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Dynamical properties of spatially non-decaying 2D Navier-Stokes flows with Kolmogorov forcing in an infinite strip

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Dedicated to Victor I. Yudovich on the occasion of his 70th birthday

Abstract

The main result of this paper is the global well-posedness of the Cauchy problem to the 2D Navier-Stokes system with the initial data $u_0 \in \mathrm{BUC}(\mathbb{O})$ and the external force $F \in \mathrm{C}\big([0,\infty), \mathrm{L}^\infty(\mathbb{O})\big)$ on the manifold $\mathbb{O} = \mathbb{S}^1 \times \mathbb{R}$, i.e., the fluid flow is supposed to be periodic in one of the spatial directions whereas in the unbounded direction only uniform boundedness is assumed. However, to obtain uniqueness we need to make assumptions which suppresses additional pressure gradients. For this aim Riesz operators on $\mathrm{L}^\infty(\mathbb{O})$ used to define $p(t) \in \mathrm{BMO}(\mathbb{O})$. For time-independent forces the solutions are shown to grow at most cubically in the time t.

Mathematical Subject Classification (2000). 35Q30, 76D03, 76D05
Keywords. 2D Navier-Stokes problem, spatially non-decaying initial data, Riesz operators, Helmholtz projection.

1 Introduction

Our study is motivated by the analysis of spatial dynamics of the time independent Navier-Stokes system near the instability threshold of a fully symmetric steady state in cylindrical domains $(x_1, y) \in \mathbb{R} \times \Omega$, where Ω is the cross section. A variety of bounded solutions to the Navier-Stokes system, that are uniformly close to the basic steady state, were found in Poiseuille, Couette-Taylor and Kolmogorov problems. The study of the Cauchy problem in the functional space that includes all solutions found in that way can not be restricted to the phase space $\mathrm{L}^p(\mathbb{R} \times \Omega)$ with $p \in [1, \infty)$ since not only a variety of x_1-periodic solutions with different periods, but also bump and (multi) pulse solutions should be considered as the initial data cf. [19, 4, 3, 5]. The natural question is to find a functional space where the Navier-Stokes system is globally well posed for any initial data and which contains all spatially inhomogeneous solutions found by the bifurcation analysis. To our knowledge the first step in this direction was done in [23], where it is shown that in the $3D$

Couette-Taylor problem slightly above the instability threshold all solutions starting L^∞ close to the Couette flow exist for all $t \in [0,\infty)$ and stay L^∞ close to it.

The main purpose of this work is to develop this program for the two-dimensional problem that goes back to the celebrated Kolmogorov question. Suppose that the fluid motion in $\mathbb{R}^2$ is generated by the action of a volume force F^*. We denote coordinates on $\mathbb{R}^2$ as (x_1, x_2). Defining the Grashof number $G := \|F^*\|_\infty \frac{L^3}{\nu^2}$, where ν is the viscosity and L is the unit length, and taking the velocity units $\frac{\nu}{L}$ we arrive at the following non-dimensional form of the governing equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u + \operatorname{grad} p = \Delta u + F, \qquad \nabla\cdot u = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{1.1}$$

where $F = Gf$ with $\|f\|_\infty = 1$ is the forcing, $u = (u_1, u_2)^\mathsf{T}$ is the velocity field and p is the pressure. A.N. Kolmogorov (see [22]) suggested to take

$$F = (Gf(x_2), 0)^\mathsf{T} \quad \text{with } f(x_2) = \sin(x_2) \tag{1.2}$$

and to study how the dynamics of the problem changes if the parameter $G \in \mathbb{R}_+$ increases. There are various reformulations of Kolmogorov's original question (e.g., [8, 12]), but it is still unclear if the Navier-Stokes equations are dynamically nontrivial on $\mathbb{T}^2$, i.e. when the additional periodicity assumptions

$$u(t, x_1, x_2) = u(t, x_1, x_2 + 2\pi), \quad u(t, x_1, x_2) = u(t, x_1 + 2\pi/\alpha, x_2) \tag{1.3}$$

and the mean flow condition

$$Q \stackrel{\text{def}}{=} \int_0^{2\pi} \int_0^{2\pi/\alpha} u \, \mathrm{d}x_1 \, \mathrm{d}x_2 = (0,0)^\mathsf{T} \tag{1.4}$$

are imposed.

The Kolmogorov flow provides us with an important hydrodynamical example of the isotropic and negative eddy viscosity, see [15]. Recall that eddy transport coefficients are used to characterize the way a given basic cellular flow responds to weak large-scale perturbations. Negative eddy viscosity has been used as an explanation of common instabilities in astrophysical and geophysical flows, for instance, for the differential rotation of the Sun.

Stability and bifurcation of the basic steady state $U_*(x_2) = (U(x_2), 0)^\mathsf{T}$, $p = \text{const}$ of the problem (1.1)-(1.4) were studied in [22, 20, 2]. It was demonstrated that for $U(x_2) = G\sin x_2$ the minimal critical Grashoff number $G_0 = \sqrt{2}$ corresponds to the wave number $\alpha = 0$ with the stability exponent $\lambda(\alpha, G_0) = 0$ and that the neutral curve of stability $\alpha \mapsto G(\alpha)$ is monotone. It was demonstrated in [2, 1] that the same is true for more general forcing terms. That is why the loss of stability and the eventual transition to turbulence is especially interesting for $\alpha = 0$, i.e. for the problem in the unbounded domain $\mathbb{O} = \mathbb{R}^1 \times \mathbb{S}^1$.

In [3],[6] the existence of a family of stationary, spatially periodic solutions of the problem (1.1), (1.3) was demonstrated that limit in pulse or front solutions. The family of these stationary solutions $u(\varepsilon; x_1, x_2)$ can be described by the expansion $u(\varepsilon; x_1, x_2) = U_*(x_2) + \varepsilon\gamma(\varepsilon x_1)(\sqrt{2}\sin x_2, 1)^\mathsf{T} + \mathcal{O}(\varepsilon^2)$, where $\varepsilon^2 = G - G_0$ and $\gamma(z)$ is any bounded solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\ddot{\gamma} + \frac{2}{3}\gamma - \frac{2}{9}\gamma^3\right) = 0. \tag{1.5}$$

Notice that the investigation of stability of homoclinic and heteroclinic solutions found in that way requires the study of Navier-Stokes dynamics in the space of spatially non-decaying solutions in the domain $\mathbb{O}$.

Since stationary solutions mentioned above are not only bounded, but also smooth it is reasonable to study the dynamics generated by the Navier-Stokes system in the space $\mathrm{BUC}(\mathbb{O})$ of bounded uniformly continuous functions or in $\mathrm{L}^\infty(\mathbb{O})$. For these phase spaces the existence of the global-in-time solutions for the Kolmogorov or Poiseuille problem, where $\Omega = \mathbb{S}^1$ and $\Omega = [-1, 1]$ respectively, was not established so far. In such open systems there is of course the problem that the mean flow is not defined from the equation, see Section 4.4 for non-uniqueness results due to this effect. This means that the proper choice of the function space for u and p is a delicate matter: it should include all physically interesting solutions and it should exclude any kind of unreasonable examples.

A recent survey on the existence of the spatially decaying solutions to the Navier-Stokes problem in various bounded and unbounded domains in $\mathbb{R}^n, n \geq 2$, is given in [7]. Most of the results are devoted either to L^p theory, $p \in (1, \infty)$, or to the exterior problem, where solutions with finite Dirichlet integral are considered. Asymptotical self-similar decay to 0 for all small initial data in $\mathbb{R}^2$ is shown in [14]. None of them covers the Kolmogorov problem on $\mathbb{R}^2$ or $\mathbb{O}$. Recently, several articles on the local existence, regularity and decay properties of solutions to the Navier-Stokes problem in $\mathbb{R}^n$ were published (see [21] and the references therein). In these papers it is supposed that the fluid fills the entire space and is not subjected to any external force. The specific features of such problems are the absence of the determining parameters like Reynolds or Grashof numbers and the presence of the full symmetry group of the Navier-Stokes system. That is why shift and dilation invariant estimates play the crucial role in the analysis.

Moreover, the Cauchy problem for the Navier-Stokes system without a forcing term was studied in [17, 16, 18] for the initial data in $\mathrm{BUC}(\mathbb{R}^n), n = 2, 3$. These papers and [11], where a localized force $F \in \mathrm{L}^2(\mathbb{R}^2)$ is allowed, are most closely related to our study. They provide global well-posedness and an a priori bound of the type $C \exp(C \exp(Ct))$ for spatially localized forcing. Our main result is the similar global well-posedness of the Cauchy problem to the Navier-Stokes system (1.1) with $u_0 \in \mathrm{BUC}(\mathbb{O})$ and $F \in \mathrm{C}\big([0, \infty), \mathrm{L}^\infty(\mathbb{O})\big)$, but our a priori bounds will be polynomial for bounded forcing.

To obtain these bounds and uniqueness we have to make assumptions on the pressure which suppresses additional pressure gradients. As in [17] we use Riesz operators on $\mathrm{L}^\infty(\mathbb{O})$ to define $p(t) \in \mathrm{BMO}(\mathbb{O})$, see Section 2. In the 2D situation a very helpful tool is the vorticity equation

$$\partial_t\omega - \Delta\omega + (u\cdot\nabla)\omega = \operatorname{rot} F, \quad \text{with} \quad \omega\big|_{t=0} = \operatorname{rot} u_0,$$

where $\omega = \partial_2 u_1 - \partial_1 u_2$, which yields an $\mathrm{L}^\infty(\mathbb{O})$ bound for ω by the maximum principle, namely

$$\|\omega(t)\|_\infty \leq \|\omega(0)\|_\infty + \int_0^t \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s. \tag{1.6}$$

The problem is to return back to the velocity field $u : \mathbb{O} \mapsto \mathbb{R}^2$ via

$$\operatorname{div} u = 0, \quad \operatorname{rot} u = \omega.$$

Using the explicit form of Green's function of the Laplace operator $\Delta_{\mathbb{O}}$ we are able to estimate u, except for the cross-sectional mean flow

$$m(t, x_1) = Pu(t, x_1, \cdot) = \int_{\mathbb{S}^1} u(t, x_1, x_2)\,\mathrm{d}x_2 := \frac{1}{2\pi}\int_0^{2\pi} u(t, x_1, x_2)\,\mathrm{d}x_2. \tag{1.7}$$

Using $\|u-m\|_\infty \le C\|\omega\|_\infty$ and the Navier-Stokes equation we finally obtain

$$\|u(t)\|_\infty \le C\big(\|u_0\|_{\mathrm{C}^1}, \|F\|_{\mathrm{C}([0,t],\mathrm{L}^\infty(\mathbb{O}))}, \|\operatorname{rot} F\|_{\mathrm{C}([0,t],\mathrm{L}^\infty(\mathbb{O}))}\big)(1+t)^3.$$

If $\int_0^\infty \|F(s)\|_\infty + \|\operatorname{rot} F(s)\|_\infty \, \mathrm{d}s \le \infty$, then we obtain $\|u(t)\|_\infty \le C(1+t)$. We do not know whether these estimates are sharp. However, in Section 3.2 we show that it is easy to find solutions with $F(t) \equiv F \neq 0$ such that $\|u(t)\|_\infty = t$.

2 Basic estimates in $\mathbb{O}$

We study the Cauchy problem for the 2D Navier-Stokes system. Since we suppose that the volume force and the solutions are 2π periodic in x_2, the problem is posed on the manifold $\mathbb{O} = \mathbb{R}^1 \times \mathbb{S}^1$. Let us consider the problem with the Cauchy data $u_0 \in \mathrm{BUC}(\mathbb{O})$. Recall that for a measurable function $u : \mathbb{O} \mapsto \mathbb{R}$ the $\mathrm{L}^\infty(\mathbb{O})$ norm is defined as

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{O}} |u(x)|.$$

With this norm $\mathrm{L}^\infty(\mathbb{O})$ is a Banach algebra with respect to multiplication. The closed subalgebra of bounded, uniformly continuous functions is denoted by $\mathrm{BUC}(\mathbb{O})$. We choose

$$\begin{aligned}\mathcal{Z} &= \Big\{\ \phi\ \in \mathrm{C}^\infty(\mathbb{O},\mathbb{R})\ \Big|\ \forall\, k,m,n \in \mathbb{N}_0\ \exists\, C>0 : (1+|x_1|)^k |\partial_{x_1}^n \partial_{x_2}^m \phi(x)| \le C\ \Big\} \\ &= \Big\{\ \sum_{n\in\mathbb{Z}} T_n s\ \Big|\ s \in \mathcal{S}(\mathbb{R}^2)\ \Big\}, \quad \text{where} \quad T_n s(x_1,x_2) = s(x_1, x_2 - 2\pi n),\end{aligned}$$

as the set of test functions that is dense in $\mathrm{L}^p(\mathbb{O})$, $p \in [1,\infty)$. These functions are periodic in x_2, decay faster then any rational function in x_1 and play the same role as Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ in the theory of distributions on $\mathbb{R}^n$.

The manifold $\mathbb{O}$ is the locally compact Abelian group. Therefore for suitably decaying functions ϕ and ψ the convolution with respect to the invariant measure, which is the Lebesgue measure, is defined via

$$(\phi * \psi)(x) = \int_{\mathbb{O}} \phi(y)\psi(x-y)\,\mathrm{d}y.$$

In particular, for all $q \in [1,\infty]$ we have Young's inequality

$$\|\phi * \psi\|_q \le \|\phi\|_1 \|\psi\|_q. \tag{2.1}$$

We will use $\langle f, g\rangle = \int_{\mathbb{O}} f(y)g(y)\,\mathrm{d}y$ to denote the duality pairing between $\mathrm{L}^p(\mathbb{O})$ and $\mathrm{L}^q(\mathbb{O})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.1
We call (u,p) a weak solution of the Navier-Stokes problem on $[0,T) \times \mathbb{O}$ with initial data $u_0 \in \mathrm{L}^\infty(\mathbb{O})$ if (u,p) satisfies

$$\begin{gathered}\nabla\cdot u = 0 \quad \textit{for a.e. } t \in [0,T) \quad \textit{and} \\ \int_0^T \{\langle u(s), \partial_s\phi\rangle + \langle u(s), \Delta\phi\rangle + \langle u\otimes u, \nabla\phi\rangle + \langle p, \nabla\cdot\phi\rangle - \langle f, \phi\rangle\}\,\mathrm{d}s = -\langle u_0, \phi(0)\rangle\end{gathered} \tag{2.2}$$

for all $\phi \in \mathrm{C}^1([0,T]\times\mathbb{O})$ with $\phi(t,\cdot) \in \mathcal{Z}^2 = \mathcal{Z}\times\mathcal{Z}$ for all $t \in [0,T]$ and $\phi(T,\cdot) = 0$.

Usually the Navier-Stokes system is transformed to an integral equation. This step in the analysis is classical and goes back to C.W. Oseen and J. Leray. Recall that by the summation convention $\nabla\cdot(u \otimes u) := \sum_j \partial_j(u_i u_j)$. Applying the operator $\nabla\cdot$ to the first equation in (1.1) and using $\nabla\cdot u = 0$ and $(u \cdot \nabla)u = \nabla\cdot(u \otimes u)$ we arrive at

$$\Delta p = \nabla\cdot\big(F - \nabla\cdot(u \otimes u)\big). \tag{2.3}$$

Therefore $\nabla p = \nabla\Delta^{-1}\nabla\cdot(F - \nabla\cdot(u \otimes u))$ and we get

$$u_t - \Delta u = \mathbb{P}(F - \nabla\cdot(u \otimes u)), \tag{2.4}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ is the Helmholtz projection. The proper definition of Δ^{-1} will be given later. Using the variation-of-constants formula (also called Duhamel's principle) we obtain the integral equation

$$u(t) = \mathrm{e}^{t\Delta}u_0 + \widehat{F}(t) + B(u,u)(t) \tag{2.5}$$

with $\widehat{F}(t) = \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}F(s)\,\mathrm{d}s$ and

$$B(u,v)(t) = -\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}(\nabla\cdot[u(s) \otimes v(s)])\,\mathrm{d}s.$$

The operator $\mathrm{e}^{t\Delta}$ denotes the convolution with the heat kernel. The next step is to use Picard iterations which converge for small t to the solution of (2.5) in the suitable functional space. The pressure can be recovered via (2.3) a posteriori.

The last step is to relate (u,p) found in this way to the original problem (1.1). Notice, however, that the equivalence of (2.5) and the original Navier-Stokes system is rather subtle since choosing function spaces and defining the Helmholtz projection $\mathbb{P}$ exactly involves certain choices about the behavior at infinity. Already under periodic boundary conditions there are different Helmholtz projections, one corresponding to a periodic pressure and the other to a periodic pressure gradient that may have a non-zero mean value, see [10].

Another way to see this problem is to look at the symmetry group associated with the Navier-Stokes system. It is well-known that if $(u,p) : (0,t)\times\mathbb{R}^n \mapsto \mathbb{R}^n\times\mathbb{R}$ is a solution of the Navier-Stokes system without forcing, then

$$\big(\widetilde{u}(t,x), \widetilde{p}(t,x)\big) = \Big(u(t,x{+}\phi(t)) - \phi'(t),\ p(t,x{+}\phi(t)) + (\phi''(t),x)_{\mathbb{R}^n} + \psi(t)\Big), \tag{2.6}$$

where $(\cdot,\cdot)_{\mathbb{R}^n}$ is the scalar product in $\mathbb{R}^n$, is a solution as well for arbitrary $\phi \in \mathrm{C}^2(\mathbb{R},\mathbb{R}^n)$ and $\psi \in \mathrm{C}(\mathbb{R},\mathbb{R})$. In particular, this formula includes Galilean invariance by choosing $\phi(t) = x_0 + ct$. However, choosing ϕ and ϕ' to be zero for $t \le 0$ and nonzero for $t > 0$ we immediately see that uniqueness of solutions of the Cauchy problem breaks down if a spatially growing pressure is allowed. This problem still occurs in $\mathbb{O}$ since a pressure gradient in x_1 direction may change the mean flux $m_1 = \int_{\mathbb{S}^1} u_1\,\mathrm{d}x_2$ at any time.

To achieve the uniqueness of solutions of the Cauchy problem we have to introduce pressure conditions. Our choice for p is explicit and is given in terms of Riesz operators on $\mathbb{O}$. The definition of $\mathbb{P}$ as bounded projection on $\mathrm{BMO}(\mathbb{O})$ is obtained as in [17], [24] via duality, since $\mathcal{Z}$ is dense in the Hardy space $\mathcal{H}^1(\mathbb{O})$. For the definition and the properties of $\mathrm{BMO}(\mathbb{R}^2)$, the space of functions of bounded mean oscillations, we refer to [24]. It is equipped with the norm

$$\|\varphi\|_{\mathrm{BMO}} = \sup\Big\{ \frac{1}{|C|}\int_C \Big|\varphi(x) - \frac{1}{|C|}\int_C \varphi(y)\,\mathrm{d}y\Big|\,\mathrm{d}x \ \Big|\ C \subset \mathbb{R}^2 \text{ finite square } \Big\}$$

and thus functions in $\mathrm{BMO}(\mathbb{R}^2)$ are defined only up to a constant. Our space $\mathrm{BMO}(\mathbb{O})$ is the subset of functions in $\mathrm{BMO}(\mathbb{R}^2)$ that are 2π-periodic in the x_2-direction. In particular, $\mathrm{BMO}(\mathbb{O})$ is the dual space of $\mathcal{H}^1(\mathbb{O})$ and satisfies $\mathrm{L}^\infty(\mathbb{O}) \subset \mathrm{BMO}(\mathbb{O}) \subset \mathrm{L}^p_{\mathrm{loc}}(\mathbb{O})$ for all $p \in (1,\infty)$.

The matrix elements of the projector $\mathbb{P}$ are $(\mathbb{P})_{i,j} = \delta_{i,j} + R_i R_j$, where the Riesz operators R_j are formally given via $(-\Delta_{\mathbb{O}})^{-1/2}\partial_j$. They have a proper definition on the group $\mathbb{O} = \mathbb{R} \otimes \mathbb{S}^1$ in the same way as on $\mathbb{R}^n$, see [24]. We only need the products $R_i R_j$ which are well-defined via Fourier transform on $\mathcal{Z}$. For $\phi \in \mathcal{Z}$ we write $\phi(x) = \sum_{\mathbb{Z}} \phi_k(x_1)\mathrm{e}^{\mathrm{i}kx_2}$ such that $\phi_k : \mathbb{R} \to \mathbb{C}$ lies in the space $\mathcal{S}$ of the Schwartz functions. We then have

$$\begin{aligned}
(\mathbb{P}_{11}\phi)(x) &= \sum_{k\neq 0} (G_k * \phi_k'')(x_1)\mathrm{e}^{\mathrm{i}kx_2}, \\
(\mathbb{P}_{12}\phi)(x) &= \sum_{k\neq 0} \mathrm{i}k(G_k * \phi_k')(x_1)\mathrm{e}^{\mathrm{i}kx_2}, \qquad\qquad \mathbb{P}_{21} = \mathbb{P}_{12} \\
(\mathbb{P}_{22}\phi)(x) &= \phi(x) - \sum_{k\neq 0} k^2(G_k * \phi_k)(x_1)\mathrm{e}^{\mathrm{i}kx_2},
\end{aligned}$$

where $G_k(\xi) = \frac{1}{2\pi|k|}\mathrm{e}^{-|k\xi|}$. Note that for functions $\phi : \mathbb{O} \mapsto \mathbb{R}^2$ which are independent of x_2 we have

$$\mathbb{P}\big(\phi_1(t,x_1), \phi_2(t,x_1)\big)^\mathsf{T} = (0, \phi_2(t,x_1))^\mathsf{T}. \tag{2.7}$$

This remark suggests that it is convenient to restrict the attention to divergence free forces F that satisfy the additional condition

$$(PF_1)(t,x_1) := \int_{\mathbb{S}^1} F_1(t,x_1,x_2)\,\mathrm{d}x_2 = 0 \quad \text{for all } t \geq 0 \text{ and } x_1 \in \mathbb{R}. \tag{2.8}$$

For general F we may define $\widetilde{F} = F - (PF_1)(1,0)^\mathsf{T}$ and

$$\widetilde{p}(t,x) = p(t,x) - \int_0^{x_1} (PF_1)(t,\xi)\,\mathrm{d}\xi.$$

Then, (u,p) solves (2.2) with forcing F if and only if $(u,\widetilde{p})$ solves (2.2) with forcing $\widetilde{F}$. Thus without loss of generality we impose (2.8) and fix p by the restriction $p(t) \in \mathrm{BMO}(\mathbb{O})$.

Since the delta distribution $\delta_{\mathbb{O}}(x)$ on the manifold $\mathbb{O}$ is

$$\delta_{\mathbb{O}}(x) = \delta_{\mathbb{R}}(x_1) \otimes \delta_{\mathbb{S}^1}(x_2) = \delta_{\mathbb{R}}(x_1) \sum_{k=-\infty}^{\infty} \delta(x_2 + 2\pi k) = \frac{1}{2\pi}\delta_{\mathbb{R}}(x_1) \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx_2},$$

the fundamental solution $K \in \mathcal{Z}'$ of the Laplacian $(-\Delta)_{\mathbb{O}}$ on the manifold $\mathbb{O}$ has the Fourier series representation

$$K(x) = \frac{1}{4\pi}|x_1| + \sum_{k\neq 0} \frac{1}{2\pi|k|}\mathrm{e}^{-|kx_1|}\mathrm{e}^{\mathrm{i}kx_2}$$

that can be simplified to

$$K(x) = \frac{1}{4\pi} \ln\big(\cosh x_1 - \cos x_2\big).$$

Note that the explicit choice of $R_i R_j$ and of the fundamental solution K reflects in the following relations for the Riesz operators, the Laplacian and the fundamental solution K associated with $\Delta_{\mathbb{O}}$.

Lemma 2.2
For all $\phi \in \mathcal{Z}$ we have

$$\text{(i) } R_1R_2\phi = R_2R_1\phi, \quad \text{(ii) } R_iR_j\Delta_{\mathbb{O}}\phi = -\partial_i\partial_j\phi, \quad \text{(iii) } \textstyle\sum_j R_iR_j\partial_j\phi = -\partial_i\phi,$$
$$\text{(iv) } R_iR_j\phi = \partial_j K * \partial_i\phi = \partial_i K * \partial_j\phi, \quad \text{(v) } (R_1^2 + R_2^2)\phi = -\phi.$$

Proof: Since $\mu_{\alpha,\beta} : x \mapsto \alpha{+}\beta x_1$ is in the kernel of $\Delta_{\mathbb{O}}$, any $\widetilde{K} = K + \mu_{\alpha,\beta} \in \mathcal{Z}'$ is a fundamental solution as well. It is the explicit choice of the fundamental solution K and projection $\mathbb{P}$ that make (iv) valid. The constant α is irrelevant, but the constant β is set to 0 to satisfy (iv). □

Notice that the local behavior of K near $x = 0$ is the same as that of the fundamental solution of $(-\Delta)$ on all of $\mathbb{R}^2$, namely $x \mapsto \frac{-1}{2\pi}\ln|x|$. The behavior of K at infinity strongly depends on the domain. For the manifold $\mathbb{O}$ we obtain a linear growth like $|x_1|$. Nevertheless

$$\partial_2 K(x) = \frac{\sin x_2}{4\pi(\cosh x_1 - \cos x_2)}$$

is uniquely defined and $\partial_2 K \in \mathrm{L}^1(\mathbb{O})$. This is in contrast to $\partial_1 K$ which is not in $\mathrm{L}^1(\mathbb{O})$. However, extracting the mean over $\mathbb{S}^1$ defined by (1.7) eliminates the difficulty. A straightforward calculation gives

$$\rho(x) := (I{-}P)\partial_1 K(x) = -\frac{\operatorname{sign} x_1}{4\pi} + \frac{\sinh x_1}{4\pi(\cosh x_1 - \cos x_2)}.$$

The second component of the velocity can be split into $u_2(t,x) = m_2(t,x_1){+}v_2(t,x)$, where

$$m = (m_1, m_2)^\mathsf{T} = Pu.$$

To find the pointwise estimates of u_1 and v_2 we will use the following $\mathrm{L}^1(\mathbb{O})$ estimates of $(I{-}P)\partial_1 K(x)$ and $\partial_2 K(x)$.

Lemma 2.3
For all $\phi \in \mathrm{L}^\infty(\mathbb{O})$ we have the estimates

$$\|(I{-}P)\partial_1 K * \phi\|_\infty \le C_1\|\phi\|_\infty \quad \text{and} \quad \|\partial_2 K * \phi\|_\infty \le C_2\|\phi\|_\infty, \tag{2.9}$$

where $C_1 = \|\rho\|_1 = 2\ln 2 \approx 1.39$ and $C_2 = \|\partial_2 K\|_1 = \frac{2}{\pi}(\frac{\pi^2}{3} - \int_1^2 \frac{\ln s}{1+s}\,\mathrm{d}s) \approx 1.57$.

Proof: These estimates follow from Young's inequality (2.1). It remains to calculate the L^1-norms of ρ and $\partial_2 K(x)$. By studying the signs of these functions and using the fact that they are derivatives with respect to x_1 and x_2, respectively, we can integrate once in the corresponding regions. Then, the remaining one-dimensional integrals give the desired results. Let us calculate, for instance, $\|\rho\|_1$. Using the identities

$$\int_{\pi/2}^{\pi} \ln(2{-}2\cos x_2)\,\mathrm{d}x_2 = 2\mathrm{Cat}, \qquad \int_0^{\pi/2} \ln(2{-}2\cos x_2)\,\mathrm{d}x_2 = -2\mathrm{Cat},$$

where Cat is the Catalan number, we finally get

$$\|(I{-}P)\partial_1 K\|_1 = \frac{-4}{2\pi}\int_0^{\pi/2} \ln\Big[2\cos x_2\,\big(\cosh[\ln(\cos x_2)]{-}\cos x_2\big)\Big]\,\mathrm{d}x_2 = 2\ln 2.$$

□

As a consequence the semi-group of the heat equation on $\mathbb{O}$ is easily controlled on $\mathrm{L}^p(\mathbb{O})$ for all $p \in [1,\infty]$. The following proposition shows that the composition of the heat kernel and the Helmholtz projector $\mathrm{e}^{t\Delta}\mathbb{P} = \mathbb{P}\mathrm{e}^{t\Delta}$ can be estimated as well.

Proposition 2.4
There exist constants C_3, C_4 and C_5 such that for all $F \in \mathrm{L}^\infty(\mathbb{O})$ we have

$$\|\mathrm{e}^{t\Delta}\mathbb{P}F\|_\infty \leq \Big(C_3 + \frac{C_4}{\sqrt{t}}\Big)\|F\|_\infty \quad \textit{for} \quad t>0; \tag{2.10}$$

$$\|\nabla\mathrm{e}^{t\Delta}\mathbb{P}F\|_\infty \leq \Big(\frac{C_5}{\sqrt{t}}\Big)\|F\|_\infty \quad \textit{for all} \quad t>0. \tag{2.11}$$

In fact, we have $C_3 = \frac{1+2\pi}{2\pi} \approx 1.16$, $C_4 = (C_1{+}C_2)/\sqrt{\pi} \approx 1.67$.
Proof: By the $(\mathrm{L}^\infty, \mathrm{L}^1)$ duality and the density of $\mathcal{Z}$ in $\mathrm{L}^1(\mathbb{O})$ we have

$$\begin{aligned} \|\mathrm{e}^{t\Delta}\mathbb{P}F\|_\infty &= \sup\{\, |\langle \mathrm{e}^{t\Delta}\mathbb{P}F, \phi\rangle| \;\big|\; \|\phi\|_1 = 1,\ \phi\in\mathcal{Z}^2 \,\} \\ &= \sup\{\, |\langle F, \mathbb{P}\,\mathrm{e}^{t\Delta}\phi\rangle| \;\big|\; \|\phi\|_1 = 1,\ \phi\in\mathcal{Z}^2 \,\} \\ &\leq \|F\|_\infty \sup\{\, \|\mathbb{P}\,\mathrm{e}^{t\Delta}\phi\|_1 \;\big|\; \|\phi\|_1 = 1,\ \phi\in\mathcal{Z}^2 \,\} \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the action of the distribution on a test function. The last identity uses that $\mathbb{P}$ and $\mathrm{e}^{t\Delta}$ are symmetric in the sense of operators acting on distributions. For $\phi \in \mathcal{Z}$ we let

$$\psi(t) = \mathrm{e}^{t\Delta}\phi \in \mathcal{Z}$$

and have to estimate $\|\mathbb{P}\psi(t)\|_1$ in terms of ϕ and $t>0$. For this purpose we use that the heat kernel $H_\mathbb{O}(t,\cdot) = \mathrm{e}^{t\Delta}$ on $\mathbb{O}$ is given in the form $H_\mathbb{O}(t,x_1,x_2) = H_\mathbb{R}(t,x_1)H_{\mathbb{S}^1}(t,x_2)$ with the usual properties of heat kernels on $\mathbb{R}$ and $\mathbb{S}^1$: $H_\mathbb{R}, H_{\mathbb{S}^1} \geq 0$, $\int_\mathbb{R} H_\mathbb{R}(t,x_1)\,\mathrm{d}x_1 = \int_{\mathbb{S}^1} H_{\mathbb{S}^1}(t,x_2)\,\mathrm{d}x_2 = 1$, $H_\mathbb{R}'(t,x_1) = -H_\mathbb{R}'(t,-x_1) \leq 0$ for $x_1 \geq 0$ and $H_{\mathbb{S}^1}'(t,x_2) = -H_{\mathbb{S}^1}'(t,-x_2) \leq 0$ for $x_2 \in [0,\pi]$. In particular, we easily find $\|H_\mathbb{O}(t)\|_1 = 1$ and

$$\|\partial_1 H_\mathbb{O}(t)\|_1 = 2H_\mathbb{R}(t,0) = \frac{1}{\sqrt{\pi t}}.$$

From the representation of the heat kernel on $\mathbb{S}^1$ via periodization

$$H_{\mathbb{S}^1}(t,x_2) = \frac{1}{2\sqrt{\pi t}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{\frac{-(x_2-2\pi n)^2}{4t}}$$

follows that

$$\|\partial_2 H_\mathbb{O}(t)\|_1 = 2(H_{\mathbb{S}^1}(t,0){-}H_{\mathbb{S}^1}(t,\pi)) \leq \frac{1}{\sqrt{\pi t}},$$

for all $t>0$. By Young's inequality we conclude

$$\|\psi_j(t)\|_1 \leq \|\phi_j\|_1 \quad \text{and} \quad \|\partial_i\psi_j\|_1 \leq \frac{1}{\sqrt{\pi\, t}}\|\phi_j\|_1. \tag{2.12}$$

With the Lemmas 2.3 and 2.2 (iv) we arrive at

$$\begin{aligned} \|\mathbb{P}\psi\|_1 &\leq \|\psi\|_1 + \|\partial_1 K * \nabla\psi_1\|_1 + \|\partial_2 K * \nabla\psi_2\|_1 \\ &\leq \frac{2\pi{+}1}{2\pi}\|\psi\|_1 + \sum_{i,j=1}^{2} C_j\|\partial_i\psi_j\|_1 \\ &\leq (C_3 + C_4/\sqrt{t})\|\phi\|_1 \end{aligned}$$

with C_3 and C_4 as given above. Combining these results with $\|\phi\|_1 = \|\phi_1\|_1 + \|\phi_2\|_1 = 1$ gives (2.10).

Using inequalities (2.12) we arrive at the estimates (2.11) in a similar way. Notice that estimate (2.11) follows directly from estimate (2.8) in [17] since for $F \in \mathrm{L}^\infty(\mathbb{O}) \subset \mathrm{L}^\infty(\mathbb{R}^2)$ we have $\mathbb{P}_{\mathrm{GIM}}\nabla{\cdot}F = \mathbb{P}\nabla{\cdot}F$ where $\mathbb{P}_{\mathrm{GIM}}$ is the Helmholtz projection defined in [17] on $\mathrm{L}^\infty(\mathbb{R}^2)$. $\square$

3 Estimates for solutions of the Navier-Stokes system in $\mathbb{O}$

Now we show how the above estimates can be used to establish local and then global estimates in $\mathrm{BUC}(\mathbb{O})$ for solutions to (3.3).

Lemma 3.1
For $u \in \mathrm{W}^{1,\infty}(\mathbb{O}, \mathbb{R}^2)$ with $\nabla \cdot u \equiv 0$ and $t > 0$ the estimates

$$\|\mathrm{e}^{t\Delta}\mathbb{P}((u\cdot\nabla)u)\|_\infty \le \Big(C_3 + \frac{C_4}{\sqrt{t}}\Big)\|\operatorname{rot} u\|_\infty \|u\|_\infty \quad \textit{and} \tag{3.1}$$

$$\|\nabla\mathrm{e}^{t\Delta}\mathbb{P}((u\cdot\nabla)u)\|_\infty \le \Big(\frac{C_5}{\sqrt{t}}\Big)\|\operatorname{rot} u\|_\infty \|u\|_\infty \quad \textit{for all} \quad t>0 \tag{3.2}$$

hold, where $\operatorname{rot} u = \partial_2 u_1 - \partial_1 u_2$.

Proof: The well-known identities $(u\cdot\nabla)u = \nabla\cdot(u\otimes u) = (\operatorname{rot} u)\widehat{u} + \frac{1}{2}\nabla|u|^2$, where $\widehat{u} = (u_2, -u_1)^\mathsf{T}$, together with $\mathbb{P}\nabla\psi = 0$ (see Lemma (2.2)) yield

$$\mathrm{e}^{t\Delta}\mathbb{P}((u\cdot\nabla)u) = \mathrm{e}^{t\Delta}\mathbb{P}G \quad \text{with } G = (\operatorname{rot} u)\widehat{u}.$$

With $\|G\|_\infty \le \|\operatorname{rot} u\|_\infty \|u\|_\infty$ the result follows from Proposition 2.4. With similar arguments (3.2) follows from (2.11). □

Lemma 3.2
If (u,p) is a weak solution to the Navier-Stokes system such that $u \in L^\infty([0,T]\times\mathbb{O})$ then u satisfies the integral equation

$$u(t) = \mathrm{e}^{t\Delta}u_0 + \widehat{F}(t) + B(u,u) \tag{3.3}$$

with $\widehat{F}(t) = \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}F \,\mathrm{d}s$ *and* $B(u,v) = -\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}(\nabla\cdot(u(s)\otimes v(s)))\,\mathrm{d}s$, *where $\mathbb{P}$ is the Helmholtz projector and $\mathrm{e}^{t\Delta}$ denotes the convolution with the heat kernel on $\mathbb{O}$.*

If u is a solution to (3.3) with $u(0) = u_0 \in \mathrm{BUC}(\mathbb{O})$, then (u,p) with $p = \sum_{i,j} R_iR_j(u\otimes u) + K * \nabla\cdot F$ is a weak solution of the Navier-Stokes system.

The proof is similar to the one given in Theorem 2 of [17].

Remark 3.3 Our assumption on the external force and the choice of the pressure assure that $p \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T], \mathrm{BMO}(\mathbb{O}))$ which means that $p(t) \in \mathrm{BMO}(\mathbb{R}^2)$ and is 2π-periodic in x_2. For the discussion of uniqueness questions we refer to [18].

The following result establishes local existence of solutions for initial data $u_0 \in \mathrm{BUC}(\mathbb{O})$.

Theorem 3.4
Let $F \in C([0,T], \mathrm{BUC}(\mathbb{O}))$ and $F_0 = \sup_{s\in[0,T]}\|F(s)\|_\infty$ then for any $u_0 \in \mathrm{BUC}(\mathbb{O})$ with $\nabla\cdot u_0 = 0$ there exists a unique solution $u \in \mathrm{C}([0,T_0], \mathrm{BUC}(\mathbb{O}))$ to (3.3) with $T_0 = \min\{T, \frac{K_0}{2F_0}, \frac{C_5K_0}{4F_0^2}, \frac{1}{64(C_5K_0)^2}\}$. Moreover,

(i) $t^{1/2}\nabla u \in \mathrm{C}([0,T_0], \mathrm{BUC}(\mathbb{O}))$.

(ii) $\nabla u \in \mathrm{C}^\alpha\big([\delta,T_0], \mathrm{BUC}(\mathbb{O})\big)$ *for any $\delta > 0$ and $\alpha \in (0,1/2)$.*

Proof: We use Picard's iterations for $t \in [0, T_0]$:

$$u_1(t) = \mathrm{e}^{t\Delta} u_0 + \widehat{F}(t), \quad u_{j+1}(t) = \mathrm{e}^{t\Delta} u_0 + \widehat{F}(t) + B(u_j, u_j), \quad j = 1, 2, \ldots.$$

The Young inequality gives $\|\mathrm{e}^{t\Delta} u_0\|_\infty = \|u_0\|_\infty$ and

$$\|\widehat{F}(t)\|_\infty := \| \int_0^t \mathrm{e}^{(t-s)\Delta} \mathbb{P} F \, \mathrm{d}s \|_\infty \le (C_3 t + 2C_4 \sqrt{t}) F_0.$$

The convergence of the sequence $\{u_j(t)\}, j = 1, 2, \cdots$ can be established with the use of estimates (2.10),(3.1) along the same lines as in [17]. Introduce the notations $K_0 = \|u_0\|_\infty, \widetilde{K}_0 = C_5 \|u_0\|_\infty$ and

$$K_j(T) = \sup_{t \in [0,T]} \|u_j\|_\infty, \quad \widetilde{K}_j(T) = \sup_{t \in [0,T]} t^{1/2} \|\nabla u_j\|_\infty,$$

where ∇u is the Jacobi matrix. From (2.10),(3.1) follows

$$\|B(u_j, u_j)\|_\infty \le C_5 \sqrt{T} K_j(T)^2 \quad \text{and hence}$$

$$K_{j+1}(T) \le K_0 + (C_3 T + C_4 \sqrt{T}) F_0 + C_5 T^{1/2} K_j(T)^2.$$

Therefore, if we take

$$\widehat{T}_0 \le \min \Big\{ \frac{K_0}{2C_3 F_0}, \frac{1}{64 (C_5 K_0)^2} \Big\},$$

then for $0 < t < \widehat{T}_0$ we have

$$K_j(T) \le 2K_0 \quad j = 1, 2, \cdots.$$

In the same way

$$\widetilde{K}_{j+1}(T) \le C_5 K_0 + \sqrt{T} C_5 F_0 + \sqrt{T} C_5 K_j(T) \widetilde{K}_j(T),$$

and with

$$\widetilde{T}_0 \le \min \Big\{ \frac{C_5 K_0}{4 F_0^2}, \frac{1}{64 (C_5 K_0)^2} \Big\},$$

we have

$$\widetilde{K}_j \le 2 C_5 K_0$$

for $0 < t < T_0 = \min\{\widehat{T}_0, \widetilde{T}_0\}$.

Now it is easy to prove that the sequences $\{u_j(t)\}$, $j = 1, 2, \cdots$ and $\{t^{1/2} \nabla u_j(t)\}$, $j = 1, 2, \cdots$ uniformly converge on $[0, T_0]$ in the L^∞ norm.

Finally, we set $\nabla p = (I - \mathbb{P})(\nabla \cdot (u \otimes u) - F)$. The details of the proof can be reconstructed from Theorem 1 in [17]. □

Note that the time T_0 of local existence in Theorem 3.4 only depends on the force F and the size $K_0 = \|u_0\|_\infty$ of the initial datum. Hence if we control the maximal growth of $\|u(t)\|_\infty$ we are able to establish global-in-time existence.

For this purpose we use the fact that for two dimensional flows the vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ satisfies a maximum principle and that u can be reconstructed from ω as follows. We use $\mathrm{BUC}^1(\mathbb{O})$ to denote those functions in $\mathrm{BUC}(\mathbb{O})$ whose derivative ∇u exists and lies in $\mathrm{BUC}(\mathbb{O})$.

Proposition 3.5

For $u \in \mathrm{BUC}^1(\mathbb{O})$ let $\omega = \operatorname{rot} u$. Then there exists $m \in \mathbb{R}^2$ with

$$u = m + (-\partial_2 K * \omega, [(I - P) \partial_1 K] * \omega)^\mathsf{T}.$$

The proof is obvious from the discussion in Section 2, which shows that $\partial_2 K$, $[I-P]\partial_1 K \in \mathrm{L}^1(\mathbb{O})$. Clearly, the constant part $m = (m_1, m_2)^{\mathsf{T}} \in \mathbb{R}^2$ is not seen in ω.

Since $\nabla u \in \mathrm{C}([\delta, T_0], \mathrm{BUC}(\mathbb{O}))$ for any $\delta > 0$ we may set $\widetilde{u}(t) = u(t+\delta)$. Then $\widetilde{u}(t)$ is a solution with $\widetilde{u}_{|t=0} \in \mathrm{BUC}^1(\mathbb{O})$ and $\nabla\cdot\widetilde{u}_{|t=0} = 0$. That is why we consider from now on only solutions with $u(0) \in \mathrm{BUC}^1(\mathbb{O})$. Our next aim is to use the vorticity equation to get a pointwise estimate for the velocity.

Theorem 3.6
Let $F \in \mathrm{C}\big([0,\infty), \mathrm{L}^\infty(\mathbb{O})\big)$ with $\operatorname{rot} F \in \mathrm{C}\big([0,\infty), \mathrm{L}^\infty(\mathbb{O})\big)$. Then for each $u_0 \in \mathrm{BUC}(\mathbb{O})$ with $\nabla\cdot u_0 = 0$ equation (3.3) has a global solution

$$u \in \mathrm{C}([0,\infty); \mathrm{BUC}(\mathbb{O})) \cap \mathrm{C}((0,\infty); \mathrm{BUC}^1(\mathbb{O})).$$

In particular, if $u_0 \in \mathrm{BUC}^1(\mathbb{O})$ we have an estimate

$$\|u(t)\|_\infty \le \delta(t) \quad \text{for all} \quad t \ge 0, \tag{3.4}$$

where $\delta : [0,\infty) \mapsto \mathbb{R}$ is explicitly given through the data as follows

$$\begin{aligned} \delta(t) \quad = \quad & \|u_0\|_\infty + \int_0^t \|F_2(s)\|_\infty \,\mathrm{d}s + (C_1 + C_2)\big(\|\omega(0)\|_\infty + \int_0^t \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s\big) \\ & + C_2 \int_0^t \big(\|\omega(0)\|_\infty + \int_0^s \|\operatorname{rot} F(\tau)\|_\infty \,\mathrm{d}\tau\big)^2 \,\mathrm{d}s. \end{aligned}$$

If additionally F and $\operatorname{rot} F$ are in $\mathrm{L}^\infty\big([0,\infty) \times \mathbb{O}\big)$ then there exists $\widetilde{C} > 0$ such that

$$\delta(t) \le \widetilde{C}(1+t)^3.$$

Proof: Since $\nabla u \in C([0,T]; \mathrm{BUC}(\mathbb{O}))$ we may define the vorticity $\omega(t,x) = \operatorname{rot} u(t,x)$. For convenience we summarize all the corresponding relations in the system.

$$\left\{ \begin{aligned} \partial_t \omega - \Delta\omega + (u\cdot\nabla)\omega \quad &= \quad \operatorname{rot} F, \quad \text{with} \quad \omega_{|t=0} = \operatorname{rot} u_0, \\ u_1(t) \quad &= \quad m_1(t) - \partial_2 K * \omega(t), \\ u_2(t) \quad &= \quad m_2(t) + [(I-P)\partial_1 K] * \omega(t), \\ \partial_t u + \nabla\cdot(u \otimes u) + \nabla p \quad &= \quad \Delta u + F, \quad \nabla\cdot u = 0. \end{aligned} \right. \tag{3.5}$$

Recall that the function $m_2(t, x_1) = P u_2(t,x)$ is the transversal and $m_1(t)$ is the axial mean flow. To find the governing equation for m_2 we have to go back to the Navier-Stokes system and apply the averaging operator P to its second component. We obtain

$$\begin{aligned} \partial_t m_2 \quad &= \quad P\Big[- u_1(\omega + \partial_2 u_1) - \frac{1}{2}\partial_2 u_2^2 - \partial_2 p + \Delta(v_2 + m_2) + F_2 \Big] \\ &= \quad \partial_1^2 m_2 + P[F_2 - u_1\omega]. \end{aligned} \tag{3.6}$$

The function m_1 is a constant which is defined by the initial data $u_{|t=0} = u_0$. To demonstrate this, notice that the averaging over $\mathbb{S}^1$ applied to the first component of (2.4) implies

$$\partial_t m_1 - \partial_{x_1}^2 m_1 = P[\mathbb{P}\big(F - \nabla\cdot(u \otimes u)\big)]_1. \tag{3.7}$$

From (2.7) follows $P[\mathbb{P}\big(F - \nabla\cdot(u\otimes u)\big)]_1 = 0$ where $[v]_j$ denotes the j-component of the vector v. It is left to observe that the divergence-free condition yields

$$\partial_{x_1} m_1 := \partial_{x_1} \int_{\mathbb{S}^1} u_1 \,\mathrm{d}x_2 = - \int_{\mathbb{S}^1} \partial_{x_2} u_2 \,\mathrm{d}x_2 = 0,$$

and hence (3.7) implies additionally $\partial_t m_1(t) = 0$. Thus, $m_1(t, x_1) = m_1(0)$.

Let us use system (3.5) to get an a priori estimate of the solution $u(t, x)$. To use the maximum principle for the vorticity equation

$$\partial_t \omega - \Delta\omega + (u\cdot\nabla)\omega = \operatorname{rot} F,$$

note that $\widetilde{\omega}$ with $\widetilde{\omega}(t, x) = \omega(t, x) - \int_0^t \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s$ satisfies $\partial_t\widetilde{\omega} - \Delta\widetilde{\omega} + (u\cdot\nabla)\widetilde{\omega} = \operatorname{rot} F - F_0 \leq 0$ which gives $\sup_{x\in\mathbb{O}} \widetilde{\omega}(t, x) \leq \sup_{x\in\mathbb{O}} \widetilde{\omega}(0, x) \leq \|\omega(0)\|_\infty$. Similarly, one obtains the lower bound $\inf_{x\in\mathbb{O}} \omega(t, x) \geq -\|\omega(0)\|_\infty - \int_0^t \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s$. Thus we have proved

$$\|\omega(t)\|_\infty \leq \|\omega(0)\|_\infty + \int_0^t \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s. \tag{3.8}$$

Now Lemma (2.3) and Proposition (3.5) provide estimates for u, namely

$$\|u_1(t)\|_\infty \leq \|m_1(t)\|_\infty + \|\partial_2 K * \omega(t)\|_\infty \leq |m_1(0)| + C_2\|\omega(t)\|_\infty. \tag{3.9}$$

Similarly, we have

$$\|u_2(t)\|_\infty \leq \|m_2(t)\|_\infty + \|(I - P)\partial_1 K * \omega(t)\|_\infty \leq \|m_2(t)\|_\infty + C_1\|\omega(t)\|_\infty. \tag{3.10}$$

However, m_2 has to be estimated via (3.6). From the Duhamel formula follows

$$\begin{aligned} \|m_2(t)\|_\infty &\leq \|m_2(0)\|_\infty + \int_0^t (\|u_1\omega\|_\infty + \|F_2(s)\|_\infty) \,\mathrm{d}s \leq \\ &\leq C_2 \textstyle\int_0^t (\|\omega(0)\|_\infty + \displaystyle\int_0^t \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s)^2 \,\mathrm{d}t \\ &\quad + \|m_2(0)\|_\infty + \int_0^t \|F_2(s)\|_\infty \,\mathrm{d}s. \end{aligned} \tag{3.11}$$

Collecting estimates (3.9),(3.10),(3.11) we arrive at the desired result. □

As a conclusion we get from the local existence and the estimate (3.4) the statement on the global existence in time for the solutions of the Cauchy problem with the initial data in BUC($\mathbb{O}$).

Remark 3.7 *Notice that if* $\int_0^\infty \|F(s)\|_\infty + \|\operatorname{rot} F(s)\|_\infty \,\mathrm{d}s \leq \infty$, *then we obtain*

$$\|u(t)\|_\infty \leq C(1 + t). \tag{3.12}$$

We do not know whether estimates (3.11),(3.12) are sharp.

4 Special solution types

4.1 Shear flows in axial direction

As already mentioned A.N. Kolmogorov suggested to take the forcing term

$$F(t, x) = (F_1(t, x_2), 0)^\mathsf{T},$$

which gives $\nabla\cdot F \equiv 0$, i.e. $\mathbb{P}F = F$. Then, there are exact solutions of (1.1) in the form

$$u(t, x) = (u_1(t, x_2), 0)^\mathsf{T}.$$

The component u_1 has to satisfy the linear heat equation

$$\partial_t u_1(t,x_1) - \partial_1^2 u_1(t,x_2) = F_1(t,x_2) \text{ for } x_2 \in \mathbb{S}^1 \text{ and } t > 0. \tag{4.1}$$

If F_1 is time independent and $F_1(x_2) = \sum_{k\neq 0} f_k \mathrm{e}^{\mathrm{i}kx_2}$ we get the steady state solution $(\widehat{u}_1, 0)^\mathsf{T}$, $p = const$ with $\widehat{u}_1 = \sum_k \frac{-f_k}{k^2}\mathrm{e}^{\mathrm{ikx_2}}$. The explicit time dependent solution of (4.1) is given via the one-dimensional heat kernel

$$u_1(t) = H_{\mathbb{S}^1}(t) * u_1(0) + \int_0^t H_{\mathbb{S}^1}(t-\tau) * F_1(\tau)\,\mathrm{d}\tau.$$

From this expression we have the exponential convergence to a steady state

$$\|\widehat{u}_1 - u_1(t)\|_\infty \le \mathrm{e}^{-t}\|u_0\|_\infty.$$

Nevertheless the Kolmogorov flow becomes unstable in BUC($\mathbb{O}$) for sufficiently large Grashof numbers for perturbations that depend on both x_1 and x_2, see [22, 20, 2].

4.2 Shear flows in transverse direction

Inequality (3.11) gives a hint how to construct nontrivial examples of solutions with a linear growth in time. We first mention that exact shear flows are included in our function space. Assume that the forcing has the form

$$F(t,x) = (0, F_2(t,x_1))^\mathsf{T} \quad (\text{which gives } \operatorname{div} F \equiv 0, \text{ i.e., } \mathbb{P}F = F).$$

Then, there are exact solutions of (1.1) in the form

$$u(t,x) = (0, u_2(t,x_1))^\mathsf{T}.$$

The component u_2 has to satisfy the linear heat equation

$$\partial_t u_2(t,x_1) - \partial_1^2 u_2(t,x_1) = F_2(t,x_1) \text{ for } x_1 \in \mathbb{R} \text{ and } t > 0. \tag{4.2}$$

Again, the explicit solution is given via the one-dimensional heat kernel

$$u_2(t) = H_\mathbb{R}(t) * u_2(0) + \int_0^t H_\mathbb{R}(t-s) * F_2(s)\,\mathrm{d}s.$$

It is now easy to see that a constant forcing of the type $F_2(x_1) = \lambda \tanh x_1$ and the initial condition $u_2(0) \equiv 0$ lead to solutions which satisfy $\lim_{x\to\pm\infty} u_2(t,x_1) = \pm\lambda t$, since for very large $|x_1|$ the flow is accelerated uniformly in space and time. For this example we obtain the lower bound

$$\|u_2(t)\|_\infty \ge \lambda t = t\|F_2\|_\infty.$$

Moreover, there is the upper bound for all initial data with $u_1 = 0$ and bounded forces

$$\|u_2(t)\|_\infty \le \|u_2(0)\|_\infty + \int_0^t \|F_2(s)\|_\infty\,\mathrm{d}s.$$

The cubic bound given in Theorem 3.6 may correspond to solutions with $u_1 \neq 0$ and therefore is still realistic. If the vorticity is bounded (for instance this happens when $\operatorname{rot} F = 0$), then estimate (3.4) is reduced to the sharp linear form. Note, however, that from the physical viewpoint it is expected that vorticity behaves worse (is less regular) than velocity and therefore it will be interesting to get the growth faster than linear in time.

4.3 Flows on $\mathbb{T}^2$

Flows on $\mathbb{T}^2$ are widely known to have an absorbing ball. This is true if we suppose that $\int_{\mathbb{T}^2} F \, \mathrm{d}x_1 \, \mathrm{d}x_2 := (f_1, f_2) = 0$ or if we allow for a compensation of the mean force $f = (f_1, f_2)$ by the pressure gradient $\nabla p = (f_1, f_2)$. With these conditions the problem on $\mathbb{T}^2$ is imbedded into the frame of the problem on $\mathbb{O}$ and the estimate (3.4) is still valid. The temporal growth of the L^∞ norm of solutions on $\mathbb{T}^2$ can be abandoned by the additional periodicity which leads to the obvious changes in the definitions of the fundamental solution and the Helmholtz projection.

4.4 Solutions with arbitrary temporal growth

As we have seen in (2.6), it is very easy to construct solutions with arbitrary growth in time, even with finite-time blow up. For our problem on $\mathbb{O}$ with forcing $F \equiv 0$ we immediately see that for each $\phi_1 \in \mathrm{C}^2(\mathbb{R}, \mathbb{R})$ the pair (u, p) with

$$u(t, x) = (-\phi_1'(t), 0)^\mathsf{T} \quad \text{and} \quad p(t, x) = \phi_1''(t) x_1$$

is a solution for the Navier-Stokes system. Hence, $\|u(t)\|_\infty$ can have arbitrary growth or finite time blow up, if we allow for large pressure gradients. Similarly, one can construct solutions (u, p) on all of $\mathbb{R}^n$ which are not related to the extended Galilean symmetry group and for which u is linear in x and p is quadratic, see [25, 9]. They have the form

$$u(t, x) = (S(t) + \Omega(t))x \quad \text{and} \quad p(t, x) = \frac{1}{2}(B(t)x) \cdot x, \tag{4.3}$$

where $S = S^\mathsf{T}$, $B = B^\mathsf{T}$ and $\Omega = -\Omega^\mathsf{T}$. Then, the Navier-Stokes equations are satisfied if and only if

$$\dot{S} + S^2 + \Omega^2 + B = 0, \quad \dot{\Omega} + S\Omega + \Omega S = 0, \quad \operatorname{tr} S = 0.$$

Thus, we may choose arbitrary $S : t \mapsto S(t)$ satisfying $S(t) = S(t)^\mathsf{T}$ and the divergence free condition $\operatorname{tr} S(t) = 0$. Then, we solve the linear equation for $\Omega(t)$ and finally we adjust the pressure matrix $B(t)$ to fulfill the first equation. Again there is no uniqueness.

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References

[1] A.L. Afendikov: *The analysis of instabilities of the generalised Kolmogorov flows.* Preprint KIAM No. 126, Keldysh Inst. Appl. Math. Acad.Sci., Moscow 1995, 1-30.

[2] A.L. Afendikov, K. Babenko: Bifurcation in the presence of the symmetry group and loss of stability of some plane viscous fluid flows. *Soviet Math. Dokl.* **33**(3) (1986), 742-747.

[3] A.L. Afendikov, K. Kirchgässner: Spatial dynamics approach to Kolmogorov flows on plane torus, *Zeits. angew. Math. Mech.*, **78**, (1998) (Supl.1): S211–S212.

[4] A.L. Afendikov, A. Mielke: Bifurcation of the Poiseuille flow between parallel plates: three–dimensional solutions with large spanwise wavelength, *Arch. Rat. Mech. Analysis* **129** (1995), 101–127.

[5] A.L. Afendikov, A. Mielke: Multi–pulse solutions to the Navier–Stokes problem between parallel plates, *Zeits. angew. Math. Physik (ZAMP)* **51** (2001), 79-101.

[6] A.L. Afendikov, B. Fiedler, S. Liebscher: *Plane Kolmogorov flows, spatial reversibilities, and bifurcation without parameters.* (2004) Manuscript.

[7] H. Amann: On the strong solvability of the Navier–Stokes equations, *J. Math. Fluid Mech. (JMFM)* **2** (2000), 16-98.

[8] V.I. Arnold, B. Khesin: Topological methods in hydrodynamics. *Applied Math. Sci. 125* Springer-Verlag, New-York 1998.

[9] S. Childres, G.R. Ierley, E.A. Spiegel, W.R. Young: Blow up of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation-point form, *J. Fluid Mech.* **203** (1989), 1-22.

[10] P. Chossat, G. Iooss: *The Couette-Taylor problem.* Appl.Math.Sci v. 102, Springer Verlag, NY 1994.

[11] P. Collet: *A global existence result for the Navier-Stokes equation in the plane.* (1998) Manuscript

[12] W. E, Ya. Sinai: New results in mathematical and statistical hydrodynamics, (Russian) *Uspekhi Mat. Nauk* **55** (4) (2000) 25-58; translation in *Russian Math. Surveys* **55**, 4, (2000), 635-666.

[14] T. Gallay, C. E. Wayne: Invariant manifilds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$, *Arch. Rat. Mech. Analysis* **163** (2002), 209–258.

[15] S. Gama, M. Vergassola, U. Frisch: Negative eddy viscosity in isotropically forced two-dimensional flow: linear and nonlinear dynamics, *J. Fluid Mech.* **260**(1994), 95–126.

[16] Y. Giga, S. Matsui, O. Sawada: Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity, *J. Math. Fluid Mech.* **3**(3)(2001), 302-315.

[17] Y. Giga, K. Inui, S. Matsui: On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data. in "*Advances in Fluid Dynamics*, Quad. Mat., 4, Aracne, Rome" 1999, 27-68.

[18] Y. Giga, K. Inui, J. Kato, S. Matsui: Remarks on the uniqueness of bounded solutions for the Navier-Stokes equations, *Nonlinear Analysis.* **47** (2001), 4151-4156.

[19] G. Iooss, A. Mielke, Y. Demay: Theory of the steady Ginzburg-Landau equation in hydrodynamic stability problems. *Eur. J. Mech. B/Fluids*, **8** n 3, (1989), 229-268.

[20] V. Iudovich: An example of the loss of stability of laminar viscous incompressible fluid flow and the birth of the secondary stationary or periodic flows, (Russian) *Prikl. Math. Mech.* **29** (1965), 453-467.

[21] P.G. Lemarie-Rieusset: *Recent developments in the Navier-Stokes problem.* Research Notes in Math. 341, CRC Press, 2002.

[22] L. Meshalkin, J. Sinai: Investigation of stability of the stationary solution of the problem of the motion of some plane incompressible fluid flow, (Russian) *Prikl. Math. Mech.* **25** (1961), 1140-1143.

[23] G. Schneider: Global existence results for pattern formation processes in infinite cylindrical domains. Applications to 3D Navier-Stokes problem, *J. Mathem. Pure et Applique* **78** (1999), 265-312.

[24] E.M. Stein: *Harmonic analysis.* Princeton Univ. Press. New Jersey 1993.

[25] J.T. Stuart: Nonlinear Euler partial differential equations: singularities in their solution. In *Symposium to Honor C.C. Lin*, D.J. Benney, F.H. Shu, C. Yuan, eds, World Scientific Publishing Co., Singapore 1988, pp. 81-95.

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