The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors*

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Abstract

Using weighted L^p–norms we derive new bounds on the long–time behavior of the solutions improving on the known results of the polynomial growth with respect to the instability parameter. These estimates are valid for quite arbitrary, possibly unbounded domains. We establish precise estimates on the maximal influence of the boundaries on the dynamics in the interior. For instance, the attractor $\mathcal{A}_\ell$ for the domain $(-\ell, \ell)^d$ with periodic boundary conditions is upper semicontinuous to $\mathcal{A}_\infty$.

1 Introduction

Dynamics of parabolic equations on large or unbounded domains has features which are completely different from dynamics on small domains. In particular, the typical spatial patterns are not dominated by boundary effects. In this paper we want to develop a theoretical framework which allows us to study such situations. We have restricted the whole theory to the complex Ginzburg–Landau equation (CGL)

$$u_t = (1 + i\alpha)\Delta u + Ru - (1 + i\beta)|u|^2 u, \ t > 0, \ x \in \Omega \subset \mathbb{R}^d$$

which on the one hand plays a central role in the theory of amplitude equations (see [NPL93, MS96]) and on the other hand has its one interest in studying turbulence [BC*90].

The main idea we want to propagate is the use of weighted energy estimates using

$$\|u\|_{p,\rho} = (\textstyle\int_\Omega \rho(x)|u(x)|^p \, dx)^{1/p}$$

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where $\rho > 0$ is a suitable weight with $|\nabla\rho(x)| \leq \rho_0\rho(x)$ and $\rho_I = \int_{I\!R^n} \rho(x)\,dx < \infty$. The usage of such a weight for problems on unbounded domains was introduced in [CE90] and further developed in [Col94]. The advantage compared to the weight $\rho \equiv 1$ is three fold. First, we are able to consider bounded nondecaying functions on arbitrary (unbounded) domains $\Omega \subset I\!R^d$. Second, even on bounded domains the weight can be chosen to have a width $\frac{1}{\rho_0}$ which corresponds to typical length scales of the problem, e.g. $\rho_0 = \sqrt{R(1+\alpha^2)}$ for CGL. Third, the decaying weight ρ screens effects which are far away such that their influence on nearby points can be estimated properly.

As a major result, the weighted norms will allow us to derive a–priori estimates which are essentially independent of the underlying domain. Thus, we will be able to exploit the scaling invariance of CGL

$$(t, x, u, R, \alpha, \beta) \to (\ell^{-2}t, \ell^{-1}x, \ell u, \ell^2 R, \alpha, \beta).$$

Note, that this scaling stretches the domain when the instability parameter R is reduced (i.e. $\ell < 1$).

In Section 2 we show how the classical energy estimates generalize to the weighted case. All estimates are independent of Ω, and the weight ρ only appears through ρ_0 and ρ_I. In Section 3 we introduce the uniformly local spaces $L^p_{\rm lu}(\Omega)$ with the norm

$$\|u\|_{p,{\rm lu}} = \sup\{\, \|u\|_{p,T_y\rho} \;:\; y \in I\!R\,\} \tag{1.1}$$

where $T_y\rho(x) = \rho(x-y)$ is the translated weight. Applying Gronwall's estimate to $\|u(t)\|_{p,T_y\rho}$ for each $y \in I\!R^n$ we find, whenever $2\lambda_Q = p - (p-2)\sqrt{1+\alpha^2} > 0$,

$$\|u(t)\|^p_{p,{\rm lu}} \leq e^{-2\widetilde{R}t}\|u(0)\|^p_{p,{\rm lu}} + \left(1 - e^{-2\widetilde{R}t}\right)\widetilde{R}^p\rho_I,$$

where $\widetilde{R} = R + \rho_0^2(1+\alpha^2)/(4\lambda_Q)$.

Note that CGL is invariant under the scalings $(t, x, u, R) \to (\delta^2 t, \delta t, \delta u, \delta^2 R)$ which allows us to reduce the analysis to the case $R \in (0, 1)$ when the domain Ω is made suitably large. In Section 4 we derive then global existence result in the same parameter sets $(\alpha, \beta) \in \mathcal{P}(d)$ as in the classical case, namely

$$\begin{aligned}&\mathcal{P}(1) = \mathcal{P}(2) = I\!R^2, \quad \mathcal{P}(3) = \{\, (\alpha,\beta) \in I\!R^2 \;:\; |\alpha| < \sqrt{8} \text{ or } -(1+\alpha\beta) < \sqrt{3}|\alpha-\beta|\,\},\\ &\text{and } \mathcal{P}(d) = \{\, (\alpha,\beta) \in I\!R^2 \;:\; |\alpha| < 2\sqrt{d-1}/(d-2)\,\} \text{ for } d \geq 4.\end{aligned} \tag{1.2}$$

We follow the approach in [LO96] which uses on ideas of [BC*90] for the energy estimates and of [Wei80] for the regularity theory. For this purpose we establish the semigroup properties of (e^{At}), with $Au = (1+i\alpha)\Delta u + Ru$, on the space $L^p_{\rm lu}(\Omega)$ for general Ω.

Using these global existence results in $L^{2d}_{\rm lu}(\Omega)$ and $W^{1,2d}_{\rm lu}(\Omega)$, the global boundedness independent of Ω, as well as the scaling property (1.1) we conclude

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_\infty \leq C_\infty(\alpha,\beta,d)(1+\sqrt{R})$$

for $(\alpha, \beta) \in \mathcal{P}(d)$ and arbitrary domains. Thus, we improve considerably the result in [BC*90] where the growth rates were estimated by $R^{e(\alpha,\beta,d)}$ with $e > 1/2$. Similar results for L^p–norms are given in Section 3.

In Section 5 we consider the influence of the boundaries on the dynamics on the interior. To this end we use a compactly supported weight ρ_* according to [Col94], which satisfies $|\nabla \rho_*| \le C\,\rho_*^\gamma$ for some $\gamma \in (0,1)$. We find an estimate

$$\int_{\Omega_*} \rho_* |u_1(t) - u_2(t)|^p \, dx \le C\, e^{C\,t} \left(\int_{\Omega_*} \rho_* |u_1(0) - u_2(0)|^p \, dx + D_* \right)$$

where $\Omega_* = \mathrm{support}(\rho_*) \subset B(0, r_*),\ \rho(x) = e^{-|x|}$ for $|x| \le r_* - 1$ and $D_* = r_*^{d-1} e^{-r_*}$. This enables us to estimate the maximal difference between two solutions which may correspond to different domains, different initial conditions, or boundary conditions outside of Ω_*. The maximal influence of the dynamics outside of Ω_* is controlled by D_* which decays exponentially with large r_*.

Moreover, two semiflows (S_t^1) and (S_t^2) corresponding to the parameters (R, α, β) but to two different domains Ω_1 and Ω_2, respectively, are close to each other in the norm $\|\cdot\|_{p,\rho_*}$ if the common domain $\Omega_* = B(x_*, r_*) \cap \Omega_1 = B(x_*, r_*) \cap \Omega_2$ has sufficiently large r_*. In particular, true orbits $u_1(t) = S_t^1(u_1^0)$ can be approximated in Ω_* by (T, δ) pseudo–orbits for (S_t^2).

In Section 6 we develop the notion of attractors of CGL on general unbounded domains. For bounded domains the existence of compact attractors with finite Hausdorff dimension is well–known. For unbounded domain we cannot expect compactness in the strong topology induced by $\|\cdot\|_{p,\mathrm{lu}}$. This difficulty can be overcome by using a weaker topology for the attrativity property, see e.g. [BV90]. We follow the ideas in [MS95], which originate in [Fei95], and use the compactness with respect to the localized topology induced by $\|\cdot\|_{p,\rho}$. Thus, we establish existence and uniqueness of an attractor $\mathcal{A} \subset L^p_{\mathrm{lu}}(\Omega)$ for each (S_t). Here, $\mathcal{A}$ is bounded and closed in $L^p_{\mathrm{lu}}(\Omega)$, invariant under S_t and compact in $L^p_\rho(\Omega)$. The attraction takes place in $L^p_\rho(\Omega)$ as well, namely for each bounded $B \subset L^p_{\mathrm{lu}}(\Omega)$ we have

$$\mathrm{dist}_{p,\rho}(S_t(B), \mathcal{A}) = \sup_{b \in B} \inf_{a \in A} \|S_t(b) - a\|_{p,\rho} \to 0 \quad \text{for } t \to \infty.$$

Finally we compare two attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ associated to two semigroups (S_t^j) on domains $\Omega_j,\ j = 1, 2$, in the case of a large joint domain Ω_* as above. We find

$$\mathrm{dist}_{p,\rho_*}(\mathcal{A}_1, \mathcal{A}_2) \le \psi_2(r_*)$$

with $\psi_2(r_*) \to 0$ for $r_* \to \infty$. Moreover, every orbit in $\mathcal{A}_1$ can be approximated by a (T, δ) pseudo–orbit inside $\mathcal{A}_2$ if r_* is sufficiently large.

2 Weighted estimates

Throughout this work we consider domains $\Omega \subset I\!R^d$ which are sufficiently smooth in order to allow for the application of the divergence theorem. (In fact in Section 3 we assume C^2 smoothness of the boundary.) Moreover, the boundary conditions associated to CGL are such that partial integration does not generate boundary integrals. This is the case if either Dirichlet ($u=0$) or Neumann boundary conditions ($\nabla u \cdot \nu = 0$ with ν the unit outward normal vector on $\partial\Omega$) are prescribed. In addition we may have periodic boundary conditions, see Definiton 3.2 below.

Since the domain we are interested in large and even unbounded domains we want to allow for solutions which are bounded are not necessarily decaying at infinity. To control such functions we introduce a localizing weighted norm as well as a uniform norm. Consider a weight function $\rho : I\!R^d \to (0,\infty)$ with $|\nabla \rho(x)| \le \rho_0 \rho(x)$ and $\int_{I\!R^d} \rho \, dx = \rho_I < \infty$. As a consequence we have $\rho(x+y) \ge e^{-\rho_0|y|}\rho(x)$. We denote the weighted L_p–Norm by

$$\|u\|_{p,\rho} = \Big(\int_\Omega \rho(x)|u(x)|^p \, dx \Big)^{1/p}.$$

Frequently the index $p=2$ will be omitted.

Using the translates $T_y\rho$ by $(T_y\rho)(x) = \rho(x-y)$, the uniformly local L^p–norm on Ω is given as

$$\|u\|_{p,\mathrm{lu}} = \sup\{ \, \|u\|_{p,T_y\rho} \, : \, y \in I\!R^n \, \},$$

where $\|u\|_{p,T_y\rho}^p = \int_\Omega \rho(x-y)|u(x)|^p \, dy$. We define the space $\widetilde{L}^p_{\mathrm{lu}}(\Omega) = \{ \, u \in L^p_{\mathrm{loc}}(\Omega) \, : \, \|u\|_{p,\mathrm{lu}} < \infty \, \}$ and the associated Sobolev spaces

$$\widetilde{W}^{s,p}_{\mathrm{lu}}(\Omega) = \{ \, u \in \widetilde{L}^p_{\mathrm{lu}}(\Omega) \, : \, D^q u \in \widetilde{L}^p_{\mathrm{lu}}(\Omega) \text{ for all } q \in I\!N_0^d \text{ with } q_1 + \ldots + q_d \le s \, \}$$

for integers s. The uniformly local Sobolev spaces are then defined as

$$W^{s,p}_{\mathrm{lu}}(\Omega) = \text{closure of } \mathcal{C}^\infty_{\mathrm{bdd}}(\Omega) \text{ in } \widetilde{W}^{s,p}_{\mathrm{lu}}(\Omega),$$

where $\mathcal{C}^\infty_{\mathrm{bdd}}(\Omega)$ is the set of all $\mathcal{C}^\infty$–functions which have all its derivatives bounded in Ω. This construction is necessary for unbounded domains in order to ensure density of $W^{s+1,p}_{\mathrm{lu}}(\Omega)$ in $W^{s,p}_{\mathrm{lu}}(\Omega)$, see [MS95]. Note the embeddings $C_{\mathrm{bdd,unif}}(\overline{\Omega}) \subset L^q_{\mathrm{lu}}(\Omega) \subset L^p_{\mathrm{lu}}(\Omega)$ for $1 \le p < q < \infty$ with norm estimates $\|u\|_{L^p,\mathrm{lu}} \le \rho_I^{(q-p)/(qp)} \|u\|_{L^q,\mathrm{lu}} \le \rho_I^{1/p}\|u\|_\infty$.

Throughout this work we will work with strong solutions u of CGL which lies simultaneous in the spaces $\mathcal{C}^0([0,T], L^p_{\mathrm{lu}}(\Omega))$, $\mathcal{C}^1((0,T], L^p_{\mathrm{lu}}(\Omega))$, $\mathcal{C}^0((0,T], W^{1,p}_{\mathrm{lu}}(\Omega))$, and $\mathcal{C}^0((0,T], W^{2,2}_{\mathrm{lu}}(\Omega))$ for some $p > d$. Therefore we have $\mathcal{C}^0((0,T], L^\infty(\Omega))$. Existence of such solutions will be established in Section 4. Here we derive a–priori bounds for such solutions.

For any strong solution of CGL with boundary conditions as above we obtain for general $p \ge 2$

$$\begin{aligned} \tfrac{1}{p}\tfrac{d}{dt}\|u\|_{p,\rho}^p &= \mathrm{Re} \int_\Omega \rho |u|^{p-2}\, \overline{u}\, \partial_t u \, dx \\ &= -\mathrm{Re}\, [(1+i\alpha)\int_\Omega |u|^{p-2}\, \overline{u}\, \nabla\rho \cdot \nabla u \, dx] - Q_{\alpha,p}(u) + \int_\Omega \rho\, |u|^p (R - |u|^2)\, dx. \end{aligned}$$

Here and further on we use the scalar product $\binom{a_1}{a_2} \cdot \binom{b_1}{b_2} = a_1 b_1 + a_2 b_2$ where no complex conjugation is involved, such that $|a|^2 = a \cdot \overline{a}$. Moreover, we use the abbreviation

$$Q_{\alpha,p}(u) = \frac{1}{2} \int_\Omega \rho |u|^{p-4} \begin{pmatrix} \overline{u}\, \nabla u \\ u\, \nabla \overline{u} \end{pmatrix} \cdot M(\alpha,p) \begin{pmatrix} u\, \nabla \overline{u} \\ \overline{u}\, \nabla u \end{pmatrix} dx$$

with $M(\alpha,p) = \frac{1}{2}\begin{pmatrix} p & (1+i\alpha)(p-2) \\ (1-i\alpha)(p-2) & p \end{pmatrix}$ whose smaller eigenvalue is denoted by $\lambda_Q(\alpha,p) = p/2 - |p/2-1|\sqrt{1+\alpha^2}$. Hence, $M(\alpha,p)$ is positive semi–definite for $|\alpha| \leq 2\sqrt{p-1}/|p-2|$. As long as $\lambda_Q > 0$ we can estimate

$$\int_\Omega \rho|u|^{p-4}(\rho_\alpha|u|^3|\nabla u| - \lambda_Q|u|^2\,|\nabla u|^2)\,dx \leq \frac{\rho_\alpha^2}{4\lambda_Q}\|u\|_{p,\rho}^p,$$

where we have maximized the integrand with respect to $|\nabla u|$. We continue to use the abbreviation $\rho_\sigma = \rho_0\sqrt{1+\sigma^2}$. Thus, with $\widetilde{R} = R + \rho_\alpha^2/(4\lambda_Q)$ we have

$$\begin{aligned} \tfrac{1}{p}\tfrac{d}{dt}\|u\|_{p,\rho}^p &\leq \textstyle\int_\Omega \rho|u|^p(R_\sigma - |u|^2)\,dx \;\leq\; \int_\Omega \rho\frac{\widetilde{R}}{p}(\widetilde{R}^{p/2} - |u|^p)\,dx \\ &= (2/p)\widetilde{R}^{1+p/2}\rho_I - (2/p)\widetilde{R}\|u\|_{p,\rho}^p. \end{aligned}$$

Hence, Gronwall's estimate implies

$$\|u(t)\|_{p,\rho}^p \leq e^{-2\widetilde{R}t}\|u(0)\|_{p,\rho}^p + (1-e^{-2\widetilde{R}t})\widetilde{R}^{p/2}\rho_I \;\text{ and }\; \limsup_{t\to\infty}\|u(t)\|_{p,\rho} \leq \widetilde{R}^{1/2}\rho_I^{1/p}. \quad (2.3)$$

Thus, the weighted L^p–norm of a classical solution cannot blow up if (α,σ) satisfies $\lambda_Q(\alpha,\sigma) > 0$.

We may similarly estimate the weighted H^1–norm, however it is advantageous to estimate the linear combination $F_\delta = \int_\Omega \rho(\frac{1}{2}|\nabla u|^2 + \frac{\delta}{4}|u|^4)dx$ with a suitable $\delta > 0$. In order to handle the nonlinear terms efficiently we follow the approach in [LO96] and use in addition to $Q_{\alpha,\sigma}(u)$ the expression

$$V(N,u) = \int_\Omega \rho\begin{pmatrix}|u|^2u \\ \Delta u\end{pmatrix}\cdot N\begin{pmatrix}|u|^2\overline{u} \\ \Delta\overline{u}\end{pmatrix}dx$$

which is real for $N = \overline{N}^\top$. Using the boundary conditions we know

$$\begin{aligned} 0 &= \operatorname{Re}\left[(1+i\gamma)\textstyle\int_{\partial\Omega}\rho|u|^2\overline{u}\nabla u\cdot\nu\,da(x)\right] \\ &= \operatorname{Re}\left[(1+i\gamma)\textstyle\int_\Omega \operatorname{div}(\rho|u|^2\overline{u}\nabla u)\,dx\right] \\ &= \operatorname{Re}\left[(1+i\gamma)\textstyle\int_\Omega |u|^2\overline{u}\,\nabla\rho\cdot\nabla u\,dx\right] + Q_{\gamma,2}(u) + V(N_{1+i\gamma},u) \end{aligned}$$

with $N_z = \frac{1}{2}\begin{pmatrix}0 & \overline{z} \\ z & 0\end{pmatrix}$. Since $Q_{\gamma,2}(u) \geq 0$ for $|\gamma| \leq \sqrt{3}$, we immediately obtain

$$\rho_\gamma\|\nabla u\|_\rho\|u\|_{6,\rho}^3 - V(N_{1+i\gamma},u) \geq 0 \text{ for } |\gamma| \leq \sqrt{3}. \quad (2.4)$$

For F_δ we now obtain the estimate

$$\frac{d}{dt}F_\delta \leq R(\|\nabla u\|_\rho^2 + \delta\|u\|_{4,\rho}^4) + (\rho_\alpha\|\Delta u\|_\rho + \rho_\beta\|u\|_{6,\rho}^3)\|\nabla u\|_\rho + V(\widetilde{N},u)$$

with $\widetilde{N} = \begin{pmatrix} -2 & 1+\delta+i(\beta-\delta\alpha) \\ 1+\delta-i(\beta-\delta\alpha) & -2\delta \end{pmatrix}$. We multiply (2.4) with any non-negative μ and add it in order to estimate for any $\kappa \in (0,1)$:

$$\begin{aligned} \tfrac{d}{dt}F_\delta \;\le\; & R(\|\nabla u\|_\rho^2 + \delta\|u\|_{4,\rho}^4) + (\rho_\alpha\|\Delta u\|_\rho + (\rho_\beta + \mu\rho_\gamma)\|u\|_{6,\rho}^3)\|\nabla u\|_\rho \\ & -(1-\kappa)(\|\Delta u\|_\rho^2 + \delta\|u\|_{6,\rho}^6) + V(\widehat{N}, u), \end{aligned} \tag{2.5}$$

with $\widehat{N} = \begin{pmatrix} -2\kappa & 1+\delta-\mu+i(\beta-\delta\alpha+\mu\gamma) \\ * & -2\kappa\delta \end{pmatrix}$.

Lemma 2.1
Let $\kappa \in [0,1)$ be fixed. Then there exist $\delta > 0$, $\mu \ge 0$ and $\gamma \in [-\sqrt{3}, \sqrt{3}]$ such that $\widehat{N}$ is negative semidefinite if and only if

$$(\alpha,\beta) \in \Lambda(\kappa) = \{\, (\alpha,\beta) \in \mathbb{R}^2 \;:\; -(4\kappa - 3 + \alpha\beta) < |\alpha-\beta|\sqrt{3}\,\}.$$

Proof: We define the sector $S = \{\, \mu(1+i\gamma) \in \mathbb{C} \;:\; \mu \ge 0,\ |\gamma| \le \sqrt{3}\,\} = \{\, z \in \mathbb{C} \;:\; |\arg z| \le \pi/3\,\}$. Then, $\widehat{N}$ is negative semidefinite if and only if

$$\operatorname{dist}(z(\delta), S) \le 4\kappa^2\delta, \text{ where } z(\delta) = 1 + i\beta + \delta(1 - i\alpha). \tag{2.6}$$

Thus, we have to find $\delta \in (0,\infty)$ such that (2.6) is satisfied. For $\alpha\beta \ge 0$ this is easy, since $z(\frac{\beta}{\alpha}) \in S$. If $\alpha\beta < 0$ we may restrict to the case $\alpha > 0 > \beta$, as $\beta > 0 > \alpha$ is similar by complex conjugation. In the cases $\alpha \in (0,\sqrt{3})$ or $\beta \in (-\sqrt{3},0)$ we choose δ sufficiently small or large, respectively, and find $z(\delta) \in S$.

The remaining, more difficult case is $\alpha \ge \sqrt{3}$ and $\beta \le -\sqrt{3}$, since the half line $\{\, z(\delta) : \delta \in (0,\infty)\,\}$ does not intersect S any more, except for $(\alpha,\beta) = (\sqrt{3}, -\sqrt{3})$. However, we have $\operatorname{dist}(z(\delta),S) = \operatorname{Re}(\frac{1}{2}(i-\sqrt{3})z(\delta)) = \frac{1}{2}(-\beta-\sqrt{3}+\delta(\alpha-\sqrt{3}))$. Letting $\delta = \varepsilon^2$ it is then easy to show that $\operatorname{dist}(z(\varepsilon^2),S) - 2\kappa\varepsilon$ attains its minimum at $\varepsilon = 2\kappa/(\alpha-\sqrt{3})$ and this minimum is nonpositive if and only if $(\alpha-\sqrt{3})(|\beta|-\sqrt{3}) \le 4\kappa$. For fixed κ we obtain $-(4\kappa-3) - \beta\alpha < \sqrt{3}|\alpha-\beta|$ as the set of possible (α,β). □

From now on we assume that $\kappa \in (0,1)$ is fixed and $(\alpha,\beta) \in \Lambda(\kappa)$. Moreover, we assume that δ, μ and γ are chosen suitably. The maximal set of (α,β) where the above approach with F_δ works is given as $\operatorname{int}(\Lambda(1))$, which coincides with the analysis given in [BC*90, LO96]. We return to (2.5) where the term $V(\widehat{N},u)$ can now be dropped, since we assume $(\alpha,\beta) \in \Lambda(\kappa)$. By partial integration we obtain $-\|\nabla u\|_\rho^2 + \rho_0\|u\|_\rho\|\nabla u\|_\rho + \|u\|_\rho\|\Delta u\|_\rho \ge 0$ and hence we can add $-\varepsilon\|\nabla u\|_\rho^2 + \varepsilon\rho_0^2\|u\|_\rho^2 + 2\varepsilon\|u\|_\rho\|\Delta u\|_\rho \ge 0$ with $\varepsilon > 0$ to (2.5) which yields, after maximizing the right–hand side with respect to $\|\Delta u\|_\rho$,

$$\begin{aligned} \tfrac{d}{dt}F_\delta \;\le\; & \left(R + \tfrac{\rho_\alpha^2}{2(1-\kappa)} + \tfrac{\rho_\beta+\mu\rho_\gamma}{4\theta} - \varepsilon\right)\|\nabla u\|_\rho^2 + \left(\tfrac{2\varepsilon^2}{1-\kappa} + \varepsilon\rho_0^2\right)\|u\|_\rho^2 \\ & +\delta R\|u\|_{4,\rho}^4 + [\theta(\rho_\beta+\mu\rho_\gamma) - \delta(1-\kappa)]\|u\|_{6,\rho}^6. \end{aligned} \tag{2.7}$$

We are able to make the prefactors of $\|\nabla u\|_\rho^2$ and $\|u\|_{6,\rho}^6$ negative by choosing

$$\theta = \frac{\delta(1-\kappa)}{2(\rho_\beta+\mu\rho_\gamma)} \quad \text{ and } \quad \varepsilon = \frac{1}{2}\widetilde{\varepsilon} + a \text{ with } a = R + \frac{\rho_\alpha^2}{2(1-\kappa)} + \frac{(\rho_\beta+\mu\rho_\gamma)^2}{2\delta(1-\kappa)}.$$

Estimating all the nonlinear terms under a single integral, we arrive at

$$\tfrac{d}{dt}F_\delta \leq -\widetilde{\varepsilon}F_\delta + C^*\|u\|_\rho^2, \quad \text{with } C^* = (\widetilde{\varepsilon}^2(1+\delta)+2a^2+R^2\delta)/(1-\kappa). \tag{2.8}$$

Using (2.3) for $p=2$ and setting $\widetilde{\varepsilon}^2 = (2a^2+R^2\delta)/(1+\delta))$ Gronwall's inequality yields

$$\limsup_{t\to\infty} F_\delta(t) \leq \frac{2(1+\delta)^{1/2}}{(1-\kappa)}(2a^2+R^2\delta)^{1/2}\left(R+\rho_\alpha^2/4\right)\rho_I. \tag{2.9}$$

Since $\delta > 0$, the estimate (2.8) for F_δ implies that for $(\alpha,\beta)\in \operatorname{int}(\Lambda(1))$ the norms $\|\nabla u(t)\|_{2,\rho}$ and on $\|u(t)\|_{4,\rho}$ can not blow up, and moreover, (2.9) provides a–priori bounds for these norms.

As a first major result we can use the fact that the weight ρ in can be replaced by $T_y\rho$ for any $y\in I\!R$. Since all estimates depend on ρ only through ρ_0 and ρ_I we obtain from (2.3) estimates in $L^p_{\text{lu}}(\Omega)$:

$$\|u(t)\|^p_{p,\text{lu}} \leq e^{-2\widetilde{R}t}\|u(0)\|^p_{p,\text{lu}} + (1-e^{-2\widetilde{R}t})\widetilde{R}^{p/2}\rho_I.$$

Together with (2.9) we obtain the following a–priori bounds

$$\begin{array}{ll}
\limsup_{t\to\infty}\|u(t)\|^p_{p,\text{lu}} \leq \left(R+\frac{\rho_0^2(1+\alpha^2)}{2(p-(p-2)\sqrt{1+\alpha^2})}\right)^{p/2}\rho_I & \text{for } p\in\left[2,2+\frac{2}{\sqrt{1+\alpha^2}-1}\right); \\
\limsup_{t\to\infty}\|u(t)\|^4_{4,\text{lu}} \leq C_1(\alpha,\beta)(R^2+\rho_0^2)\rho_I & \text{for } (\alpha,\beta)\in\operatorname{int}(\Lambda(1)); \\
\limsup_{t\to\infty}\|\nabla u\|^2_{2,\text{lu}} \leq C_2(\alpha,\beta)(R^2+\rho_0^2)\rho_I & \text{for } (\alpha,\beta)\in\operatorname{int}(\Lambda(1)).
\end{array}$$

3 Scalings and optimal growth rates

The estimates of the previous section work for all strong solutions. However, in order to show that these solutions do not blow up in finite time we have to control the $L^\infty(\Omega)$–norm of $u(t,\cdot)$. In dimension $d=1$ it is clear that the $W^{1,2}_{\text{lu}}(\Omega)$–bound for $(\alpha,\beta)\in\operatorname{int}(\Lambda(1))$ implies an L^∞–bound. In [MS95] an alternative method was introduce in order to derive bounds in $H^1_{\text{lu}}(\Omega)$ for all (α,β), see Theorem 3.1 below. However, for higher dimensions it is necessary to use a scale of the spaces $L^{p_m}_{\text{lu}}(\Omega)$ for $p_m\in[1,\infty)$, see Section 4.

Before we go into the construction of global strong solutions, we want to show how the scaling properties of CGL can be exploited if we are able to prove a–priori bounds independent of the underlying spatial domain $\Omega\subset I\!R^d$. Consider for $\ell>0$ the rescaling

$$(t,x,u,R,\alpha,\beta)\to(\ell^{-2}t,\ell^{-1}x,\ell u,\ell^2R,\alpha,\beta), \tag{3.10}$$

which leaves CGL invariant. Since α and β are not changed we may adjust ℓ in order to make $\widehat{R}=\ell^2R$ arbitrarily, e.g. less or equal to 1. However, this is done on the expense of scaling Ω. Previous work [BC*90, LO96] uses the unit cube $\Omega_{\text{uc}}=(0,1)^d$ with periodic boundary conditions and keeps R as the large instability parameter. It is shown in [BC*90], that for $(\alpha,\beta)\in\mathcal{P}(d)$ (see (1.2)) the estimates

$$\limsup_{t\to\infty}\|u(t,\cdot)\|_\infty \leq C(\alpha,\beta,d)R^{e(\alpha,\beta,d)} \tag{3.11}$$

hold, where $e(\alpha,\beta,d) > 1/2$, cf. [BC*90], pp. 428–430.

Our method immediately implies that (3.11) is valid with $e(\alpha,\beta,d) \equiv 1/2$ for all $(\alpha,\beta) \in \mathcal{P}(d)$. The argument is as follows: we scale the functions $u \in L^p(\Omega_{\rm uc})$ corresponding to a given R by using the scaling (3.10) with $\ell = R^{-1/2}$ to the case $\widehat{\Omega}_\ell = (0,1/\ell)^d$ and $\widehat{R} = 1$. The results in the next section imply a bound $\limsup_{\widehat{t}\to\infty} \|\widehat{u}(\widehat{t})\|_\infty \le \widehat{C}(\alpha,\beta,d)$ which is independent of ℓ. Using the inverse scaling the result is established, since $\|u\|_{L^\infty(\Omega_{\rm uc})} = R^{1/2}\|\widehat{u}\|_{L^\infty(\widehat{\Omega}_\ell)}$.

For the case $d = 1$ we make this more precise by the following result which partly relies on a method in [MS96] and which is repeated in Appendix B.

Theorem 3.1
Let $\Omega = I\!R$, then each solution of CGL satisfies

$$\limsup_{t\to\infty} \|u(t)\|_\infty \le C(1 + |\alpha|^{1/2} + \sigma^{1/2} + |\alpha|\sigma^{1/2})R^{1/2} \tag{3.12}$$

where $\sigma = \max\{0, \sqrt{1+\beta^2} - 3\}$ and C is a numerical constant. Moreover, in the case $\alpha\beta > 0$ we have the better estimate

$$\limsup_{t\to\infty} \|u(t)\|_\infty \le C(|\alpha|^{1/2} + |\beta/\alpha|^{1/3} + |\alpha/\beta|^{1/4} + |\alpha^7\beta^5|^{1/24})R^{1/2} \tag{3.13}$$

Proof: With Theorem B.1 below we have the estimates

$$\begin{array}{rcl} \limsup_{t\to\infty} \|u(t)\|_{\rm lu}^2 & \le & C\Delta_0 \ \text{ with } \Delta_0 = R + 1 + \alpha^2, \\ \limsup_{t\to\infty} \|\partial_x u(t)\|_{\rm lu}^2 & \le & C\Delta_1 \ \text{ with } \Delta_1 = \Delta_0^2(1 + \sigma + \sigma^2\Delta_0). \end{array}$$

Using $\Delta_0 \le \Delta_1$ and the one–dimensional Sobolev embedding (A.2) in the case $p = 2$ we find

$$\limsup_{t\to\infty} \|u(t)\|_\infty \le C(\Delta_0\Delta_1)^{1/4} = C\,\widetilde{\Delta}_\infty(\alpha,\beta,R).$$

However, this result was derived with a fixed weight $\rho(x) = e^{-|x|}$ with $\rho_0 = 1$. Using the rescaling, the proper estimate has the form $\Delta_\infty(\alpha,\beta,R) = \Sigma(\alpha,\beta)R^{1/2}$ and we find $\Sigma(\alpha,\beta) \le \inf\{ R^{-1/2}\widetilde{\Delta}_\infty(\alpha,\beta,R) \ : \ R > 0 \}$ which yields the desired result (3.12).

In the case $\alpha\beta > 0$ we improve on these results by the estimates for F_δ which were derived in Section 2. We consider (2.9) for the parameters $\delta = \beta/\alpha$, $\gamma = 0$, $\mu = 1 + \delta$, $\kappa = 0$, and the weight $\rho(x) = e^{-|x|}$. Then, we have

$$\limsup_{t\to\infty} F_\delta(t) \le C\Delta_F \ \text{ with } \Delta_F = (1+\delta)^{1/2}\Delta_0(\delta^{1/2}R + \Delta_0 + m),$$

where $m = |\alpha/\beta|(1 + |\beta/\alpha| + \beta^2)^2$.

The bounds $\limsup \|u(t)\|_{2,\rm lu} \le C\Delta_0^{1/2}$, $\limsup \|u(t)\|_{4,\rm lu} \le C(\Delta_F/\delta)^{1/4}$, $\limsup \|\partial_x u(t)\|_{2,\rm lu} \le C\Delta_F^{1/2}$, and $\|u\|_{3,\rm lu}^3 \le \|u\|_{2,\rm lu}\|u\|_{4,\rm lu}^2$ are now inserted into the Sobolev embeddings (A.2) for $p = 2$ and $p = 3$, respectively, resulting in

$$\limsup_{t\to\infty} \|u(t)\|_\infty \le C\min\{(\Delta_0\Delta_F)^{1/4}, \delta^{-1/6}\Delta_F^{1/3}\},$$

where $\Delta_0 \leq \Delta_F$ was used. Employing the scaling to each of the terms in the minimum delivers

$$\limsup_{t\to\infty} \|u(t)\|_\infty \leq C\min\Big\{(1+\delta)^{1/8}[(1+\delta)^{1/2}(1+\alpha^2)+m]^{1/4}, \\ (1+\delta)^{-1/6}[(1+\delta)^{1/2}(1+\alpha^2)+m]^{1/3}\Big\},$$

which is the desired bound (3.13). □

Note that the bound (3.13), which holds for $\alpha\beta > 0$, is much better than the general result (3.12) for $|\beta|$ large. Thus, we find a principal difference between the cases $\alpha\beta \geq 0$ and $\alpha\beta < 0$ in the bounds for $\|u(t)\|$ as a function of α and β, however we do not know whether these bounds are optimal. The only existing lower bound is $\sqrt{R}$ which is attained by rotating waves of the form $u(t,x) = \sqrt{R-|k|^2}e^{i(\omega t + k\cdot x)}$.

Considering Sobolev norms rather than the L^∞ norm we need a little more care, since the norm $\|\cdot\|_{p,\mathrm{lu}}$ is not scaling invariant (when the weight is not scaled). For simplicity we restrict the argument to the unit cube $\Omega_{\mathrm{uc}} = (0,1)^d$, the generalization is straight forward. In Appendix C, see (C.2), we establish, for weights with $|\nabla\rho(x)| \leq \rho_0\rho(x)$, the estimate

$$\int_{\Omega_{\mathrm{uc}}} |u|^p\, dx \leq C_d(1+\rho_0^d)\|u\|^p_{p,\mathrm{lu}}. \tag{3.14}$$

Assume now that the case $\widehat{R} = 1$ with weight $\widehat{\rho}(x) = e^{-|x|}$ yields a bound

$$\limsup_{\widehat{t}\to\infty} \|\widehat{u}(t)\|_{p,\mathrm{lu}} \leq \widehat{C}_p(\alpha,\beta,\widehat{R}=1,d)$$

independent of the domain. Then, a bound for $u(t) \in L^p(\Omega_{\mathrm{uc}})$ can be obtained from $\|u\|^p_{p,\rho} = R^{(p-d)/2}\|\widehat{u}\|^p_{p,\widehat{\rho}}$ where $\widehat{u}(\widehat{x}) = R^{-1/2}u(R^{-1/2}\widehat{x}) \in L^p((0,R^{-1/2})^d)$ only if the weight ρ satisfies $\rho(x) = \widehat{\rho}(\sqrt{R}x)$. Together with 3.14 we conclude that if global bounds exist they have the form

$$\limsup_{t\to\infty} \int_{\Omega_{\mathrm{uc}}} |u(t,x)|^p dx \leq C_p(\alpha,\beta,d)\, R^{(p-d)/2}(1+R^{d/2}).$$

Especially, for the functionals $\overline{F}_{(n)}(t) = \int_{\Omega_{\mathrm{uc}}}(|\nabla^{n-1}u(t)|^2 + a_n|u|^{2n})dx$, which were used heavily in [BC*90], we find

$$\limsup_{t\to\infty} \overline{F}_{(n)}(t) \leq \overline{C}_{(n)}(\alpha,\beta,d)\, R^n(1+R^{-d/2})$$

and again our growth rate R^n is optimal, independent of (α,β,d), and lies below the ones in [BC*90]. There might still be essential differences between different regions in the (α,β) plane; yet these differences can only occur in the constant $\overline{C}_{(n)}(\alpha,\beta,d)$ as functions of α and β.

From the above considerations it became clear that it is very helpful to have a–priori bounds which are independent of the underlying domain Ω. Only then it is possible to exploit the scaling invariance of CGL properly. Thus, we have to define a suitable set of domains which allows for these uniform estimates. As we have to do also regularity theory these domains must have a uniformly $\mathcal{C}^2$–smooth boundary. More precisely, we define the set of admissable $\Omega \subset I\!R^d$ as follows.

Definition 3.2
A domain $\Omega \subset I\!R^d$ and boundary conditions are called admissable, if (a) or (b) holds:

- *(a) Ω has a $\mathcal{C}^2$–boundary and for each point $x \notin \partial\Omega$ there exists a ball B of radius 1 such that $x \in B$ and $B \cap \partial\Omega = \emptyset$. On each connected component of $\partial\Omega$ we have either Dirichlet conditions ($u = 0$) or Neumann condition ($\nabla u \cdot \nu = 0$).*
- *(b) Ω is of the form $R_n \times \widetilde{\Omega}$ with $R_n = (a_1, b_1) \times \cdots \times (a_n, b_n)$ where $n \leq d$ and $\widetilde{\Omega} \subset I\!R^{d-n}$ is as in (a). The boundary conditions on $R_n \times \partial\widetilde{\Omega}$ are as in (a) while those on $\partial R_n \times \widetilde{\Omega}$ are periodicity in x_i, $i = 1, \ldots, n$ with period $b_i - a_i$, respectively.*

The condition in (a) involving the ball of radius 1 guarantees that the uniform cone condition holds from inside and outside, moreover it bounds the curvature of the boundary. Note that stretching (i.e. $\ell \in (0, 1)$ in (3.10)) leaves the set of admissable domains invariant. Hence, further on we may restrict the parameter R to the interval $(0, 1]$ without loss of generality. All estimates in subsequent sections will depend only on the parameters (α, β) but not on the admissable domain Ω nor on the instability parameter $R \in (0, 1]$.

4 Regularity and global existence

We use the following result of [LO96], Thm. 5.5, which builds on the theory in [Wei80].

Theorem 4.1
Let X, Y, and Z be Banach spaces with $Y \subset Z \subset X$ and let $(e^{At})_{t\geq 0}$ be a holomorphic semigroup on X with

$$\|e^{At}u\|_Y \leq Ct^{-\gamma}\|u\|_X \text{ for all } u \in X, \quad \|e^{At}u\|_Y \leq Ct^{-\delta}\|u\|_Z \text{ for all } u \in Z$$

for $t \in (0, 1]$. Moreover, let $N : Y \to Z$ be a nonlinear mapping with

$$\|N(u_1) - N(u_2)\|_Z \leq C(\|u_1\|_Y^\sigma + \|u_2\|_Y^\sigma)\|u_1 - u_2\|_Y \text{ for all } u_1, u_2 \in Y.$$

Assume that the relations $0 \leq \delta < 1$, $0 \leq (\sigma + 1)\gamma < 1$, $\delta + \gamma\sigma < 1$ hold, then for each $M > 0$ there exists a time $T > 0$ such that the integral equation

$$u(t) = e^{At}u_0 + \textstyle\int_0^t e^{A(t-\tau)}N(u(\tau))\,d\tau$$

has for each initial condition $u_0 \in X$ with $\|u_0\|_X \leq M$ a unique solution $u \in \mathcal{C}([0, T], X) \cap \mathcal{C}((0, T], Y)$. Moreover, the mapping from u_0 to $u(\cdot)$ is locally Lipschitz continuous from X to $\mathcal{C}([0, T], X)$.

In order to apply this result we establish the existence of the semigroup $(e^{At})_{t\geq 0}$ for the linear part of CGL, i.e. $Au = (1+i\alpha)\Delta u + Ru$.

Theorem 4.2
Let $p \in [2,\infty)$ and define the operator $A : D(A) \subset L^p_{\rm lu}(\Omega) \to L^p_{\rm lu}(\Omega)$ with

$$Au = (1+i\alpha)\Delta u + Ru, \quad D(A) = W^{2,p}_{\rm lu}(\Omega) \cap \{\textit{boundary conditions}\}.$$

Then, there exist functions $C_0(\alpha)$ and $C_1(\alpha,p)$ such that for each z in the sector

$$S(\alpha,\widetilde{R},p) = \{\, z \in \mathbb{C} \,:\, z \neq \widetilde{R},\ |\arg(z-\widetilde{R})| \leq \tfrac{3\pi}{4} - \tfrac{1}{2}|\arg(1+i\alpha)|\,\} \ \textit{ with } \widetilde{R} = R + C_0(\alpha)\rho_0^2,$$

and for all admissable domains Ω the resolvent $(A-z)^{-1} : L^p_{\rm lu}(\Omega) \to D(A)$ exists and satisfies the estimate

$$\|(A-z)^{-1}f\|_{p,\rm lu} \leq \frac{C_1(\alpha,p)}{|z-\widetilde{R}|}\|f\|_{p,\rm lu} \ \textit{for all } f \in L^p_{\rm lu}(\Omega).$$

Proof: It is sufficient to show that the Laplace operator $A_0 : u \mapsto \Delta u$ has it resolvent set in $\mathbb{C} \setminus (-\infty, b]$ for some b and satisfies the analogous estimate on each sector $S(b,\theta) = \{\, z \in \mathbb{C} \,:\, |\arg(z-b)| \leq \theta \,\}$ with $\theta \in [0,\pi)$.

We use Agmon's trick [AN63] which uses regularity theory on a higher dimensional domain $\widetilde{\Omega}$. Let $\widetilde{x} = (x,\xi) \in \widetilde{\Omega} = \Omega \times \mathbb{R}$, $\widetilde{\rho}(\widetilde{x}) = \rho(x)/\cosh(\rho_0\xi)$, and define $\widetilde{\Delta}_\gamma \widetilde{u} = \Delta\widetilde{u} + e^{i\gamma}\,\partial_\xi^2\widetilde{u} - b\widetilde{u}$ for $\widetilde{u} \in W^{2,p}_{\rm lu}(\widetilde{\Omega})$, where $|\gamma| \leq \theta/2 < \pi/2$ and $b = (\rho_0/\cos(\theta/2))^2$. Then, $-\widetilde{\Delta}_\gamma\widetilde{u} = \widetilde{f}$ is equivalent to the weak form

$$B_\gamma(\widetilde{u},\widetilde{v}) = \textstyle\int_{\widetilde{\Omega}} \widetilde{\rho}\,\widetilde{f}\,\overline{\widetilde{v}}\,dx \ \text{ for all } \widetilde{v} \tag{4.15}$$

where $B_\gamma(\widetilde{u},\widetilde{v}) = \int_{\widetilde{\Omega}} \widetilde{\rho}\{\nabla\widetilde{u}\cdot[\nabla\overline{\widetilde{v}} + (\overline{\widetilde{v}}/\widetilde{\rho})\nabla\widetilde{\rho}] + e^{i\gamma}\partial_\xi\widetilde{u}[\partial_\xi\overline{\widetilde{v}} + (\overline{\widetilde{v}}/\widetilde{\rho})\partial_\xi\widetilde{\rho}] + b\,\widetilde{u}\,\overline{\widetilde{v}}\}d\widetilde{x}$. We have coercivity of B_γ because of

$$\begin{aligned} |B_\gamma(\widetilde{u},\widetilde{u})| &\geq \big|\,\|\nabla\widetilde{u}\|^2_{\widetilde{\rho}} + e^{i\gamma}\|\partial_\xi\widetilde{u}\|^2_{\widetilde{\rho}} + b\|\widetilde{u}\|^2_{\widetilde{\rho}}\,\big| - \rho_0\|\widetilde{u}\|_\rho\|\widetilde{\nabla}\widetilde{u}\|^\rho \\ &\geq \tfrac{1}{2}\cos\tfrac{\theta}{2}(\|\nabla\widetilde{u}\|^2_{\widetilde{\rho}} + 2\|\partial_\xi\widetilde{u}\|^2_{\widetilde{\rho}} + \widehat{b}\|\widetilde{u}\|^2_{\widetilde{\rho}}), \end{aligned}$$

where $|\gamma| \leq \theta/2$ and with $\widehat{b} = 2b - (\rho_0/\cos\frac{\theta}{2})^2 = b$ was used.

For any $\widetilde{f} \in L^2_{\rm lu}(\widetilde{\Omega})$ the Lax–Milgram theorem can be applied in the weighted Hilbert space $W^{1,2}_{\widetilde{\rho}}$, where

$$L^2_{\widetilde{\rho}}(\widetilde{\Omega}) = \{\, \widetilde{u} \in L^2_{\rm loc}(\widetilde{\Omega}) \,:\, \|\widetilde{u}\|_{\widetilde{\rho}} < \infty \,\}, \quad W^{1,2}_{\widetilde{\rho}}(\widetilde{\Omega}) = \{\, \widetilde{u} \in L^2_{\widetilde{\rho}}(\widetilde{\Omega}) \,:\, \nabla\widetilde{u} \in L^2_{\widetilde{\rho}}(\widetilde{\Omega}) \,\}.$$

This yields a unique solution $\widetilde{u} \in W^{1,2}_{\widetilde{\rho}}(\widetilde{\Omega})$, however, since $\widetilde{\rho}$ can be replaced by all its translates without losing the estimates we conclude $\widetilde{u} \in W^{1,2}_{\rm lu}(\widetilde{\Omega})$. By classical regularity theory we have $\widetilde{u} \in W^{2,2}_{\rm lu}(\widetilde{\Omega})$ with $\|\widetilde{u}\|_{2,\rm lu} \leq C\|f\|_{0,\rm lu}$, where C does not depend on u nor on the admissable domain Ω. Here we use that the regularity theory can be made uniform in the set of admissable domains if the uniform cone condition holds and the curvature of the boundary is uniformly bounded.

For $p \geq 2$ we use $L^p_{\rm lu}(\widetilde{\Omega}) \subset L^2_{\rm lu}(\widetilde{\Omega})$ and obtain again a unique solution in $W^{2,2}_{\rm lu}(\widetilde{\Omega})$ which again by classical regularity arguments satisfies

$$\|\widetilde{u}\|_{2,p,{\rm lu}} \leq C(\theta,p)\|\widetilde{f}\|_{p,{\rm lu}}. \tag{4.16}$$

We now return to our original problem on Ω. In order to solve $\Delta u - zu = f \in L^p_{\rm lu}(\Omega)$ we write $z \in S(b,\theta)$ as $z = b+\omega^2 e^{i2\gamma}$ with $|\gamma| \leq \theta/2$. Moreover, we let $\widetilde{f}(x,\xi) = e^{i\omega\xi} f(x) \in L^p_{\rm lu}(\widetilde{\Omega})$ and find that the unique solution $\widetilde{u}$ which must have the form $\widetilde{u}(x,\xi) = e^{i\omega\xi}u(x)$. A simple calculation shows $\Delta u - zu = f$ and (4.16) yields

$$\|u\|_{W^{2,p}_{\rm lu}(\Omega)} + |\omega|\,\|u\|_{W^{1,p}_{\rm lu}(\Omega)} + \omega^2\|u\|_{L^p(\Omega),{\rm lu}} \leq C_1\|\widetilde{u}\|_{W^{2,p}_{\rm lu}(\widetilde{\Omega})} \leq C_2\|\widetilde{f}\|_{L^p_{\rm lu}(\widetilde{\Omega})} = C_3\|f\|_{L^p_{\rm lu}(\Omega)}.$$

Recalling $\omega^2 = |z-b|$ the desired estimate is established. □

Theorem 4.2 proves exactly the assumptions that are needed to show that $A = (1+i\alpha)\Delta + R$ is the generator of an analytic semigroup $(e^{At})_{t\geq 0}$, such that

$$\|e^{At}u\|_{W^{s,p}_{\rm lu}(\Omega)} \leq C\left(1+\frac{1}{t^{s/2}}\right)e^{bt}\|u\|_{L^p_{\rm lu}(\Omega)}$$

for $t>0$, $s\in\{0,1,2\}$, $p\in[2,\infty)$, and all u; see [Kat76].

We use the Sobolev embeddings in the Gagliardo–Nirenberg form (see Appendix A) to show that e^{At} maps $W^{s,p}_{\rm lu}(\Omega)$ into $W^{s,q}_{\rm lu}(\Omega)$ for $q>p$:

$$\|e^{At}u\|_{s,q,{\rm lu}} \leq C\,\|e^{At}u\|^{\theta}_{s,p,{\rm lu}}\,\|e^{At}u\|^{1-\theta}_{s+1,p,{\rm lu}} \leq C\,t^{-(d/p-d/q)/2}\|u\|_{s,p,{\rm lu}} \text{ for } t\in(0,1], \tag{4.17}$$

where $s=\{0,1\}$ and the Gagliardo–Nirenberg interpolation needs $\theta = 1-d/p+d/q$, see Appendix (A). Similarly, for $r\in[dp/(p+d),p]$ we have

$$\|e^{At}u\|_{s+1,r,{\rm lu}} \leq C\,\|e^{At}u\|^{\theta}_{s,p,{\rm lu}}\,\|e^{At}u\|^{1-\theta}_{s+1,p,{\rm lu}} \leq C\,t^{-(1-\theta)/2}\|u\|_{s,p,{\rm lu}} \text{ for } t\in(0,1], \tag{4.18}$$

where $\theta = d/r-d/p$ and $s\in\{0,1\}$.

With $N(u) = -(1+i\beta)|u|^2u$ all strong solutions of CGL satsify the variations of the constants formula

$$u(t) = e^{A(t-t')}u(t') + \int_{t'}^{t} e^{A(t-\tau)}N(u(\tau))\,d\tau.$$

The above to theorems can now be used to improve a–priori bounds in p to bounds in q. To this end we apply Theorem 4.1 with $X = L^p_{\rm lu}(\Omega)$, $Y = L^q_{\rm lu}(\Omega)$, $Z = L^{q/3}_{\rm lu}(\Omega)$, and $\sigma = 2$. Using (4.17) we find $\gamma = (d/p-d/q)/2$ and $\delta = d/q$, which satisfies $\delta\in[0,1)$, $\gamma\in[0,\frac{1}{3})$, and $\delta+2\gamma<1$ if and only if $p>d$ and $q\in[p,p^*)$ where $p^* = 3pd/(3d-2p)$ for $p\in(d,3d/2)$ and $p^*=\infty$ else. Thus, having a suitable starting value $p_0>d$ we can iteratively increase $p_{m+1}=q_m=3p_m$ to reach any value $p\in(d,\infty)$.

If $p>3d/2$ is reached we use Theorem 4.1 again with $X = L^p_{\rm lu}(\Omega)$, $Y = Z = W^{1,r}_{\rm lu}(\Omega)$, $\sigma=2$, $\delta=0$, and $\gamma = (1-d/r+d/p)/2$, according to (4.18) with $s=0$. Here we need $r>d$ in order to have the Lipschitz condition for N, and the remaining condition $\gamma<1/3$ is satisfied whenever $r<3dp/(p+3d)$. As in the L^p–case we can now improve bounds in

$W^{1,r}$ with $r = r_m$ to bound with $r = 3r_m$, and therefore reach any $r \in (1, \infty)$. In a final step the theorem is used with $X = W^{1,r}_{\rm lu}(\Omega)$, where $r > 3d/2$, $Y = Z = W^{2,d}_{\rm lu}(\Omega)$, $\sigma = 2$, $\delta = 0$, and $\gamma = d/(2r)$ (use (4.18) with $s = 1$ and (r,p) replaced by (d,r)).

As a conclusion we obtain the following result, which states the global existence of strong solutions.

Theorem 4.3
Let the parameters satisfy $(\alpha, \beta) \in \mathcal{P}(d)$ *with*

$$\mathcal{P}(1) = \mathcal{P}(2) = I\!R^2, \quad \mathcal{P}(3) = \{\, (\alpha,\beta) \in I\!R^2 \;:\; |\alpha| < \sqrt{8} \text{ or } -(1+\alpha\beta) < \sqrt{3}|\alpha - \beta| \,\},$$
$$\text{and } \mathcal{P}(d) = \{\, (\alpha,\beta) \in I\!R^2 \;:\; |\alpha| < 2\sqrt{d-1}/(d-2) \,\} \text{ for } d \geq 4.$$

Then, for each admissable domain $\Omega \subset I\!R^d$ *and boundary conditions, any* $p > d$*, and all initial conditions* $u^0 \in L^p_{\rm lu}(\Omega)$ *there is a unique strong global solution* $u(t) = S_t(u^0)$*.*

Moreover, fix the weight $\rho(x) = e^{-|x|}$*, then there is a constant* $\mathcal{C}(\alpha, \beta, d)$ *such that for all admissable domains* $\Omega \subset I\!R^d$ *and boundary conditions and all initial conditions* $u^0 \in L^p_{\rm lu}(\Omega)$ *we have*

$$\limsup_{t\to\infty} \|S_t(u^0)\|_{1,2d,{\rm lu}} \leq \mathcal{C}(\alpha, \beta, d).$$

Note that in the last estimate we have chosen $p = 2d > d$ without loss of generality.

This results enables us to construct an absorbing set in $W^{1,2d}_{\rm lu}(\Omega)$ as follows. Let $\mathcal{B}_0 = \{\, u \in W^{1,p}_{\rm lu}(\Omega) \;:\; \|u\|_{1,2d,{\rm lu}} \leq 2\mathcal{C}(\alpha,\beta,d) \,\}$, then

$$\mathcal{B}_{\rm abs}(\alpha,\beta,\Omega) = \bigcup_{t>0} S_t(\mathcal{B}_0) \subset W^{1,2d}_{\rm lu}(\Omega) \tag{4.19}$$

is a bounded, invariant set, since the union can also be aken over a finite time interval. The above estimates imply that every bounded set in $L^p_{\rm lu}(\Omega)$ with $p > d$ is absorbed in finite time into $\mathcal{B}_{\rm abs}$.

All the considerations in the subsequent sections will be restricted to this set. Moreover, we let

$$\begin{aligned} \mathcal{C}_{\rm abs}(\alpha,\beta,d) &= \sup\{\, \|u\|_{1,2d,{\rm lu}} \;:\; \exists\, \Omega \text{ admissable} : u \in \mathcal{B}_{\rm abs}(\alpha,\beta,\Omega) \,\}, \\ \mathcal{C}_{\infty}(\alpha,\beta,d) &= \sup\{\, \|u\|_{\infty} \;:\; \exists\, \Omega \text{ admissable} : u \in \mathcal{B}_{\rm abs}(\alpha,\beta,\Omega) \,\}, \end{aligned} \tag{4.20}$$

to have universal constants to estimate the norms in $\mathcal{B}_{\rm abs}$.

5 The influence of the boundary

In this section we are going to study some aspects of the dependence of the dynamics on the underlying physical domain $\Omega \subset I\!R^d$. In particular, we are concerned with large domains where, as is common believe, the boundary should not influence the dynamics away from the boundary too much. It is our aim to give a rigorous meaning to this rule of thumb.

To this end we consider two admissable domains Ω_1 and Ω_2 with associated boundary conditions as above. Moreover, we fix one $(\alpha, \beta) \in \mathcal{P}(d)$ of parameters, such that the semiflows $(S_t^j)_{t\geq 0}$ on $L^p_{\rm lu}(\Omega_j)$ are well defined. We want to compare the dynamics of the two systems on a subdomain Ω_* of the intersection $\Omega_1 \cap \Omega_2$. The main problem is that we cannot control the boundary values of $S_t^j(u)$ on $\partial\Omega_*$, as we should not use information on u from outside of Ω_*. The same problem even occurs if $\Omega_* = \Omega_1 = \Omega_2$ but $(S_t^1)_{t\geq 0}$ and $(S_t^2)_{t\geq 0}$ correspond to different boundary conditions. In this situation the theory in [Col94] is helpful, since it allows us to use weight functions which vanish at $\partial\Omega_*$. Previously, we used $|\nabla\rho(x)| \leq \rho_0\rho(x)$ in order to estimate $\int_\Omega |\nabla\rho|\,|\nabla u|\,|u|^2\,dx \leq \rho_0\|\nabla u\|_\rho \|u\|_\rho$. However, we may use that $u \in L^4_{\rm lu}(\Omega)$ and obtain by Hölder's inequality

$$\int_\Omega |\nabla\rho|\,|\nabla u|\,|u|\,dx \leq \|\nabla u\|_\rho \|(1/\rho)|\nabla\rho|\,|u|\|_\rho \leq \|\nabla u\|_\rho \|u\|_{4,\rho}\|(1/\rho)|\nabla\rho|\,\|_{4,\rho}.$$

Recall that $|\nabla\rho| \leq \rho_0\rho$ implies $\rho(x) \geq e^{-\rho_0|x|}\rho(0) > 0$, but for $\rho(x) = (1-|x|^2)^\gamma$ for $|x| \leq 1$ and $\rho(x) = 0$ else the weaker condition

$$\|(1/\rho)|\nabla\rho|\,\|^4_{4,\rho} = \int_\Omega \rho^{-3}|\nabla\rho|^4\,dx < \infty$$

is also satisfied whenever $\gamma > 3$. Our approach combines the feature of exponential decay on the one hand with the vanishing on $\partial\Omega_*$ on the other hand.

Throughout we consider the situation, that the ball $B(x_*, r_*)$ intersects the sets Ω_1 and Ω_2 in the same set Ω_*, i.e. $\Omega = \Omega_j \cap B(x_*, r_*)$ for $j = 1$ and 2. We let $\Gamma_* = \partial B(x_*, r_*) \cap \partial\Omega_*$ which is that part of the boundary which does not belong to $\partial\Omega_j$ for $j = 1, 2$. On each Ω_j there is defined a semigroup $(S_t^j)_{t\geq 0}$ on X^j which corresponds to the same (α, β). Moreover, if $\Gamma_{1,2} = \partial\Omega_* \setminus \Gamma_*$ is nonempty we assume that the boundary conditions associated to the semigroups (S_t^j) coincide on $\Gamma_{1,2}$.

We now define for $\rho_0 > 0$ the weight $\rho_* : \mathbb{R}^d \to [0, \infty)$ as

$$\rho_*(x) = \begin{cases} e^{-\rho_0|x-x_*|} & \text{for } |x-x_*| \leq r_* - 1, \\ (r_* - |x-x_*|)^{p+2} e^{-\rho_0|x-x_*|} & \text{for } |x-x_*| \in [r_*-1, r_*], \\ 0 & \text{for } |x-x_*| \geq r_*. \end{cases}$$

For $x \in B(x_*, r_*)$ we have the estimate

$$|\nabla\rho_*(x)|/\rho_*(x) \leq \rho_0 + (p+2)m_*(x)/(r_* - |x-x_*|), \tag{5.21}$$

where $m_*(x) = 1$ for $|x - x_*| \in (r_* - 1, r_*)$ and 0 else. All subsequent considerations, except of Remark 5.3, will be done for the case $\rho_0 = 1$. As an important feature we will use that for $u_j(t) = S_t^j(u_j^0)$ the difference $w(t) = u_2(t)|_{\Omega_*} - u_1(t)|_{\Omega_*}$ is well–defined in $L^p(\Omega_*)$ such that all partial integration hold without boundary terms, e.g.,

$$\int_{\Omega_*} \rho_*\,\overline{w}\,\Delta w\,dx = -\int_{\Omega_*} \rho_*|\nabla w|^2\,dx - \int_{\Omega_*} \overline{w}\,\nabla\rho_*\,\nabla w\,dx. \tag{5.22}$$

Proposition 5.1
Assume $(\alpha,\beta)\in\mathcal{P}(d)$ *and* $(p-2)|\alpha|<\sqrt{2d-1}$. *Then, there is a positive constant* $C=C(\alpha,\beta,d)$ *such that for all* $u_j^0\in\mathcal{B}_{\rm abs}(\alpha,\beta,\Omega_j)$ *the estimate*

$$\|S_t^1(u_1^0)-S_t^2(u_2^0)\|_{p,\rho_*}\le e^{Ct}\left(\|u_1^0-u_2^0\|_{p,\rho_*}+CD_*^{1/p}\right)$$

holds, where $D_*=r_*^{d-1}e^{-r_*}$ *and* $\|w\|_{p,\rho_*}^p=\int_{\Omega_*}\rho_*(x)|w(x)|^p dx$. *Here the constant* C *is independent of the domains* Ω_j *and of* r_*.

Remark 5.2 We control the difference between $S_t^1(u_1)|_{\Omega_*}$ and $S_t^1(u_2)|_{\Omega_*}$ solely by the difference of the initial conditions on Ω_* and some constant term which must dominate all possible influences from the solutions on $\Omega_j\setminus\Omega_*$.

Proof: We let $u_j(t)=S_t^j(u_j)$ and $w(t)=u_2(t)|_{\Omega_*}-u_1(t)|_{\Omega_*}$. For $N(u)=-(1+i\beta)|u|^2u$ we have

$$\mathrm{Re}\,\overline{w}[N(u_1+w)-N(u_1)]\le 5(1+\beta^2)|u_1|^2|w|^2-\frac{1}{2}|w|^4$$

by elementary estimates. Thus, the weighted L^p–energy estimate applied to $\partial_t w=\partial_t u_2-\partial_t u_1=(1+i\alpha)\Delta w+Rw+N(u_1+w)-N(u_1)$ gives as in Section 2

$$\begin{aligned}\tfrac{1}{p}\tfrac{d}{dt}\|w\|_{p,\rho_*}^p &\le -Q_{\alpha,p}(w)+\sqrt{1+\alpha^2}\textstyle\int_{\Omega_*}|w|^{p-1}|\nabla\rho_*||\nabla w|\,dx\\ &\quad +R\|w\|_{p,\rho_*}^p+\textstyle\int_{\Omega_*}\rho_*|w|^{p-2}\overline{w}(N(u_1+w)-N(u_1))\,dx\\ &\le \tfrac{1+\alpha^2}{4\lambda_Q}\textstyle\int_{\Omega_*}\rho_*|w|^p|\nabla\rho_*|^2/\rho_*^2\,dx\\ &\quad +(R+5(1+\beta^2)\|u_1\|_\infty^2)\|w\|_p^p-\tfrac{1}{2}\|w\|_{p+2}^{p+2}.\end{aligned}$$

Using (5.21) with $\rho_0=1$ the integral involving $\nabla\rho_*$ can be estimated by

$$\begin{aligned}&2\|w\|_{p,\rho_*}^p+2(p+2)^2\textstyle\int_{\Omega_*}\rho_*|w|^p\left(\frac{m_*}{r-|x-x_*|}\right)^2dx\\ &\le 2\|w\|_{p,\rho_*}^p+2(p+2)^2\|w\|_{p+2,\rho_*}^p\left(\textstyle\int_{\Omega_*}e^{-|x-x_*|}m_*\,dx\right)^{2/(p+2)}\\ &\le 2\|w\|_{p,\rho_*}^p+\tfrac{1}{2}\|w\|_{p+2}^{p+2}+C_0(p,d)D_*.\end{aligned}$$

Together with $\lambda_Q(\alpha,p)>0$ we find $\frac{d}{dt}\|w\|_{p,\rho_*}^p\le C(1+\mathcal{C}_\infty^2)\|w\|_{p,\rho_*}^p+CD_*$, which yields the result by using Gronwall's estimate. □

Remark 5.3 In order to illustrate the relevance of Proposition 5.1, we estimate the difference $S_t^1(u_1^0)|_{\Omega_*}-S_t^2(u_2^0)|_{\Omega_*}$ on the subdomain $\Omega_r=\Omega_*\cap B(x_*,r)$ for $r<r_*-1$. To see the full effect of the theory we note that the result of Proposition 5.1 can be generalized to the case of general ρ_0 easily, namely

$$\|S_t^1(u_1^0)-S_t^2(u_2^0)\|_{p,\rho_*}\le e^{C(1+\rho_0^2)t}\left(\|u_1^0-u_2^0\|_{p,\rho_*}+CD_*(\rho_0)^{1/p}\right),$$

where $D_*(\rho_0)=r_*^{d-1}e^{-\rho_0(r_*-1)}$ and C does not depend on ρ_0. Then, on Ω_r with weight $\rho\equiv 1$ we have

$$\begin{aligned}\|u_1(t)-u_2(t)\|_{L^p(\Omega_r)}^p &= \textstyle\int_{\Omega_r}|u_1(t,x)-u_2(t,x)|^p\,dx\\ &\le e^{\rho_0 r}\textstyle\int_{\Omega_r}e^{-\rho_0|x-x_*|}|u_1(t,x)-u_2(t,x)|^p dx\\ &\le e^{\rho_0 r}\|u_1(t)-u_2(t)\|_{p,\rho_*}^p\le e^{\rho_0 r}e^{C(1+\rho_0^2)\,t}(\|u_1^0-u_2^0\|_{p,\rho_*}^p+D_*(\rho_0)).\end{aligned}$$

If $\|u_1^0 - u_2^0\|_{p,\rho_*} = 0$ we may optimize the right-hand side with respect to ρ_0 which yields

$$\|u_1(t) - u_2(t)\|^p_{L^p(\Omega_r)} \leq C\, r_*^{d-1} e^{(C\,t - \frac{(r_* - r - 1)^2}{2Ct})} \text{ for } t > 0.$$

Thus, we have some control how fast the influence from the boundary Γ_* can penetrate into the interior of Ω_*. Note that the right–hand side attains the constant δ at the hyperbolas

$$2C^2t^2 + \mu t = (r_* - r - 1)^2 \text{ with } r \in (0, r_* - 1)$$

where $\mu = 2C \log(Cr_*^{d-1}/\delta)$.

Unfortunately, our result in Proposition 5.1 is not enough to cover the important case $d = 3$ properly. The restriction $(p-2)|\alpha| < 2\sqrt{p-1}$ would force us to use $p \leq 3 = d$ as soon as $|\alpha| \geq \sqrt{8}$. Thus, we are not able to use these results together with the global existence results. Like in the case of the a–priori estimates the remaining part of $\{ -(1+\alpha\beta) < \sqrt{3}|\alpha - \beta| \}$ can not be covered by doing the L^p–estimate alone. We believe that it is possible for those (α, β) to derive similar estimates by using a functional $G_\delta(t) = \int_{\Omega_*} \rho_* \{ \frac{1}{2}|\nabla w|^2 + \frac{\delta}{4}|w|^4 \}\, dx$, yet we leave this for future research.

Instead we derive a weaker estimate by interpolation. For any (α, β) we have a good L^2–estimate for w by Proposition 5.1 and in L^∞ we have a rough bound by $2\mathcal{C}_\infty$. Note that we will lose Lipschitz continuity by this procedure, but the result is still be sufficient to establish estimates for pseudo–orbits and the distance between attractors.

Theorem 5.4
Let $(\alpha, \beta) \in \mathcal{P}(d)$. Then for every $p \geq 2$ there is a constant $C = C(\alpha, \beta, p, d)$ such that for any Ω_1 and Ω_2 as above and all $u_j^0 \in \mathcal{B}_{\text{abs}}(\alpha, \beta, \Omega_j)$ we have the estimate

$$\|S_t^1(u_1^0) - S_t^1(u_2^0)\|_{p,\rho_*} \leq Ce^{Ct}(\|u_1^0 - u_2^0\|_{p,\rho_*}^{2/p} + D_*^{1/p})$$

for all $t > 0$.

Proof: The case $p = 2$ is proved in Proposition 5.1. For $p > 2$ we use the obvious relations $\|w\|_{p,\rho_*}^p \leq \|w\|_\infty^{p-2}\|w\|_{2,\rho_*}^2$ and $\|w\|_{2,\rho_*} \leq \|1\|_{p/(p-2),\rho_*}\|w\|_{p,\rho_*}$, where $\|1\|_{p/(p-2),\rho_*} \leq \rho_I^{(p-2)/p}$. According to Section 4 we have for $w(t) = S_t^1(u_1^0) - S_t^1(u_2^0)$ the bound $\|w\|_\infty \leq \|S_t^1(u_1^0)\|_\infty + \|S_t^2(u_2^0)\|_\infty \leq 2\mathcal{C}_\infty$. Thus, employing the result for $p = 2$ yields

$$\|w(t)\|_{p,\rho_*}^p \leq (2\mathcal{C}_\infty)^{p-2}\|w(t)\|_{2,\rho_*}^2 \leq C_1e^{Ct}(\|w(0)\|_{2,\rho_*}^2 + D_*) \leq C_2e^{Ct}(\|w(0)\|_{p,\rho_*}^2 + D_*),$$

which gives the desired estimate after taking the p^{th} root. □

Now we want to show that the dynamics of two systems (S_t^1) and (S_t^2), which coincide on $\Omega_* = \Omega_j \cap B(x_*, r_*)$, are close to each other. To this end we introduce the notion of pseudo–orbits, c.f. also [Sch94].

Definition 5.5
Let (S_t) be a semigroup on the normed space $(X, \|\cdot\|)$, then a function $u : [0,\infty) \to X$ is called a (T,δ) pseudo–orbit if for each $n \in I\!N$ and $\tau \in [0,T)$ the relations

$$u((n-1)T+\tau) = S_\tau(u((n-1)T)) \text{ and } \|u(nT) - S_T(u((n-1)T))\| \le \delta$$

hold.

Thus, we have true orbits on $[(n-1)T, nT)$ with jumps of maximal size δ at $t_n = nT$. A more general notion could allow for irregular jump time t_n, as long as $t_{n+1} - t_n \ge T$. Our next result states that orbits $u_1(t) = S_t^1(u_1^0)$ can be well approximated in the domain Ω_* by (T,δ) pseudo–orbits of (S_t^2) if the radius r_* is sufficiently large. Of course, we use the weighted norm $\|\cdot\|_{p,\rho_*}$ which measures differences in the middle of Ω_* much stronger than those close to Γ_*.

Theorem 5.6
Let $(\alpha,\beta) \in \mathcal{P}(d)$, $p \ge 2$, and let the positive numbers T, δ and ε be given. Then, there is an $\widehat{r}_* > 0$ such that the following holds: if $r_* \ge \widehat{r}_*$ and $\Omega_1 \cap B(x_*, r_*) = \Omega_2 \cap B(x_*, r_*)$ is valid in addition to the above assumptions on Ω_*, then for every $u_1^0 \in \mathcal{B}_{\rm abs}(\alpha,\beta,\Omega_1)$ there exists a (T,δ) pseudo–orbit $u_2 : [0,\infty) \to (L^{2d}_{\rm lu}(\Omega_2), \|\cdot\|_{2d,\rho})$ for (S_t^2) (i.e. $\|S_T^2(u_2((n-1)T)) - u_2(nT)\|_{2d,\rho} \le \delta$) such that $\|u_1(t) - u_2(t)\|_{2d,\rho_*} \le \varepsilon$ for all $t \ge 0$.

Proof: We define the mapping $E_* : L^{2d}_{\rm lu}(\Omega_j) \to L^{2d}_{\rm lu}(\Omega_k)$ as $(E_*u)(x) = \min\{1, r_*+1-|x-x_*|\}u(x)$ and $(E_*u)(x) = 0$ elsewhere. Obviously, E_* has norm 1, and

$$\|(I-E_*)u_j\|_{2d,\rho} \le C r_* e^{-r_*/(2d)} \tag{5.23}$$

where $\|u_j\|_{2d,{\rm lu}} \le \mathcal{C}_{\rm abs}$ and the estimate (C.3) with $\widetilde{\Omega} = \Omega_j \setminus B(x_*, r_*)$ was used.

The pseudo–orbit u_2 is simply defined by $u_2(nT) = E_* u_1(nT)$ for $n = 0,1,2,\ldots$ and it remains to prove the approximation and the jump condition. (Note that that $E_* : W^{1,2d}_{\rm lu}(\Omega_1) \to W^{1,2d}_{\rm lu}(\Omega_2)$ by a fixed constant.) Because of $\|u_1(nT) - u_2(nT)\|_{p,\rho_*} = 0$ for $n \in I\!N$, Theorem 5.4 is applicable on $[nT, (n+1)T)$ leading to $\|u_1(nT+\tau) - u_2(nT+\tau)\|_{p,\rho_*} \le Ce^{C\tau} D_*^{1/p}$. For sufficiently large r_* this yields the approximation property.

The jump condition on u_2 is obtained by using $u_2(nT) = E_* u_1(nT)$:

$$\begin{aligned}\|u_2((n+1)T) - S_T^2(u_2(nT))\|_{p,\rho} &= \|E_* S_T^1(u_1(nT)) - S_T^2(u_2(n)T))\|_{p,\rho} \\ &\le Ce^{CT} D_*^{1/p} + \|(I-E_*)S_T^2(u_2(nT))\|_{p,\rho}\end{aligned}$$

which is also small because of (5.23). Hence, for sufficiently large r_* the jumps are less than δ. □

The use of pseudo–orbits is well known for chaotic systems, however our philosophy is quite different. We do not construct true orbits close given pseudo–orbits which is one of the difficult tasks in the theory of chaotic dynamical systems. We only show that for each true orbit $u_1(t) = S_t^1(u_1^0)$ there is an approximating pseudo–orbit for (S_t^2). This result is relevant here, as it is one way to say that the dynamical system (S_t^2) is a *small perturbation* (S_t^1) when smallness is measured in the semi–norm $\|\cdot\|_{p,\rho_*}$. Of course, in the result of Theorem 5.6 the indices 1 and 2 can be interchanged showing that (S_t^1) is a small perturbation of (S_t^2).

6 Global attractors

The results from the previous section imply that CGL with $(\alpha, \beta) \in \mathcal{P}(d)$ defines a global semiflow on $X = L^{2d}_{\text{lu}}(\Omega)$ for any admissable domain $\Omega \subset \mathbb{R}^d$ and boundary conditions. Moreover, we have shown that each bounded set $B \subset X$ is absorbed in finite time into the absorbing set $\mathcal{B}_{\text{abs}}(\alpha, \beta, \Omega)$, cf. (4.19).

However, as we allow for unbounded Ω this set is in general not compact in $L^p_{\text{lu}}(\Omega)$. To handle the non–compactness due to the unboundedness of Ω we use a weaker topology on $L^{2d}_{\text{lu}}(\Omega)$, namely the weighted norm $\|u\|_{2d,\rho}$ from Section 2. The normed space $(L^{2d}_{\text{lu}}(\Omega), \|\cdot\|_{2d,\rho})$ will be denoted by X_ρ, in order to distinguish it from the uniform space $X = (L^{2d}_{\text{lu}}(\Omega), \|\cdot\|_{2d,\text{lu}})$. While X is a complete normed space, the same is not true for X_ρ, as the completion of $L^{2d}_{\text{lu}}(\Omega)$ in the norm $\|\cdot\|_{2d,\rho}$ would be $L^{2d}_\rho(\Omega)$ as introduced in the proof of Theorem 4.2.

Given any translation subgroup G of $\mathbb{R}^d$ there are three different distance measures on X. For $B \subset X$ and $u \in X$ define

$$\begin{aligned}
\text{dist}_\rho(u, B) &= \inf\{ \|u - v\|_{2d,\rho} : v \in B \}, \\
\text{dist}_{\rho,G}(u, B) &= \sup\{ \text{dist}_{T_g\rho}(u, B) : g \in G \}, \\
\text{dist}_{\text{lu}}(u, B) &= \inf\{ \|u - v\|_{2d,\text{lu}} : v \in B \}.
\end{aligned}$$

For any of these distances and two sets A, B we let $\text{dist}(A, B) = \sup\{ \text{dist}(u, B) : u \in A \}$. We have the relations

$$\text{dist}_\rho(A, B) \le \text{dist}_{\rho,G}(A, B) \le \text{dist}_{\text{lu}}(A, B).$$

Convergence in the first distance relates to L^{2d}–convergence on bounded subsets of Ω. The intermediate distance will be applicable to domains Ω which are translation invariant, namely $\Omega = g + \Omega$ for all $g \in G$. The convergence in $\text{dist}_{\rho,G}$ then means convergence in $\Omega \cap B(g, r)$ uniformly with respect to $g \in G$.

The following example shows that all distances are different (cf [MS95]). Let $\Omega = G = \mathbb{R}^1$, $B = \{ c : |c| \le 1 \} \subset L^2_{\text{lu}}(\mathbb{R})$, with $\rho(x) = e^{-|x|}$, and

$$u(x) = \begin{cases} 0 & \text{for } x \le \delta \\ \varepsilon(x - \delta) & \text{for } x \in [\delta, \delta + \frac{1}{\varepsilon}] \\ 1 & \text{for } x \ge \delta + \frac{1}{\varepsilon} \end{cases}$$

Then, $\text{dist}_{\text{lu}}(u, B) = 1$ and $\text{dist}_{\rho,G}(u, B) = 2\varepsilon[1 - e^{-1/2\,\varepsilon}]$ for all δ in $\mathbb{R}$ and $\varepsilon > 0$, while $\text{dist}_\rho(u, B) = \mathcal{O}(e^{-|\delta|})$ for $|\delta| \to \infty$.

Definition 6.1
Let $S_t : X \to X$ be a continuous semigroup. A set $\mathcal{A} \subset X$ is called a (X, X_ρ)–attractor for S if

(i) $\mathcal{A}$ is a non–empty, bounded closed set in X,

(ii) $S_t(\mathcal{A}) = \mathcal{A}$ for all $t \ge 0$,

(iii) and for each bounded set $\mathcal{B}$ in X we have $\mathrm{dist}_\rho(S_t(\mathcal{B}),\mathcal{A}) \to 0$ *for* $t\to\infty$.

Note that the attraction takes place only in the weaker distance. In order to prove the existence of such an attractor we need to essential features of the semiflow S_t. The first is continuity of $S_t : X_\rho \to X_\rho$ and the second is compactness of S_{t_0} for some $t_0 > 0$.

Lemma 6.2
a) For all $p \in [2,2d]$ *and all* $u^0, v^0 \in X$ *there is a constant* $C = C(\|u^0\|_{2d,\mathrm{lu}}, \|v^0\|_{2d,\mathrm{lu}}, p, d)$ *such that*

$$\|S_t(u^0) - S_t(v^0)\|_{p,\rho} \le C\, e^{Ct}\|u^0 - v^0\|_{p,\rho}^{2/p}.$$

b) For any bounded set B *in* X *and any* $t > 0$ *the set* $S_t(B)$ *is precompact in* X_ρ.

Proof: a) Let $N(u) = -(1+i\beta)|u|^2u$, then $|N(u)-N(v)| \le 3\sqrt{1+\beta^2}(|u|^2+|v|^2)|u-v|$. With $\|u(t)\|_\infty, \|v(t)\|_\infty \le C_\infty(1+t^{-1/4})$ we conclude by a weighted energy estimate $w(t) = u(t)-v(t) = S_t(u^0)-S_t(v^0)$

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|w(t)\|_\rho^2 &\le -\|\nabla w\|_\rho^2 + \sqrt{1+\alpha^2}\|w\|_\rho\|\nabla w\|_\rho + \textstyle\int_\Omega \rho|w||N(u)-N(v)|dx\\ &\le [1+\alpha^2+3\sqrt{1+\beta^2}(\|u\|_\infty^2+\|v\|_\infty^2)]\|w\|_\rho^2 \le C(1+t^{-1/2})\|w\|_\rho^2.\end{aligned}$$

Gronwall's estimate yields estimate in a) for the case $p=2$. The case $p \in (2,2d]$ follows by interpolation as in the proof of Theorem 5.4, since $u(t)$ and $v(t)$ are bounded in X.

ad b) The a–priori estimates in Section 4 imply that for each C_0 and $t>0$ there is a C_1 such that $\|u\|_{2d,\mathrm{lu}} \le C_0 \Rightarrow \|S_t(u)\|_{1,2d,\mathrm{lu}} \le C_1$. Hence, for bounded sets $B \subset X$ the set $B_t = S_t(B)$ is bounded in $W^{1,2d}_{\mathrm{lu}}(\Omega)$. Precompactness of B_t means that for every $\varepsilon > 0$ there exists a finite number of balls $B_\rho(u_j,\varepsilon) = \{\, u \in X \ :\ \|u-u_j\|_{2d,\rho} \le \varepsilon\,\}$ such that their union covers B_t.

For any x_* and large r_* we use the mapping E_* as in the proof of Theorem 5.6. Then, on the one hand $(I-E_*)B_t \subset B_\rho(0,\varepsilon/2)$ for sufficiently large r_*, according to (5.23). On the other hand E_*B_t is bounded in $W^{1,2d}(\Omega_{r_*+1})$, where $\Omega_{r_*+1} = \Omega \cap \{\, x \in \mathbb{R}^d : |x-x_*| < r_*+1\,\}$, and by Rellich's compactness theorem it is precompact in $L^{2d}(\Omega_{r_*+1})$. Since the standard L^{2d}–norm and $\|\cdot\|_{2d,\rho}$ are equivalent on Ω_{r_*+1} we find $u_j \in X$, $j=1,\dots,N$ such that $E_*B_t \subset \bigcup_1^N B_\rho(u_j,\varepsilon/2)$. Obviously, this implies $B_t \subset \bigcup_1^N B_\rho(u_j,\varepsilon)$, and the lemma is proved. □

Theorem 6.3
(Existence of global attractors)
Let $\Omega \subset \mathbb{R}^d$ *be uniformly* $\mathcal{C}^2$ *and let* (α,β) *be admissable parameters and* $p > d$ *with* $p \ge 2$. *Then, the semiflow* (S_t) *of CGL has a unique global* (X_ρ, X)*–attractor* $\mathcal{A}$ *with the following additional properties.*

(a) We have $P_\theta\mathcal{A} = \mathcal{A}$ *for each* $\theta \in [0,2\pi)$, *where* $P_\theta u = e^{i\theta}u$ *for all* $u \in X$.

(b) If Ω *is invariant under the action of a subgroup* $\mathcal{R}$ *of the Euclidean Group* $\mathcal{E}(d)$, *then so is* $\mathcal{A}$ *and for* $\mathcal{B} \subset X$ *we have* $\mathrm{dist}_{\rho,\mathcal{R}}(S_t(\mathcal{B}),\mathcal{A}) \to 0$ *for* $t\to\infty$.

Remark 6.4 An element of $\mathcal{E}(d)$ is denoted by (R, r) and acts on $\mathbb{R}^d$ by $x \mapsto Rx + r$. If Ω is invariant under such actions for all $(R, r) \in \mathcal{R}$, then these elements (R, r) act on $u \in X = L^{2d}_{\text{lu}}(\Omega)$ as $T_{(R,r)} : u(x) \mapsto u(Rx + r)$. The attractor $\mathcal{A}$ is called invariant under the action of $\mathcal{R}$ if $T_{(R,r)}\mathcal{A} = \mathcal{A}$ for all $(R, r) \in \mathcal{R}$. The symmetry distance $\text{dist}_{\rho,\mathcal{R}}$ was defined as $\text{dist}_{\rho,\mathcal{R}}(u, A) = \sup\{\, \text{dist}_{T_{(R,r)}\rho}(u, A) \,:\, (R, r) \in \mathcal{R}\,\}$.

Proof: We follow closely the arguments in [MS95], Thm 2.6.

(1) According to Section 4 there is an absorbing set $\mathcal{B}_{\text{abs}} \subset X$ such that for every bounded $B \subset X$ there is a $t_1 = t_1(B)$ such that $S_{t_1}(B) \subset \mathcal{B}_{\text{abs}}$. (In fact, [LO96] show that t_1 can be taken independent of B.) Hence, it suffices to show that $\mathcal{B}_{\text{abs}}$ is attracted to some attractor. The candidate $\mathcal{A}$ for the attractor is defined as

$$\mathcal{A} = \textstyle\bigcap_{t\geq 0} A_t \text{ with } A_t = \text{closure of } S_t(\mathcal{B}_1) \text{ in } X_\rho.$$

Here, A_t forms a decreasing set of compact subsets of X_ρ and thus $\mathcal{A}$ is non–empty and compact in X_ρ. Since $\mathcal{A} \subset \mathcal{B}_{\text{abs}}$ and closedness in X_ρ implies closedness in X, we have proved part (i) of Definition 6.1.

(2) To prove the time invariance let $v \in S_t(\mathcal{A})$, i.e. $v = S_t(u)$ where $u = \lim\limits_{t_n\to\infty} S_{t_n}(u_n)$ in X_ρ where $u_n \in \mathcal{B}_1$. The continuity in X_ρ according to Lemma 6.2(a) gives

$$v = S_t(u) \leftarrow S_t(S_{t_n}(u_n)) = S_{t+t_n}(u_n) \text{ in } X_\rho,$$

which implies $v \in \mathcal{A}$ and thus $S_t(\mathcal{A}) \subset \mathcal{A}$.

For the opposite direction let $v \in \mathcal{A}$ and $t > 0$. We have to find a $u \in \mathcal{A}$ with $S_t(u) = v$. There is a increasing sequence $t_n > t$ and $v_n \in \mathcal{B}_{\text{abs}}$ with $v = \lim_{t_n\to\infty} S_{t_n}(v_n)$ in X_ρ. The compactness result of Lemma 6.2(b) shows that the set $\{\, S_{t_n-t}(v_n) \,:\, n \in \mathbb{N}\,\} \subset S_{t_1-t}(\mathcal{B}_1)$ is precompact and hence there is a subsequence (n_k) with $S_{t_{n_k}-t}(v_{n_k}) \to u$ in X_ρ. Again by continuity of S_t in X_ρ we find $S_t(u) = v$ since

$$v \leftarrow S_{t_{n_k}}(v_{n_k}) = S_t(S_{t_{n_k}-t}(v_{n_k})) \to S_t(u) \text{ in } X_\rho.$$

By construction we have $u \in \mathcal{A}$, and thus $\mathcal{A} \subset S_t(\mathcal{A})$ is proved.

(3) We assume that $\text{dist}_\rho(S_t(\mathcal{B}_1), \mathcal{A}) \not\to 0$ in order to generate a contradiction. Then, there exists $t_n \to \infty$ and $u_n \in \mathcal{B}_{\text{abs}}$ with $\text{dist}_\rho(S_{t_n}(u_n), \mathcal{A}) \geq \delta > 0$. By compactness of $\mathcal{B}_{\text{abs}}$ in X_ρ there is a subsequence (n_k) with $S_{t_{n_k}}(u_{n_k}) \to w$ in X_ρ and $w \in \mathcal{A}$. But now $\text{dist}_\rho(S_{t_{n_k}}(u_{n_k}), \mathcal{A}) \leq \text{dist}_\rho(S_{t_{n_k}}(u_{n_k}), \{w\}) \to 0$ which contradicts the above assumption.

(4) It remains to prove the uniqueness and the properties (a) and (b). Assume there are two attractors $\mathcal{A}_1$ and $\mathcal{A}_2$. By invariance and attractivity we find $\text{dist}_\rho(\mathcal{A}_1, \mathcal{A}_2) = \text{dist}_\rho(\mathcal{A}_2, \mathcal{A}_1) = 0$ and the closedness yields $\mathcal{A}_1 = \mathcal{A}_2$.

Moreover, CGL is invariant under multiplication with $e^{i\theta}$, hence $S_t \circ R_\theta = R_\theta \circ S_t$ for all $\theta \in [0, 2\pi]$. Hence, $R_\theta\mathcal{A}$ is attracted to $\mathcal{A}$ as well as $\mathcal{A}$ is attracted to $R_\theta\mathcal{A}$. By uniqueness we have $R_\theta\mathcal{A} = \mathcal{A}$.

(5) Assume the symmetry actions as described in the remark. Obviously, we have $T_{(R,r)}\mathcal{A} = \mathcal{A}$ for $(R, r) \in \mathcal{R}$ by the same argument as in (4). We also want to infer the stronger decay measure using $\text{dist}_{\rho,\mathcal{R}}$. Using that $\|u\|_{2d,\rho} = \|T_{(R,r)}u\|_{2d,T_{(R,r)}\rho}$ we see that

$T_{\mathcal{R}}\mathcal{B} = \bigcup_{(R,r)\in\mathcal{R}} T_{(R,r)}\mathcal{B}$ is bounded in X whenever $\mathcal{B}$ is bounded in X. Now we estimate as follows

$$\begin{aligned} \mathrm{dist}_{\rho,\mathcal{R}}(S_t(\mathcal{B}),\mathcal{A}) &\leq \mathrm{dist}_{\rho,\mathcal{R}}(S_t(T_{\mathcal{R}}\mathcal{B}),T_{\mathcal{R}}\mathcal{A}) \\ &= \sup\{\, \mathrm{dist}_{T_{(R,r)}\rho}(S_t(T_{\mathcal{R}}\mathcal{B}),T_{\mathcal{R}}\mathcal{A}) \,:\, (R,r)\in\mathcal{R}\,\} \\ &= \sup\{\, \mathrm{dist}_{\rho}(T_{(R,r)^{-1}}S_t(T_{\mathcal{R}}\mathcal{B}),T_{(R,r)^{-1}}T_{\mathcal{R}}\mathcal{A}) \,:\, (R,r)\in\mathcal{R}\,\} \\ &= \mathrm{dist}_{\rho}(S_t(T_{\mathcal{R}}\mathcal{B}),\mathcal{A}) \to 0 \end{aligned}$$

for $t\to\infty$, since $T_{(R,r)^{-1}}T_{\mathcal{R}}A = T_{\mathcal{R}}A$ and $T_{\mathcal{R}}\mathcal{A} = \mathcal{A}$. □

For later use we introduce the decay function $a_{\mathcal{A}} : [0,\infty) \to (0,\infty)$ which measures the distance of $S_t(\mathcal{B}_{\rm abs})$ from $\mathcal{A}$ in the norm $\|\cdot\|_{2d,\rho}$;

$$a_{\mathcal{A}}(t) = \mathrm{dist}_{\rho}(S_t(\mathcal{B}_{\rm abs}),\mathcal{A}).$$

As proved above this function decays to 0 for $t\to\infty$.

Finally we consider two systems (S_t^1) and (S_t^2) posed on Ω_1 and Ω_2, respectively, and compare their attractors on the joint set $\Omega_* = \Omega_j \cap B(x_*,r_*)$ as in Section 5. From above we know the existence of global attractors $\mathcal{A}_j \subset X^j = L^{2d}_{\rm lu}(\Omega_j)$ such that $a_j(t) = \mathrm{dist}_\rho(S_t^j(\mathcal{B}_{\rm abs}(\Omega_j),\mathcal{A}_j) \to 0$. Recall $\rho(x) = e^{-|x-x_*|}$ and x_* is the center of $\Omega_* = B(x_*,r_*)\cap\Omega_j$. Using our control on $S_t^1(u_1^0)|_{\Omega_*} - S_t^2(u_2^0)|_{\Omega_*}$ we derive an upper bound on the distance between $\mathcal{A}_1|_{\Omega_*}$ and $\mathcal{A}_2|_{\Omega_*}$.

Theorem 6.5
Use the notations from Theorem 5.4 and define $\psi_j(r_*) = \min\{\, Ce^{C\tau}D_*^{1/(2d)} + a_j(\tau) \,:\, \tau > 0\,\}$. *Then, we have the estimates*

$$\mathrm{dist}_{\rho_*}(\mathcal{A}_1,\mathcal{A}_2) \leq \psi_2(r_*) \text{ and } \mathrm{dist}_{\rho_*}(\mathcal{A}_2,\mathcal{A}_1) \leq \psi_1(r_*),$$

where $\mathrm{dist}_{\rho_*}(A,B) = \sup\{\, \inf\{\, \|a|_{\Omega_*} - b|_{\Omega_*}\|_{2d,\rho_*} \,:\, b\in B\,\} \,:\, a\in A\,\}$.

Proof: Let $u_1 \in \mathcal{A}_1$, then for each $\tau > 0$ there is a $u_1^{-\tau} \in \mathcal{A}_1$ with $S_\tau^1(u_1^{-\tau}) = u_1$. Now consider $u_2^{-\tau} = E_* u_1^{-\tau} \in X^2$, then

$$\mathrm{dist}_{\rho_*}(u_1,\mathcal{A}_2) \leq \|S_\tau^1(u_1^{-\tau}) - S_\tau^2(u_2^{-\tau})\|_{2d,\rho_*} + \mathrm{dist}_{2d,\rho_*}(S_\tau^2(u_2^{-\tau}),\mathcal{A}_2) \leq Ce^{C\tau}D_*^{1/p} + a_2(\tau).$$

By choosing τ optimal the assertion $\mathrm{dist}_{2d,\rho_*}(\mathcal{A}_1,\mathcal{A}_2) \leq \psi_2(\rho_0,r_*)$ is proved and the opposite case follows by interchanging 1 and 2. □

It is easy to see that $\psi_j(r_*) \to 0$ for $r_*\to\infty$, since $D_* = r_*^{d-1}e^{-r_*} \to 0$. Hence, the further we push the boundaries away the less they influence the long–time behavior in the middle of the domain. There is a major problem in applying the result of Theorem 6.5, since the decay functions a_j may depend sensitively on the domain Ω_j. Thus, we cannot derive results for sequences of domains. However, if Ω_2 is fixed while Ω_1 varies arbitrarily we still obtain a one–sided estimate.

Corollary 6.6
For $X_{I\!R^d} = L^{2d}_{\rm lu}(I\!R^d)$ and $(\alpha,\beta) \in \mathcal{P}(d)$ let $(S_t^{I\!R^d})$, $\mathcal{A}_{I\!R^d} \subset X_{R^d}$, and $\psi_{I\!R^d}$ be the associated semigroup, attractor, and the function defined in Theorem 6.5, respectively.

Then, for any admissable domain $\Omega_1 \subset I\!R^d$ and boundary conditions with $B(x_*, r_*) \subset \Omega_1$, the attractor $\mathcal{A}_1 \subset X^1$ of (S_t^1) satisfies

$$\mathrm{dist}_{\rho_*}(\mathcal{A}_1, \mathcal{A}_{I\!R^d}) \leq \psi_{I\!R^d}(r_*).$$

It is important to note that we cannot reverse the order of the arguments in the distance function. In particular, if we have a family of admissable domains $\Omega_{(k)} \supset B(x_*, r_*^{(k)})$ with $r_*^{(k)} \to \infty$ we are not able to show $\mathrm{dist}_{\rho_{*,(k)}}(\mathcal{A}_{I\!R^d}, \mathcal{A}_{(k)}) \to 0$ whereas $\mathrm{dist}_{\rho_{*,(k)}}(\mathcal{A}_{(k)}, \mathcal{A}_{I\!R^d}) \leq \psi_{I\!R^d}(r_*^{(k)}) \to 0$ is trivial. The latter convergence is called upper–semicontinuity of the attractors $\mathcal{A}_{(k)}$ to the limit $\mathcal{A}_{I\!R^d}$. The opposite convergence would be lower–semicontinuity which is much harder to show. The problem is that $\mathrm{dist}_{\rho_{*,(k)}}(\mathcal{A}_{I\!R^d}, \mathcal{A}_{(k)})$ can only be estimated by the attraction functions $a_{(k)}$, and in general there is no control on their behavior as $k \to \infty$.

By Theorem 5.6 we know that the orbits of (S_t^1) can be approximated by pseudo–orbits for $(S_t^{I\!R^d})$ and vice–versa. In the present situation we can say more. The orbits in $\mathcal{A}_1 \subset X^1$ can be approximated by pseudo–orbits of $(S_t^{I\!R^d})$ lying completely in $\mathcal{A}_{I\!R^d}$.

Theorem 6.7
Let the assumptions of Corollary 6.6 be satisfied. Then, for each positive T, ε, and δ there is an $\widehat{r}_* > 0$ such that for all $r_* \geq \widehat{r}_*$ the following holds: for any admissable domain $\Omega_1 \subset I\!R^d$ with $B(x_*, r_*) \subset \Omega_1$ and all $u_1^0 \in \mathcal{A}_1 \subset L^{2d}_{\rm lu}(\Omega_1)$ there exists a (T,δ) pseudo–orbit v for $(S_t^{I\!R^d})$ in $(L^p_{\rm lu}(I\!R^d), \|\cdot\|_{2d,\rho})$ such that

$$v(t) \in \mathcal{A}_{I\!R^d}, \quad \textit{and } \|S_t^1(u_1^0) - v(t)\|_{2d,\rho_*} \leq \varepsilon.$$

Proof: We proceed as in the proof of Theorem 5.6 but take care that $v(nT)$ lies in $\mathcal{A}_{I\!R^d}$.

For any $\kappa > 0$ there is $\widehat{r}_*$ with $\mathrm{dist}_{\rho_*}(\mathcal{A}_1, \mathcal{A}_{I\!R^d}) < \kappa$ whenever $r_* \geq \widehat{r}_*$. Hence, we choose $v(nT) \in \mathcal{A}_{I\!R^d}$ with $\|u(nT) - v(nT)\|_{2d,\rho_*} \leq \kappa$ and control the distance on $[nT, (n+1)T)$ by Theorem 5.4: $u(nT+\tau) - v(nT+\tau)\|_{2d,\rho_*} \leq Ce^{C\tau}\left(\kappa^{1/d} + D_*^{1/(2d)}\right)$. Since $\tau \leq T$ we can make κ small and $\widehat{r}_*$ large in order to obtain that the difference is less than $\min\{\,\varepsilon, \delta/2\,\}$.

For the jump we estimate

$$\begin{aligned} \|v((n+1)T) - S_T^{I\!R^d}(v(nT))\|_{2d,\rho} &\leq \|v((n+1)T) - u((n+1)T)\|_{2d,\rho} \\ &\quad + \|S_T^1(u(nT)) - S_T^{I\!R^d}(v(nT))\|_{2d,\rho_*} \\ &\leq \kappa + Ce^{CT}\left(\kappa^{1/d} + D_*^{1/(2d)}\right) \;\leq\; \kappa + \delta/2. \end{aligned}$$

For large enough $\widehat{r}_*$ we have $\kappa \leq \delta/2$ and the result is proved. □

It would be more desirable to approximate the solutions in $\mathcal{A}_{I\!R^d}$ by pseudo–orbits in $\mathcal{A}_1$ in cases where Ω_1 is bounded. In such a case the attractor $\mathcal{A}_1$ is the classical compact attractor in $L^p(\Omega_1)$ as studied in [Tem88]. In particular the Hausdorff dimension of $\mathcal{A}_1$ is finite and can be estimated by the parameters (R, α, β). Of course, there are pseudo–orbits $u_1(t)$ of (S_t^1) which approximate the solutions $v(t)$ in $\mathcal{A}_{\mathbb{R}^d}$. However, we cannot guarantee that these solutions are contained in $\mathcal{A}_1$.

An interesting case occurs when Ω is the interval $\Omega_\ell = (-\ell, \ell) = (-\ell_1, \ell_1) \times \ldots \times (-\ell_d, \ell_d)$ with periodic boundary conditions. Then $X^{(\ell)} = L^{2d}_{\text{lu}}(\Omega_\ell) = L^{2d}(\Omega_\ell)$ can be embedded into $X^{\mathbb{R}^d}$ by continuing each function periodically. Obviously, $S_t^{(\ell)} = S_t^{\mathbb{R}^d}|_{X^{(\ell)}}$ and moreover $\mathcal{A}_{(\ell)} = \mathcal{A}_{\mathbb{R}^d} \cap X^{(\ell)}$. The last identity follows since $\mathcal{A}_{(\ell)}$ is attracted to $\mathcal{A}_{\mathbb{R}^d}$ in $X^{\mathbb{R}^d}$, is invariant and contained in $X^{(\ell)}$, hence $\mathcal{A}_{(\ell)} \subset \mathcal{A}_{\mathbb{R}^d} \cap X^{(\ell)}$. The inverse inclusion holds since $\mathcal{A}_{\mathbb{R}^d} \cap X^{(\ell)}$ is invariant under $(S_t^{(\ell)})$. Thus, we have $\mathcal{A}_{(\ell)} \subset \mathcal{A}_{\mathbb{R}^d}$ with $\text{dist}_\rho(\mathcal{A}_{(\ell)}, \mathcal{A}_{\mathbb{R}^d}) \to 0$ for $\ell_{\min} \to \infty$, where $\ell_{\min} = \min\{\ell_1, \ldots, \ell_d\}$. Additionally, we know

$$\text{dist}_{\rho_*}(\mathcal{A}_{\mathbb{R}^d}, X^{(\ell)}) \le C D_*(\ell_{\min})^{1/(2d)}, \quad \text{where } D_*(l) = l^{d-1}e^{-l},$$

which is a consequence of the boundedness of $\mathcal{A}_{\mathbb{R}^d}$ in $L^{2d}_{\text{lu}}(\mathbb{R}^d)$ and the estimate (C.3). Nevertheless we were not able to show the lower semi–continuity $\text{dist}_{\rho_*}(\mathcal{A}_{\mathbb{R}^d}, \mathcal{A}_{(\ell)}) \to 0$ for $\ell_{\min} \to \infty$. It would be equivalent to the conjecture

$$\mathcal{A}_{\mathbb{R}^d} = \text{closure of } \Big[\textstyle\bigcup_{\ell \in (0,\infty)^d} \mathcal{A}_{(\ell)} \Big] \text{ in } X_\rho,$$

which was posed as an open problem already in [MS95].

A Sobolev embeddings

For the fixed weight $\rho(x) = e^{-|x|}$ we have in one space dimension the explicit estimate

$$\|u\|_\infty^2 \le \|u\|_{2,\text{lu}}(\|u\|_{2,\text{lu}} + 2\|\partial_x u\|_{2,\text{lu}}). \tag{A.1}$$

We give an elementary proof of the Gagliardo–Nirenberg estimate in the one–dimensional case. For any weight ρ with $|\nabla\rho(x)| \le \rho_0\rho(x)$ we have

$$\rho(0)|u(y)|^p = \textstyle\int_{-\infty}^y \frac{d}{dx}(\rho(x-y)|u(x)|^p)dx \le \rho_0\|u\|_{p,\rho}^p + p\|u\|_{2p-2,\rho}^{p-1}\|\partial_x u\|_{2,\rho}$$

Thus, for the weight $\rho(x) = e^{-|x|}$ we find

$$\|u\|_\infty^p \le \|u\|_{p,\text{lu}}^p + p\|u\|_{2p-2,\text{lu}}^{p-1}\|\partial_x u\|_{2,\text{lu}}. \tag{A.2}$$

For a proof of the general d–dimensional case we refer to the literatur, e.g. [Ad75, Tem88]. We need the following result.

Theorem A.1
Let $s_1, s_2 \in \mathbb{N}_0$, and $p_1, p_2 \in (1,\infty)$ such that $s_1 \le s_2 + 1$ and $s_1 - d/p_1 < s_2 + 1 - d/p_2$. Moreover, assume that $\theta \in [0,1]$ satisfies $s_1 - d/p_1 < \theta(s_2 - 1s_2 - d/p_2) + (1-\theta)(s_2 + 1 - d/p_2)$. Then there exists a constant depending only on d, s_1, s_2, p_1, and p_2, such that for all admissable domains $\Omega \subset \mathbb{R}^d$ and all $u \in W^{s_2+2,p_2}_{\text{lu}}(\Omega)$ the estimate

$$\|u\|_{s_1,p_1,\text{lu}} \le C\|u\|_{s_2,p_2,\text{lu}}^\theta C\|u\|_{s_2+1,p_2,\text{lu}}^{1-\theta}$$

holds.

The only nontrivial part here is the claim that the constant does not depend on the domain. However, for admissable domains we may use a uniform partition of unity with sets of diameter less than 1/2. On each of these sets the estimate holds with a uniform constant.

B Another a–priori estimate

We provide an alternative a-priori estimate which works in the case of dimesension $d = 1$. This method was introduced in [MS96], but unfortunately there is an omission in one estimate in the last quarter of the proof, which led to a wrong result. We repeat the analysis for the reader's convenience.

Theorem B.1
For all $(\alpha, \beta) \in I\!R^2$ the solutions $u = u(t, x)$ of CGL exist globally and satisfy the estimates

$$\begin{aligned} \|u(t)\|^2_{2,\mathrm{lu}} &\le e^{-2\widetilde{R}t}\|u(0)\|^2_{2,\mathrm{lu}} + (1 - e^{-2\widetilde{R}t})2\widetilde{R}, \\ \limsup_{t\to\infty} \|\partial_x u(t)\|^2_{2,\mathrm{lu}} &\le 56\widetilde{R}^2(1 + 10\sigma^2\widetilde{R}), \end{aligned}$$

where $\widetilde{R} = R + (1 + \alpha^2)/4$, $\sigma = \max\{0, \sqrt{1 + \beta^2} - 2\}$.

Proof: The first estimate is exactly (2.3) for $p = 2$. As in Section 2 we have

$$\tfrac{d}{dt} \int_{I\!R} \rho|\partial_x u|^2 du \le -\int_{I\!R} \rho|\partial_x^2 u|^2 dx + 2\int_{I\!R} \rho(\widetilde{R}|\partial_x u|^2 + \sigma|u|^2|\partial_x u|^2)\, dx. \tag{B.1}$$

In order to shorten the following formulae we abbreviate $r_j(t) = (\int_{I\!R} \rho|\partial_x^j u(t)|^2 dx)^{1/2}$ and $e_j(t) = \|\partial_x^j u(t)\|_{2,\mathrm{lu}}$ for $j = 0, 1$, and 2. From partial integration we find $r_1^2 \le r_0(r_1 + r_2)$ and (A.2) reads $\|u\|^2_\infty \le e_0(e_0 + 2e_1)$. With these stipulations (B.1) takes the form

$$\begin{aligned} \tfrac{d}{dt} r_1^2 &\le -r_2^2 - (\gamma + 1)r_1^2 + (\gamma + 1 + 2\widetilde{R} + 2\sigma e_0(e_0 + 2e_1))r_0(r_1 + r_2) \\ &\le -(\gamma + 1)r_1^2 + s(t)r_1 + s^2(t)/4 \;\le\; -\gamma r_1^2 + s^2(t)/2, \end{aligned}$$

where $\gamma > 0$ is arbitrary, and $s(t) = (\gamma + 1 + 2\widetilde{R} + 2\sigma e_0(e_0 + 2e_1))r_0$. Applying Gronwall's inequality and using the same estimate for all translated weights we obtain

$$\begin{aligned} e_1^2(t) &\le e^{-\gamma t}e_1^2(0) + \textstyle\int_0^t e^{-\gamma(t-\tau)}s^2(\tau)/2\, d\tau \\ &\le e^{-\gamma t}e_1^2(0) + \textstyle\int_0^t e^{-\gamma(t-\tau)}\Big([\gamma{+}1{+}2\widetilde{R}{+}2\sigma e_0^2]^2 e_0^2 + 16\sigma^2 e_0^4 e_1^2\Big) d\tau. \end{aligned}$$

Note that γ is still arbitrary in the above estimate. We may now use Lemma B.2 below and $\limsup_{t\to\infty} e_0^2(t) \le \widetilde{R}$ in order to see that $e_1(t)$ is also bounded and satisfies

$$\limsup_{t\to\infty} e_1^2(t) \le \frac{[\gamma + 1 + 2\widetilde{R}(1 + 2\sigma)]^2 2\widetilde{R}}{\gamma - 64\sigma^2\widetilde{R}^2}.$$

Choosing γ optimum we obtain the desired result when using $\widetilde{R} \ge 1$. □

The following lemma is proved in [MS96].

Lemma B.2
Let $\mu, \nu \in \mathcal{C}^0([0,\infty), I\!R)$ be bounded functions with $\nu(t) \geq 0$ for all t. Assume that $\gamma > \overline{\nu} = \limsup_{t\to\infty} \nu(t)$ and that the continuous function $A \in \mathcal{C}^0([0,\infty), I\!R)$ satisfies

$$A(t) \leq A(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t-\tau)}[\mu(\tau) + \nu(\tau)A(\tau)]\, d\tau,$$

for all $t \geq 0$. Then, A is bounded on $[0,\infty)$ and satisfies

$$\limsup_{t\to\infty} A(t) \leq \frac{1}{\gamma - \overline{\nu}} \limsup_{\tau\to\infty} \mu(\tau).$$

C Uniform versus local norm

Lemma C.1
Assume $|\nabla\rho(x)| \leq \rho_0\rho(x)$, $\rho(0) = 1$ and define for $\widetilde{\Omega} \subset I\!R^d$ the neighborhood $\widetilde{\Omega}_{\rho_0} = \{\, x \in I\!R^d \,:\, \mathrm{dist}(x, \widetilde{\Omega}) < 1/\rho_0 \,\}$. Let $\widetilde{\rho} : \widetilde{\Omega}_{\rho_0} \to (0,\infty)$ be a function with $|\nabla\widetilde{\rho}(x)| \leq \kappa\, \widetilde{\rho}(x)$ on $\widetilde{\Omega}_{\rho_0}$ and $\int_{\widetilde{\Omega}_{\rho_0}} \widetilde{\rho}\, dx < \infty$. Then for each $u \in L^p_{\mathrm{lu}}(I\!R^d)$ we have

$$\int_{\widetilde{\Omega}} \widetilde{\rho}|u|^p\, dx \leq e^{1+\kappa/\rho_0}(\rho_0\sqrt{d})^d \int_{\widetilde{\Omega}_{\rho_0}} \widetilde{\rho}\, dx\, \|u\|^p_{p,\mathrm{lu}}$$

where $\|u\|^p_{p,\mathrm{lu}} = \sup\{ \int_{I\!R^d} \rho(x+y)|u(x)|^p\, dx \,:\, y \in I\!R^d \}$.

Proof: We consider cubical d–dimensional intervals Q_j with side length $1/(\rho_0\sqrt{d})$ such that $Q_j \cap Q_k = \emptyset$ for $j \neq k$ and $\widetilde{\Omega} \subset \bigcup_{j\in J} Q_j \subset \widetilde{\Omega}_{\rho_0}$. Define

$$d_j = \inf\{ \sup\{ \widetilde{\rho}(x)/T_y\rho(x) \,:\, x \in Q_j \} \,:\, y \in I\!R^d \},\ a_j = \int_{Q_j} \widetilde{\rho}(x)\, dx,$$

then we can estimate

$$\begin{aligned} \int_{\widetilde{\Omega}} \widetilde{\rho}|u|^p\, dx &= \sum_{j\in J} \int_{Q_j} \tfrac{\widetilde{\rho}}{T_{y_j}\rho} T_{y_j}\rho|u|^p\, dx \leq \sum_{j\in J}(1+\varepsilon)d_j\|u\|^p_{p,\mathrm{lu}} \\ &= (1+\varepsilon)\sum_{j\in J} \tfrac{d_j}{a_j} \int_{Q_j} \widetilde{\rho}\, dx\, \|u\|^p_{p,\mathrm{lu}} \leq (1+\varepsilon)C \int_{\widetilde{\Omega}_{\rho_0}} \widetilde{\rho}\, dx\, \|u\|^p_{p,\mathrm{lu}} \end{aligned}$$

where $C = \sup\{ d_j/a_j \,:\, j \in J \}$.

Using the estimate on the gradients and $\mathrm{diam}(Q_j) \leq 1/\rho_0$ we find $d_j \leq (\max_{Q_j} \widetilde{\rho})\, e$ and $a_j \geq (\min_{Q_j} \widetilde{\rho})(\rho_0\sqrt{d})^{-d}$ and hence $d_j/a_j \leq e^{1+\kappa/\rho_0}(\rho_0\sqrt{d})^d$ which is the result. □

One typical application of this result is obtained when $\widetilde{\Omega}$ is bounded and $\widetilde{\rho} \equiv 1$ (i.e., $\kappa = 0$):

$$\int_{\widetilde{\Omega}} |u|^p\, dx \leq e(\rho_0\sqrt{d})^d\, \mathrm{vol}(\widetilde{\Omega}_{\rho_0})\, \|u\|^p_{p,\mathrm{lu}}. \tag{C.2}$$

Note that the right–hand side does not tend to zero for $\rho_0 \to 0$ as $\mathrm{vol}(\widetilde{\Omega}_{\rho_0}) \sim \rho_0^{-d}$.

Another example occurs when $\widetilde{\rho} = \rho = e^{-\rho_0|x|}$ (i.e. $\kappa = \rho_0$) and $\widetilde{\Omega} = I\!R^d \backslash \{ x : |x| \leq r_* \}$. Then we find

$$\int_{\widetilde{\Omega}} \rho|u|^p\, dx \leq e^2 d^{d/2}\omega_d I_{d-1}(\rho_0 r_* - 1)\|u\|^p_{p,\mathrm{lu}} \tag{C.3}$$

where ω_d is the surface of $\{ x \in I\!R^d : |x| = 1 \}$ and $I_d(s) = \int_s^\infty t^d\, e^{-t}\, dt = e^{-s} d! \sum_{j=0}^{d} s^j/(j!) \leq (d+s)^d\, e^{-s}$.

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