On the rate-independent limit of systems with dry friction and small viscosity

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19 October 2005

1 Introduction

Rate-independent evolution models originate as limits of systems with strongly separated time scales. Typically a system with fast internal time scales is driven by an external loading on a much slower time scale. We want to describe the model on the latter slow time scale, in which viscous transitions are seen as instantaneous jumps. However, effects of dry friction, which are rate-independent, will lead to nontrivial continuous solution behavior. The purpose of this work is to present a model which is able to account for viscous as well as for dry-friction effects and is still rate-independent. We do this by reparametrization of the slow time variable and thus blowing up the time scale in viscous regimes.

Our theory is based on a purely energetic approach as introduced in [CoV90, MiT04, MTL02]. This theory is rather flexible and allows us to deal with fully nonlinear, nonsmooth systems in the infinite dimensional setting, see [Mie05] for a recent survey. However for simplicity and clarity of this work, we restrict ourselves to the case that X is a finite-dimensional Banach space. The evolution is defined via an energy functional $I : [0,T]\times X \to \mathbb{R}$; $(t,z) \mapsto I(t,z)$, where $t \in [0,T]$ is the slow process time and $z \in X$ is the state of the system. Moreover, on the tangent space $\mathrm{T}X$ of X (which equals $X\times X$ in our case) there is given a dissipation functional $\Delta : \mathrm{T}X \to \mathbb{R}$, such that $\Delta\big(z(t), \dot z(t)\big) \geqslant 0$ describes the dissipation (rate of dissipated energy). The energetic evolution law takes the form

$$0 \in \partial_{\dot z}\Delta\big(z(t), \dot z(t)\big) + \mathrm{D}_z I\big(t, z(t)\big) \quad \text{a.e. on } [0,T]\,. \tag{1.1}$$

Here we assume that $\Delta(z,\cdot) : X \to [0,\infty)$ is convex and

$$\partial_{\dot z}\Delta(z,v) = \big\{ \sigma \in X^* \,\big|\, \forall w \in X : \Delta(z,w) \geqslant \Delta(z,v) + \langle \sigma, w-v\rangle \big\}$$

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is the subdifferential with respect to the second argument. Throughout this work we assume that $\Delta(z,v)=\Delta(v)$ which simplifies the presentation considerably. For a treatment of the case that Δ depends on the state $z\in X$ and X is infinite dimensional but $I(t,\cdot)$ is convex, we refer to [MiR05].

Note that (1.1) contains gradient flows (viscous case) if we set $\Delta(v)=\frac{1}{2}\langle Gv,v\rangle$ for some symmetric and positive definite G (Riemannian case). The rate independent case is obtained if Δ is homogenous of degree one, i.e. $\Delta(\alpha v)=\alpha\Delta(v)$ for all $\alpha>0$. In fact, many of the existing rate-independent hysteresis models can be written in the form of (1.1), e.g. Moreau's sweeping processes [Mor77] or linearized elastoplasticity [Mor76, HaR95]. See [KrP89, BrS96, Vis94] for the general theory on hysteresis. In most of these models the energy functional $I(t,\cdot):X\to\mathbb{R}$ is quadratic and coercive or at least uniformly convex. In that case (1.1) is in fact equivalent to the weaker energetic form introduced in [MiT99, MiT04, MTL02]:

$$
\begin{aligned}
&\text{(S) Stability: For all } t\in[0,T] \text{ and all } \widehat{z}\in X \text{ we have}\\
&\qquad I\big(t,z(t)\big)\leqslant I\big(t,\widehat{z}\big)+\Delta\big(\widehat{z}-z(t)\big).\\
&\text{(E) Energy inequality: For } t_1\leqslant t_2 \text{ we have}\\
&\qquad I\big(t_2,z(t_2)\big)+\int_{t_1}^{t_2}\Delta\big(\dot{z}(t)\big)\,\mathrm{d}t\leqslant I\big(t_1,z(t_1)\big)+\int_{t_1}^{t_2}\partial_t I\big(t,z(t)\big)\,\mathrm{d}t.
\end{aligned}
\tag{1.2}
$$

This energetic formulation consists of the standard energy inequality (E) and a purely static stability condition (S): the gain in stored energy $I\big(t,z(t)\big)-I\big(t,\widehat{z}\big)$ is not allowed to be larger than the energy $\Delta\big(\widehat{z}-z(t)\big)$ lost by dissipation. In fact, if (S) and (E) hold, then (E) holds in fact with equality, see Lemma 3.7 in [MiT04].

Of course, the energetic formulation (1.2) can also be used in cases where $I(t,\cdot)$ is nonconvex (see for example [Efe03, MiT04, Mie05, FrM05]), however this will lead to solutions having jumps which may not correspond to the physically desired jumps. Instead of jumps the real system would switch to viscous, rate-dependent behavior until it finds a suitable nearby stable state again. Here we want to modify (1.1) and (1.2) in such a way that the viscous transition path between two stable states is still captured. However, we will not resolve the temporal behavior along this path and thus are able to obtain a rate-independent model.

One way to obtain such a limit problem is by viscous regularization. We will deal with viscous regularizations in Section 3. Indeed, assume that Δ is homogenous of degree one. Then, consider (1.1) with Δ replaced by $\Delta^\varepsilon : v\mapsto\Delta(v)+\frac{\varepsilon}{2}\|v\|^2$, where $\|\cdot\|$ is any norm on X. Standard theory provides solutions $z^\varepsilon\in \mathrm{W}^{1,2}\big([0,T],X\big)$, and the question is whether the limit $z^0(t)=\lim_{\varepsilon\to0}z^\varepsilon(t)$ exists and, if so, what limit equation z^0 satisfies. Since the limit will develop jumps in general, the model remains incomplete. Instead one can ask for convergence of the graph $\big\{\big(t,z^\varepsilon(t)\big)\,\big|\,t\in[0,T]\big\}$ in the extended phase space $[0,T]\times X$. An even stronger convergence is obtained by parametrizing this graph by arc length

$$
\tau^\varepsilon(t)=t+\int_0^t\big\|\dot{z}^\varepsilon(s)\big\|\,\mathrm{d}s
$$

and considering the rescaled functions

$$
\hat{t}^\varepsilon(\tau)=\big(\tau^\varepsilon\big)^{-1}(\tau)\quad\text{and}\quad\widehat{z}^\varepsilon(\tau)=z^\varepsilon\big(\hat{t}^\varepsilon(\tau)\big).
$$

By construction we now have $\dot{\hat{t}}^\varepsilon(\tau)+\big\|\dot{\widehat{z}}^\varepsilon(\tau)\big\| \equiv 1$ and, after choosing a subsequence, limit functions $\hat{t}=\hat{t}^0$ and $\widehat{z}=\widehat{z}^0$ exists, which are again Lipschitz continuous. In Section 3 we show that these limits satisfy the following limit equation: for a.a. $\tau \in [0,\widehat{T}]$ holds

$$\begin{aligned}
&0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\widehat{z}}(\tau)\big) + \mathrm{D}_z I\big(\hat{t}(\tau),\widehat{z}(\tau)\big),\\
&1 = \dot{\hat{t}}(\tau) + \big\|\dot{\widehat{z}}(\tau)\big\|,\\
&\hat{t}(0)=0, \quad \hat{t}\big(\widehat{T}\big)=T, \quad \widehat{z}(0)=z_0;
\end{aligned} \tag{1.3}$$

where $\widehat{T}=\lim_{\varepsilon\to 0}\tau^\varepsilon(T)$ and

$$\Delta_{\|\cdot\|}(v) = \begin{cases} \Delta(v) & \text{for } \|v\| \leqslant 1,\\ \infty & \text{for } \|v\| > 1. \end{cases} \tag{1.4}$$

Here regions with $\dot{\hat{t}}(\tau)\equiv 0$ correspond to viscous slip while $\dot{\hat{t}}(\tau) \in (0,1)$ means motion under dry friction and $\dot{\hat{t}}(\tau)\equiv 1$ corresponds to sticking $\big(\dot{\widehat{z}}(\tau)\equiv 0\big)$. In Section 2 we will derive our model (1.3) as a weak form of (1.1). We argue that (1.3) is equivalent to (1.1) under the assumption that $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$ for almost all $\tau \in [0,\widehat{T}]$ (that means no viscous slips occur).

In the series of papers [MMG94, MSGM95, GMM98, PiM03, MMP05] similar approaches were developed to resolve discontinuities in rate-independent systems. In particular, a similar time reparametrization was introduced and our limit problem, which is formulated in the arc-length parametrization, has close relations to the dissipative graph solutions in [MSGM95]. However, the regularization there does not use viscous friction but rather kinetic terms giving rise to forces via inertia. In [MMP05] a situation with quadratic and convex energy was considered and convergence of the regularized solutions to the rate-independent limit solution, which is Lipschitz continuous in time, is established.

Additionally we will show that a time-incremental problem can be used to find solutions of (1.3). The time discretization replaces the need of the regularization via $\frac{\varepsilon}{2}\|v\|^2$. Choosing $N\in\mathbb{N}$ and $h=\widehat{T}/N$ we define $\tau_j = jh$. The time-incremental problem reads:

Problem 1.1 *Let $\hat{t}_0=0$ and $\widehat{z}_0=z_0$. For $j=1,\dots,N$ find $\widehat{z}_j$ with*

$$\widehat{z}_j \in \operatorname{argmin}\left\{\, I\big(\hat{t}_{j-1},\widehat{z}\big)+\Delta\big(\widehat{z}-z_{j-1}\big) \;\big|\; \big\|\widehat{z}-z_{j-1}\big\| < h \,\right\} \tag{1.5}$$

and then let $\hat{t}_j = \hat{t}_{j-1}+h-\big\|\widehat{z}_j-\widehat{z}_{j-1}\big\|$.

In Section 4 we will show that a subsequence of the linear interpolants associated with the solutions of (1.5) also converges to a solution of (1.3). We cannot expect convergence of the full sequence, since problem (1.3) may have several solutions. Hence, different subsequences may have different limits.

One special problem in establishing (1.3) is that from the weak convergence of the approximate solutions, obtained by viscous regularization or time discretization, we only obtain the inequality

$$1 \geqslant \dot{\hat{t}}(\tau) + \|\dot{\widehat{z}}(\tau)\| =: \lambda(\tau) \quad \text{ for a.a. } \tau\in[0,\widehat{T}] \tag{1.6}$$

for the limit function $(\hat{t}, \widehat{z})$. In Example 4.4 we show that strict inequality can occur. For obtaining solutions to (1.3) there are two ways. First, we establish under rather mild assumptions (see Condition 2.5), that $\lambda(\tau) \geqslant c_* > 0$ a.e. in $[0, \widehat{T}]$. By a suitable reparametrization it is then possible to find a solution to (1.3). Second, we provide a restrictive compatibility condition between Δ and $\|\cdot\|$ (see Condition 3.4) which guarantees that any limit function $(\hat{t}, \widehat{z})$ automatically satisfies (1.6) with equality, i.e., reparametrization is not necessary.

Acknowledgments: This research was supported by the German Research Foundation DFG within the Collaborative Research Center SFB 404 "Mehrfeldprobleme in der Kontinuumsmechanik", subproject C7, at Universität Stuttgart and by the European Union under HPRN-CT-2002-00284 "Smart Systems". AM is grateful to Florian Schmid and João Martins for fruitful discussions.

2 Formulation of the problem

Let X be a finite-dimensional Banach space. We denote by X^* its dual space. Before formulating our basic task we start with some motivations. First, assuming $I \in C^1\big([0,T]\times X, \mathbb{R}\big)$ and that Δ is convex and homogenous of degree 1, we aim at solving the following problem:

$$\begin{aligned}&\text{Find } z \in \mathrm{W}^{1,1}([0,T], X) \text{ such that}\\ &0 \in \partial\Delta(\dot{z}) + \mathrm{D}_z I(t,z) \subset X^* \text{ and } \ z(0) = z_0 \in X\,.\end{aligned} \tag{2.1}$$

Proposition 2.1 *Let $z \in \mathrm{W}^{1,1}\big([0,T], X\big)$ be a solution of (2.1) and $\widehat{T} = T + \int_0^T \big\|\dot{z}(s)\big\|\,ds$. Then there exists a pair $\big(\hat{t}, \widehat{z}\big) \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}], \mathbb{R}\times X\big)$, such that for almost all (shortly a.a.) $\tau \in [0, \widehat{T}]$ we have*

$$0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\widehat{z}}(\tau)\big) + \mathrm{D}_z I\big(\hat{t}(\tau), \widehat{z}(\tau)\big)\,, \tag{2.2}$$

$$1 = \dot{\hat{t}}(\tau) + \big\|\dot{\widehat{z}}(\tau)\big\|\,, \tag{2.3}$$

$$\widehat{z}(0) = z_0\,,\ \hat{t}(0) = 0\,,\ \hat{t}\big(\widehat{T}\big) = T\,, \tag{2.4}$$

with $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$. Conversely, let a pair $(\hat{t}, \widehat{z}) \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}], \mathbb{R}\times X\big)$ be a solution of (2.2)–(2.4) with $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$ for a.a. $\tau \in [0, \widehat{T}]$, then (2.3) guarantees that $\hat{t} : [0, \widehat{T}] \to [0,T]$ has a continuous inverse $\tau : [0,T] \to [0,\widehat{T}]$ and that $z : t \mapsto \widehat{z}\big(\tau(t)\big)$ lies in $\mathrm{W}^{1,1}\big([0,T], X\big)$ and solves (2.1).

Proof: Indeed, let $z \in \mathrm{W}^{1,1}\big([0,T], X\big)$ solve (2.1). For this z we define $\tau : [0,T] \to [0,\widehat{T}]$ via

$$\tau(t) := t + \int_0^t \|\dot{z}(s)\|\,ds \text{ and } \widehat{T} = \tau(T)\,.$$

We denote the inverse of τ by $\hat{t} : [0,\widehat{T}] \to [0,T]$ and define $\widehat{z} : [0,\widehat{T}] \to X$ via

$$\widehat{z}(\tau) := z\big(\hat{t}(\tau)\big)\,.$$

Obviously $\hat{t} \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}],\mathbb{R}\big)$, $\widehat{z} \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}],X\big)$ with $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$ and $\dot{\hat{t}}(\tau)+\big\|\dot{\widehat{z}}(\tau)\big\| = 1$ for a.a. $\tau \in [0,\widehat{T}]$. Moreover $(\hat{t},\widehat{z})$ satisfies

$$0 \in \partial\Delta\left(\frac{1}{\dot{\hat{t}}(\tau)}\,\dot{\widehat{z}}(\tau)\right) + \mathrm{D}_z I\big(\hat{t}(\tau),\widehat{z}(\tau)\big)\,,$$

with $\hat{t}(0) = 0$ and $\widehat{z}(0) = z_0$. Taking into account that $\partial\Delta : X \to X^*$ is homogenous of order 0 and that $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$ for a.a. τ, we obtain

$$0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\widehat{z}}\big) + \mathrm{D}_z I\big(\hat{t}(\tau),\widehat{z}(\tau)\big) \text{ for a.a. } \tau\,,$$

where $\Delta_{\|\cdot\|} : X \to [0,\infty)$ is defined via (1.4).

Conversely let a pair $(\hat{t},\widehat{z})$ be any solution of (2.2)–(2.4) with $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$ for a.a. τ. We set $T = \hat{t}(\widehat{T})$ and define $\tau(t)$ as the inverse of $\hat{t} : [0,\widehat{T}] \to [0,T]$, which exists due to (2.3). Then it is not difficult to see that $z \in W^{1,1}\big([0,T],X\big)$, defined via $z(t) := \widehat{z}\big(\tau(t)\big)$, satisfies (2.1). This proves Proposition 2.1. ∎

Obviously, (2.2)–(2.4) is a more general problem than (2.1), since equivalence (due to Proposition 2.1) is only obtained in regions where $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$ for a.a. τ. Further on we will consider a new model, namely (2.2)–(2.4) without the restriction $\big\|\dot{\widehat{z}}(\tau)\big\| < 1$. In particular, regions with $\big\|\dot{\widehat{z}}(\tau)\big\| = 1$ correspond to fast motion which is much faster than the process time which is encoded in $\dot{\hat{t}}(\tau) = 1 - \big\|\dot{\widehat{z}}(\tau)\big\| = 0$. Note that this situation corresponds to the rescaled gradient flow at fixed process time. Indeed, to illustrate this, let us consider the following example:

Example 2.2 *Assume that $X = \mathbb{R}^n$ and that $\|\cdot\|$ is the Euclidean norm. Moreover for some $\delta > 0$ we choose $\Delta(w) = \delta\|w\|$. Then, we have*

$$\partial\Delta_{\|\cdot\|}(w) = \begin{cases} \{\,\eta \mid \|\eta\| \leqslant \delta\,\} & \text{for } w = 0\,,\\ \frac{\delta}{\|w\|}\,w & \text{for } 0 < \|w\| < 1\,,\\ \{\,\alpha\, w \mid \alpha \geqslant \delta\,\} & \text{for } \|w\| = 1\\ \emptyset & \text{for } \|w\| > 1\,. \end{cases}$$

For a solution $\widehat{z}$ of (2.2) assume that $\big\|\dot{\widehat{z}}(\tau)\big\| = 1$ for $\tau \in [\tau_1,\tau_2]$. Then $\dot{\hat{t}}(\tau) = 1 - \big\|\dot{\widehat{z}}(\tau)\big\| \equiv 0$ for $\tau \in [\tau_1,\tau_2]$, hence $\hat{t}(\tau) \equiv \hat{t}(\tau_1) =: \hat{t}_1$, where $\hat{t}_1$ is a fixed value of the process time. Hence, in this case (2.2) leads, for a.a. $\tau \in [\tau_1,\tau_2]$, to

$$0 = \alpha(\tau)\,\dot{\widehat{z}}(\tau) + \mathrm{D}_z I\big(\hat{t}_1,\widehat{z}(\tau)\big)$$

for some $\alpha(\tau) \geq \delta$, or equivalently to

$$\dot{\widehat{z}}(\tau) = \beta(\tau)\,\mathrm{D}_z I\big(t_1,\widehat{z}(\tau)\big) \in \mathbb{S}^{n-1} = \{\,z \in \mathbb{R}^n \mid \|z\| = 1\,\}\,,$$

with $\beta(\tau) = -1/\alpha(\tau)$. The latter is exactly the rescaled gradient flow at fixed process time $\hat{t}_1$.

Remark 2.3 *The problem* (2.2) *is rate-independent in the following sense. If the original functional* $I : [0,T]{\times}X \to \mathbb{R}$ *is replaced by a time rescaled one* $J : [0,S]{\times}X \to \mathbb{R}$*, i.e.* $I(t,y) = J\big(\tilde{s}(t),y\big)$ *for some strictly monotone function* $\tilde{s} : [0,T] \to [0,S]$ *with* $\tilde{s}(T) = S$*, then the rescaled problem* (2.2)–(2.4) *with* $I(t,z)$ *replaced by* $J(s,y)$ *has a solution* $\big(\hat{s},\hat{y}\big) : [0,\hat{S}] \to [0,S]{\times}X$ *which is obtained from* $(\hat{t},\widehat{z})$ *by rescaling as follows:*

$$\hat{s}(\sigma) = \tilde{s}\big(\hat{t}\big(\tilde{\tau}(\sigma)\big)\big) \quad \text{and} \quad \hat{y}(\sigma) = \hat{z}\big(\tilde{\tau}(\sigma)\big),$$

where $\tilde{\tau} : [0,\hat{S}] \to [0,\widehat{T}]$ *is the rescaling of the arc length defined via*

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\tilde{\tau}(\sigma) = \frac{1}{\dot{\tilde{s}}\big(\hat{t}(\tau)\big)\dot{\hat{t}}(\tau) + \|\dot{\widehat{z}}(\tau)\|}\Bigg|_{\tau\,=\,\tilde{\tau}(\sigma)}.$$

Remark 2.4 *We consider two solutions* $\big(\hat{t}_1,\widehat{z}_1\big) \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}_1],\mathbb{R}{\times}X\big)$ *and* $\big(\hat{t}_2,\widehat{z}_2\big) \in \mathrm{W}^{1,\infty}\big([\widehat{T}_1,\widehat{T}],\mathbb{R}{\times}X\big)$ *of* (2.2)–(2.4) *on the intervals* $\big[0,\widehat{T}_1\big]$ *and* $\big[\widehat{T}_1,\widehat{T}\big]$*, respectively. If additionally* $\hat{t}_1\big(\widehat{T}_1\big) = \hat{t}_2\big(\widehat{T}_1\big)$ *and* $\widehat{z}_1\big(\widehat{T}_1\big) = \widehat{z}_2\big(\widehat{T}_1\big)$*, then it is easy to see that the pair* $(\hat{t},\widehat{z}) : [0,\widehat{T}] \to \mathbb{R}{\times}X$ *defined by*

$$\big(\hat{t}(\tau),\widehat{z}(\tau)\big) = \begin{cases} \big(\hat{t}_1(\tau),\widehat{z}_1(\tau)\big) & \text{for } \tau \in \big[0,\widehat{T}_1\big], \\ \big(\hat{t}_2(\tau),\widehat{z}_2(\tau)\big) & \text{for } \tau \in \big[\widehat{T}_1,\widehat{T}\big], \end{cases}$$

is a solution of (2.2)–(2.4) *belonging to* $\mathrm{W}^{1,\infty}\big([0,\widehat{T}],\mathbb{R}{\times}X\big)$.

In the sequel we make the following assumptions on $\Delta(\cdot)$ and $I(t,\cdot)$.

Condition 2.5 *Assume that* $\Delta : X \to [0,\infty)$ *is convex, homogenous of degree 1 and satisfies*

$$C_\Delta\|v\|_X \leqslant \Delta(v) \leqslant C_\Delta^{-1}\|v\|_X\,, \tag{2.5}$$

for all $v \in X$*. Moreover, we assume that* $I \in C^1\big([0,T]{\times}X,\mathbb{R}\big)$ *with* $I(t,z) \geqslant 0$.

Subsequently, we assume that all our solutions are contained in a suitable large ball $B_R(0) = \big\{z \in X \,\big|\, \|z\| \leqslant R\big\}$*. We assume that the estimate*

$$\big|\partial_t I(t,z)\big| \leqslant C_I, \quad \big\|\mathrm{D}_z I(t,z)\big\| \leqslant M, \quad \text{for all } z \in B_R(0). \tag{2.6}$$

This will enable us to make most estimates more explicit.

Condition (2.6) is chosen here for convenience only. In light of the more geometric formulation in [Mie05] it could be replaced by the more general condition

$$|\partial_t I(t,z)| + \|\mathrm{D}_z I(t,z)\| \leq M_1\Big(I(t,z) + M_0\Big), \quad \text{for all } (t,z) \in [0,T]{\times}X.$$

After these preliminaries we state our main existence result, which is proved in two different ways in the subsequent sections.

Theorem 2.6 *Let Condition 2.5 be satisfied. Then there exists* $\widehat{T} > 0$*, such that problem* (2.2)–(2.4) *admits at least one solution* $\big(\hat{t},\widehat{z}\big) \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}],\,\mathbb{R}{\times}X\big)$.

3 Convergence for the regularized problem

Here we study the limit passage for the viscously regularized problem. We define $\Delta^\varepsilon(v) = \Delta(v) + \frac{\varepsilon}{2}\|v\|^2$ where $\varepsilon > 0$ is a small viscosity and $\|\cdot\|$ is again an arbitrary norm on X. The regularized problem reads

$$z(0) = z_0, \quad 0 \in \partial\Delta^\varepsilon\big(\dot z(t)\big) + \mathrm{D}_z I\big(t, z(t)\big), \quad \text{a.e. on } [0,T]. \tag{3.1}$$

Since $\Delta^\varepsilon(v) \geqslant \frac{\varepsilon}{2}\|v\|^2$, we know from [CoV90] that (3.1) has a solution $z^\varepsilon \in \mathrm{W}^{1,2}\big((0,T),X\big)$ and it satisfies, for $0 \leqslant s < t \leqslant T$, the energy estimate

$$I\big(t, z^\varepsilon(t)\big) + \int_s^t \Delta^\varepsilon\big(\dot z^\varepsilon(r)\big)\,\mathrm{d}r \leqslant I(s, z^\varepsilon(s)) + \int_s^t \partial_r I\big(r, z^\varepsilon(r)\big)\,\mathrm{d}r. \tag{3.2}$$

In fact, by convexity and $\Delta^\varepsilon(0) = 0$ we have $\Delta^\varepsilon(v) \leqslant \langle \eta, v\rangle$ for each $\eta \in \partial\Delta^\varepsilon(v)$. Hence, (3.1) implies $\Delta^\varepsilon(\dot z^\varepsilon(t)) + \langle \mathrm{D}_z I(t, z^\varepsilon(t)), \dot z^\varepsilon(t)\rangle \leqslant 0$ and integration over $[s,t]$ gives (3.2).

With (2.6) we find $\int_0^T \Delta^\varepsilon\big(\dot z^\varepsilon(t)\big)\,\mathrm{d}t \leqslant C$, where C is independent of ε. Using $\Delta^\varepsilon(v) \geqslant \Delta(v)$ and (2.5) we conclude that

$$\widehat T^\varepsilon = T + \textstyle\int_0^T \big\|\dot z^\varepsilon(t)\big\|\,\mathrm{d}t$$

is bounded by $T + C/C_\Delta$. Choosing a subsequence, we have $\widehat T^\varepsilon \to \widehat T$. We define the arc length

$$\tau^\varepsilon(t) = t + \int_0^t \big\|\dot z^\varepsilon(s)\big\|\,\mathrm{d}s$$

and the rescalings

$$\hat t^\varepsilon = \big(\tau^\varepsilon\big)^{-1}, \quad \widehat z^\varepsilon(\tau) = z^\varepsilon\big(\hat t^\varepsilon(\tau)\big).$$

By definition $\big(\dot{\hat t}^\varepsilon\big)(\tau) + \big\|\dot{\widehat z}^\varepsilon(\tau)\big\| \equiv 1$ which implies $\big(\hat t^\varepsilon, \widehat z^\varepsilon\big) \in C^{\mathrm{Lip}}\big([0,\widehat T^\varepsilon], \mathbb{R}\times X\big)$ with the uniform Lipschitz constant 1. For notational convenience, we extend $\big(\hat t^\varepsilon, \widehat z^\varepsilon\big)$ in the case $\widehat T^\varepsilon < \widehat T$ on the interval $[\widehat T^\varepsilon, \widehat T]$ with the constant value $\big(\hat t^\varepsilon(\widehat T^\varepsilon), \widehat z^\varepsilon(\widehat T^\varepsilon)\big)$, such that all functions are defined on $[0,\widehat T]$. By the Arzela-Ascoli theorem (use $\dim X < \infty$), choosing a further subsequence, we have uniform convergence on $[0,\widehat T]$, i.e.

$$\big(\hat t^\varepsilon, \widehat z^\varepsilon\big) \longrightarrow \big(\hat t, \widehat z\big) \quad \text{in } C^0\big([0,\widehat T], \mathbb{R}\times X\big) \text{ as } \varepsilon \to 0. \tag{3.3}$$

Theorem 3.1 *Any limit function $\big(\hat t, \widehat z\big)$ constructed above satisfies for a.a. $\tau \in [0,\widehat T]$ the limit problem*

$$\begin{aligned}
&0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\widehat z}(\tau)\big) + \mathrm{D}_z I\big(\hat t(\tau), \widehat z(\tau)\big) \subset X^*\,,\\
&1 \geqslant \dot{\hat t}(\tau) + \big\|\dot{\widehat z}(\tau)\big\|\,,\\
&\widehat z(0) = z_0\,,\ \hat t(0) = 0\,,\ \hat t\big(\widehat T\big) = T\,.
\end{aligned} \tag{3.4}$$

Proof: We note that $\big\|\dot{z}^\varepsilon(t)\big\|$ is finite for a.e. $t \in [0,T]$. For these t consider $\tau = \tau^\varepsilon(t)$, then $\dot{z}^\varepsilon(t) = \dfrac{1}{1 - \big\|\dot{\widehat{z}}^\varepsilon(\tau)\big\|}\big(\dot{\widehat{z}}^\varepsilon\big)(\tau)$. Moreover, with

$$g(\rho) = \begin{cases} -\rho - \log(1{-}\rho) & \text{for } \rho \in [0,1), \\ \infty & \text{for } \rho \geqslant 1, \end{cases}$$

and $\widehat{\Delta}^\varepsilon(v) = \Delta(v) + \varepsilon g\big(\|v\|\big)$ we find

$$\partial\Delta^\varepsilon\big(\dot{z}^\varepsilon(t)\big) = \partial\widehat{\Delta}^\varepsilon\big(\dot{\widehat{z}}^\varepsilon(\tau)\big).$$

Hence, the rescaled functions $\hat{t}^\varepsilon, \widehat{z}^\varepsilon$ satisfy

$$0 \in \partial\widehat{\Delta}^\varepsilon\big(\dot{\widehat{z}}^\varepsilon(\tau)\big) + \mathrm{D}_z I\big(\hat{t}^\varepsilon(\tau), \widehat{z}^\varepsilon(\tau)\big), \quad 1 = \dot{\hat{t}}^\varepsilon(\tau) + \big\|\dot{\widehat{z}}^\varepsilon(\tau)\big\|.$$

Since $\big(\hat{t}^\varepsilon, \widehat{z}^\varepsilon\big)$ converges uniformly on $[0,\widehat{T}]$ and $\mathrm{D}_z I$ is continuous, we know that

$$\sigma^\varepsilon(\tau) := -\mathrm{D}_z I\big(\hat{t}^\varepsilon(\tau), \widehat{z}^\varepsilon(\tau)\big)$$

converges uniformly as well. Moreover, since $\dim X < \infty$, we have

$$\big(\dot{\widehat{z}}^\varepsilon\big) \overset{*}{\rightharpoonup} \dot{\widehat{z}} \quad \text{in } L^\infty\big([0,\widehat{T}], X\big).$$

Using part (i) of the subsequent Lemma 3.5 we conclude that $\sigma^0 = \lim_{\varepsilon\to 0}\sigma^\varepsilon = -\mathrm{D}_z I\big(\hat{t}, \widehat{z}\big)$ satisfies $\sigma^0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\widehat{z}}\big)$ which is the inclusion in first line of (3.4). The inequality on the second line of (3.4) is an easy consequence of the weak–$*$ convergence of $\big(\hat{t}^\varepsilon, \widehat{z}^\varepsilon\big)$. ■

As we will see in Section 4 the equality $\dot{\hat{t}}(\tau) + \big\|\dot{\widehat{z}}(\tau)\big\| = 1$ for a.a. $\tau \in [0,\widehat{T}]$ does not hold automatically. However, since our problem is rate-independent, we will show through reparametrization of time, that $\big(\hat{t}, \widehat{z}\big)$ can be transformed into a true solution $\big(\tilde{t}, \tilde{z}\big)$. Indeed, for a solution $\big(\hat{t}, \widehat{z}\big)$ of (3.4) define $L(\tau) := \int_0^\tau \Big(\dot{\hat{t}}(\sigma) + \big\|\dot{\widehat{z}}(\sigma)\big\|\Big)\,\mathrm{d}\sigma$. Reparametrization is possible if L is strictly increasing, which is a consequence of our assumptions.

Lemma 3.2 *If Condition 2.5 holds, then there is $c_* > 0$ such that for all limit functions $(\hat{t}, \widehat{z})$ we have $\dot{L}(\tau) = \dot{\hat{t}}(\tau) + \big\|\dot{\widehat{z}}(\tau)\big\| \geq c_*$ a.e. on $[0,\widehat{T}]$.*

Proof: Using (2.5) and the energy inequality (3.2) we find

$$\begin{aligned} C_\Delta \int_s^t \|\dot{z}^\varepsilon(r)\|\,\mathrm{d}r &\leqslant \int_s^t \Delta^\varepsilon(\dot{z}^\varepsilon(r))\,\mathrm{d}r \\ &\leqslant I(s, z^\varepsilon(s)) - I(t, z^\varepsilon(t)) + \int_s^t \partial_r I(r, z^\varepsilon(r))\,\mathrm{d}r \\ &\leqslant M\|z^\varepsilon(t) - z^\varepsilon(s)\| + 2C_I|t-s| \end{aligned}$$

Using the rescaling together with rate-independence of the left-hand side we find, for $0 \leqslant \tau_1 < \tau_2 \leqslant \widehat{T}$,

$$\int_{\tau_1}^{\tau_2} \|\dot{\widehat{z}}^{\varepsilon}(\sigma)\| \,\mathrm{d}\sigma \le \frac{M}{C_\Delta} \|\widehat{z}^{\varepsilon}(\tau_2) - \widehat{z}^{\varepsilon}(\tau_1)\| + \frac{2C_I}{C_\Delta} \Big| \hat{t}^{\varepsilon}(\tau_2) - \hat{t}^{\varepsilon}(\tau_1) \Big|.$$

Since by construction $\tau_2 - \tau_1 = \hat{t}^{\varepsilon}(\tau_2) - \hat{t}^{\varepsilon}(\tau_1) + \int_{\tau_1}^{\tau_2} \|\dot{\widehat{z}}^{\varepsilon}(\sigma)\| \,\mathrm{d}\sigma$, we conclude

$$\tau_2 - \tau_1 \leqslant C_* \left(\hat{t}^{\varepsilon}(\tau_2) - \hat{t}^{\varepsilon}(\tau_1) + \|\widehat{z}^{\varepsilon}(\tau_2) - \widehat{z}^{\varepsilon}(\tau_1)\| \right) \text{ with } C_* = \max\{1 + 2C_I/C_\Delta, M/C_\Delta\}.$$

Using uniform convergence of $(\hat{t}^{\varepsilon}, \widehat{z}^{\varepsilon})$ to $(\hat{t}, \widehat{z})$ we pass to the limit $\varepsilon \to 0$ and find

$$\tau_2 - \tau_1 \leqslant C_* \left(\hat{t}^{\varepsilon}(\tau_2) - \hat{t}^{\varepsilon}(\tau_1) + \|\widehat{z}(\tau_2) - \widehat{z}(\tau_1)\| \right) \quad \text{for all } 0 \leqslant \tau_1 < \tau_2 \leqslant \widehat{T}.$$

This implies the desired result $\dot{L}(\tau) = \dot{\hat{t}}(\tau) + \|\dot{\widehat{z}}(\tau)\| \ge 1/C_* =: c_*$ a.e. on $[0, \widehat{T}]$. ■

Denoting the inverse of $L(\tau)$ by $L^{\vee}(\theta)$, $\theta \in [0, L(\widehat{T})]$, we set $\tilde{t}_*(\theta) = \hat{t}_*\big(L^{\vee}(\theta)\big)$, $\tilde{z}_*(\theta) = \widehat{z}_*\left(L^{\vee}(\theta)\right)$. Using $\dot{L}(\tau) > 0$ a.e., it is not difficult to see that

$$\begin{aligned} &0 \in \partial\Delta_{\|\cdot\|} \left(\dot{L}\big(L^{\vee}(\theta)\big) \dot{\tilde{z}}_*(\theta) \right) + \mathrm{D}_z I\big(\tilde{t}_*(\theta), \tilde{z}_*(\theta)\big) \,, \\ &\dot{\tilde{t}}_*(\theta) + \big\| \dot{\tilde{z}}_*(\theta) \big\| = 1 \,. \end{aligned} \tag{3.5}$$

Simple analysis together with homogeneity of order zero of $\partial\Delta : X \to X^*$ and the fact that $\partial\Delta_{\|\cdot\|}(\xi) = \partial\Delta(\xi)$, for $\|\xi\| < 1$, yield

$$\begin{aligned} &0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\tilde{z}}_*(\theta)\big) + \mathrm{D}_z I\big(\tilde{t}_*(\theta), \tilde{z}_*(\theta)\big) \,, \\ &\dot{\tilde{t}}_*(\theta) + \big\| \dot{\tilde{z}}_*(\theta) \big\| = 1 \,, \end{aligned} \tag{3.6}$$

for $\theta \in \big[0, L\big(\widehat{T}\big)\big]$. Using $\dot{L}(\tau) \geqslant c_*$ we have $L(\widehat{T}) \ge c_* \widehat{T}$, where c_* does not depend on initial data and Remark 2.4. Hence, we can extend $\big(\tilde{t}_*, \tilde{z}_*\big)$, which is a solution of (3.6) on $\big[0, L\big(\widehat{T}\big)\big]$, to all of $[0, \widehat{T}]$.

Corollary 3.3 *Problem* (2.2)–(2.4) *admits at least one solution.*

Using the following structural compatibility condition between Δ and $\|\cdot\|$ we will show that reparametrization is not necessary.

Condition 3.4 *For all $\sigma \in X^*$ the norm $\|\cdot\|$ is affine when restricted to the set*

$$M(\sigma) := \big\{ v \in X \,|\, \sigma \in \partial\Delta_{\|\cdot\|}(v) \big\}.$$

(It is easy to see that this assumption always holds when $\Delta = \delta\|\cdot\|$ for some $\delta > 0$.)

To establish the following results we recall that the subgradients $\partial\Delta_{\|\cdot\|}, \partial\widehat{\Delta}^{\varepsilon} : X \to 2^{X^*}$ define (multi-valued) maximal monotone operators, see [EkT76, Zei90]. On the Hilbert space $H = \mathrm{L}^2\big([0, \widehat{T}], X\big)$ the mapping $\partial\widehat{\Delta}^{\varepsilon}$ induces a maximal monotone operator A^{ε} via

$$\sigma \in A^{\varepsilon} v \quad \Leftrightarrow \quad \sigma(\tau) \in \partial\widehat{\Delta}^{\varepsilon}\big(v(\tau)\big) \quad \text{for a.a. } \tau \in [0, \widehat{T}].$$

Similarly, define A^0 using $\Delta_{\|\cdot\|}$.

Lemma 3.5 *Let $v^\varepsilon, \sigma^\varepsilon \in H = \mathrm{L}^2\big([0,\widehat{T}], X\big)$ with $\sigma^\varepsilon \in A^\varepsilon v^\varepsilon$. Moreover, assume $\sigma^\varepsilon \to \sigma^0$ in $C^0\big([0,\widehat{T}], X\big)$ and $\|v^\varepsilon\|_{\mathrm{L}^\infty} \leqslant 1$ and $v^\varepsilon \rightharpoonup v^0$ in H. Then,*

$$\sigma^0 \in A^0 v^0, \tag{i}$$

$$\big\|v^\varepsilon(\cdot)\big\|_X \rightharpoonup \hat{n} \quad \text{in } \mathrm{L}^2\big([0,T], \mathbb{R}\big) \text{ with } \hat{n} \geqslant \big\|v^0(\cdot)\big\|_X. \tag{ii}$$

$$\text{If Condition 3.4 holds, then } \hat{n} = \big\|v^0(\cdot)\big\|_X. \tag{iii}$$

Proof: In fact part (i) follows from the standard Browder-Minty theory together with the convergence of A^ε to A^0 (see [Bre73, EkT76, Zei90]). However, since we also want to establish (iii) we prove (i) also using Young measures. We consider the sequence $a^\varepsilon = \big(v^\varepsilon, \sigma^\varepsilon\big)_{\varepsilon>0}$ in $\mathrm{L}^2\big([0,\widehat{T}], X\times X^*\big)$. With $G^\varepsilon = \big\{(v,\sigma) \in X\times X^* \,|\, \sigma \in \partial\widehat{\Delta}^\varepsilon(v)\big\}$ we have

$$a^\varepsilon(\tau) \in G^\varepsilon \cap K \quad \text{for a.e. } \tau \in [0,\widehat{T}],$$

where K is a compact set in $X\times X^*$. Hence, after choosing a subsequence a^ε generates an L^∞-Young measure $\mu \in \mathrm{YM}\big((0,\widehat{T}), X\times X^*\big)$ where $\mu(\tau) \in \mathrm{Prob}\big(X\times X^*\big)$ (see [Dac82, Rou97, Mie99, Mie04]). Because of $\mathrm{dist}_{\mathrm{Hausdorff}}\big(G^\varepsilon, G^0\big) \to 0$ for $\varepsilon \to 0$, we conclude $\mu(\tau) \in \mathrm{Prob}(G^0)$. Moreover, $\|\sigma^\varepsilon - \sigma^0\|_{C^0} \to 0$ implies that a.e. $\mu(\tau)$ has the form $\mu(\tau) = \nu(\tau) \otimes \delta_{\sigma^0(\tau)}$, with $\nu(\tau) \in \mathrm{Prob(X)}$. Together with $\mu(\tau) \in \mathrm{Prob}(G^0)$ this implies $\nu(\tau) \in \mathrm{Prob}\big(M(\sigma^0(\tau))\big)$ where $M(\sigma)$ is defined in Condition 3.4, i.e. $M(\sigma)$ is the natural projection of $G^0 \cap \big(X\times\{\sigma\}\big)$ onto X.

Since A^0 is maximal monotone, the sets $M(\sigma)$ are closed and convex in X. By the Young measure theory we have for $v^\varepsilon \rightharpoonup v^0$ and $\big\|v^\varepsilon(\cdot)\big\| \rightharpoonup \hat{n}$ the identities

$$v^0(\tau) = \int_{\hat{v}\in X} \hat{v}\,\nu(\tau, \mathrm{d}\hat{v}) \quad \text{and} \quad \hat{n}(\tau) = \int_{\hat{v}\in X} \|\hat{v}\|\,\nu(\tau, \mathrm{d}\hat{v}). \tag{3.7}$$

Convexity of $M(\sigma^0)$ and $\nu(\tau) \in \mathrm{Prob}\big(M(\sigma^0(\tau))\big)$ imply $v^0(\tau) \in M(\sigma^0)$ and hence $\sigma^0(\tau) \in \partial\Delta_{\|\cdot\|}\big(v^0(\tau)\big)$. This proves part (i).

Part (ii) follows from (3.7) by Jensen's inequality since $\|\cdot\|_X$ is convex.

To obtain part (iii) we note that on each set $M(\sigma)$ the function $\|\cdot\|_X$ is affine, namely we have

$$\|v\| = \big\|v^0(\tau)\big\| + \big\langle \sigma^0(\tau), v - \hat{v}^0(\tau)\big\rangle \quad \text{for all } v \in M\big(\sigma^0(\tau)\big).$$

Integration with respect to the measure $\nu(\tau)$ then gives

$$\hat{n}(\tau) = \int_{v\in X} \|v\|\nu(\tau,\mathrm{d}v) = \int_{v\in X} \Big(\big\|v^0(\tau)\big\| + \big\langle\sigma^0(\tau), v - v^0(\tau)\big\rangle\Big)\nu(\tau,\mathrm{d}v) = \big\|v^0(\tau)\big\|.$$

This proves Lemma 3.5. ■

Corollary 3.6 *Let Condition 3.4 hold. Then any limit function $\big(\hat{t}, \widehat{z}\big)$ constructed above is a solution of* (2.2)–(2.4) *(i.e., reparametrization is not necessary).*

Indeed, for this purpose we apply Lemma 3.5. In our case, $A^\varepsilon = \partial\widehat{\Delta}^\varepsilon$, $A^0 = \partial\Delta_{\|\cdot\|}$, $v^\varepsilon = \dot{z}^\varepsilon$ and the compatibility condition 3.4 allow us to pass to the limit in $\dot{t}^\varepsilon + \big\|\dot{z}^\varepsilon\big\| = 1$.

4 Time discretization

Here we provide a second alternative to obtain an approximate solution whose limit will provide solutions for (2.2)–(2.4). This approach is close to the numerical treatment of the rate-independent problem. Indeed, let $0 = \tau_0 < \tau_1 < \ldots < \tau_N = \widehat{T}$ be any partition, with $\tau_j - \tau_{j-1} = h$, $j = 1, \ldots, N$. Consider the following minimization problem:

Problem 4.1 *For given $\hat{t}_0 = 0, \widehat{z}_0 = z_0$ find pairs $\big(\hat{t}_j, \widehat{z}_j\big) \in [0, \widehat{T}] \times X$, for $j = 1, \ldots, N$, such that for all $w \in X$ and with $\|w - \widehat{z}_{j-1}\| \leqslant h$*

$$I\big(\hat{t}_{j-1}, \widehat{z}_j\big) + \Delta(\widehat{z}_j - \widehat{z}_{j-1}) \leqslant I(\hat{t}_{j-1}, w) + \Delta(w - \widehat{z}_{j-1}) \tag{IP}$$

and $\hat{t}_j := \hat{t}_{j-1} + h - \big\|\widehat{z}_j - \widehat{z}_{j-1}\big\|$, for $j = 1, \ldots, N$.

For notational convenience, we use the following shorthand of (IP):

$$\begin{aligned}\widehat{z}_j &\in \operatorname{argmin}\big\{I\big(\hat{t}_{j-1}, w\big) + \Delta\big(w - \widehat{z}_{j-1}\big) \,\big|\, \big\|w - \widehat{z}_{j-1}\big\| \leqslant h\big\} \\ &= \operatorname{argmin}\left\{ I\big(\hat{t}_{j-1}, w\big) + h\Delta_{\|\cdot\|}\left(\frac{w - \widehat{z}_{j-1}}{h}\right) \,\middle|\, w \in X \right\},\end{aligned}$$

where "argmin" denotes the set of minimizers.

Proposition 4.2 *The problem* (IP) *always has a solution $\big(\hat{t}_j, \widehat{z}_j\big)_{j=1,\ldots N}$. Any solution satisfies*

$$\text{(a)} \quad I\big(\hat{t}_j, \widehat{z}_j\big) - I\big(\hat{t}_{j-1}, \widehat{z}_{j-1}\big) + \Delta\big(\widehat{z}_j - \widehat{z}_{j-1}\big) \leqslant \int_{\hat{t}_{j-1}}^{\hat{t}_j} \partial_t I\big(\sigma, \widehat{z}_j\big)\,\mathrm{d}s\,,$$

$$\text{(b)} \quad 0 \in \partial\Delta_{\|\cdot\|}\left(\frac{\widehat{z}_j - \widehat{z}_{j-1}}{h}\right) + \mathrm{D}_z I\big(\hat{t}_{j-1}, \widehat{z}_j\big)\,,$$

$$\text{(c)} \quad \frac{\hat{t}_j - \hat{t}_{j-1}}{h} + \frac{\big\|\widehat{z}_j - \widehat{z}_{j-1}\big\|}{h} = 1\,.$$

Proof: From $I(t, z) \geqslant 0$ it follows that

$$I\big(\hat{t}_{j-1}, w\big) + \Delta\big(w - \widehat{z}_{j-1}\big) \geqslant C_\Delta \big\|w - \widehat{z}_{j-1}\big\|\,.$$

Since the balls $\big\{w \in X \,\big|\, \big\|w - \widehat{z}_{j-1}\big\| \leqslant h\big\}$ are compact in X, and I and Δ are continuous, the existence of minimizers $\widehat{z}_j$ is classical. Next we prove part (a). Indeed, by construction, for all $\big\|w - \widehat{z}_{j-1}\big\| \leqslant h$, we have

$$I\big(\hat{t}_{j-1}, \widehat{z}_j\big) + \Delta\big(\widehat{z}_j - \widehat{z}_{j-1}\big) \leqslant I\big(\hat{t}_{j-1}, w\big) + \Delta\big(w - \widehat{z}_{j-1}\big)\,. \tag{4.1}$$

Taking in (4.1) $w = \widehat{z}_{j-1}$ we obtain

$$I\big(\hat{t}_{j-1}, \widehat{z}_j\big) + \Delta\big(\widehat{z}_j - \widehat{z}_{j-1}\big) \leqslant I\big(\hat{t}_{j-1}, \widehat{z}_{j-1}\big)\,, \tag{4.2}$$

or

$$I\big(\hat{t}_j, \widehat{z}_j\big) - I\big(\hat{t}_{j-1}, \widehat{z}_{j-1}\big) + \Delta\big(\widehat{z}_j - \widehat{z}_{j-1}\big) \leqslant I\big(\hat{t}_j, \widehat{z}_j\big) - I\big(\hat{t}_{j-1}, \widehat{z}_j\big), \tag{4.3}$$

which is estimate (a).

Part (b) is in fact a standard consequence of the fact that $\widehat{z}_j$ is a minimizer. Hence, 0 is in the subdifferential of any suitable version of the multivalued differentials, e.g. the Clarke differential or the Mordukhovich differential, see e.g., Thm. 4.3 in [MoS95]. For this purpose note that we minimize $w \mapsto \psi(w) + \phi(w)$ where $\psi : w \mapsto I(\hat{t}_{j-1}, w)$ is a C^1 function and $\phi : w \mapsto \frac{1}{h}\Delta_{\|\cdot\|}\left(\frac{w-\widehat{z}_{j-1}}{h}\right)$ is convex and lower semicontinuous. For the readers convenience we give a short self-contained proof.

With ψ and ϕ defined as before, we use that $\widehat{z}_j$ minimizes the sum $\psi + \phi$, i.e.,

$$0 \leqslant A(v) = \psi(v) - \psi(\widehat{z}_j) + \phi(v + \widehat{z}_j) - \phi(\widehat{z}_j) \text{ for all } v \in X.$$

Since ψ is differentiable we may also consider

$$B(v) = \langle \mathrm{D}\psi(\widehat{z}_j), v\rangle + \phi(\widehat{z}_j + v) - \phi(\widehat{z}_j).$$

Clearly, $B(v) \geqslant 0$ for all $v \in X$ is equivalent to the desired relation (b). Now assume that B attains a negative value, namely $B(v_0) < 0$. By convexity we conclude $B(\theta v_0) \leq \theta B(v_0)$ for $\theta \in [0,1]$. Using $\psi \in C^1(X, \mathbb{R})$ we easily find $A(\theta v_0) = B(\theta v_0) + o(|\theta|)$ for $\theta \to 0$. This implies $A(\theta v_0) < 0$ for $0 < \theta \ll 1$, which is a contradiction to $A(v) \geqslant 0$ for all $v \in X$. Hence, we conclude $B \geqslant 0$ and (b) is proved.

The assertion (c) is an immediate consequence of the definition of $\big(\hat{t}_j, \widehat{z}_j\big)$. This proves Proposition 4.2. ■

Based on Proposition 4.2 we are in the position to construct an approximate solution for (2.2)–(2.4). To this end, for $\tau \in \big(\tau_{j-1}, \tau_j\big]$, we define piecewise linear ("pl") and piecewise constant ("pc") interpolants via

$$\hat{t}_h^{\,\mathrm{pl}}(\tau) := \hat{t}_{j-1} + (\tau - \tau_{j-1})\frac{\hat{t}_j - \hat{t}_{j-1}}{\tau_j - \tau_{j-1}}\,, \tag{4.4}$$

$$\widehat{z}_h^{\,\mathrm{pl}}(\tau) := \widehat{z}_{j-1} + (\tau - \tau_{j-1})\frac{\widehat{z}_j - \widehat{z}_{j-1}}{\tau_j - \tau_{j-1}}\,, \tag{4.5}$$

$$\hat{t}_h^{\,\mathrm{pc}}(\tau) := \hat{t}_j\,,\ \widehat{z}_h^{\,\mathrm{pc}}(\tau) := \widehat{z}_j\,. \tag{4.6}$$

Lemma 4.3 *A pair $\big(\hat{t}_h^{\,\mathrm{pl}}(\tau), \widehat{z}_h^{\,\mathrm{pl}}(\tau)\big)$ satisfies*

$$0 \in \partial\Delta_{\|}\big(\dot{\widehat{z}}_h^{\,\mathrm{pl}}(\tau)\big) + \mathrm{D}_z I\big(\hat{t}_h^{\,\mathrm{pc}}(\tau), \widehat{z}_h^{\,\mathrm{pc}}(\tau)\big)\,, \tag{4.7}$$

$$\dot{\hat{t}}_h^{\,\mathrm{pl}}(\tau) + \big\|\dot{\widehat{z}}_h^{\,\mathrm{pl}}(\tau)\big\| = 1 \tag{4.8}$$

on each $\tau \in \big(\tau_{j-1}, \tau_j\big)$, $j = 1, \ldots, N$. Moreover we have

$$\begin{aligned} &I\big(\tau_k, \widehat{z}_h^{\,\mathrm{pl}}(\tau_k)\big) + \int_{\tau_j}^{\tau_k} \Delta\big(\dot{\widehat{z}}_h^{\,\mathrm{pl}}(s)\big)\,\mathrm{d}s \\ &\quad\leqslant I\big(\tau_j, \widehat{z}_h^{\,\mathrm{pl}}(\tau_j)\big) + \int_{\tau_j}^{\tau_k} \partial_t I\big(\hat{t}_h^{\,\mathrm{pl}}(s), \widehat{z}_h^{\,\mathrm{pc}}(s)\big)\dot{\hat{t}}_h^{\,\mathrm{pl}}(s)\,\mathrm{d}s\,. \end{aligned} \tag{4.9}$$

Assertions (4.7) and (4.8) are the best discrete replacements of (2.2)–(2.4) for the time continuous case.

Proof: Is is easy to see that (4.7) and (4.8) are an immediate consequence of Proposition 4.2(b) and (c). As to inequality (4.9) it is a consequence of the discrete energy inequality 4.2(a). Indeed, for $1 \leqslant j < k \leqslant N$ it follows from (a) that

$$I\big(\hat{t}_k, \widehat{z}_k\big) - I\big(\hat{t}_j, \widehat{z}_j\big) + \sum_{m=j+1}^{k} \Delta\big(\widehat{z}_m - \widehat{z}_{m-1}\big) \leqslant \sum_{m=j+1}^{k} \int_{\hat{t}_{j-1}}^{\hat{t}_j} \partial_t I\big(s, \widehat{z}_j\big)\, \mathrm{d}s\,. \tag{4.10}$$

This can be rewritten as

$$\begin{aligned} & I\big(\hat{t}_h^{\,\mathrm{pl}}(\tau_k), \widehat{z}_h^{\,\mathrm{pl}}(\tau_k)\big) - I\big(\hat{t}_h^{\,\mathrm{pl}}(\tau_j), \widehat{z}_h^{\,\mathrm{pl}}(\tau_j)\big) + \textstyle\sum_{m=j+1}^{k} \Delta\big(\widehat{z}_h^{\,\mathrm{pl}}(\tau_m) - \widehat{z}_h^{\,\mathrm{pl}}(\tau_{m-1})\big) \\ & \quad \leqslant \textstyle\int_{\tau_j}^{\tau_k} \frac{\partial}{\partial t} I\big(\hat{t}_h^{\,\mathrm{pl}}(\sigma), \widehat{z}_h^{\,\mathrm{pc}}(\sigma)\big) \dot{\hat{t}}_h^{\,\mathrm{pl}}(\sigma)\, \mathrm{d}s\,, \end{aligned}$$

which in turn leads to (4.9). This proves Lemma 4.3. ■

For notational convenience we write $\big(\hat{t}_h, \widehat{z}_h\big)$ instead of $\big(\hat{t}_h^{\,\mathrm{pl}}, \widehat{z}_h^{\,\mathrm{pl}}\big)$ further on. Taking into account that $\widehat{z}_h \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}], X\big)$, $\hat{t}_h \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}], \mathbb{R}\big)$ are uniformly bounded for all time increments $h = \frac{1}{N}$, $N \in \mathbb{N}$, and using the Arzela–Ascoli theorem, we can extract a subsequence from $\hat{t}_h, \widehat{z}_h$ (still denoted by $\hat{t}_h, \widehat{z}_h$) such that

$\hat{t}_h(\tau)$ converges uniformly to $\hat{t}_*(\tau)$ as $h \to 0$, and
$\widehat{z}_h(\tau)$ converges uniformly to $\widehat{z}_*(\tau)$ as $h \to 0$.

Moreover, again using $\dim X < \infty$,

$\dot{\hat{t}}_h$ converges weak$-*$ to $\dot{\hat{t}}_*$ in $\mathrm{L}^\infty\big([0,\widehat{T}], \mathbb{R}\big)$ as $h \to 0$, and
$\dot{\widehat{z}}_h$ converges weak$-*$ to $\dot{\widehat{z}}_*$ in $\mathrm{L}^\infty\big([0,\widehat{T}], X\big)$ as $h \to 0$.

In the same manner as in Section 3, one can show that any limit $\big(\hat{t}_*, \widehat{z}_*\big)$ in the above sense of $\big(\hat{t}_h, \widehat{z}_h\big) \in \mathrm{W}^{1,\infty}\big([0,\widehat{T}], \mathbb{R}{\times}X\big)$ satisfies

$$\begin{aligned} & 0 \in \partial\Delta_{\|\cdot\|}\big(\dot{\widehat{z}}_*(\tau)\big) + \mathrm{D}_z I\big(\hat{t}_*(\tau), \widehat{z}_*(\tau)\big)\,, \\ & \dot{\hat{t}}_*(\tau) + \big\|\dot{\widehat{z}}_*(\tau)\big\| \leqslant 1\,. \end{aligned} \tag{4.11}$$

Example 4.4 *Here we show that, if Assumption 3.4 is violated, then in* (4.11) *the strict inequality may hold. Indeed, let* $X = \mathbb{R}^2$ *and for* $z = (u, w)$ *we set* $I(t,z) = -u - w$, $\Delta(z) = |z|_1 := |u| + |w|$, $\|z\| = |z|_2 := \sqrt{u^2 + w^2}$. *In this case* $M\big((1,1)\big) = \big\{z = (\tilde{u}, \tilde{w}) \,\big|\, \tilde{u} \geqslant 0, \tilde{w} \geqslant 0 \text{ with } |z|_2 \leq 1\big\}$ *and* $|\cdot|_2$ *is not affine on* $M\big((1,1)\big)$. *Moreover (IP) given by*

$$\begin{aligned} & \widehat{z}_k \in \operatorname*{argmin}_{|z - \widehat{z}_{k-1}|_2 \leqslant h} \{I(\hat{t}_{k-1}, z) + |z - \widehat{z}_{k-1}|_1\} \\ & \hat{t}_k := \hat{t}_{k-1} + h - |\widehat{z}_k - \widehat{z}_{k-1}|_2 \end{aligned}$$

with $t_0 = 0$, $z_0 = (0,0)$ *has, among others, the following solution*

$$\hat{t}_k = 0, \text{ for all } k \text{ and } \widehat{z}_k = \begin{cases} \left(\frac{hk}{2}, \frac{hk}{2}\right), & \text{if } k \text{ even,} \\ \left(\frac{h(k+1)}{2}, \frac{h(k-1)}{2}\right), & \text{if } k \text{ odd.} \end{cases}$$

Clearly, $\left|\widehat{z}_k - \widehat{z}_{k-1}\right|_1 = \left|\widehat{z}_k - \widehat{z}_{k-1}\right|_2 = h$ *and the uniform limit is* $\hat{t}_* \equiv 0$, $\widehat{z}_*(\tau) = \left(\frac{\tau}{2}, \frac{\tau}{2}\right)$. *Hence,* $\dot{\widehat{z}}_*(\tau) = \left(\frac{1}{2}, \frac{1}{2}\right)$ *whereas* $\dot{\widehat{z}}_h(\tau)$ *oscillates between* $(1,0)$ *and* $(0,1)$. *In particular,*

$$\dot{\hat{t}}_h + |\dot{\widehat{z}}_h|_1 = 1 > \frac{1}{\sqrt{2}} = \dot{\hat{t}}_* + |\dot{\widehat{z}}_*|_2$$

a.e. on $\left[0, \widehat{T}\right]$. *Choosing* $\|z\| = |z|_1 = \Delta(z)$ *would allow for the same approximating solutions* $\widehat{z}_h$ *with oscillating derivative* $\dot{\widehat{z}}_h$. *But Assumption 3.4 holds and* $\left|\dot{\widehat{z}}_*\right|_1 \equiv 1$.

As in Section 3 it remains to show that we can reparametrize $\left(\hat{t}_*, \widehat{z}_*\right)$ such that the reparametrized solution $\left(\tilde{t}_*, \tilde{z}_*\right)$ solves problem (2.2)–(2.4). Again, we have to show that the limit functions satisfy $\dot{L}(\tau) = \dot{\hat{t}}(\tau) + \|\dot{\widehat{z}}(\tau)| \geq c_* > 0$ a.e. on $[0, \widehat{T}]$. In complete analogy to Lemma 3.2 we have the following result.

Lemma 4.5 *Let Condition 2.5 hold and let* $(\hat{t}, \widehat{z})$ *be any limit obtained from* $(\hat{t}_h^{\mathrm{pl}}, \widehat{z}_h^{\mathrm{pl}})$ *constructed above. Then, there exists* $c_* > 0$ *such that* $\dot{L}(\tau) \geq c_*$ *for a.a.* $\tau \in [0, \widehat{T}]$.

Proof: The discrete energy inequality (4.10) and (2.6) imply

$$I\left(\hat{t}_k, \widehat{z}_k\right) - I\left(\hat{t}_j, \widehat{z}_j\right) + D_{j,k} \leqslant C_I\left(\hat{t}_k - \hat{t}_j\right) \text{ with } D_{j,k} = \textstyle\sum_{m=j+1}^{k} \Delta\left(\widehat{z}_m - \widehat{z}_{m-1}\right),$$

for $0 \leqslant j < k \leqslant N$. Again using (2.6) we find $D_{j,k} \leqslant 2C_I\left(\hat{t}_k - \hat{t}_j\right) + M\left\|\widehat{z}_k - \widehat{z}_j\right\|$ and (2.5) leads to

$$N_{j,k} \leqslant \tfrac{1}{C_\Delta}\left(2C_I\left(\hat{t}_k - \hat{t}_j\right) + M\left\|\widehat{z}_k - \widehat{z}_j\right\|\right) \text{ with } N_{j,k} = \textstyle\sum_{m=j+1}^{k} \left\|\widehat{z}_m - \widehat{z}_{m-1}\right\|.$$

By construction (see Prop. 4.2(c)) we have $N_{j,k} + \left(\hat{t}_k - \hat{t}_j\right) = h(k-j)$ and conclude

$$\tau_k - \tau_j = h(k-j) \leq C_*\left(\left\|\widehat{z}_k - \widehat{z}_j\right\| + \left(\hat{t}_k - \hat{t}_j\right)\right) \text{ where } C_* = \max\left\{1+2C_I/C_\Delta, M/C_\Delta\right\}.$$

Taking the limit as $h \to 0$ and uniform convergence of a subsequence of $\left(\hat{t}_h(\tau), \widehat{z}_h(\tau)\right)$ to the limit $(\hat{t}, \widehat{z})$ we arrive at

$$|\hat{t}_*(\tau_2) - \hat{t}_*(\tau_1)| + \left\|\widehat{z}_*(\tau_2) - \widehat{z}_*(\tau_1)\right\| \geqslant c_*|\tau_2 - \tau_1| \text{ with } c_* = 1/C_*,$$

for all $\tau_1, \tau_2 \in [0, \widehat{T}]$. This gives $\dot{L}(\tau) = \dot{\hat{t}}_*(\tau) + \left\|\dot{\widehat{z}}_*\right\| \geqslant c_*$ and proves Lemma 4.5. ■

Thus, reparametrization works the same way as in Section 3. Moreover, the arguments involving the compatibility condition 3.4 (i.e., Lemma 3.5 and Corollary 3.6) apply for the discrete approximations in exactly the same way. Summarizing, the results of this section are the following.

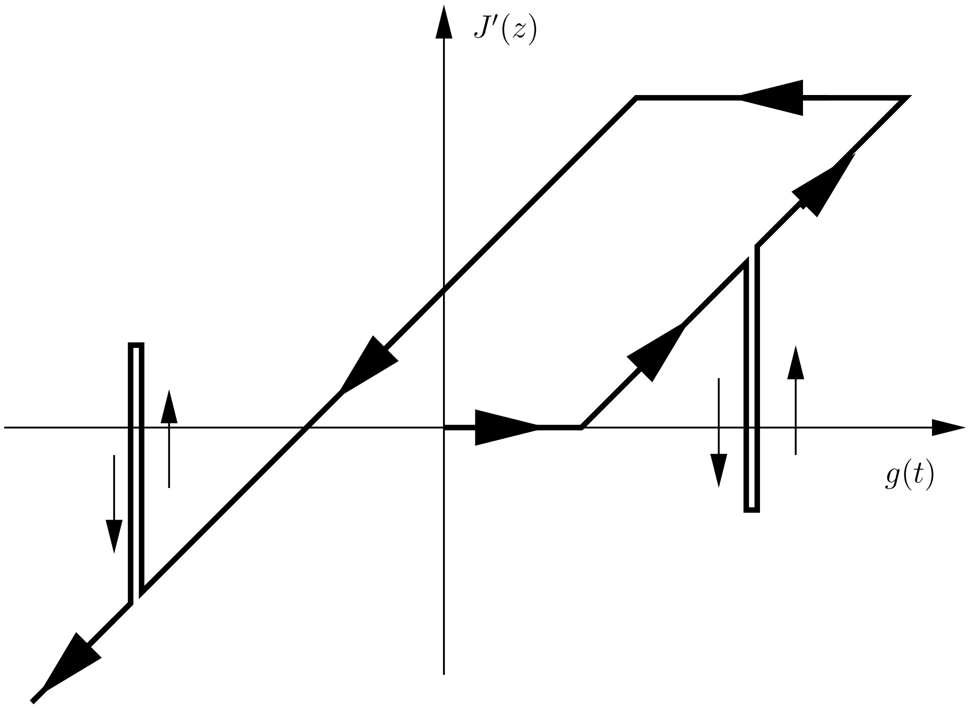

Figure 1: Stress-strain diagram for the model in Example 4.7: Horizontal lines correspond to sticking ($\dot{\widehat{z}} = 0$), lines parallel to the diagonal correspond to motion under dry friction ($0 < |\dot{\widehat{z}}| < 1$), and vertical lines correspond to viscously slipping motions ($|\dot{\hat{t}}| = 0$).

Theorem 4.6 *Let Condition 2.5 be satisfied. Then, a subsequence of the discrete approximations* $\big(\hat{t}_h^{\mathrm{pl}}, \widehat{z}_h^{\mathrm{pl}}\big)$ *converges uniformly to limit function* $\big(\hat{t}_*, \widehat{z}_*\big)$ *which is a solution of* (3.4). *The latter, after reparametrization of time, gives a solution of* (2.2)–(2.4).

Moreover, if Δ *and* $\|\cdot\|$ *satisfy the additional compatibility condition 3.4, then* $\big(\hat{t}_*, \widehat{z}_*\big)$ *solves* (2.2)–(2.4) *without reparametrization of time.*

Example 4.7 *Finally we provide a simple one-dimensional example which displays the different features of our model. Since the function* I *is nonconvex we will have sticking* ($\dot{\widehat{z}} = 0$)*, dry friction* ($0 < \|\dot{\widehat{z}}\| < 1$) *and also viscous slips* ($|\dot{\hat{t}}| = 0$ *and* $\|\dot{\widehat{z}}\| = 1$)*.*

Let $X = \mathbb{R}$ *and* $\Delta(v) = \delta|v|$, $\delta > 0$, *and* $I(t, z) = J(z) - g(t)z$, *where*

$$J(z) = \begin{cases} \frac{1}{2}(z+2)^2\,, & \text{for } z \leqslant -1, \\ 1 - \frac{1}{2}z^2\,, & \text{for } -1 \leqslant z \leqslant 1, \\ \frac{1}{2}(z-2)^2\,, & \text{for } z \geqslant 1, \end{cases} \tag{4.12}$$

and $g(t) = \min\{t, 2a - t\}$, *with* $a > 2 + \delta$. *Then direct computation of the solutions*

(2.2)–(2.4), *with* $\hat{t}(0)=0$, $\widehat{z}(0)=-2$, *leads to*

$$\begin{pmatrix}\hat{t}(\tau)\\ \widehat{z}(\tau)\end{pmatrix} = \begin{cases} \begin{pmatrix}\tau\\ -2\end{pmatrix} & \text{in } [0,\delta], \qquad \dot{\widehat{z}}\equiv 0 \quad \text{sticking}\\ \begin{pmatrix}\delta+\frac12(\tau-\delta)\\ -2+\frac12(\tau-\delta)\end{pmatrix} & \text{in } [\delta,2+\delta], \qquad \dot{\widehat{z}}\equiv \frac12 \quad \text{dry fric.}\\ \begin{pmatrix}1+\delta\\ -1+\big(\tau-(2+\delta)\big)\end{pmatrix}, & \text{in } [2+\delta,4+\delta], \qquad \dot{\widehat{z}}\equiv 1 \quad \text{viscous}\\ \begin{pmatrix}1+\delta\\ 1+\big(\tau-(4+\delta)\big)\end{pmatrix} & \text{in } [4+\delta,6+\delta], \qquad \dot{\widehat{z}}\equiv 1 \quad \text{viscous}\\ \begin{pmatrix}1+\delta+\frac12\big(\tau-(6+\delta)\big)\\ 3+\frac12\big(\tau-(6+\delta)\big)\end{pmatrix} & \text{in } [6+\delta,2a+4-\delta], \qquad \dot{\widehat{z}}\equiv \frac12 \quad \text{dry fric.}\\ \begin{pmatrix}a+\big(\tau-(2a+4-\delta)\big)\\ a+2-\delta)\end{pmatrix} & \text{in } [2a+4-\delta,2a+4+\delta], \qquad \dot{\widehat{z}}\equiv 0 \quad \text{sticking}\\ \begin{pmatrix}2\delta+\frac12\big(\tau-(4+\delta)\big)\\ 2-\delta-\frac12\big(\tau-(4+\delta)\big)\end{pmatrix} & \text{in } [2a+4+\delta,4a+6-\delta], \qquad \dot{\widehat{z}}\equiv -\frac12 \quad \text{dry fric.}\\ \begin{pmatrix}2a+\delta+1\\ 1-\big(\tau-(4a+6-\delta)\big)\end{pmatrix} & \text{in } [4a+6-\delta,4a+8-\delta], \qquad \dot{\widehat{z}}\equiv -1 \quad \text{viscous} \end{cases}$$

which show possible hysteretic behavior of $\big(g\big(\hat{t}(\tau)\big), J'\big(\widehat{z}(\tau)\big)\big)$, *see Figure 1.*

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