Exercise Sheet 14

Exercise 40. Quasiconvexity implies weak lower semicontinuity. For $p > q \geq 1$ consider a quasiconvex function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ such that

$$\exists\, C > 0\ \forall\, A, B \in \mathbb{R}^{m\times d} : \quad |f(A) - f(B)| \leq C\big(1 + |A| + |B|\big)^{q-1}|A{-}B|.$$

Define the functional $I_A(u) = \int_\Omega f(A{+}\nabla u(x))\,\mathrm{d}x$.

(a) Show that $I_A(u) \geq I_A(0)$ for all $u \in \mathrm{W}^{1,q}_0(\Omega;\mathbb{R}^m)$ (which is the closure of $\mathrm{C}^\infty_\mathrm{c}(\Omega;\mathbb{R}^m)$).

(b) It can be used without proof that the cut-off function

$$\chi_\varepsilon : \Omega \to [0,1];\ \ x \mapsto \min\big\{1, \max\{0, (\mathrm{dist}(x,\partial\Omega){-}\varepsilon)/\varepsilon\}\big\}$$

lies in $\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^m)$ and satisfies $\|\nabla\chi_\varepsilon\|_{\mathrm{L}^\infty} = 1/\varepsilon$. Show that for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$ we have $\chi_\varepsilon u \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^d)$ and

$$\|\nabla(\chi_\varepsilon u) - \nabla u\|_{\mathrm{L}^q} \leq C\varepsilon^{1/r}\Big(\|\nabla u\|_{\mathrm{L}^p} + \frac{1}{\varepsilon}\,\|u\|_{\mathrm{L}^p}\Big)$$

for a suitable constant C, where $r \in\,]1,\infty[$ is given by $1/q = 1/p + 1/r$.

(c) For $u_k \rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$ show the estimate $\liminf_{k\to\infty} I_A(u_k) \geq I_A(0)$. (*Hint: Recall $p > q$ and consider $\chi_{\varepsilon_k}u_k$ for a good sequence $\varepsilon_k \to 0$.*)

(d) Why is the assumption "$p > q$" bad for the direct method in the calculus of variations?

Exercise 41. Counterexample concerning Reshetnyak's theorem. Take $m = d = p = 2$ and $\Omega = \,]{-1},1[^2$ and the sequence

$$u^k(x_1,x_2) = \frac{1}{\sqrt{k}}(1{-}|x_2|)^k\big(\sin(kx_1), \cos(kx_1)\big).$$

We will show that $\nabla u^k \rightharpoonup 0$ in $\mathrm{L}^2(\Omega)$ but $\det(\nabla u^k) \not\rightharpoonup 0$ in $\mathrm{L}^1(\Omega)$.

(a) Show that $u^k \rightharpoonup 0$ in $\mathrm{H}^1(\Omega;\mathbb{R}^2)$.

(b) Prove that $\int_\Omega \det(\nabla u^k)\varphi\,\mathrm{d}x \to 0$ for all $\varphi \in \mathrm{C}_\mathrm{c}(\Omega)$.

(c) Show that $\det(\nabla u^k)$ does not converge weakly to 0 in $\mathrm{L}^1(\Omega)$. (*Hint: Consider suitable $\varphi \in \mathrm{L}^\infty(\Omega)$ in (b).*)

Exercise 42. Cofactor matrix and adjugate matrix.
(Auf deutsch: Kofaktormatrix und adjunkte Matrix)
For a quadratic matrix $A \in \mathbb{R}^{d\times d}$ define $\mathrm{cof}\, A \in \mathbb{R}^{d\times d}$ such that $(\mathrm{cof}\, A)_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the determinant of the $(d{-}1)\times(d{-}1)$ matrix obtained after deleting column i and row j. Moreover, $\mathrm{adj}(A) = \mathrm{cof}(A)^\top$.

(a) For $f(A) = \det A$ show $\mathrm{D}f(A)[B] = \mathrm{cof}(A){:}B = \mathrm{tr}(\mathrm{adj}(A)B)$.

(b) Prove the formula $\mathrm{cof}(A)A^\top = \mathrm{adj}(A)A = \det(A)I$. Relate this to Cramer's rule and to Euler's formula $qf(A) = \langle \mathrm{D}f(A), A\rangle$ for q-homogeneous functions.

(c) For $d = 2$ and $d = 3$ show $\det(A{+}B) = \det A + \mathrm{cof}(A){:}B + \det B$ and $\det(A{+}B) = \det A + \mathrm{cof}(A){:}B + A{:}\,\mathrm{cof}(B) + \det B$, respectively.