Exercise Sheet 6

Exercise 18. A generalization of the Riemann-Lebesgue lemma. Consider a function $p \in \mathrm{L}^\infty(\mathbb{R}^d)$ that is 1-periodic in all directions, i.e.

$$p(y+k) = p(y) \quad \text{for all } y \in \mathbb{R}^d \text{ and all } k \in \mathbb{Z}^d,$$

and define $p_\varepsilon(x) = p\big(\frac{1}{\varepsilon}x\big)$ and $p_{\mathrm{av}} = \int_{[0,1]^d} p(y)\,\mathrm{d}y$.

(a) Show that for all functions $\phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ we have $\int_{\mathbb{R}^d} p_\varepsilon(x)\phi(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} p_{\mathrm{av}}\phi(x)\,\mathrm{d}x$.
(*Hint: Approximate ϕ by a suitable piecewise constant function.*)

(b) Show the weak-star convergence

$$p_\varepsilon \overset{*}{\rightharpoonup} p_{\mathrm{av}} \;\text{ in } \mathrm{L}^\infty(\mathbb{R}^d), \quad \text{i.e. } \forall\, f \in \mathrm{L}^1(\Omega): \quad \int_{\mathbb{R}^d} p_\varepsilon f\,\mathrm{d}x \to \int_{\mathbb{R}^d} p_{\mathrm{av}} f\,\mathrm{d}x.$$

Exercise 19. R1Cvx = QCvx for quadratic densities. For quadratic densities $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ show that f is rank-one convex if and only if it is quasiconvex.
(*Hint: Fourier transform gives $\mathcal{F}(\nabla w) = \mathrm{i}(\mathcal{F}w)\otimes\xi$.*)

**Two versions of "Mazur's separation theorem"
= geometric version of the Hahn-Banach theorem.** For a Banach space X denote by $X^* = \mathrm{Lin}(X;R)$ its dual space. We consider two non-empty, convex sets A and B such that $A \cap B = \emptyset$. Then, the following two statements hold.

(I) If A is open then there exists $\ell \in X^*$ with $\ell \neq 0$ and $\gamma \in \mathbb{R}$ such that

$$\forall\, a \in A\ \forall\, b \in B: \quad \ell(a) \geq \gamma \geq \ell(b).$$

(II) If A is closed and B is compact, then there exists $\ell \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\forall\, a \in A\ \forall\, b \in B: \quad \ell(a) \geq \gamma_1 > \gamma_2 \geq \ell(b).$$

Exercise 20. Convex sets.

(a) Use one of the above separation theorems to show that a closed, convex $M \subset X$ is also weakly (sequentially) closed, i.e. $u_k \in M$ and $u_k \rightharpoonup u_*$ implies $u_* \in M$.

(b) Give an example for X and a closed M such that M is not weakly (seq.) closed.

(c) Consider the Hilbert space $\ell^2(\mathbb{N})$ and the sets

$$A = \Big\{\, (u_k)_{k\in\mathbb{N}} \,\Big|\, \sum\nolimits_{k\in\mathbb{N}} k^2 u_k^2 < 1 \,\Big\} \;\text{ and }\; B = \{\mathbf{b} = (b_k)_{k\in\mathbb{N}}\}$$

where $\sum_{k\in\mathbb{N}} k^2 b_k^2 = 1$. Give one case of $\mathbf{b}$ where A and B can be separated as in "(I)" and one case of $\mathbf{b}$ where the separation is not possible.