Exercise Sheet 7

Definition: The *epigraph* of a function $I : X \to \mathbb{R}_\infty$ is defined as

$$\mathrm{epi}(I) := \big\{ (u,\alpha) \in X \times \mathbb{R} \,\big|\, I(u) \le \alpha \big\} \subset X \times \mathbb{R}.$$

Exercise 21. Estimates via affine functions for convex functions. Consider a proper, convex, and lower semicontinuous functional $I : X \to \mathbb{R}_\infty$.

(a) Show that for all u with $I(u) < \infty$ and all $\varepsilon > 0$ there exists $\xi \in X^*$ such that $I(u{+}v) \ge I(u) - \varepsilon + \langle \xi, v\rangle$ for all $v \in X$. (*Hint: Use* epi(I) *and separate it in* $X \times \mathbb{R}$ *from a suitable set.*)

(b)* Show that for all u with $I(u) = \infty$ and all $M \in \mathbb{R}$ there exists $\xi_M \in X^*$ such that $I(u{+}v) \ge M + \langle \xi_M, v\rangle$ for all $v \in X$. (*Hint: Work in* $X \times \mathbb{R}$ *and construct a line segment connecting* (u, M) *and* $(u_1, I(u_1){-}1)$ *that does not intersect* epi(I).)

(c) Conclude from (a) and (b) (without using sublevels) that I is weakly lower semicontinuous.

Exercise 22. Bounded convex functions are Lipschitz continuous. Let $I : X \to \mathbb{R}_\infty$ be proper, convex, and lsc. Assume further that

$$\exists\, M, K \in \mathbb{R}\ \forall\, u \in B_R(u_*) : \quad K \le I(u) \le M.$$

Show that I restricted to $B_r(u_*)$ with $r \in \,]0, R[$ is Lipschitz continuous with a Lipschitz constant that only depends on $M{-}K$ and r/R.

Exercise 23. Continuity points of convex functionals. For a proper, lower semicontinuous convex functional $I : X \to \mathbb{R}_\infty$ on a Banach space X the domain is defined via

$$\mathrm{dom}(I) := \{ u \in X \mid I(u) < \infty \} \ \ne\ \emptyset.$$

(a) Show that for $u_1 \in \mathrm{dom}(I)$ the following conditions are equivalent:

(i) $\exists\, \delta > 0 : \quad \sup\{ I(u) \mid u \in B_\delta(u_1) \} < \infty$;

(ii) I is continuous in u_1.

(b) Show that I is continuous on $A := \mathrm{int}\big(\mathrm{dom}(I)\big)$, if I is continuous at one $u_1 \in A$.

(c) Assume that I is continuous at one $u_1 \in A$. Find a supporting hyperplane for all $u \in A$, i.e. there exists $\beta \in X^*$ such that $I(u{+}v) \ge I(u) + \langle \beta, v\rangle$ for all $v \in X$.
(*Hint: Use the "open epigraph"* $\big\{ (u,\alpha) \in X \times \mathbb{R} \,\big|\, u \in A,\ I(u) \lneqq \alpha \big\}$.)

Exercise 24. Sobolev embeddings. Let $\Omega = B_1(0) \subset \mathbb{R}^d$.

(a) Consider the function $u : \Omega \to \mathbb{R}$ with $u(x) = |x|^\alpha$ for $x \ne 0$ and $u(0) = 0$. For which p do we have $u \in \mathrm{L}^p(\Omega)$ and for which $u \in \mathrm{W}^{1,p}(\Omega)$?

(b) Consider the function $u(x) = \big(1 {-} \log|x|\big)^\beta$ with $\beta \in \mathbb{R}$. For which β and $p \in [1, \infty]$ do we have $u \in \mathrm{L}^p(\Omega)$ and for which $u \in \mathrm{W}^{1,p}(\Omega)$?

(c) For the case $d \ge 2$ give a function $u \in \mathrm{W}^{1,d}(\Omega) \setminus \mathrm{L}^\infty(\Omega)$.