k &\le e_{k-1} + (e_{k-1} + c_0^E)(\exp(c_1^E(t_k - t_{k-1})) - 1) \\ &= (e_{k-1} + c_0^E)\exp(c_1^E(t_k - t_{k-1})) - c_0^E. \end{aligned} \tag{15}$$

Since $\delta_k \ge 0$ we may drop it and induction over $j = 1, \dots, k$ gives (11a), namely

$$e_k + c_0^E \le (e_0 + c_0^E)\prod_{j=1}^{k} \exp(c_1^E(t_j - t_{j-1})) \le (e_0 + c_0^E)\exp(c_1^E T).$$

Observe that $e_k + c_0^E$ is positive and we may deduce that

$$\sum_{j=1}^{k} \delta_j \le e_0 - e_k + \sum_{j=1}^{k}(e_{j-1} + c_0^E)(\exp(c_1^E(t_j - t_{j-1})) - 1). \tag{16}$$

Carrying (15) into (16), we obtain

$$\begin{aligned} \sum_{j=1}^{k} \delta_j &\le (e_0 + c_0^E) - (e_k + c_0^E) + (e_0 + c_0^E)\sum_{j=1}^{K}(\exp(c_1^E t_j) - \exp(c_1^E t_{j-1})) \\ &\le (e_0 + c_0^E)\exp(c_1^E t_k), \end{aligned}$$

from which the desired result (11a) follows. □

Step 2: Selection of subsequences. We choose a sequence of partitions $0 = t_0^N < t_1^N < \dots < t_N^N = T$ whose fineness $\max_{k=1,\dots,N}(t_k^N, t_{k-1}^N)$ tends to zero when N tends to ∞. Let $\bar{q}^N : [0,T] \to \mathcal{Q}$, $\bar{q}^N(t) = (\bar{\varphi}^N(t), \bar{z}^N(t)) \in \mathcal{F} \times \mathcal{Z} = \mathcal{Q}$, be a piecewise constant interpolant with $\bar{q}(t) = q_{k-1}^N$ for t belonging to $[t_{k-1}^N, t_k^N)$. We have

$$\mathcal{E}(t, \bar{q}^N(t)) \le (\mathcal{E}(0, q_0) + c_0^E)\exp(c_1^E t - c_0^E) \text{ for all } t \in [0,T], \tag{17a}$$

$$(\mathcal{E}(0, q_0) + c_0^E)\exp(c_1^E T) \ge \sum_{j=1}^{N} \mathcal{D}(\bar{z}^N(t_{j-1}^N), \bar{z}^N(t_j^N)) = \mathrm{Diss}_{\mathcal{D}}(\bar{z}^N; [0,T]). \tag{17b}$$

Recall that the classical Helly's theorem states that a bounded sequence of monotone functions on the real line always has a subsequence that converges pointwise everywhere. Then using a suitable version of Helly's selection principle (see e.g., [Mon93, MT04, MM05]), it is possible to find a sequence $(\bar{z}^{N_k})_{k\in\mathbb{N}}$ and a limit function $z : [0,T] \to \mathcal{Z}$ and $\delta : [0,T] \to [0,\infty)$ monotone such that

$$\forall t \in [0,T] : \ \bar{z}^{N_k}(t) \to z(t), \tag{18a}$$

$$\forall t \in [0,T] : \ \mathrm{Diss}_{\mathcal{D}}(\bar{z}^{N_k}; [0,t]) \to \delta(t), \tag{18b}$$

$$\forall 0 \le r \le t \le T : \ \mathrm{Diss}_{\mathcal{D}}(z; [r,t]) \le \delta(t) - \delta(r). \tag{18c}$$

Here we use (D1)-(D3) essentially.

Note that the function $p_k(t) = \partial_t \mathcal{E}(t, \bar{q}^{N_k}(t))$ form a bounded sequence in $\mathrm{L}^\infty([0,T])$. This follows from the estimate (E2), i.e.,$|\partial_t \mathcal{E}(s,q)| \le c_1^E(\mathcal{E}(s,q) + c_0^E)$ together with (17a). We conclude that

$$p^k \rightharpoonup p^* \text{ weakly } * \text{ in } \mathrm{L}^\infty([0,T]), \tag{19}$$

at least for a subsequence. Moreover, we define $p^{\sup} : [0,T] \to \mathbb{R}$ through

$$p^{\sup}(t) = \limsup_{k\to\infty} p_k(t).$$

Observe that by application of Fatou's Lemma, we obtain $p^*(t) \le p^{\sup}(t)$ a.e. $t \in [0,T]$.

On the other hand, for fixed $t \in [0,T]$ we choose a t-dependent subsequence $(N_{k_j}^t)_{j\in\mathbb{N}}$ such that

$$p^{k_j^t}(t) \to p^{\sup}(t) \text{ for } j \to \infty,$$
$$\bar{\varphi}_{k_j^t}(t) \to \varphi(t) \text{ for } j \to \infty \text{ in } \mathcal{F}.$$

It remains to prove that $q(t) = (\varphi(t), z(t)) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ is a solution.

Step 3: Stability of the limit process. For all $t \in [0,T]$ we have $\bar{q}^{N_{k_l}^t}(t) = \bar{q}^{N_{k_l}^t}(t_j)$ $t_j = t_j^{N_{k_l}^t} \le t < t_{j+1}^{N_{k_l}^t}$. By Theorem 3.8, cf (11a) we find $\bar{q}^{N_{k_l}^t}(t_j) \in \mathcal{S}(t_j)$. We deduce from (CCb) that $\varphi(t) \in \mathcal{S}(t)$ since $t_j \to t$ and $\bar{q}^{N_{k_l}^t}(t) \to q(t)$.

Step 4: Upper energy estimate. Let us define the functions $\bar{e}_k(t) = \mathcal{E}(t, \bar{q}^{N_{k_l}^t}(t))$ and $\bar{\delta}(t) = \mathrm{Diss}_{\mathcal{D}}(\bar{q}^{N_{k_l}^t}(t); [0,t])$. We have the following inequality

$$\bar{e}_k(t) + \bar{\delta}_k(t) \le \bar{e}(0) + \int_0^t p^k(s)\, ds + c\Psi(\xi, t), \tag{20}$$

where $\Psi(\xi, t^N)$ is the fineness of partition and c bounds $\partial_t \mathcal{E}$ on the considered solutions. When k tends to zero, we observe that $\bar{e}_k(t)$ tends to the limit that is larger than $\mathcal{E}(t, q(t))$, by Helly's Theorem, $\bar{\delta}(t)$ tends to $\delta(t) \ge \mathrm{Diss}_{\mathcal{D}}(q; [0,T])$, and thanks to (19), we may pass to the limit in (20) and since $p^*(t) \le p^{\sup}(t)$ a.e., we finally obtain

$$\begin{aligned}\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q; [0,T]) &\le \mathcal{E}(0, q(0)) + \int_0^t p^*(s)\, ds \\ &\le \mathcal{E}(0, q(0)) + \int_0^t p^{\sup}(s)\, ds.\end{aligned}$$

Since $p^{\sup}(t) = \partial_t \mathcal{E}(t, q(t))$, we infer from (21) that

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q; [0,T]) \le \mathcal{E}(0, q(0)) + \int_0^t \partial_s \mathcal{E}(s, q(s))\, ds.$$

Step 5: Lower energy estimate. The lower estimate for the energy balance is a direct consequence of stability (S) and (E3). This finishes the proof of theorem 3.4.

3.5 A sufficient condition for (CCa) and (CCb)

The following proposition describes a sufficient condition such that the compatibility conditions (CCa) and (CCb) hold.

Proposition 3.9 *Assume* (E1)–(D3). *If* $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ *is sequentially continuous, then the compatibility conditions* (CCa) *and* (CCb) *are satisfied. Moreover,*

$$\mathcal{E} : \mathcal{S}_{[0,T]} = \cup_{t \in [0,T]} \{t\} \times \mathcal{S}(t) \to \mathbb{R}_\infty$$

is sequentially continuous.

Note that the dissipation functional in linear elastoplasticity is not sequentially continuous on $\mathcal{Z} = L^2(\Omega)_{\text{weak}}$.

Proof. The continuity of $\mathcal{E}$ on $\mathcal{S}_{[0,T]}$ can be seen as follows: Condition (E1) implies that $\mathcal{E}$ is lower semi-continuous, i.e. we have

$$\mathcal{E}(t_*, q_*) \le \liminf_{j \to \infty} \mathcal{E}(t_j, q_j)$$

for every sequence $(t_j, q_j)_{j \in \mathbb{N}}$ with $(t_j, q_j) \to (t_*, q_*)$. Let now $(t_j, q_j) \in \mathcal{S}(t_j)$ with $(t_j, q_j) \to (t_*, q_*) \in \mathcal{S}(t_*)$ and assume for simplicity that $t_j = t_*$ for every j (the case with $t_j \neq t_*$ follows easily by employing (E2)). By stability and the continuity of $\mathcal{D}$ we obtain

$$\limsup_{j \to \infty} \mathcal{E}(t_*, q_j) \overset{\text{(S)}}{\le} \limsup_{j \to \infty} \big(\mathcal{E}(t_*, q_*) + \mathcal{D}(q_j, q_*) \big) = \mathcal{E}(t_*, q_*).$$

This proves the continuity of $\mathcal{E}$ on $\mathcal{S}_{[0,T]}$. Let now $(t_j, q_j)_{j \in \mathbb{N}}$ be a stable sequence with $(t_j, q_j) \to (t_*, q_*)$. For condition (CCb) we have to show that $(t_*, q_*) \in S(t_*)$. For simplicity we assume that $t_j = t_*$ for every j. The lower semi-continuity of $\mathcal{E}$ and the continuity of $\mathcal{D}$ imply that

$$\mathcal{E}(t_*, q_*) \le \liminf_{j \to \infty} \mathcal{E}(t_*, q_j) \overset{\text{(S)}}{\le} \liminf_{j \to \infty} \big(\mathcal{E}(t_*, \tilde{q}) + \mathcal{D}(q_j, \tilde{q}) \big) = \mathcal{E}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q})$$

for every $\tilde{q} \in \mathcal{Q}$. This proves (CCb). Condition (CCa) follows from the next lemma. $\square$

Lemma 3.10 *Let* $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$ *satisfy* (E1)– (E3). *The following implication holds true:*

$$\left. \begin{array}{c} q_j \to q_*, \\ \mathcal{E}(t_j, q_j) \to \mathcal{E}(t_*, q_*) < \infty \end{array} \right\} \quad \Longrightarrow \quad \partial_t \mathcal{E}(t_j, q_j) \to \partial_t \mathcal{E}(t_*, q_*).$$

This result is a variant of the well-known fact in nonlinear elasticity that stresses converge if the deformation and the energy converge. More precisely, if $\mathcal{I}$ is lower semi-continuous and Gâteaux-differentiable on a reflexive Banach space $\mathcal{X}$, then

$$\left. \begin{array}{c} q_j \overset{\mathcal{X}}{\to} q_* \\ \mathcal{I}(q_j) \to \mathcal{I}(q_*) \end{array} \right\} \quad \Longrightarrow \quad \mathrm{D}_q \mathcal{I}(q_j) \overset{\mathcal{X}^*}{\rightharpoonup} \mathrm{D}_q \mathcal{I}(q_*),$$

see e.g., [BKK00].

Proof. For simplicity we assume again that $t_j = t_*$ for every j. For the general case allowing for $t_j \neq t_*$ we refer to [FM06, MRS06].

By assumption (E3) there exists a modulus of continuity ω_E (i.e. $\omega_E : [0,T] \to [0,\infty)$ nondecreasing with $\omega_E(h) \to 0$ for $h \to 0$) such that for every j and every $h > 0$ we have

$$\left| \frac{1}{h}\big(\mathcal{E}(t_* \pm h, q_j) - \mathcal{E}(t_*, q_j)\big) \mp \partial_t \mathcal{E}(t_*, q_j) \right| \leq \omega_E(h). \tag{21}$$

The lower semi-continuity of $\mathcal{E}$ and the convergence of $\mathcal{E}(t_*, q_j)$ lead to

$$\liminf_{j\to\infty} \tfrac{1}{h}\big(\mathcal{E}(t_* \pm h, q_j) - \mathcal{E}(t_*, q_j)\big) \geq \tfrac{1}{h}\big(\mathcal{E}(t_* \pm h, q_*) - \mathcal{E}(t_*, q_*)\big) \tag{22}$$

for every $h > 0$. Thus,

$$\begin{aligned}
\limsup_{j\to\infty} \partial_t \mathcal{E}(t_*, q_j) &\overset{(21)}{\leq} \limsup_{j\to\infty} \tfrac{1}{h}\big(- \mathcal{E}(t_* - h, q_j) + \mathcal{E}(t_*, q_j)\big) + \omega_E(h) \\
&= \omega_E(h) - \liminf_{j\to\infty} \tfrac{1}{h}\big(\mathcal{E}(t_* - h, q_j) - \mathcal{E}(t_*, q_j)\big) \\
&\overset{(22)}{\leq} \omega_E(h) - \tfrac{1}{h}\big(\mathcal{E}(t_* - h, q_*) - \mathcal{E}(t_*, q_*)\big) \\
&\overset{(21)}{\leq} 2\omega_E(h) + \partial_t \mathcal{E}(t_*, q_*).
\end{aligned}$$

Since $\omega_E(h) \to 0$ for $h \to 0$, we have finally shown that $\limsup_{j\to\infty} \partial_t \mathcal{E}(t_*, q_j) \leq \partial_t \mathcal{E}(t_*, q_*)$. The "lim inf"–case can be treated in the same way using the "+" version of inequality (21).

□

4 Examples and applications

4.1 An example with discontinuous solutions

We consider a system consisting of a mass and a spring, where the mass is pulled via the spring over a rough surface (see figure 7). The energy can be stored in the spring and it can be dissipated due to friction. Let $\ell(t)$ be the position of the right end of the spring at time t and let $z(t)$ denote the position of the mass. The energy, which is stored in the spring, is given by

$$\hat{\mathcal{E}}(t, z) = \tfrac{1}{2}\big(\ell(t) - z\big)^2 = \tfrac{1}{2}z^2 - \ell(t)z + \tfrac{1}{2}\ell(t)^2,$$

whereas the dissipation potential is modeled by

$$\mathcal{R}(z, \dot{z}) = \mu(z)\,|\dot{z}|\,.$$

The coefficient $\mu(z)$ is the friction coefficient of the surface and depends on z. Since the term $\frac{1}{2}\ell(t)^2$ in $\hat{\mathcal{E}}$ does not depend on z, we can skip it and formulate our problem with $\mathcal{E}$ here below:

$$\mathcal{E}(t, z) = \tfrac{1}{2}z^2 - \ell(t)z.$$

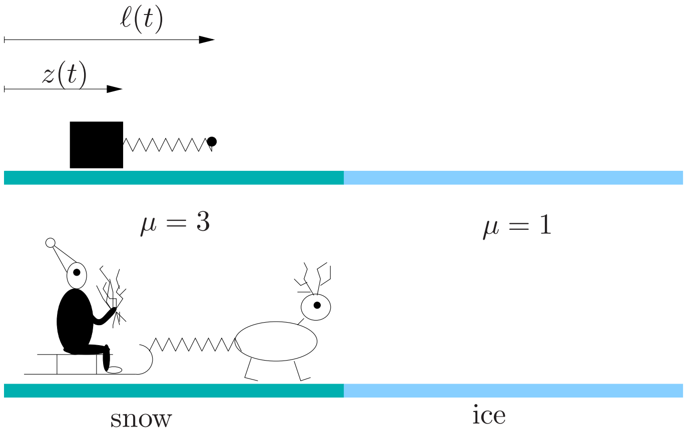

Figure 7: Mass with spring pulled over a rough surface (above) and real world application (below).

For our example we choose

$$\mu(z) = \begin{cases} 3 & \text{for } z \leq 0 \text{ (rough surface)}, \\ 1 & \text{for } z > 0 \text{ (slippery surface)}. \end{cases}$$

The dissipation distance is given by

$$\mathcal{D}(z_0, z_1) = \inf \Big\{ \int_0^1 \mathcal{R}(z(s), \dot{z}(s)) \mathrm{d}s,\ z \in W^{1,1}(\mathbb{R}),\ z(0) = z_0,\ z(1) = z_1 \Big\} = |A(z_0) - A(z_1)|\,,$$

where $A(z) = 2z - |z|$. Altogether we are looking for an energetic solution $z : [0, T] \to \mathbb{R} = \mathcal{Q}$ with $z(0) = z_0$ which for every $t \geq 0$ and $v \in \mathbb{R}$ satisfies

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, v) + \mathcal{D}(z(t), v), \tag{23}$$

$$\mathcal{E}(t, z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [0, t]) = \mathcal{E}(0, z(0)) - \int_0^t \dot{\ell} z(s) \mathrm{d}s. \tag{24}$$

Since the function $v \mapsto \mathcal{D}(z, v)$ is convex if and only if $z \geq 0$, see fig. 8, the energetic formulation (23)–(24) is not equivalent to (EVI). But we can show that the force balance is satisfied on smooth parts of solutions and that a certain jump relation has to be satisfied otherwise.

Force balance: Let $\ell, z \in \mathrm{C}^1([s_0, s_1], \mathbb{R})$ and assume that z satisfies (23) and (24) for every $t \in [s_0, s_1]$. For smooth paths, the dissipated energy is given by

$$\mathrm{Diss}_{\mathcal{D}}(z, [s_0, s_1]) = \int_{s_0}^{s_1} \mathcal{R}(z(s), \dot{z}(s)) \mathrm{d}s.$$

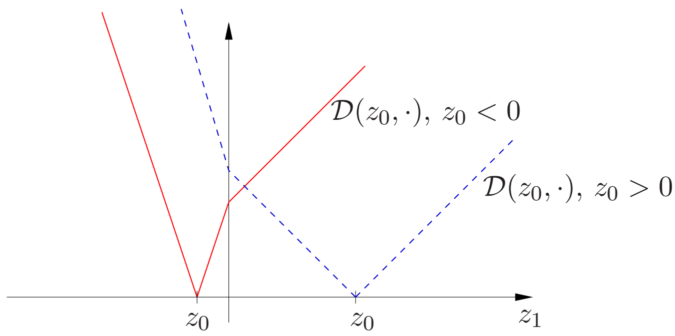

Figure 8: Graph of $\mathcal{D}(z_0,\cdot)$ for $z_0<0$ and $z_0>0$. The slopes are from the set $\{\pm 1,\pm 3\}$. If $z_0<0$, then $\mathcal{D}(z_0,\cdot)$ is not convex.

Assume first that $z(t)\leq 0$ for every $t\in[s_0,s_1]$. In this case we have $\mathcal{R}(z(t),v-z(t))\geq \mathcal{D}(z(t),v)$ for every $v\in\mathbb{R}$ and obtain from (23)–(24)

$$\mathcal{E}(t,z(t))\leq \mathcal{E}(t,v)+\mathcal{R}(z(t),v-z(t)), \tag{25}$$

$$\mathcal{E}(t,z(t))+\int_{s_0}^t \mathcal{R}(z(s),\dot z(s))\mathrm{d}s=\mathcal{E}(s_0,z(s_0))-\int_{s_0}^t \dot\ell(s)z(s)\mathrm{d}s \tag{26}$$

for every $t\in[s_0,s_1]$ and every $v\in\mathbb{R}$. By similar arguments as in Section 2.1 it follows that z satisfies the force balance

$$0\in\mu(z(t))\,\mathrm{Sign}(\dot z(t))+z(t)-\ell(t) \tag{27}$$

for every $t\in[s_0,s_1]$.
Assume now that $z(t)>0$ for every $t\in[s_0,s_1]$ and let $\mathcal{E}_\ell(z)=\frac{1}{2}z^2-\ell z$. For every $z>0$ and $\ell\in\mathbb{R}$ the functions

$$F_{z,\ell}:\mathbb{R}\to\mathbb{R},\quad v\mapsto \mathcal{E}_\ell(v)+\mathcal{D}(z,v), \tag{28}$$

$$G_{z,\ell}:\mathbb{R}\to\mathbb{R},\quad v\mapsto \mathcal{E}_\ell(v)+\mathcal{R}(z,v-z) \tag{29}$$

are continuous and strictly convex. Moreover, z is the unique minimizer of $F_{z,\ell}$ (by assumption (23)). For every $z>0$ there exists an open neighborhood U_z of z with $F_{z,\ell}(v)=G_{z,\ell}(v)$ for every $v\in U_z$. Thus, z is at least a local minimizer of $G_{z,\ell}$ and, due to the strict convexity of $G_{z,\ell}$, it is even a global minimizer of $G_{z,\ell}$. Altogether it follows for smooth paths $z:[s_0,s_1]\to(0,\infty)$ satisfying (23) and (24) that (25)–(26) are fulfilled as well. Like in the case $z\leq 0$ we conclude that z satisfies the force balance (27).
Summarizing the above considerations we have shown that smooth parts of solutions to (23)–(24) satisfy

$$0\in\partial_v\mathcal{R}(z(t),\dot z(t))+\mathrm{D}_v\mathcal{E}(s,z(s))$$

for every t.

Jumps: From condition (24) we deduce a jump relation for piecewise smooth solutions. Let again $\ell \in \mathrm{C}^1$. Let furthermore $z : [s_0, s_1] \to \mathbb{R}$ with $z\big|_{[s_0,s_J]} \in \mathrm{C}^1$ and $z\big|_{[s_J,s_1]} \in \mathrm{C}^1$ for some $s_J \in (s_0, s_1)$ and assume that z satisfies the energy balance (24) for every $t \in [s_0, s_1]$. We define $z_+ = \lim_{t\searrow s_J} z(t)$ and $z_- = \lim_{t\nearrow s_J} z(t)$. For every $s < s_J$ and $t > s_J$, the energy balance (24) implies that

$$\mathcal{E}(t, z(t)) + \mathrm{Diss}_{\mathcal{D}}(z, [s, t]) = \mathcal{E}(s, z(s)) - \int_s^t \dot{\ell}(\tau) z(\tau) \mathrm{d}\tau.$$

For $s \nearrow s_J$ and $t \searrow t_J$, this relation converges to the jump relation

$$\mathcal{E}(s_J, z_+) + \mathcal{D}(z_-, z_+) = \mathcal{E}(s_J, z_-). \tag{30}$$

The previous relation (30) means that the spring energy, which is released due to the jump, equals to the energy, which is dissipated due to the jump.

Based on the force balance (27) and the jump relation (30) we can now construct piecewise smooth functions which satisfy the energy balance (24). We then have to find those functions, which satisfy the stability condition (23) as well.

Let $z_0 = -2$ and $\ell(t) = t$. The initial datum and ℓ are chosen in such a way that the mass is pulled from the region with high friction to the region with low friction. By the force balance (27) we obtain

$$z_{s_J}(t) = \begin{cases} -2 & \text{if } 0 \le t \le 1, \\ t - 3 & \text{if } 1 \le t \le s_J, \end{cases} \tag{31}$$

as long as $z(t) \le 0$, i.e. for $s_J \le 3$. Furthermore, relation (27) implies for $z \ge 0$ that

$$z_{s_J}(t) = \begin{cases} z_+ & \text{if } s_J < t \le z_+ + 1, \\ t - 1 & \text{if } t > z_+ + 1. \end{cases} \tag{32}$$

The parameters $s_J \le 3$ and $z_+ \ge 0$ have to be adjusted in such a way that the jump relation (30) is satisfied. With $z_- = s_J - 3 \le 0$, relation (30) reads as follows :

$$\tfrac{1}{2} z_+^2 - s_J z_+ - \tfrac{1}{2} z_-^2 + s_J z_- + (z_+ - 3z_-) = 0.$$

Taking into account that $z_+ \ge s_J - 1$ we obtain

$$z_+(s_J) = s_J - 1 + 2\sqrt{s_J - 2},$$

which makes sense for $s_J \ge 2$, only. Thus we have constructed a family z_{s_J} of discontinuous functions which satisfy the balance of forces (27) and the energy balance (24): for $s_J \in [2, 3]$

$$z_{s_J}(t) = \begin{cases} -2 & \text{if } 0 \le t \le 1, \\ t - 3 & \text{if } 1 \le t \le s_J, \\ z_+(s_J) & \text{if } s_J < t \le z_+(s_J) + 1, \\ t - 1 & \text{if } t > z_+(s_J) + 1. \end{cases}$$

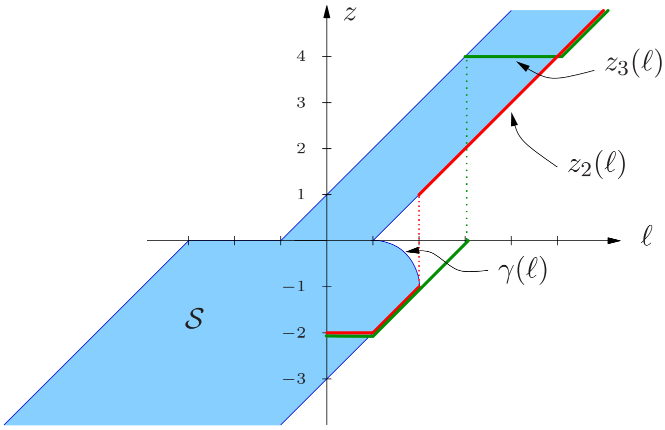

Figure 9: Set $\mathcal{S}$ of admissible states, solution z_2 and function z_3 plotted over ℓ

From this family we select now those functions, which satisfy the stability condition (23), as well. Let $\mathcal{S} = \{ (\ell, z) \in \mathbb{R}^2 \mid z \text{ minimizes } F_{z,\ell} \}$ be the set of admissible states ($F_{z,\ell}$ from (28)). Energetic solutions have to satisfy $(\ell(t), z(t)) \in \mathcal{S}$ for every t. It is technical (but straight forward) to calculate $\mathcal{S}$ and to show that that z_{s_J} satisfies the stability condition (23) if and only if $s_J = 2$. Thus z_2 is a (discontinuous) energetic solution of (23)–(24). The set $\mathcal{S}$ of admissible states, the solution z_2 (red) and the function z_3 (green), which is not a solution but satisfies the force balance and the energy balance, are plotted in figure 9. The curve γ is defined through $\gamma(\ell) = \ell - 3 + 2\sqrt{2-\ell}$.

This example shows that discontinuous solutions may occur. Furthermore, the example shows that the force balance (27) in combination with the energy balance is necessary but not sufficient to find piecewise smooth energetic solutions of (S) & (E) (or (23)–(24)). Note that the solution z_2 of (23)–(24) jumps before the mass reaches the point, where the friction coefficient changes. This is not what is observed in the experiment. The function z_3 seems to coincide best with the real experiment, since the jump takes place in the moment, where the friction coefficient changes. Note that for $t \in (2, 4)$ the point $z_2(t)$ is a global minimizer of the function $\mathcal{E}(t, \cdot) + \mathcal{D}(z_2(t), \cdot)$, whereas the point $z_3(t)$ is a local minimizer of $\mathcal{E}(t, \cdot) + \mathcal{D}(z_3(t), \cdot)$, only. With formulation (23) we are looking for a global minimizer, whereas nature seems to prefer local minimization.

It is questionable whether the rate independent formulation (23)–(24) is an appropriate model for describing the problem in a proper way since the jump violates the assumption that the process is very slow.

Remark 4.1 *The functions $\{ z_{s_J} \mid s_J \in [2,3] \}$ are solutions of (E) and ($\tilde{S}$), where*

$$\begin{gathered} \textit{For every } v \in \mathbb{R},\ t \geq 0 \\ \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, v) + \mathcal{R}(q(t), v - q(t)). \end{gathered} \tag{$\tilde{\mathrm{S}}$}$$

Remark 4.2 *Discontinuous solutions may also occur in the case of a smooth friction coefficient with steep gradient. If $\mu \in \mathrm{C}^1$, then any smooth solution should satisfy* $0 =$

$\mu(z(t)) + z(t) - l(t)$. *Hence, we see that* $\mu'(z) \leq -1$ *leads to troubles in solving this implicit equation, see Example 3.5 in [Mie05] and [SM06].*

Remark 4.3 *It is possible to modify* $\mathcal{D}$ *in such a way that the jump takes place at* $t_* = 3$ *from* $z_- = 0$ *to* $z_+ = t_* - 1$. *In this case,* $\mathcal{D}$ *is unsymmetric and not continuous.*

4.2 Quasistatic magnetostriction

Here we present a rate-independent model for the interaction of magnetization and elasticity, see [Mie06a, Sect. 5.6]. It is based on quite related work by Kružík and Roubíček (see [Kru02, RK04]) and on a similar model for ferro-electricity in [MT06a, MT06b], which is used in engineering, see [KW03, RS05].

For a body $\Omega \subset \mathbb{R}^3$ let $u : \Omega \to \mathbb{R}^3$ be the displacement field, $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ the linearized strain tensor, $m : \Omega \to \mathbb{S}^2 = \partial B_1(0) \subset \mathbb{R}^3$ the magnetization and $B : \mathbb{R}^3 \to \mathbb{R}^3$ the magnetic flux. The constitutive law, which connects the magnetic flux with the magnetic field $H : \mathbb{R}^3 \to \mathbb{R}^3$, is given by

$$B = \mu_0(H + m_{\text{ext}}) \quad \text{in } \mathbb{R}^3,$$

where m_{ext} is the extension of m with 0 to $\mathbb{R}^3 \backslash \Omega$. Furthermore, the static Maxwell equations shall be satisfied in $\mathbb{R}^3$:

$$\text{div} B = 0, \qquad \text{curl} H = 0.$$

We consider the following energy functional:

$$\begin{aligned} \mathcal{E}(t, u, B, m) = &\int_\Omega W(x, e(u), m)\, \mathrm{d}x + \int_\Omega \frac{\rho}{2} |\nabla m|^2\, \mathrm{d}x - \int_\Omega B \cdot m\, \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \frac{1}{2\mu_0} |B|^2\, \mathrm{d}x - \langle \begin{pmatrix} l_{\text{mech}}(t) \\ l_{\text{magn}}(t) \end{pmatrix}, \begin{pmatrix} u \\ B \end{pmatrix} \rangle. \end{aligned} \tag{33}$$

The first term is the stored elastic energy, the second term describes the exchange energy, the third term gives the interaction between magnetization and the magnetic flux, the fourth term is the energy which is stored in the magnetic field. Note that $\mathrm{D}_u \mathcal{E}(t, u, B, m) = 0$ corresponds to the weak formulation of the elastic equilibrium and $\mathrm{D}_B \mathcal{E}(t, u, B, m) = 0$ is the weak form of the condition $\text{curl} H = 0$ since $\text{div} B \equiv 0$. We choose $\mathcal{Q} = \mathcal{F} \times \mathcal{Z}$, where

$$\begin{aligned} \mathcal{F} &= H^1_{\Gamma_{\text{Dir}}}(\Omega, \mathbb{R}^3) \times H(\mathbb{R}^3, \text{Div}) \ni (u, B), \\ H(\mathbb{R}^3, \text{Div}) &= \{\, B \in L^2(\mathbb{R}^3) \mid \text{Div} B = 0 \,\}, \\ \mathcal{Z} &= H^1(\Omega, \mathbb{S}^2) \ni m. \end{aligned}$$

In all spaces we consider the weak topology.

Assume that W is a Carathéodory function and that $W(x, \cdot, m) : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$ is convex and bounded from below by a function with quadratic growth. It is not hard to describe further "natural" assumptions on W, $l_{\mathrm{mech}}(t)$ and $l_{\mathrm{magn}}(t)$ such that conditions (E1)–(E3) are satisfied.

In order to model magnetic hysteresis we introduce a (macroscopically motivated) dissipation distance on $\mathbb{S}^2$

$$D(x,\cdot,\cdot) : \mathbb{S}^2 \times \mathbb{S}^2 \to [0,\infty).$$

It is assumed that D satisfies the triangle inequality and that there is a constant $c > 0$ such that

$$\tfrac{1}{c}\,|m_1 - m_2| \leq D(x,m_1,m_2) \leq c\,|m_1 - m_2|\,.$$

A possible choice for D is

$$D(x,m_1,m_2) = c_1(x)\arccos(m_1\cdot m_2) + c_2(x)\,|\hat{e}(x)\cdot(m_1 - m_2)|\,,$$

where $\hat{e}(x)$ is a preferred direction. The dissipation distance on $\mathcal{Z}\times\mathcal{Z}$ is then given by

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty], \quad \mathcal{D}(m_1,m_2) = \int_\Omega D(x,m_1(x),m_2(x))\,\mathrm{d}x. \tag{34}$$

The assumptions on D imply that $\mathcal{D}$ satisfies the triangle inequality as well. Moreover, we have

$$\tfrac{1}{c}\,\|m_1 - m_2\|_{L^1(\Omega)} \leq \mathcal{D}(m_1,m_2) \leq c\,\|m_1 - m_2\|_{L^1(\Omega)}$$

and therefore, $\mathcal{D}$ is strongly continuous with respect to the strong L^1–topology. Since $\mathcal{Z} = H^1(\Omega)$ is compactly embedded in $L^1(\Omega)$, $\mathcal{D}$ is weakly continuous in H^1 and therefore continuous with respect to the topology in $\mathcal{Z}$. Thus, $\mathcal{D}$ satisfies (D1)–(D3) and the compatibility conditions (CCa), (CCb) due to Proposition 3.9.

Under the above assumptions, the existence theorem is applicable to the energetic formulation based on $\mathcal{E}$ from (33) and $\mathcal{D}$ from (34).

4.3 Small-strain model for shape-memory alloys

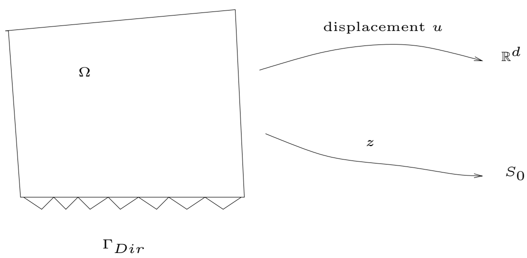

Figure 10: Domain Ω with displacement u

Let the displacement u be defined on a domain Ω with Dirichlet boundary Γ_{Dir}, i.e. u satisfies Dirichlet boundary conditions

$$u(t,\cdot) = u_{\mathrm{Dir}}(t,\cdot) \quad \text{on} \quad \Gamma_{\mathrm{Dir}} \subset \partial\Omega.$$

The displacement u is split additively as $u(t,\cdot) = u_{\mathrm{Dir}}(t,\cdot) + \tilde{u}$, where $\tilde{u} \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d)$. The energy potential is defined as

$$\mathcal{E}(t,\tilde{u},z) = \int_\Omega W(e(u_{\mathrm{Dir}}(t)+\tilde{u}) - z) + h(z) + \frac{\sigma}{2}|\nabla z|^2 \,\mathrm{d}x$$

with

- $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$,
- $z \in S_0 := \{A \in \mathbb{R}^{d\times d} | A = A^\top, \mathrm{tr}\ A = 0\}$ mesoscopic transformation strain,
- $W(e) = \frac{1}{2} e : \mathbb{C} : e$, for example $W(e) = \frac{\lambda}{2}(\mathrm{tr}\, e)^2 + \mu|e|^2$ in the isotropic case,
- $\frac{\sigma}{2}|\nabla z|^2$ a mathematical regularization with $\sigma > 0$.

Following a model of [SMZ98, AP02, AS04, AP04] we set

$$h(z) = \begin{cases} c_1|z| + \frac{c_2}{2}|z|^2 & \text{if } |z| \le c_3, \\ \infty & \text{else} \end{cases} \tag{35}$$

and define the dissipation through

$$\mathcal{D}(z_0,z_1) \;=\; \int_\Omega \kappa|z_1(x) - z_0(x)| \,\mathrm{d}x = \kappa\|z_1 - z_0\|_{L^1(\Omega)}.$$

For $\sigma > 0$ we can define the state space $\mathcal{Q} = \mathcal{F}\times\mathcal{Z}$ with $\mathcal{F} = \mathrm{H}^1_{\mathrm{Dir}}(\Omega,\mathbb{R}^d)$ and $\mathcal{Z} = \mathrm{H}^1(\Omega, S_0)$. The state space is equipped with the weak topology. Note that $\mathrm{H}^1(\Omega, S_0)$ is compactly embedded in $\mathrm{L}^1(\Omega, S_0)$ and hence $\mathcal{D}$ is continuous on $\mathcal{Z}\times\mathcal{Z}$.

The power of external forces is given by

$$\begin{aligned} \partial_t \mathcal{E}(t, \underbrace{\tilde{u}, z}_{\text{fixed}}) &= \int_\Omega \partial_e W(e(u_{\mathrm{Dir}}(t)+\tilde{u}) - z)[e(\dot{u}_{\mathrm{Dir}}(t))] \,\mathrm{d}x \\ &= \int_\Omega \big(e(u_{\mathrm{Dir}}(t)+\tilde{u}) - z\big) : \mathbb{C} : e(\dot{u}_{\mathrm{Dir}}) \,\mathrm{d}x. \end{aligned}$$

With the assumption $u_{Dir} \in \mathrm{C}^1([0,T], \mathrm{H}^1(\Omega;\mathbb{R}^d))$ we find the estimate

$$|\partial_t \mathcal{E}(t,\tilde{u},z)| \le c \underbrace{\|e(\dot{u}_{Dir})\|_{L^2}}_{\le \tilde{c}} \underbrace{\|e(u_{Dir}+\tilde{u}) - z\|_{L^2}}_{\le c_1(\mathcal{E}(t,\tilde{u},z)+c_0)^{1/2}}$$

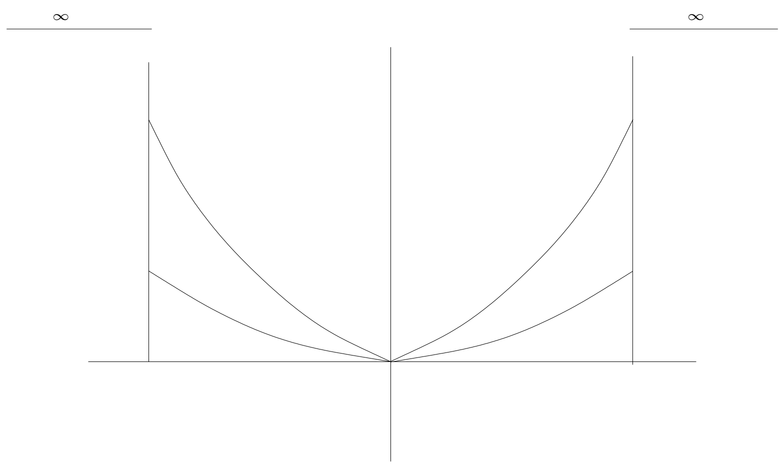

Figure 11: $h(z)$

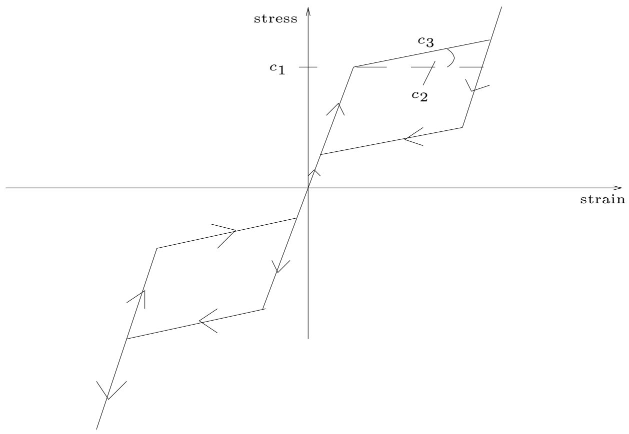

Figure 12: Hysteresis

Theorem 4.4 (Existence Result) *For all stable initial conditions there exists an energetic solution*

$$(\tilde{u}, z) \in \mathrm{L}^\infty([0,T], \mathrm{H}^1(\Omega;\mathbb{R}^d) \times \mathrm{H}^1(\Omega, S_0)).$$

Moreover, since $\mathcal{E}$ *is uniformly convex, there holds* $(\tilde{u}, z) \in \mathrm{C}^{Lip}([0,T], \mathrm{H}^1(\Omega;\mathbb{R}^d)\times\mathrm{H}^1(\Omega, S_0))$.

The uniqueness is still an open problem. However, if h defined in (35) is replaced by an approximation $h \in \mathrm{C}^3(S_0, \mathbb{R})$, which satisfies suitable growth bounds, then $\mathcal{E}(t, \cdot)$ will be three times continuously differentiable on $\mathrm{H}^1(\Omega;\mathbb{R}^d) \times \mathrm{H}^1(\Omega, S_0)$ and the methods in [MT04] provide uniqueness results.

4.4 Large-strain model for shape-memory alloys

Given a deformation φ and the deformation gradient $F = \nabla\varphi \in \mathbb{R}^{d\times d}$, we assume in the following $\det \nabla\varphi > 0$ to avoid local self-interpenetration.

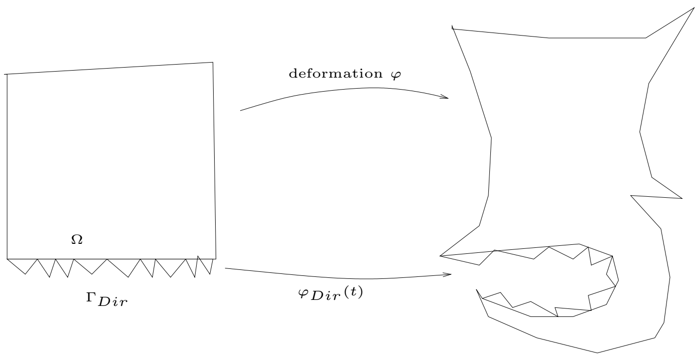

Figure 13: Deformation

We use J.M. Ball's polyconvex material laws and define

$$W(F,z) = \begin{cases} a(z)|F|^p + \frac{b(z)}{(\det F)^r} + c(z) & \text{for } \det F > 0 \\ \infty & \text{for } \det F \leq 0, \end{cases}$$

It is assumed that $p > d$ and $a, b, r > 0$. (Ogden type material)

The energy potential $\mathcal{E} : \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \times \mathrm{H}^1(\Omega;\mathbb{R}^m) \to \mathbb{R}_\infty$ is defined through

$$\mathcal{E}(\varphi, z) := \int_\Omega W(\nabla\varphi, z) + \frac{\sigma}{2}|\nabla z|^2 \,\mathrm{d}x.$$

$\mathcal{E}$ attains the value $+\infty$ on a dense set, namely for each φ having $\det\nabla\varphi \leq 0$ on a set of positive measure.
How do we implement time-dependent Dirichlet conditions?

First we try an ansatz with additive decomposition as above, i.e. $\varphi(t,x) = \varphi_{Dir}(t,x) + u$, where φ denotes the desired solution, φ_{Dir} the given data and $u \in \mathrm{W}^{1,p}_{\Gamma_{Dir}}(\Omega) = \mathcal{F}$ has to be calculated. We have the time dependent energy potential $\bar{\mathcal{E}}(t,u,z) = \mathcal{E}(\varphi_{Dir}(t) + u, z)$ with

$$\partial_t \bar{\mathcal{E}}(t,u,z) = \int_\Omega \partial_F W(\nabla\varphi_{Dir}(t) + \nabla u, z)[\nabla\dot{\varphi}_{Dir}]\, dx,$$

where

$$\begin{aligned} \partial_F W(F,z) &= a(z)p|F|^{p-2}F - \frac{b(z)r}{(\det F)^{r+1}}\mathrm{cof}F \\ &= ap|F|^{p-2} - \frac{br}{(\det F)^r}F^{-T} \end{aligned}$$

with $\mathrm{cof}\, F = (\det F)F^{-T}$ if $\det F \neq 0$. Here, $\partial_F W(\nabla\varphi)$ is the 1. Piola-Kirchhoff stress tensor. The assumption $\mathcal{E}(\varphi, z) < \infty$ yields $W(\nabla\varphi, z) \in L^1(\Omega)$, and so

$$\|\nabla\varphi\|_{L^p} < \infty, \text{ and } \|\frac{1}{\det\nabla\varphi}\|_{L^r} < \infty.$$

Unfortunately, this does not imply that the first Piola-Kirchhoff stress tensor is an element of $L^1(\Omega)$.

Instead we introduce the additional assumption that the Kirchhoff stress tensor, which is defined through

$$K(F) = \partial_F W(F) F^\top, \tag{36}$$

is bounded from above by the energy [Bal02]:

$$|K(F)| \le c_1^W (W(F) + c_0^W). \tag{37}$$

Our example satisfies this condition since

$$\begin{aligned} |K(F)| &= \left| ap|F|^{p-2} F F^\top - \frac{br}{(\det F)^r} \mathbb{I} \right| \\ &\le \max\{p, r\} W(F). \end{aligned}$$

Remark 4.5 *The Kirchhoff tensor K can be seen as a left-invariant derivative on the Lie group $GL_+(d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}$. While the 1. Piola-Kirchhoff tensor $\partial_F W(F)$ lies in $T_F^* GL_+(d)$ the Kirchhoff tensor $K(F)$ lies in $gl(D)^* = T_{\mathbb{I}}^* GL_+(d)$. We refer to [Mie02, Mie06b] for more information on these Lie group aspects.*

From the observation above it turns out that the additive decomposition, which is natural from the physical point of view, fails for mathematical reasons. Therefore, we try multiplicative decomposition in a second ansatz, i.e.

$$\varphi(t, x) = \varphi_{\mathrm{Dir}}(t, \psi(t, x)),$$

where ψ has to be calculated.

On the Dirichlet boundary Γ_{Dir} there holds $\varphi = \varphi_{\mathrm{Dir}}$, and equivalently $\psi = \mathrm{id}$ on Γ_{Dir}. Therefore, we choose

$$\mathcal{F} = \left\{ \psi \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^d);\ \psi|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id} \right\}$$

and

$$\tilde{\mathcal{E}}(t, \psi, z) = \mathcal{E}(\varphi_{\mathrm{Dir}}(t) \circ \psi, z).$$

By the chain rule we obtain

$$\nabla\varphi = \nabla\varphi_{\mathrm{Dir}}(t, \psi(x)) \nabla\psi(x), \quad \det\nabla\varphi = (\det\nabla\varphi_{\mathrm{Dir}})(\det\nabla\varphi).$$

With $A = \nabla\varphi_{\mathrm{Dir}}\nabla\psi$ and $B = \nabla\dot{\varphi}_{\mathrm{Dir}}\nabla\psi$ we see that

$$\begin{aligned} \partial_F W(A)[B] &= \mathrm{tr}(\partial_F W(A) B^\top) \\ &= \mathrm{tr}\left(\partial_F W(A) \nabla\psi^\top \nabla\dot{\varphi}_{\mathrm{Dir}}^\top\right) \\ &= \mathrm{tr}\left(\partial_F W(A) \nabla\psi^\top \nabla\varphi_{\mathrm{Dir}}^\top \nabla\varphi_{\mathrm{Dir}}^{-\top} \nabla\dot{\varphi}_{\mathrm{Dir}}^\top\right) \\ &= K(A) : (\nabla\dot{\varphi}_{\mathrm{Dir}} \nabla\varphi_{\mathrm{Dir}}^{-1}). \end{aligned}$$

And therefore

$$\begin{aligned}\partial_t\tilde{\mathcal{E}}(t,\psi,z) &= \int_\Omega \partial_F W(\nabla\varphi_{\rm Dir}\nabla\psi, z)[\nabla\dot{\varphi}_{\rm Dir}(t,\psi)\nabla\psi]\,dx \\ &= \int_\Omega K(\nabla\varphi_{\rm Dir}\nabla\psi, z) : (\nabla\dot{\varphi}_{\rm Dir}\nabla\varphi_{\rm Dir}^{-1})\,dx.\end{aligned}$$

If $\tilde{\mathcal{E}}(t,\psi,z) < \infty$ then the assumption (36) implies $K(\nabla\varphi_{\rm Dir}\nabla\psi, z) \in \mathrm{L}^1(\Omega)$. If in addition the given data satisfies $(\nabla\dot{\varphi}_{\rm Dir}\nabla\varphi_{\rm Dir}^{-1}) \in \mathrm{C}^0(\bar{\Omega})$, then $\partial_t\tilde{\mathcal{E}}(t,\psi,z)$ is well defined.

This leads to the following Proposition:

Proposition 4.6 *If $\varphi_{Dir} \in \mathrm{C}^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ and $\nabla\varphi_{Dir}, \nabla\varphi_{Dir}^{-1} \in \mathrm{C}^1([0,T]\times\mathbb{R}^d,\mathbb{R}^{d\times d})$, then (E2) and (E3) hold for the above defined W.*

Thus, defining a suitable dissipation distance $\mathcal{D}(z_0,z_1) = \int_\Omega \mathrm{D}(x, z_0(x), z_1(x))\mathrm{d}x$, it is possible to derive existence of energetic solutions, see [FM06].

4.5 Damage, delamination and fracture

In the following the damage variable is denoted by $z \in [0,1]$, where $z = 1$ means no damage, whereas $z = 0$ stands for totally damaged material. We choose the dissipation metric as

$$R(\dot{z}) = \begin{cases} \underbrace{-\rho\dot{z}}_{\geq 0} & \text{if } \dot{z} \leq 0 \\ +\infty & \text{if } \dot{z} > 0, \end{cases} \tag{38}$$

which gives the dissipation

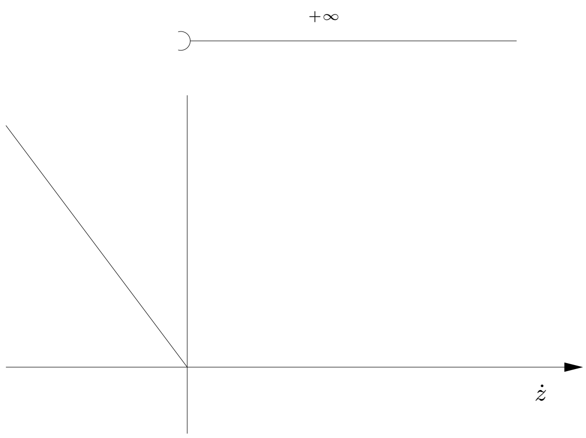

Figure 14: Dissipation metric $R(\dot{z})$

$$\mathcal{D}(z_0,z_1) = \begin{cases} \int_\Omega \rho(z_0(x) - z_1(x))\,\mathrm{d}x & \text{if } z_1 \leq z_0 \text{ a.e.} \\ \infty & \text{else} \end{cases}$$

from which the dissipation distance can be determined via

$$\mathrm{Diss}_{\mathcal{D}}(z,[s,t]) = \sup_{\text{finite partitions}} \sum_{j=1}^{N} \mathcal{D}(z(\tau_{j-1}), z(\tau_j)).$$

We have $\mathrm{Diss}_{\mathcal{D}}(z,[s,t]) < \infty$ if and only if $z(s)$ is a monotone path. In this case it is $\mathrm{Diss}_{\mathcal{D}}(z,[s,t]) = \mathcal{D}(z(s), z(t))$. For details see [MR06b].

The dissipation $\mathcal{D}$ is not continuous, even not in the strong $\mathrm{W}^{1,p}(\Omega)$ topology and therefore, we cannot apply proposition 3.9 to show that (CCa) and (CCb) of the existence theorem 3.4 are satisfied. We can compensate the missing continuity of $\mathcal{D}$ by showing that the sets of stable states are closed.

For (CCb) we have to prove that for every stable sequence (t_j, q_j) with $q_j \in \mathcal{S}(t_j)$, $\sup \mathcal{E}(t_j, q_j) < \infty$ and $(t_j, q_j) \to (t_*, q_*)$ we have

$$q_* \in \mathcal{S}(t_*).$$

This is equivalent to prove that $\forall \hat{q} \in Q : -\mathcal{E}(t_*, q_*) + \mathcal{E}(t_*, \hat{q}) + \mathcal{D}(q_*, \hat{q}) \geq 0$.
By assumption we have for every j and for every $\tilde{q} \in Q$ that $-\mathcal{E}(t_j, q_j) + \mathcal{E}(t_j, \tilde{q}) + \mathcal{D}(q_j, \tilde{q}) \geq 0$.
Property (CCb) follows if there exists a joint recovery sequence (JRS) $\hat{q}_j \to \hat{q}$ such that

$$\begin{aligned} 0 &\overset{q_j \in \mathcal{S}(t_j)}{\leq} \limsup(-\mathcal{E}(t_j, q_j) + \mathcal{E}(t_j, \hat{q}_j) + \mathcal{D}(q_j, \hat{q}_j)) \\ &\overset{\text{JRS}}{\leq} \mathcal{E}(t_*, \hat{q}) + \mathcal{D}(t_*, \hat{q}) - \mathcal{E}(t_*, q_*) \end{aligned}$$

The second inequality motivates the assumption on the existence of a joint recovery sequence.

Proposition 4.7 (Existence of joint recovery sequences) *Assume that for all stable sequences $(t_j, q_j)_{j\in\mathbb{N}}$ with $(t_j, q_j) \to (t_*, q_*)$ and for all $\hat{q} \in \mathcal{Q}$ there exists a sequence $(\hat{q}_j)_{j\in\mathbb{N}}$ with $\hat{q}_j \to \hat{q}$ and*

$$\limsup_{j\to\infty} \left(\mathcal{E}(t_j, \hat{q}_j) + \mathcal{D}(q_j, \hat{q}_j) - \mathcal{E}(t_j, q_j)\right) \leq \mathcal{E}(t_*, \hat{q}) + \mathcal{D}(q_*, \hat{q}) - \mathcal{E}(t_*, q_*)\,. \tag{39}$$

Then $q_* \in \mathcal{S}(t_*)$.

A simple damage model

The following nonlocal damage model is treated in [MR06b]. It is inspired by engineering models in [FN96, Fré02, HS03].

We consider the energy functional

$$\mathcal{E}(t,u,z) = \int_\Omega W_1(e(u)) + zW_2(e(u)) + \frac{\sigma}{r}|\nabla z|^r \,\mathrm{d}x - \langle l(t), u\rangle\,.$$

Like in the previous examples the displacements are denoted by $u \in \mathcal{F} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d)$ and $z \in \mathcal{Z} := \{z \in \mathrm{W}^{1,r}(\Omega) \mid z(x) \in [0;1]\}$ is the damage variable. The density W_1

corresponds to the unbreakable matrix while W_2 describes the breakable fibers. We assume $W_1(e) \geq c|e|^2 - c$, i.e. coercivity and $W_2(e)$ to be non-negative. The regularization term is necessary to have the embedding $\mathrm{W}^{1,r}(\Omega) \hookrightarrow \mathrm{C}^0(\bar{\Omega})$. Thus, we have to assume $r > d$, where d is the space dimension. Finally the dissipation distance is

$$\mathcal{D}(z_0, z_1) = \int_\Omega \mathcal{R}(z_1(x) - z_0(x))\,\mathrm{d}x\ ,$$

where $\mathcal{R}$ is defined like in (38), see also Figure 14.

We are going to prove that the JRS-condition holds. Given a sequence $(t_j, u_j, z_j) \to (t_*, u_*, z_*) \in \mathbb{R} \times H^1_{\text{weak}} \times W^{1,r}_{\text{weak}}$ and an arbitrary test state $(\hat{u}, \hat{z})$ we want to construct a recovery sequence $(\hat{u}_j, \hat{z}_j)$ such that (39) holds. Note that $(\hat{u}_j, \hat{z}_j) \to (\hat{u}, \hat{z})$ actually is not needed.

Observe that only $\mathcal{D}(q_*, \hat{q}) = \mathcal{D}(z_*, \hat{z}) < \infty$ is interesting. Hence we may assume $\hat{z} \leq z_*$ almost everywhere. To find $(\hat{z}_j)_{j\in\mathbb{N}}$ such that $\mathcal{D}(z_j, \hat{z}_j) \to \mathcal{D}(z_*, \hat{z})$ we need $\hat{z}_j \leq z_j$ almost everywhere and $\hat{z}_j \to \hat{z}$ in $W^{1,r}$. For instance, we may take

$$\hat{z}_j = \max\{0, \hat{z} - \|z_j - z_*\|_\infty\}\ .$$

Then $\hat{z}_j \to \hat{z}$ strongly (since $r > d$) and $\mathcal{E}(t_j, \hat{u}_j, \hat{z}_j) \to \mathcal{E}(t, \hat{u}, \hat{z})$ for a suitable choice of $\hat{u}_j$. For instance we may choose $\hat{u}_j$ to be the minimizer of $\mathcal{E}(t_j, \cdot, \hat{z}_j)$. Thus (39) is valid.

Crack propagation in brittle fracture

As next example we consider crack propagation in an elastic body. This is an example without a Banach space structure and is investigated in detail in [FM93, FM98, DT02, FL03, DFT05]. We will only state the problem and refer to the above references for the details.

A crack Γ is defined to be a closed subset of the considered domain $\bar{\Omega}$ with finite Hausdorff measure $\mathcal{H}^{d-1}$. We do not distinguish two crack if the symmetric difference has zero Hausdorff measure and introduce equivalent classes via

$$\Gamma_0 \sim \Gamma_1 :\Longleftrightarrow \mathcal{H}^{d-1}\left((\Gamma_0 \setminus \Gamma_1) \cup (\Gamma_1 \setminus \Gamma_0)\right)\ .$$

We take

$$\mathcal{Z}_\sim := \left\{\Gamma_\sim \mid \Gamma \subset \bar{\Omega} \text{ closed}\ , \mathcal{H}^{d-1}(\Gamma) < \infty\right\}$$

and consider the topology defined by the Hausdorff distance $d_\sim(\Gamma_0, \Gamma_1) := \mathrm{dist}_{\mathrm{H}}(\Gamma_0, \Gamma_1)$. Note that $(\mathcal{Z}_\sim, d_\sim)$ is a compact metric space [].

The dissipation distance is defined by

$$\mathcal{D}(\Gamma_0, \Gamma_1) = \begin{cases} \mathcal{H}^{d-1}(\Gamma_1 \setminus \Gamma_0) & \text{if } \Gamma_0 \subset \Gamma_1 \\ \infty & \text{else} \end{cases}$$

and the energy functional is

$$\mathcal{E}(t, u, \Gamma) = \int_{\Omega\setminus\Gamma} W(e(u))\,\mathrm{d}x - \langle l(t), u\rangle\ .$$

The admissible set is given by

$$\mathcal{Q} = \{(u,\Gamma) \mid \Gamma \in \mathcal{Z}_\sim \, , u \in \mathrm{GSBV}(\Omega),\ J(u) \subset \Gamma\} \ ,$$

where GSBV(Ω) denotes the space of generalized special functions of bounded variations and $J(u)$ denotes the jump set of u.

The main difficulty of this problem consists in the construction of the joint recovery sequence. To handle this the *jump transfer lemma* is essential, see Francfort and Larsen in [FL03].

5 Γ-convergence of energetic formulations

This last section deals with the convergence of sequences of rate-independent evolutionary problems and is based on [MRS06]. Since energetic solutions are defined via functionals it turns out that Γ-convergence provides an appropriate setting. General introductions to (static) Γ-convergence can be found in [Dal93, Bra02].

In the following we consider a sequence of the two functionals,

$$\begin{aligned} \mathcal{E}_n : \ [0;T] \times \mathcal{Q} &\longrightarrow \mathbb{R}_\infty \\ \mathcal{D}_n : \quad \mathcal{Z} \times \mathcal{Z} &\longrightarrow [0;\infty] \ . \end{aligned} \tag{40}$$

We assume that the assumptions (E1)-(D3), (CCa) and (CCb) hold for all $n \in \mathbb{N}$. Thus, due to Theorem 3.4, for every $n \in \mathbb{N}$ and arbitrary initial conditions $q_n^0 \in \mathcal{S}_n(0)$ there exists an energetic solution $q_n : [0;T] \to \mathcal{Q}$ associated with (40).

We consider the situation that the sequences of the two functionals and the solutions, respectively, converge to some limit,

$$\mathcal{E}_n \rightsquigarrow \mathcal{E} \ , \qquad \mathcal{D}_n \rightsquigarrow \mathcal{D} \ , \qquad q_n \rightsquigarrow q \ .$$

Here "$\rightsquigarrow$" denotes the convergence in a suitable sense. Now the question is if the limit function q solves the energetic formulation of the problem associated with $\mathcal{E}$ and $\mathcal{D}$.

A typical application for the convergence of energetic formulations is a finite element approximation. In that case we have

$$\mathcal{E}_n(t,q) = \begin{cases} \mathcal{E}(t,q) & \text{if } q \in \mathcal{Q}_n \\ \infty & \text{else} \end{cases} \ , \qquad \mathcal{D}_n = \mathcal{D} \ .$$

where for the sequence of state spaces holds

$$\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_n \subset \ldots \mathcal{Q} \qquad \text{and} \qquad \overline{\bigcup_{n\in\mathbb{N}} \mathcal{Q}_n} = \mathcal{Q} \ .$$

We will discuss this problem in more detail in section 5.3

5.1 Γ-convergence of functionals

At first we recall the notion of Γ-convergence.

Definition 5.1 (Γ-convergence of functionals) *A sequence of functionals* $(\mathcal{E}_n)_{n\in\mathbb{N}}$ *on* $\mathcal{Q}$ Γ*-converges to some functional* $\mathcal{E} : \mathcal{Q} \to \mathbb{R}_\infty$*, i.e.* $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$*, if the following two conditions hold*

(Γ_1) $\liminf$-*estimate:*

$$q_n \rightsquigarrow q \qquad \Longrightarrow \qquad \mathcal{E}(q) \leq \liminf_{n\to\infty} \mathcal{E}_n(q_n)$$

(Γ_2) $\limsup$-*estimate/existence of a recovery sequence:*

$$\forall\, \hat{q} \in \mathcal{Q}\ \exists\, \hat{q}_j \rightsquigarrow \hat{q} :\ \mathcal{E}(\hat{q}) \geq \limsup_{n\to\infty} \mathcal{E}_n(\hat{q}_n)\ .$$

Here "$\rightsquigarrow$" again denotes the convergence in a suitable sense. In many applications this is convergence in the weak sense. To illustrate the definition we will only give two simple examples for the application of Γ-convergence. For further details we refer to the book [Bra02] which provides a proper introduction to the topic appropriate for PhD students.

A numerical example

We consider the energy functional

$$\mathcal{E}(t, u, z) = \frac{1}{2}\langle \mathcal{A}q; q\rangle - \langle l(t); u\rangle$$

for $q \in \mathcal{Q}$ some Hilbert space and a linear operator $\mathcal{A}$ with $0 < \alpha \leq \mathcal{A} = \mathcal{A}^* \leq \beta$. Assume $(\mathcal{Q}_n)_{n\in\mathbb{N}}$ to be a sequence of finite dimensional subspaces of $\mathcal{Q}$ such that

$$\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_n \subset \ldots \mathcal{Q} \qquad \text{and} \qquad \overline{\bigcup_{n\in\mathbb{N}} \mathcal{Q}_n} = \mathcal{Q}\ .$$

We define

$$\mathcal{E}_n(t, q) = \begin{cases} \mathcal{E}(t, q) & \text{if } q \in \mathcal{Q}_n \\ \infty & \text{else} \end{cases}.$$

If $\mathbb{P}_n : \mathcal{Q} \to \mathcal{Q}_n$ denotes the orthogonal projections then of course $\mathbb{P}_n\hat{q} \to \hat{q}$ for all $\hat{q} \in \mathcal{Q}$.

Now we check the two conditions for Γ-convergence. Let $q_n \rightharpoonup q$. Due to the definition of $\mathcal{E}_n$ we have $\mathcal{E}_n(q_n) \geq \mathcal{E}(q_n)$. Since $\mathcal{E}$ is w.l.s.c. we obtain $\mathcal{E}(q) \leq \liminf_{n\to\infty} \mathcal{E}_n(q_n)$. Thus we have ($\Gamma_1$). Given $\hat{q} \in \mathcal{Q}$ we choose $\hat{q}_n := \mathbb{P}_n q$. Then $\mathcal{E}_n(\hat{q}_n) = \mathcal{E}(\hat{q}_n) \to \mathcal{E}(\hat{q})$ since $\hat{q}_n \to \hat{q}$ strongly. This proves (Γ_2).

A homogenization problem

For a second example we consider a typical homogenization problem. We define $\mathcal{Q} = \mathrm{H}^1((a;b);\mathbb{R}^m)$ and

$$\mathcal{E}_n(u) = \int_a^b (A(nx)\cdot u'(x))\cdot u'(x)\,\mathrm{d}x\ ,$$

where $A(y) \in \mathbb{R}^{m\times m}$. We assume $0 < \alpha \le A(y) = A^*(y) \le \beta$. Then, for $n \to \infty$ we obtain $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$ where

$$\mathcal{E}(u) = \int_a^b (A_{\mathrm{eff}}\cdot u'(x))\cdot u'(x)\,\mathrm{d}x\ , \qquad A_{\mathrm{eff}} = \Big(\int_a^b A(y)^{-1}\,\mathrm{d}y\Big)^{-1}\ .$$

5.2 Convergence Result

We consider $(\mathcal{E}_n, \mathcal{D}_n)_{n\in\mathbb{N}}$, assume that (E1)-(D3) holds uniformly and

$$\mathcal{E}_n(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}(t,\cdot)\ , \quad \mathcal{D}_n \xrightarrow{\Gamma} \mathcal{D} \qquad \text{as } n\to\infty\ .$$

As compatibility condition we need the following (joint recovery sequences):

For every stable sequence $(t_n, q_n)_{n\in\mathbb{N}}$ (i.e. $q_n \in \mathcal{S}_n(t_n)$) with $(t_n, q_n) \to (t_*, q_*)$, the relations ($\Gamma$-CCa), ($\Gamma$-CCb) here below are satisfied:

$$\partial_t\mathcal{E}_n(t_n, q_n) \to \partial_t\mathcal{E}(t_*, q_*), \tag{Γ-CCa}$$

$$q_* \in \mathcal{S}(t_*). \tag{Γ-CCb}$$

Theorem 5.2 *Under the above assumptions the following holds: If $q_n : [0,T] \to \mathcal{Q}$ is an energetic solution to $(\mathcal{E}_n, \mathcal{D}_n)$ for all $n \in \mathbb{N}$,*

$$q_n(0) \longrightarrow q^0 \qquad \text{and} \qquad \mathcal{E}(0, q_n(0)) \longrightarrow \mathcal{E}(0, q^0)$$

then there exists a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ and a function $q : [0,T] \to \mathcal{Q}$ such that

1. *q is an energetic solution to $(\mathcal{E}, \mathcal{D})$ with $q(0) = q^0$*
2. *for all $t \in [0,T]$ holds*

$$\begin{aligned} \mathcal{E}_{n_k}(t, q_{n_k}(t)) &\longrightarrow \mathcal{E}(t, q(t)) \\ \mathrm{Diss}_{\mathcal{D}}(q_{n_k}, [0,T]) &\longrightarrow \mathrm{Diss}_{\mathcal{D}}(q, [0,T]) \\ z_{n_k}(t) &\longrightarrow z(t)\ . \end{aligned} \tag{42}$$

As in Section 3 we have a simple sufficient condition for the compatibility condition (Γ-CC).

Proposition 5.3 *If $\mathcal{D}_n \xrightarrow{c} \mathcal{D}$, then* (CCa) *and* (CCa) *hold.*

Here "$\mathcal{D}_n \xrightarrow{c} \mathcal{D}$" denotes continuous convergence which is defined as

$$(z_n, \tilde{z}_n) \longrightarrow (z, \tilde{z}) \qquad \Longrightarrow \qquad \mathcal{D}_n(z_n, \tilde{z}_n) \longrightarrow \mathcal{D}(z, \tilde{z}) \; .$$

Note that this is different from uniform convergence.

The following analog to Proposition 4.7 is proved in [MRS06].

Proposition 5.4 *If for all stable sequences* $(t_n, q_n)_{n\in\mathbb{N}}$ *with* $(t_n, q_n) \to (t_*, q_*)$ *and for all* $\hat{q} \in \mathcal{Q}$ *there exists a joint recovery sequence* $(\hat{q}_n)_{n\in\mathbb{N}}$ *such that*

$$\limsup_{n\to\infty} \mathcal{E}_n(t_n, \hat{q}_n) + \mathcal{D}_n(q_n, \hat{q}_n) - \mathcal{E}_n(t_n, q_n) \leq \mathcal{E}(t_*, \hat{q}) + \mathcal{D}(q_*, \hat{q}) - \mathcal{E}(t_*, q_*) \; , \tag{43}$$

then (CCb) *holds.*

5.3 Numerical methods for linearized elasticity

Here we treat the simplest nontrivial case and refer to [MR06a] for more general situations. In this section we apply the convergence result stated above to numerical methods for linearized elastoplasticity. We have

$$\mathcal{Q} = \mathcal{F} \times \mathcal{Z} \; , \qquad \mathcal{F} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d) \; , \quad \mathcal{Z} = \mathrm{L}^2(\Omega, S_0)$$

for $\Omega \subset \mathbb{R}^d$ and $S_0 := \{A \in \mathbb{R}^{d\times d} \mid A = A^\top, \mathrm{tr}\, A = 0\}$ and we consider an energy formulation defined by the following energy and dissipation functionals

$$\begin{aligned} \mathcal{E}(t, u, z) &= \frac{1}{2} \left\langle \mathcal{A} \begin{pmatrix} u \\ z \end{pmatrix} ; \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle - \langle l(t); u \rangle, \\ \mathcal{D}(z_0, z_1) &= \mathcal{R}(z_1 - z_0) = \int_\Omega R(z_1(x) - z_0(x))\, \mathrm{d}x \; . \end{aligned} \tag{44}$$

The linear operator $\mathcal{A}$ is assumed to be symmetric and bounded from above and below and l is assumed to be continuous, see also section 2.3. Note that the functional $\mathcal{R}$ is in general not continuous, but only lower semi-continuous since it may take the value "$+\infty$".

We choose a sequence of finite dimensional subspaces of $\mathcal{Q}$ corresponding to triangulations $\mathcal{T}_n$ of Ω:

$$\mathcal{Q}_n = \mathcal{F}_n \times \mathcal{Z}_n$$

where $\mathcal{F}_n$ consists of piecewise linear and $\mathcal{Z}_n$ of piecewise constant functions. We assume that

$$\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots \subset \mathcal{Q}_n \subset \ldots \mathcal{Q} \qquad \text{and} \qquad \overline{\bigcup_{n\in\mathbb{N}} \mathcal{Q}_n} = \mathcal{Q} \; .$$

Let furthermore

$$\mathbb{P}_n : \; \mathcal{Q} \longrightarrow \mathcal{Q}_n \tag{45}$$

be a not necessarily orthogonal projector with

$$\mathbb{P}_n q \longrightarrow q \,, \qquad R(\mathbb{P}_n q) \longrightarrow R(q)$$

where we assume strong convergence for $n \to \infty$. As above we define the sequences of energy and dissipation functionals according to

$$\mathcal{E}_n(t,q) = \begin{cases} \mathcal{E}(t,q) & q \in \mathcal{Q}_n \\ \infty & \text{else} \end{cases} \,, \qquad \mathcal{D}_n = \mathcal{D} \,.$$

Due to the conditions stated above we have $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$, see the example in section 5.1.

In this case Proposition 5.3 is not applicable since $\mathcal{R}$ is only lower semi continuous. We need Proposition 5.4. For a sequence $(q_n)_{n\in\mathbb{N}}$ with $q_n \rightharpoonup q$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}} \times \mathrm{L}^2$ and arbitrary $\hat{q}$ with $\mathcal{R}(\hat{q}-q) = \mathcal{D}(q,\hat{q}) < \infty$ we define the joint recovery sequence by

$$\hat{q}_n = q_n + \mathbb{P}_n(\hat{q}-q) \,.$$

Then we have

$$\begin{aligned} \mathcal{D}(q_n,\hat{q}_n) &= \mathcal{R}(\hat{q}_n - q_n) = \mathcal{R}(\mathbb{P}_n(\hat{q}-q)) \\ &\to \mathcal{R}(\hat{q}-q) = \mathcal{D}(q,\hat{q}) \end{aligned} \tag{46}$$

Note that $\hat{q}_n \rightharpoonup q$ only converges weakly, but not strongly. In general we have $\mathcal{E}(t,\hat{q}) \leq \liminf_{n\to\infty} \mathcal{E}(t,\hat{q}_n)$ where "=" holds if and only if the convergence is strong. But due to the quadratic structure of $\mathcal{E}$ we have

$$\begin{aligned} \mathcal{E}_n(t_n,\hat{q}_n) - \mathcal{E}_n(t_n,q_n) &= \mathcal{E}(t_n,\hat{q}_n) - \mathcal{E}(t_n,q_n) \\ &= \frac{1}{4}\langle \mathcal{A}(\hat{q}_n - q_n), \hat{q}_n + q_n\rangle - \langle l(t_n), \hat{q}_n - q_n\rangle \\ &\to \frac{1}{4}\langle \mathcal{A}(\hat{q}-q), \hat{q}+q\rangle - \langle l(t), \hat{q}-q\rangle \\ &= \mathcal{E}(t,\hat{q}) - \mathcal{E}(t,q) \,. \end{aligned}$$

Here we used that $\hat{q}_n - q_n = \mathbb{P}_n(\hat{q}-q) \to \hat{q}-q$ (strong), $\hat{q}_n + q_n \rightharpoonup \hat{q}+q$ (weak), and $l(t_n) \to l(t)$ (strong). Together with (46) we finally find that the condition (43) is satisfied and theorem 5.2 provides the convergence of our numerical scheme.

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