Exercise Sheet 4

Exercise 14. Mollification = Smoothing.
Consider a Lipschitz function $\widetilde{u} : \mathbb{R}^d \to \mathbb{R}^m$, i.e. for $L = \mathrm{Lip}(\widetilde{u})$ we have

$$\forall\, x, y \in \mathbb{R}^d : \quad |\widetilde{u}(x){-}\widetilde{u}(y)| \leq L|x{-}y|.$$

(a) Take the Dirac sequence ψ_δ from the lectures and define

$$u_\delta = \widetilde{u} * \psi_\delta : \; x \mapsto \int_{\mathbb{R}^d} \widetilde{u}(y)\psi_\delta(x{-}y)\,\mathrm{d}y.$$

Show that $\mathrm{Lip}(u_\delta) \leq L$ and $\|\widetilde{u} - u_\delta\|_{\mathrm{C}^0} \leq L\delta$.

(b) For any $w \in \mathrm{C}^1(\mathbb{R}^d; \mathbb{R}^m)$ establish the identity

$$\mathrm{Lip}_{B_R(x_0)}(w) = \sup\{\, \|\nabla w(y)\|_{\mathbb{R}^{m\times d}} \,|\, y \in \mathrm{B}_R(x_0)\,\},$$

where the expression in left-hand side indicates the smallest Lipschitz constant of $w|_{B_R(x_0)}$.
(*Hint: For estimating $w(x){-}w(y)$ consider w on the connecting line.*)

(c) Conclude $\|\nabla u_\delta\|_{\mathrm{C}^0} \leq L = \mathrm{Lip}(\widetilde{u})$.

Exercise 15. Second variation Consider the functional $I : \mathrm{C}^1(\overline{\Omega}; \mathbb{R}^m) \to \mathbb{R}$ with $I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$, where $f \in \mathrm{C}^2(\overline{\Omega}{\times}\mathbb{R}^m{\times}\mathbb{R}^{m\times d})$. For $\gamma_1, \gamma_2 > 0$ assume the estimates

$$\int_\Omega \partial_A^2 f(x, u_0(x), \nabla u_0(x))[\nabla w, \nabla w]\,\mathrm{d}x \geq \gamma_1 \int_\Omega |\nabla w|^2\,\mathrm{d}x, \tag{Eq.1}$$

$$\mathrm{D}^2 I(u_0)[w, w] \geq \gamma_2 \int_\Omega |w|^2\,\mathrm{d}x. \tag{Eq.2}$$

(a) Use (Eq.1) and suitable estimates for $\partial_A\partial_u f$ and $\partial_u^2 f$ to find C^* such that

$$\mathrm{D}^2 I(u_0)[w, w] \geq \gamma_1/2 \int_\Omega |\nabla w|^2\,\mathrm{d}x - C^*|w|^2\,\mathrm{d}x \text{ for all } w.$$

(b) Combine (Eq.2) and (Eq.1) to find $\gamma_3 > 0$, such that

$$\mathrm{D}^2 I(u_0)[w, w] \geq \gamma_3 \int_\Omega |\nabla w|^2{+}|w|^2\,\mathrm{d}x \quad \text{for all } w \in C^1(\overline{\Omega}; \mathbb{R}^m).$$

Exercise 16. Anisotropic elasticity theory. The functional $I : \mathrm{C}^1(\overline{\Omega}; \mathbb{R}^d) \to \mathbb{R};\; u \mapsto \int_\Omega f(\nabla u)\,\mathrm{d}x$ is defined via

$$f(A) = \frac{\lambda}{2}(\mathrm{spur} A)^2 + \frac{\mu}{4}\left|A + A^T\right|^2 + \frac{\delta}{2}A_{11}^2.$$

(a) Establish the formula $\partial_A^2 f(A)[B, B] = 2f(B)$ for all $A, B \in \mathbb{R}^{d\times d}$.

(b) For which $\lambda, \mu, \delta \in \mathbb{R}$ do we have $f(A) \geq 0$ for all $A \in \mathbb{R}^{d\times d}$ (which is equivalent to convexity)? Try first to solve the case $d = 2$.
(*Hint: For testing the positivity, it essentially suffices to consider diagonal matrices.*)

(c) For which $\lambda, \mu, \delta \in \mathbb{R}$ does f satisfy the Legendre–Hadamard condition? Try first to solve the case $d = 2$.
(*Hint: Write $\partial_A^2 f(x, u, A)[\mathbf{b} \otimes \boldsymbol{\eta}, \mathbf{b} \otimes \boldsymbol{\eta}] \geq 0$ in the form $\mathbb{A}(\boldsymbol{\eta})\mathbf{b} \cdot \mathbf{b} \geq 0$ with $\mathbb{A}(\boldsymbol{\eta}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.*)