Existence theory for finite-strain crystal plasticity with gradient regularization

Alexander Mielke

Abstract We provide a global existence result for the time-continuous elastoplasticity problem using the energetic formulation. The strain tensor is decomposed multiplicatively into an elastic part and the plastic tensor P, which is driven by the plastic slip strain rates $\dot{p}_j$. We allow for self-hardening as well as cross-hardening. The strain gradients ∇p_j and ∇P are used to regularize the problem, thus introducing a length scale and preventing the formation of microstructure.

1 Introduction

Elastoplasticity at finite strain is usually based on the multiplicative decomposition $\nabla\varphi = F = F_{\rm el}F_{\rm pl}$, introduced in [Lee69]. This decomposition reflects the Lie group structure of $\mathrm{GL}^+(d) \overset{\rm def}{=} \{ F \in \mathbb{R}^{d\times d} \mid \det F > 0 \}$, where the elastic part $F_{\rm el}$ will contribute to the energy storage whereas the plastic tensor $P = F_{\rm pl}$ evolves according to a plastic flow rule. The plastic tensor maps the material frame (crystallographic lattice) onto itself and is usually assumed to lie in the special linear group $\mathrm{SL}(d) \overset{\rm def}{=} \{ P \in \mathbb{R}^{d\times d} \mid \det P = 1 \}$.

In this paper we combine the formal ideas for single-crystal plasticity from [OrR99, Mie03] with the recent analytical developments in [MaM08] proving a global-in-time existence result for solution in finite-strain elastoplasticity. The difficulty is to find a formulation that allows us to use functional analytical tools that are compatible with the strong nonlinearities generated by the Lie group structures resulting from $\mathrm{GL}^+(d)$ and $\mathrm{SL}(d)$. We use here the theory of energetic solutions for rate independent systems as developed in [MTL02, Mie05]. The recently developed geometric formulation on abstract topological spaces (cf. [FrM06, MRS08, Mie08]) was strongly motivated by the present application and, thus, provides the first math-

Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstraße 39, D-10117 Berlin
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ematical foundation to treat the existence theory for time-dependent finite-strain elastoplasticity.

To be more specific we introduce some notations. Let $\varphi:\Omega\to\mathbb{R}^d$ denote the deformation, $P:\Omega\to\mathrm{SL}(d)$ the plastic tensor, and $p:\ \Omega\to[0,\infty[^m$ is the vector of slip strains. Then, we assume that the stored-energy functional takes the form

$$\mathcal{E}(t,\varphi,P,p)=\int_\Omega W(x,\nabla\varphi P^{-1},p,\nabla P,\nabla p)\,\mathrm{d}x-\langle\ell(t),\varphi\rangle.$$

Here $F_{\mathrm{elast}}=\nabla\varphi P^{-1}$ represents the multiplicative decomposition. The gradients $(\nabla P,\nabla p)$ introduce a length scale and will be essential to provide compactness, thus preventing the formation of microstructure, cf. [CHM02, BC*04]. Such regularizing terms are also common in engineering models, cf. [DiK70, MüA91, FlH97, Gur00, Gur02, BaJ02].

In our quasistatic setting we will assume that

$$\begin{array}{l}\varphi(t) \text{ minimizes the energy } \mathcal{E}(t,\cdot,P(t),p(t))\\ \text{subject to } \varphi(t,x)=g_{\mathrm{Dir}}(t,x) \text{ for } x\in\Gamma_{\mathrm{Dir}},\end{array}\tag{1}$$

which provides the usual elastic equilibrium equation $\operatorname{div}\sigma=f_{\mathrm{vol}}$ in Ω and $\sigma\cdot\nu=f_{\mathrm{tract}}$ on the Neumann part of the boundary $\partial\Omega$, where $\sigma=\partial_F W$ is the first Piola-Kirchhoff stress tensor.

The evolution of the plastic variables P and p is governed by the plastic flow rule which will be assumed to be formulated by a dissipation potential $R(x,P,p,\dot P,\dot p)$ such that

$$0\in\partial^{\mathrm{sub}}_{(\dot P,\dot p)}R(x,P,p,\dot P,\dot p)+\begin{pmatrix}\partial_P W(\cdots)-\operatorname{div}\big(\partial_{\nabla P}W(\cdots)\big)\\ \partial_p W(\cdots)-\operatorname{div}\big(\partial_{\nabla p}W(\cdots)\big)\end{pmatrix}.\tag{2}$$

It would be possible to supplement $\mathcal{E}$ by a surface integral involving the plastic variables, namely

$$\int_{\partial\Omega}\rho(x,P(x),p(x))\,\mathrm{d}x,$$

where $\rho:\partial\Omega\times\mathrm{SL}(d)\to\mathbb{S}^{d-1}\to\mathbb{R}$ is a nonnegative Caratheodory function. This term could be used to account for surface effects due to plasticity (i.e., accumulation of dislocation). The boundary conditions associated with (2) are

$$\partial_{\nabla P}W(\cdots)\nu+\partial_P\rho=0,\quad \partial_{\nabla p}W(\cdots)\nu+\partial_p\rho=0.$$

where ν is the outer normal vector. To simplify the presentation we omit this term.

In (2) $R(x,P,p,\cdot,\cdot)$ is convex on the tangent space and $\partial^{\mathrm{sub}}_{(\dot P,\dot p)}R$ denotes the corresponding subdifferential. This flow rule is rate independent if $R(x,P,p,\cdot,\cdot)$ is positively homogeneous of degree 1, i.e., $R(x,P,p,\lambda(\dot P,\dot p))=\lambda R(x,P,p,\dot P,\dot p)$. By the proper choice of R we will guarantee that this flow rule contains the essential kinematic relation between the plastic tensor and the slip strains, namely

$$\dot{P} = \Big(\sum_{\alpha=1}^{m} \dot{p}_\alpha S_\alpha\Big)P, \qquad \text{where } S_\alpha = m^\alpha \otimes n^\alpha,\ \alpha = 1,\dots,m, \tag{3}$$

are the the slip systems. Here $n^\alpha \in \mathbb{R}^d$ is the (unit) normal vector of the slip system S_α and $m^\alpha \in \mathbb{R}^m$ is the slip direction satisfying $m^\alpha \cdot n^\alpha = 0$.

A major step for deriving an existence theory is the replacement of the dissipation potential R by the associated dissipation distance D, see (5). The dissipation functional

$$\mathcal{D}(P_0,p_0,P_1,p_1) = \int_\Omega D(x,P_0(x),p_0(x),P_1(x),p_1(x))\,\mathrm{d}x$$

measures the minimal amount of energy dissipated when going from the state (P_0,p_0) to (P_1,p_1). An important fact is that $\mathcal{D}$ satisfies the (unsymmetric) triangle inequality. A major difficulty arises from the fact that D has only logarithmic growth because of plastic invariance, see (6). As a consequence $\mathcal{D}$ cannot be coercive on linear function spaces. The energetic approach for rate-independent systems $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is exactly suited for this situation. However, we still will have extra work to establish coercivity of the energy, see Section 3.1.

The energetic formulation of rate-independent systems provides a weak form of the system (1) and (2). For this we choose a state space $\mathcal{Q}$ for $\boldsymbol{q} = (\varphi,P,p)$ by identifying suitable weakly closed subsets of Sobolev spaces over Ω. A mapping $\boldsymbol{q} = (\varphi,P,p):\ [0,T] \to \mathcal{Q}$ is called *energetic solution*, if for all $t \in [0,T]$ the *stability condition* (S) and the *energy balance* (E) hold:

$$\begin{aligned}&\text{(S)}\ \ \mathcal{E}(t,\boldsymbol{q}(t)) \le \mathcal{E}(t,\widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}(t),\widehat{\boldsymbol{q}}) \text{ for all } \widehat{\boldsymbol{q}} \in \mathcal{Q},\\ &\text{(E)}\ \ \mathcal{E}(t,\boldsymbol{q}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q};[0,t]) = \mathcal{E}(0,\boldsymbol{q}(0)) + \textstyle\int_0^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s.\end{aligned} \tag{4}$$

Here $\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{q};[r,s]) = \sup\sum_1^N \mathcal{D}(P(\tau_{j-1}),p(\tau_{j-1}),P(\tau_j),p(\tau_j))$, where the supremum is taken over all partitions of $[r,s]$. IN the case of external loadings and time-independent boundary conditions we have $\partial_t\mathcal{E}(t,q) = -\langle\dot{\ell}(t),\varphi\rangle$.

However, if g_{Dir} depends on time the power of the displacement loadings is more difficult to express in a mathematically correct way, since the stresses on the boundary are not well defined. Following [FrM06, MaM08] we write the unknown displacement as a composition $\varphi(t,x) = g_{\mathrm{Dir}}(t,y(t,x))$, where $y:\Omega \to \mathbb{R}^d$ is the new unknown satisfying $y(t,x) = x$ for $x \in \Gamma_{\mathrm{Dir}}$. With $q = (y,z)$ we write $\widehat{\mathcal{E}}(t,q) = \mathcal{E}(t,g_{\mathrm{Dir}}(t)\circ y,z)$ and find that $\partial_t\mathcal{E}(t,q)$ can be expressed in terms of the Kirchhoff stress tensor and a convected derivative.

In Section 2 we follow [Mie03] for discussing the mechanical modeling of elastoplasticity and for explaining why the concept of energetic solutions can be seen as a weak version of the classical plasticity formulation. The major advantage of (S) and (E) is that it avoids derivatives and is based solely on the functionals $\mathcal{E}$ and $\mathcal{D}$, which need not be smooth or even continuous. In Section 2.2 we formulate precise assumptions on W,D, and g_{Dir} that allow us to construct solutions in suitable Sobolev spaces. The main result is Theorem 1 which states the global existence of energetic solutions for single-crystal plasticity. For the cases of kinematical hardening and isotropic hardening we refer to [MaM08].

2 Modeling assumptions and results

We first provide an exact description of the mechanical model in terms of the constitutive functions, namely the stored-energy density W and the dissipation potential R. Here we discuss the main symmetries and the basic kinematic relations. Next we discuss the assumptions that are necessary to develop a mathematical existence theory. Finally, this section closes by stating the main existence result and the underlying abstract theory developed in [MaM08].

2.1 Mechanical modeling

We recall the multiplicative decomposition $\nabla\varphi = F = F_{\mathrm{el}}F_{\mathrm{pl}}$, where the plastic tensor $P = F_{\mathrm{pl}} \in \mathrm{SL}(d)$ maps the material space crystallographic lattice onto itself. The slip strains p_j are combined into a vector $p \in [0,\infty[^m$. To simplify notations we let $z = (P,p) \in Z = \mathrm{SL}(d)\times[0,\infty[^m$ and use A as a place holder for $\nabla z = (\nabla P, \nabla p)$.

The stored-energy density $W = \widetilde{W}(x,F,P,p,A)$ and the dissipation potential $R = \widetilde{R}(x,P,p,\dot{P},\dot{p})$ have to satisfy the following symmetry properties:

(Sy1) *Objectivity (frame indifference)*
$$\widetilde{W}(x,QF,P,p,A) = \widetilde{W}(x,F,P,p,A) \text{ for all } Q \in \mathrm{SO}(d);$$
(Sy2) *Plastic indifference*
$$\left.\begin{array}{r} \widetilde{W}(x,F\widetilde{P},P\widetilde{P},p,A) = \widetilde{W}(x,F,P,p,A) \\ \widetilde{R}(x,P\widetilde{P},p,\dot{P}\widetilde{P},\dot{p}) = \widetilde{R}(x,P,p,\dot{P},\dot{p}) \end{array}\right\} \text{ for all } \widetilde{P} \in \mathrm{SL}(d);$$
(Sy3) *Material symmetry*
$$\left.\begin{array}{r} \widetilde{W}(x,F,PS,\pi_S p,\Pi_S A) = \widetilde{W}(x,F,P,p,A) \\ \widetilde{R}(x,PS,\pi_S p,\dot{P}S,\pi_S\dot{p}) = \widetilde{R}(x,P,p,\dot{P},\dot{p}) \end{array}\right\} \text{ for all } S \in \mathfrak{S} \subset \mathrm{O}(d).$$

In (Sy3) the group $\mathfrak{S}$ is the material-symmetry group which acts on the plastic strain by a permutation $\pi_Q : p \mapsto (p_{\pi_Q(1)},\ldots,p_{\pi_Q(m)})$, see [Mie03, Sect. 3.4.4], and $\Pi_S(\nabla P,p) = \nabla(PS,\pi_S p)$. In the sequel we will drop the explicit dependence on x for notational simplicity. However, the whole theory is still valid if $\widetilde{W}$ and $\widetilde{R}$ depend on $x \in \Omega$, which would be the case for polycrystals.

A consequence of (Sy2) is that $\widetilde{W}$ and $\widetilde{R}$ can be written in a reduced form via
$$\widetilde{W}(F,P,p,A) = W(FP^{-1},p,A) \;\text{ and }\; \widetilde{R}(P,p,\dot{P},\dot{p}) = R(p,\dot{P}P^{-1},\dot{p}),$$
where $\xi = \dot{P}P^{-1} \in \mathrm{sl}(d) = \mathrm{T}_{\mathbf{1}}\mathrm{SL}(d) = \{\, \xi \in \mathbb{R}^{d\times d} \mid \mathrm{tr}\,\xi = 0 \,\}$. We now define the dissipation distance $D(\cdot,\cdot)$ on $Z\times Z$ via
$$D(z_0,z_1) = \inf\{ \int_0^1 \widetilde{R}(z(s),\dot{z}(s))\,\mathrm{d}s \mid z \in \mathrm{C}^1([0,1],Z), z(0) = z_0, z(1) = z_1 \}. \tag{5}$$
Thus, D has the dimension of an energy density and measures the amount of energy per volume that has to be spent to transform a material point from the internal state

z_0 into z_1. The plastic indifference (Sy2) implies that the dissipation distance D is right-invariant, namely

$$D(P_1,p_1,P_2,p_2) = D(\mathbf{1},0,P_2P_1^{-1},p_2{-}p_1) \;\text{ for all } P_1,P_2,p_1,p_2. \tag{6}$$

We specify R further in such a way that the slip kinematics (3) is enforced automatically by $R(p,\xi,\nu) < \infty$, namely

$$R(p,\xi,\nu) = \begin{cases} \sum_{\alpha=1}^m \kappa_\alpha \nu_\alpha & \text{for } \xi = \sum_{\alpha=1}^m \nu_\alpha S_\alpha \text{ and } \nu \in [0,\infty[^m, \\ \infty & \text{otherwise,} \end{cases} \tag{7}$$

where the threshold parameters κ_α (cf. [OrR99, Gur00] are assumed to be bounded positive constants. Note that the slip strain behave monotonically and are not allowed to decrease. However, often $S_{\alpha+m/2} = -S_\alpha$ for $\alpha \le m/2$, then $\dot P$ may take any value.

Since $\nu = \dot p$ and $\dot P = \xi P$, the flow rule (2) implies the slip kinematics (3), because the subdifferential $\partial_{(\dot P,\dot p)}\widetilde R$ is nonempty if and only if R is finite.

We assume that the set of slip systems $\mathfrak{S} \overset{\text{def}}{=} \{\, S_\alpha \,|\, \alpha = 1,\dots,m \,\}$ is large enough to generate the whole group $\mathrm{SL}(d)$. More precisely, a slip system S_α has to be considered as an element of $\mathrm{sl}(d) = \mathrm{T}_{\mathbf{1}}\mathrm{SL}(d)$, such that $P_\alpha(\tau) = \mathrm{e}^{\tau S_\alpha} = \mathbf{1}{+}\tau S_\alpha$ is a simple shear. We say that $\mathrm{SL}(d)$ is generated by $\mathfrak{S}$, if each $P \in \mathrm{SL}(d)$ can be written in the form $P_{\alpha_1}(\tau_1)\cdots P_{\alpha_N}(\tau_N)$, where $N \in \mathbb{N}$, $\alpha_k \in \{1,\dots,m\}$, and $\tau_k \in \mathbb{R}$. By the standard theory of Lie groups and their Lie algebras this is equivalent to saying that $\mathrm{sl}(d)$ is the smallest Lie algebra containing $\mathfrak{S}$ (with respect to the standard Lie bracket $[\xi_1,\xi_2] = \xi_1\xi_2 - \xi_2\xi_1$). Obviously, $\mathfrak{S}$ generates $\mathrm{SL}(d)$, if the linear hull of $\mathfrak{S}$ equals $\mathrm{sl}(d)$, and this is the case in many cases of crystal plasticity, see [CoO05]. However, this is by far not necessary, for an example consider $\mathfrak{S} = \{e_1{\otimes}e_2, e_2{\otimes}e_1\}$, which generates $\mathrm{SL}(2)$, see [HMM03].

Subsequently we will not write down this condition on $\mathfrak{S}$, since it is not essential. If it is not satisfied, we just have to replace $\mathrm{SL}(d)$ by the smaller Lie group $\mathcal{G} \subset \mathrm{SL}(d)$ that is generated by $\mathfrak{S}$. The whole theory will still hold for any such subgroup.

2.2 Precise mathematical assumptions

For notational simplicity we restrict to the case of displacement boundary conditions that are independent of time and use volume and surface forces to drive the system. We refer to [FrM06, MaM08] to the case of time-dependent boundary conditions.

The domain $\Omega \subset \mathbb{R}^d$ is bounded and has a Lipschitz boundary. The Dirichlet part Γ_{Dir} of the boundary is assumed to have positive surface measure. For g_{Dir} we assume that it can be extended to all of $\mathbb{R}^d$ as follows:

$$\begin{aligned} &g_{\mathrm{Dir}} \in \mathrm{C}^1([0,T]{\times}\mathbb{R}^d;\mathbb{R}^d), \quad \nabla g_{\mathrm{Dir}} \in \mathrm{BC}^1([0,T]{\times}\mathbb{R}^d, \mathrm{Lin}(\mathbb{R}^d;\mathbb{R}^d)) \\ &\text{and } |\nabla g_{\mathrm{Dir}}(t,x)^{-1}| \le C \text{ for all } (t,x) \in [0,T]{\times}\mathbb{R}^d, \end{aligned} \tag{8}$$

where "BC[1]" stands for bounded and once continuously differentiable. Thus, for each $t \in [0,T]$ the mapping $g_{\mathrm{Dir}}(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a global diffeomorphism.

We seek for $\varphi(t,\cdot)$ in the form

$$\varphi(t,x) = g_{\mathrm{Dir}}(t,y(x)) \;\text{ with } y \in \mathcal{Y} \stackrel{\mathrm{def}}{=} \{\, y \in \boldsymbol{Y} \mid y|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id} \,\} \text{ and } \boldsymbol{Y} = \mathrm{W}^{1,q_{\boldsymbol{Y}}}(\Omega;\mathbb{R}^d).$$

We set $\boldsymbol{q} = (y,z)$ and $\widehat{\mathcal{E}}(t,y,P,p) = \mathcal{E}(t, g_{\mathrm{Dir}}(t,\cdot)\circ y, P, p)$. Since no confusion can arise, we denote $\widehat{\mathcal{E}}$ again by $\mathcal{E}$.

The internal variable is $z = (P,p) \in Z \stackrel{\mathrm{def}}{=} \mathrm{SL}(d)\times[0,\infty[^m$, where the space $\mathcal{Z}$ of internal states is chosen as

$$\begin{aligned}&\mathcal{Z} \stackrel{\mathrm{def}}{=} \{\, (P,p) \in \boldsymbol{Z} \mid (P(x),p(x)) \in Z \text{ and } D(\mathbf{1},0,P(x),p(x)) < \infty \text{ a.e. in } \Omega \,\},\\ &\text{where } \boldsymbol{Z} = \mathrm{W}^{1,r}(\Omega;\mathbb{R}^{d\times d}\times\mathbb{R}^m) \text{ with } r > d.\end{aligned}$$

The stored-energy functional $\mathcal{E}$ and the dissipation distance $\mathcal{D}$ take the forms

$$\begin{aligned}\mathcal{E}(t,y,z) &\stackrel{\mathrm{def}}{=} \int_\Omega W(\nabla g_{\mathrm{Dir}}(t,y(x))\nabla y(x)P(x)^{-1}, z(x), \nabla z(x))\,\mathrm{d}x,\\ \mathcal{D}(z_0,z_1) &\stackrel{\mathrm{def}}{=} \int_\Omega D(z_0(x),z_1(x))\,\mathrm{d}x,\end{aligned}$$

where D is defined in (5) via R in (7).

The conditions on W are much more involved. In particular, they include coercivity assumptions and convexity assumptions to obtain lower semicontinuity. To shorten notation we let $L^{(d,m)} \stackrel{\mathrm{def}}{=} \mathbb{R}^{d\times d\times d}\times\mathbb{R}^{m\times d}$ and use A as a placeholder for $\nabla z = (\nabla P, \nabla p) \in L^{(d,m)}$. The function $\mathbb{M} : \mathbb{R}^{d\times d} \to \mathbb{R}^{\mu_d}$ with $\mu_d = \sum_{s=1}^d \binom{d}{s}^2 = \binom{2d}{d} - 1$ maps a matrix to all its minors (subdeterminants). The Kirchhoff stress tensor is defined via $K(F,p,A) = \partial_F W(F,p,A)F^\mathsf{T}$. We impose the following:

$$\begin{aligned}&\text{there exists } \mathbb{W} : \mathbb{R}^{\mu_d}\times\mathbb{R}^m\times L^{(d,m)} \to \mathbb{R}_\infty :\\ &\quad\text{(i)}\quad \mathbb{W} \text{ is lower semicontinuous,}\\ &\quad\text{(ii)}\quad W(F,p,A) = \mathbb{W}(\mathbb{M}(F),p,A),\\ &\quad\text{(iii)}\quad \mathbb{W}(x,\cdot,p,\cdot) : \mathbb{R}^{\mu_d}\times L^{(d,m)} \to \mathbb{R}_\infty \text{ is convex;}\end{aligned} \tag{9a}$$

$$\begin{aligned}&\text{there exist } c>0,\ q_{\boldsymbol{Y}},\ r>d,\ q_{\boldsymbol{p}}>1 \text{ such that}\\ &\quad W(F,p,A) \geq c\left(|F|^{q_{\boldsymbol{Y}}} + |p|^{q_{\boldsymbol{p}}} + |A|^r\right) - 1/c.\end{aligned} \tag{9b}$$

$$\begin{aligned}&\text{there exist } c_0^W,\ c_1^W, \text{ and } \alpha \in (0,1] \text{ such that for } |E| \leq 1/(2d)\\ &\quad |K(F,p,A)| \leq c_1^W(W(F,p,A) + c_0^w)\\ &\quad |K((\mathbf{1}+E)F,p,A) - K(F,p,A)| \leq c_1^W(W(F,p,A) + c_0^w)|E|^\alpha.\end{aligned} \tag{9c}$$

Thus, (9a) implies that the mapping $F \mapsto W(x,F,z,A)$ is polyconvex, cf. [Bal76]. Condition (9b) implies the necessary coercivity, which includes (self or cross) hardening via the lower bound $c|p|^{q_{\boldsymbol{p}}}$. Note that we do not assume a coercivity in P. Condition (9c) will be used to control the power of the time-dependent Dirichlet boundary data.

2.3 Statement of the result

We now formulate our existence result, which will be proved in Section 3.

Theorem 1. *Let the spaces* $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z} \subset \boldsymbol{Y}\times\boldsymbol{Z} = \boldsymbol{Q}$ *and the functionals* $\mathcal{E}$ *and* $\mathcal{D}$ *be defined as above such that the conditions* (8), (11), (9) *hold.*

Let $\boldsymbol{q}_0 = (y_0, z_0) \in \mathcal{Q} \cap (\mathcal{Y}\times\mathcal{Z})$ *be a stable initial condition, i.e.,*

$$\mathcal{E}(0,\boldsymbol{q}_0) < \infty \ \textit{ and } \ \mathcal{E}(0,\boldsymbol{q}_0) \le \mathcal{E}(0,\widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q}_0,\widehat{\boldsymbol{q}}) \textit{ for all } \widehat{\boldsymbol{q}} \in \mathcal{Q}.$$

Then, there exists an energetic solution $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ *for* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *with* $\boldsymbol{q}(0) = \boldsymbol{q}_0$ *and* $\boldsymbol{q} \in \mathrm{L}^\infty([0,T];\boldsymbol{Y}\times\boldsymbol{Z})$.

For similar results involving kinematic or isotropic hardening models in finite-strain plasticity, we refer to [MaM08]. All these existence result are based on the abstract theory of energetic solutions for rate-independent processes on topological spaces developed in [MaM05, FrM06, MiR08]

We consider two reflexive and separable Banach spaces $\boldsymbol{Y}$ and $\boldsymbol{Z}$ and weakly closed subsets $\mathcal{Y}$ and $\mathcal{Z}$, respectively. The state space for the full system is then given by $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z} \subset \boldsymbol{Q} \overset{\text{def}}{=} \boldsymbol{Y}\times\boldsymbol{Z}$, and the states are denoted by $\boldsymbol{q} = (y,z)$. The evolution is described in terms of the stored-energy functional $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and the dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$. The set in which $\mathcal{E}$ takes finite values is denoted by

$$\mathrm{dom}\,\mathcal{E} \overset{\text{def}}{=} \{\, (t,\boldsymbol{q}) \in [0,T]\times\mathcal{Q} \mid \mathcal{E}(t,\boldsymbol{q}) < \infty \,\}.$$

The triple $(\mathcal{Q},\mathcal{E},\mathcal{D})$ is called a *rate-independent energetic system*.

For the stored-energy functional $\mathcal{E}$ impose two general conditions:

Compactness of energy sublevels:
for all $t \in [0,T]$ and $E > 0$ the sublevels $\{\, \boldsymbol{q} \in \mathcal{Q} \mid \mathcal{E}(t,\boldsymbol{q}) \le E \,\}$ are bounded and weakly closed in $\boldsymbol{Q}$. (E1)

Uniform control of the power $\partial_t\mathcal{E}$*:*
there exist $c_0^E, c_1^E > 0$ such that for all $(t_*,\boldsymbol{q}) \in \mathrm{dom}\,\mathcal{E}$:
$\mathcal{E}(\cdot,\boldsymbol{q}) \in \mathrm{C}^1([0,T])$ and $|\partial_t\mathcal{E}(t,\boldsymbol{q})| \le c_1^E(c_0^E + \mathcal{E}(t,\boldsymbol{q}))$ for all t. (E2)

For the dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ we impose two general conditions:

Extended quasi-distance:
(i) $\forall z_1, z_2 \in \mathcal{Z}:\ \mathcal{D}(z_1,z_2) = 0 \Longleftrightarrow z_1 = z_2,$ (D1)
(ii) $\forall z_1, z_2, z_3 \in \mathcal{Z}:\ \mathcal{D}(z_1,z_3) \le \mathcal{D}(z_1,z_2) + \mathcal{D}(z_2,z_3).$

Weak lower semi-continuity:
$z_k \rightharpoonup z,\ \widehat{z}_k \rightharpoonup \widehat{z} \quad \Longrightarrow \quad \mathcal{D}(z,\widehat{z}) \le \liminf_{k\to\infty} \mathcal{D}(z_k,\widehat{z}_k).$ (D2)

To formulate the existence result we need to impose additional conditions which provide a suitable compatibility between the two functionals $\mathcal{E}$ and $\mathcal{D}$. For this we define the *set of stable states at time t* via

$$\mathcal{S}(t) \stackrel{\text{def}}{=} \{\, \boldsymbol{q} \in \mathcal{Q} \mid \mathcal{E}(t,\boldsymbol{q}) < \infty,\ \mathcal{E}(t,\boldsymbol{q}) \le \mathcal{E}(t,\widehat{\boldsymbol{q}}) + \mathcal{D}(\boldsymbol{q},\widehat{\boldsymbol{q}}) \text{ for all } \widehat{\boldsymbol{q}} \,\}.$$

Moreover, we define the notion of a *stable sequence* $(t_k,\boldsymbol{q}_k)_{k\in\mathbb{N}}$ via $\sup_{k\in\mathbb{N}} \mathcal{E}(t_k,\boldsymbol{q}_k) < \infty$ and $\boldsymbol{q}_k \in \mathcal{S}(t_k)$ for all $k \in \mathbb{N}$. A function $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ is called an *energetic solution* of $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if $t \mapsto \partial_t \mathcal{E}(t,\boldsymbol{q}(t))$ is integrable and if for all $t \in [0,T]$ we have global stability (S) and energy balance (E) in (4).

Theorem 2. *Let $\mathcal{E}$ and $\mathcal{D}$ satisfy conditions (E) and (D). Moreover, let the following compatibility condition hold:*

For all stable seq. $(t_j,\boldsymbol{q}_j)_{j\in\mathbb{N}}$ with $(t_j,\boldsymbol{q}_j) \rightharpoonup (t_*,\boldsymbol{q}_*)$:

$$\partial_t \mathcal{E}(t_*,\boldsymbol{q}_j) \to \partial_t \mathcal{E}(t_*,\boldsymbol{q}_*), \tag{C1}$$

$$\boldsymbol{q}_* \in \mathcal{S}(t_*). \tag{C2}$$

Then, for each $\boldsymbol{q}_0 \in \mathcal{S}(0)$ there exists a solution $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ of the rate-independent energetic system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfying $\boldsymbol{q}(0) = \boldsymbol{q}_0$. Moreover, the solution can be chosen such that $\boldsymbol{q} : [0,T] \to \boldsymbol{Q}$ is measurable.

3 Coercivity and lower semicontinuity

In this section we show that the assumptions in Section 2.2 for the elastoplastic problem are sufficient to establish the abstract assumption (E) for the stored-energy functional $\mathcal{E}$, (D) for the dissipation distance $\mathcal{D}$, and the compatibility conditions (C). Having done this, the Existence Theorem 1 for the elastoplastic problem is a direct consequence of the abstract existence result in Theorem 2.

3.1 Stored energy potential

To establish the coercivity of $\mathcal{E}$ we note that we always use the matrix norm $|F| \stackrel{\text{def}}{=} (F{:}F)^{1/2}$. In particular, we have $|AB| \le |A|\,|B|$, which implies

$$|\nabla g_{\mathrm{Dir}} \nabla y P^{-1}| \ge |\nabla y|/(|\nabla g_{\mathrm{Dir}}|\,|P|) \ge c|\nabla y|/|P|,$$

where here and in the sequel c and C denote small and large positive constants that may vary from occurrence to occurrence. These constants only depend on the data and are independent of the states $\boldsymbol{q}$.

Integrating the last estimate we obtain, for all $\boldsymbol{q} \in \mathcal{Q}$, the estimate

$$\begin{aligned} \|\nabla g_{\mathrm{Dir}} \nabla y P^{-1}\|_{\mathrm{L}^{q_Y}}^{q_Y} &\ge c\|\nabla y\|_{\mathrm{L}^{q_Y}}^{q_Y} / \|P\|_{\mathrm{L}^\infty}^{q_Y} \ge c\|\nabla y\|_{\mathrm{L}^{q_Y}}^{q_Y}\, \mathrm{e}^{-C\|p\|_{\mathrm{L}^\infty}} \\ &\ge c\log\big(\|\nabla y\|_{\mathrm{L}^{q_Y}}\big) - C\|p\|_{\mathrm{L}^\infty} - C, \end{aligned}$$

where we used (8) and (11e), which can be applied since $(\mathbf{1},0,P(x),p(x))\in\mathbb{D}$ by the definition of $\mathcal{Z}$. The last estimate follows from the rough lower estimate $\mathrm{e}^{\beta}\geq\beta$. It is the missing coercivity in P that forces us to use such weak logarithmic estimates. Using the coercivity (9b) of W and the embedding $\mathrm{W}^{1,r}(\Omega)\subset\mathrm{C}(\overline{\Omega})$ we obtain

$$\begin{aligned}\mathcal{E}(t,\boldsymbol{q}) &\geq c\log\big(\|\nabla y\|_{\mathrm{L}^{q_{\boldsymbol{Y}}}}\big)-C\|p\|_{\mathrm{L}^{\infty}}+c\|p\|_{\mathrm{L}^{q_{\boldsymbol{p}}}}^{q_{\boldsymbol{p}}}+c\|(\nabla P,\nabla p)\|_{\mathrm{L}^r}^r-C\\ &\geq c\log\big(\|\nabla y\|_{\mathrm{L}^{q_{\boldsymbol{Y}}}}\big)+c\log\big(\|P\|_{\mathrm{L}^{\infty}}\big)+c\|p\|_{\mathrm{L}^{\infty}}^{q_{\boldsymbol{p}}}+c\|(\nabla P,\nabla p)\|_{\mathrm{L}^r}^r-C,\end{aligned}$$

where we used (11e) once again. This proves coercivity, since $\|\boldsymbol{q}_k\|_{\boldsymbol{Q}}\to\infty$ implies $\mathcal{E}(t,\boldsymbol{q}_k)\to\infty$.

The weak sequential lower semi-continuity of $\mathcal{E}(t,\cdot)$ follows similarly as in [MaM08, Thm. 5.2]. In fact, the proof is even simpler, since the weak convergence $\boldsymbol{q}_k\rightharpoonup\boldsymbol{q}$ implies the uniform convergence of $(P_k,p_k)\to(P,p)$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d}\times\mathbb{R}^m)$. Thus, the convexity semi-conditions (9a) for W (via $\mathbb{W}$) allow us to use the standard techniques developed in [Bal76]. Thus, we have established the following result, which means that the abstract assumption (E1) holds.

Lemma 1. *Assume* (9) *and* (11e) *hold. Then the functional* $\mathcal{E}(t,\cdot)$ *restricted to* $\mathcal{Y}\times\mathcal{Z}$ *is weakly lower semicontinuous and coercive.*

Finally, we investigate the differentiability of $\mathcal{E}(t,\boldsymbol{q})$ with respect to time. For this we recall the definition of the Kirchhoff stress tensor from (9), namely $K(F,p,A)=\partial_F W(F,p,A)F^{\mathsf{T}}\in\mathrm{sl}(d)=\mathrm{T}_{\mathbf{1}}\mathrm{SL}(d)$. For $\boldsymbol{q}=(y,P,p)\in\mathcal{Q}$ with $\mathcal{E}(0,\boldsymbol{q})<\infty$ we introduce the abbreviation

$$\mathbb{K}_{\boldsymbol{q}}(x,F)\stackrel{\mathrm{def}}{=}\partial_F W(x,FP(x)^{-1},P(x),p(x),\nabla P(x),\nabla p(x))(FP(x)^{-1})^{\mathsf{T}}.$$

The following result was established in [MaM08] (by combining Propositions 4.3 and 4.4 with Theorem 5.3 there) and using the property (12) established below.

Lemma 2 (Power of the boundary conditions). *If assumption* (8) *and* (9) *hold, then* $\mathcal{E}$ *satisfies* (E2) *and* (C1). *In particular, there exist constants* $c_0^E\in\mathbb{R}$ *and* $c_1^E>0$ *and a modulus of continuity* ω *such that the following holds:*

For $(t,\boldsymbol{q})\in\mathrm{dom}\,\mathcal{E}$ *we have* $\mathcal{E}(\cdot,\boldsymbol{q})\in\mathrm{C}^1([0,T])$ *with*

$$\partial_t\mathcal{E}(t,\boldsymbol{q})=\int_\Omega\mathbb{K}_{\boldsymbol{q}}(x,\nabla g_{\mathrm{Dir}}(t,y(x))\nabla y(x)):V(t,y(x))\,\mathrm{d}x,\tag{10a}$$

$$\textit{where } V(t,y)=\big(\nabla g_{\mathrm{Dir}}(t,y)\big)^{-1}\frac{\partial}{\partial t}\nabla g_{\mathrm{Dir}}(t,y),$$

$$|\partial_t\mathcal{E}(t,\boldsymbol{q})|\leq c_1^{\mathcal{E}}\big(\mathcal{E}(t,\boldsymbol{q})+c_0^E\big),\ \ \textit{and}\tag{10b}$$

$$|\partial_t\mathcal{E}(t_1,\boldsymbol{q})-\partial_t\mathcal{E}(t_2,\boldsymbol{q})|\leq\omega(|t_2-t_1|)\big(\mathcal{E}(t_1,\boldsymbol{q})+c_0^E\big).\tag{10c}$$

The importance of formula (10a) is that $\mathbb{K}_{\boldsymbol{q}}$ is in $\mathrm{L}^1(\Omega)$ for all $\boldsymbol{q}\in\mathrm{dom}\,\mathcal{E}(t,\cdot)$, whereas V lies in $\mathrm{C}(\overline{\Omega})$ because of the smoothness of the given boundary data g_{Dir}.

3.2 Dissipation potential $\mathcal{D}$

The first result provides some elementary properties for the dissipation distance D defined via (7) and (5). We let $\mathbb{D} \stackrel{\text{def}}{=} \{ (P_0, p_0, P_1, p_1) \,|\, D(P_0, p_0, P_1, p_1) < \infty \}$, which is a closed subset of $(\mathrm{SL}(d)\times\mathbb{R}^m)^2$, and $\eta = (1,1,...,1) \in \mathbb{R}^m$.

Lemma 3. *Assume that R has the form* (7)*. Then D defined* (5) *satisfies*

$$D : Z\times Z \to [0,\infty] \textit{ is lower semicontinuous}; \tag{11a}$$

$$D : \mathbb{D} \to [0,\infty[\textit{ is continuous}; \tag{11b}$$

$$D(z_1,z_2) = 0 \iff z_1 = z_2; \tag{11c}$$

$$D(z_1,z_3) \le D(z_1,z_2) + D(z_2,z_3); \tag{11d}$$

$$\begin{aligned}&\textit{there exist constants } c_1, c_2 > 0 \textit{ such that}\\ &(P_0, p_0, P_1, p_1) \in \mathbb{D} \implies |P_1 - P_0| \le c_1\big(\mathrm{e}^{c_2|p_1-p_0|} - 1\big);\end{aligned} \tag{11e}$$

$$\begin{aligned}&\textit{for each } \varepsilon > 0 \textit{ there exists } P_\varepsilon \in \mathrm{SL}(d) \textit{ and } \rho_\varepsilon > 0 \textit{ such that}\\ &(\mathbf{1}, 0, P, \varepsilon\eta) \in \mathbb{D} \ \textit{ for all } P \in \mathrm{SL}(d) \textit{ with } |P - P_\varepsilon| \le \rho_\varepsilon.\end{aligned} \tag{11f}$$

While the proof of the properties (11a)–(11e) is standard, see [Mie02, MaM08], the property (11f) is not so obvious. To show this, we recall the implicit assumption that $\{S_1, ..., S_m\}$ generates $\mathrm{SL}(d)$. Moreover, we let

$$\begin{aligned}&\mathcal{A}_\varepsilon = \{ v \in \mathrm{C}^1([0,1];\mathbb{R}^m \,|\, \dot{v}_\alpha \ge 0,\ v(1) - v(0) = \varepsilon\eta \},\\ &\mathcal{P}_\varepsilon = \{ P(1) \,|\, P \in \mathrm{C}^1([0,1];\mathrm{SL}(d)),\ P(0) = \mathbf{1},\ \dot{P}(\cdot)P(\cdot)^{-1} \in \mathcal{A}_\varepsilon \},\\ &\Xi = \textstyle\sum_{\alpha=1}^m S_\alpha,\ \ \kappa_* = \sum_{\alpha=1}^m \kappa_\alpha,\ \ \text{and } N_\varepsilon = \exp(\varepsilon\Xi),\end{aligned}$$

and obtain $N_\varepsilon \in \mathcal{P}_\varepsilon$ and $D(\mathbf{1}, 0, P, \varepsilon\eta) \le \varepsilon\kappa_* < \infty$ for all $P \in \mathcal{P}_\varepsilon$.

Now the control theory on non-commutative Lie groups shows that N_ε is in fact an interior point of the reachable set $\mathcal{P}_\varepsilon$. Thus we may set $P_\varepsilon = N_\varepsilon$ and have found $\rho_\varepsilon > 0$, such that (11f) holds.

Condition (11a) implies that $\mathcal{D}$ is well defined and the positivity (D1)(i) follows from (11c). Integrating the pointwise triangle inequality (11d) we see that (D1)(ii) holds.

Using again that $z_k \rightharpoonup z$ in $\boldsymbol{Z}$ implies $z_k \to z$ in $\mathrm{C}^0(\overline{\Omega})$ and that D is nonnegative and lower semicontinuous in both z-variables, the classical lower semicontinuity theory implies the lower semicontinuity of $\mathcal{D}$, namely (D2).

3.3 Compatibility conditions (C2)

To apply Theorem 2 it remains to establish the compatibility condition (C2), which states that weak limits of stable sequences are stable again. We establish this by constructing so-called joint recovery sequences, cf. [MRS08].

Assume that a stable sequence $(t_j, \boldsymbol{q}_j)_{j\in\mathbb{N}}$ with $t_j \to t_*$ and $\boldsymbol{q}_j \rightharpoonup \boldsymbol{q}_*$ is given. We have to show $\boldsymbol{q}_* \in \mathcal{S}(t_*)$. For any given test state $\widehat{\boldsymbol{q}}$ we have to show $\mathcal{E}(t_*, \boldsymbol{q}_*) \leq \mathcal{E}(t_*, \widehat{\boldsymbol{q}}) + \mathcal{D}(z_*, \widehat{z})$. If $\mathcal{D}(z_*, \widehat{z}) = \infty$ the estimate holds and nothing needs to be shown.

For the case $\mathcal{D}(z_*, \widehat{z}) < \infty$ we establish this condition by construction a joint recovery sequence $(\widehat{\boldsymbol{q}}_j)_{j\in\mathbb{N}}$ that satisfies

$$\text{(a)}\ \ \mathcal{E}(t_j, \widehat{\boldsymbol{q}}_j) \to \mathcal{E}(t_*, \widehat{\boldsymbol{q}}), \qquad \text{(b)}\ \ \mathcal{D}(z_j, \widehat{z}_j) \to \mathcal{D}(z_*, \widehat{z}). \tag{12}$$

From these conditions the desired stability of $\boldsymbol{q}_*$ follows by using the stability of $\boldsymbol{q}_j$, i.e., $\mathcal{E}(t_j, \boldsymbol{q}_j) \leq \mathcal{E}(t_j, \widehat{\boldsymbol{q}}_j) + \mathcal{D}(z_j, \widehat{z}_j)$. Passing to the limit $j \to \infty$ the left-hand side can be estimated by weak lower semi-continuity and the right-hand side converges to the desired limit.

The problem in deriving (12b) is the lack of continuity of the integrand D of $\mathcal{D}$. Hence, we have to choose $\widehat{z}_j = (\widehat{P}_j, \widehat{p}_j)$ carefully. For this we use property (11f) where we additionally observe that $(P_\varepsilon, \rho_\varepsilon)$ must satisfy $P_\varepsilon \to \mathbf{1}$ and $\rho_\varepsilon \to 0$ because of (11e). We let $\delta_j = \|z_j - z\|_{\mathrm{L}^\infty}$, which satisfies δ_j and choose a sequence $(\varepsilon_j)_j$ such that $\delta_j < \rho_{\varepsilon_j} \to 0$. We set

$$\widetilde{P}_j = N_{\varepsilon_j} P,\ \widetilde{p}_j = p + \varepsilon_j \eta,\ \widehat{P}_j = N_{\varepsilon_j} \widehat{P},\ \widehat{p}_j = \widehat{p} + \varepsilon_j \eta, \tag{13}$$

and find by the triangle inequality

$$\begin{aligned}
&\mathcal{D}(P_j, p_j, \widehat{P}_j, \widehat{p}_j) \leq \mathcal{D}(P_j, p_j, \widetilde{P}_j, \widetilde{p}_j) + \mathcal{D}(\widetilde{P}_j, \widetilde{p}_j, \widehat{P}_j, \widehat{p}_j) \\
&= \int_\Omega D(\mathbf{1}, 0, N_{\varepsilon_j} P P_j^{-1}, p - p_j + \varepsilon_j \eta)\,\mathrm{d}x + \mathcal{D}(P, p, \widehat{P}, \widehat{p}),
\end{aligned}$$

where we have used plastic invariance for the second term. By construction the integrand of the first term can be estimated by $\kappa_* \varepsilon_j$ and we obtain $\limsup_{j\to\infty} \mathcal{D}(z_j, \widehat{z}_j) \leq \mathcal{D}(z, \widehat{z})$. Since the opposite estimate follows by lower semi-continuity we have established (12b). The convergence (12a) follows easily by setting $\widehat{y}_j = \widehat{y}$ and applying Lebesgue's dominated convergence theorem.

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