Exercise Sheet 8

Exercise 25. A weak version of Rademacher's theorem. Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and a function $u \in \mathrm{C}^{\mathrm{Lip}}(\Omega)$, i.e. $|u(x){-}u(y)| \leq L|x{-}y|$ for all $x, y \in \Omega$. Show that the weak gradient exists and lies in $\mathrm{L}^\infty(\Omega;\mathbb{R}^d)$.
(*Hint: Consider the difference quotients* $v_n^{(i)}(x) = n\big(u(x{+}\frac{1}{n}e_i){-}u(x)\big)$ *and take the limit* $n \to \infty$.)

Exercise 26. L^∞ **bound for supercritical** $p > d$**.** Construct a constant C depending only on d and $p \in \,]d, \infty[$ such that

$$\forall\, u \in \mathrm{W}^{1,p}(\mathbb{R}^d): \quad \|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}.$$

(*Hint: Write* $u(x) = u(y) + \int_0^1 \nabla u(x + s(y{-}x))(x{-}y)\,\mathrm{d}s$ *and average over* y.)