Inhomogenous diffusion equations

$$\gamma_\varepsilon(x)\dot u(t,x) = \operatorname{div}\big(A_\varepsilon(x)\nabla u\big) - f_\varepsilon(x,u(t,x)), \qquad t>0,\ x\in\Omega$$

(& suitable BC)

L^2-type gradient system $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathbb{G}_\varepsilon)$ with $\boldsymbol{X}=\mathrm{L}^2(\Omega)$

$$(\mathbb{G}_\varepsilon v)(x) = \gamma_\varepsilon(x)v(x) \quad\Rightarrow\quad \mathcal{R}_\varepsilon(v) = \int_\Omega \tfrac12\gamma_\varepsilon(x)v(x)^2\,\mathrm{d}x$$

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \tfrac12\nabla u\cdot A_\varepsilon(x)\nabla u + F_\varepsilon(x,u(x))\,\mathrm{d}x \qquad F_\varepsilon(x,u)=\int_0^u f_\varepsilon(x,w)\,\mathrm{d}w$$

3. Motivating examples

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  Aim: $\mathcal{R}_{\text{eff}}(v) = \int_\Omega \frac{\gamma_{\text{eff}}}{2}v^2\,\mathrm{d}x$ and $\mathcal{E}_{\text{eff}}(u) = \int_\Omega \frac12\nabla u\cdot A_{\text{eff}}\nabla u + F_{\text{eff}}(u)\,\mathrm{d}x$

3. Motivating examples

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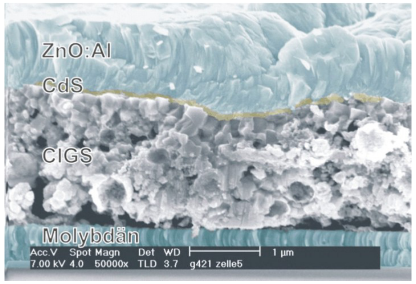
- Dimension reduction (modeling of active interfaces)

$$x\in\,]{-1},1[\,\subset\mathbb{R}^1 \qquad A_\varepsilon(x) = \begin{cases}\alpha & \text{for } |x|>\varepsilon/2,\\ \beta\varepsilon & \text{for } |x|<\varepsilon/2\end{cases}$$

$$\mathcal{E}_{\mathsf{eff}}(u) = \int_{-1}^0 \tfrac{\alpha}{2}u_x^2\,\mathrm{d}x + \underbrace{\frac{\beta}{2}\big(u(0^-)-u(0^+)\big)^2}_{\text{gives interface conditions}} + \int_0^1 \tfrac{\alpha}{2}u_x^2\,\mathrm{d}x$$



Fundamental work on evolutionary Γ-convergence:

Sandier-Serfaty 2004 (Comm. Pure Appl. Math.):
Gamma-convergence of gradient flows with applications to Ginzburg-Landau

more recent, nicely readable survey:

Serfaty 2011: Gamma-convergence of gradient flows on Hilbert spaces and metric spaces and applications.

Ginzburg-Landau vortices: $\psi(t,\cdot):\Omega\to\mathbb{C}=\mathbb{R}^2$

$$\boxed{\text{(GLe)}\ c_\varepsilon\dot\psi=\Delta\psi+\tfrac{1}{\varepsilon^2}(1{-}|\psi|^2)\psi\ \text{\& Neum.BC}}$$

(GLe) is induced by the gradient system $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with $\boldsymbol{X}=\mathrm{H}^1(\Omega;\mathbb{C})$,
$\mathcal{E}_\varepsilon(\psi)=\int_\Omega\frac12|\nabla\psi|^2+\frac{1}{4\varepsilon^2}(1{-}|\psi|^2)^2\,\mathrm{d}x$, and $\mathcal{R}_\varepsilon(\dot\psi)=\frac{1}{2\log(1/\varepsilon)}\int_\Omega|\dot\psi|^2\,\mathrm{d}x$

Evol.Γ-limit for $\varepsilon\to0$ $\rightsquigarrow$ ODE for vortex positions

Chemical reaction systems with detailed balance

E.g. $\quad 2B + C \overset{k_1}{\rightleftharpoons} 2A \qquad A + 2C \overset{1/\varepsilon}{\rightleftharpoons} B \qquad A + B + C \overset{k_3}{\rightleftharpoons} 3C$

Fast reaction versus slow reactions $k_1,\ k_3 = O(1)$

$$\begin{pmatrix} \dot c_A \\ \dot c_B \\ \dot c_C \end{pmatrix} = k_1(c_B^2 c_C - c_A^2) \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + \frac{1}{\varepsilon}(c_A c_C^2 - c_B) \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + k_3(c_A c_B c_C - c_C^3) \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

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Energy = relative entropy $\mathcal{E}(\boldsymbol{c}) = \sum_{i=A,B,C} \lambda_{\mathrm{Bz}}(c_i) \qquad \lambda_{\mathrm{Bz}}(z) = z\log z - z + 1$

$$\dot{\boldsymbol{c}} = -\mathbb{K}(\boldsymbol{c})\mathrm{D}\mathcal{E}(\boldsymbol{c}) \text{ with } \mathbb{K}_\varepsilon(\boldsymbol{c}) = \mathbb{K}_{1,3}(\boldsymbol{c}) + \frac{1}{\varepsilon}\,\frac{c_Ac_C^2-c_B}{\log(c_Ac_C^2/c_B)}\begin{pmatrix}1&-1&2\\-1&1&-2\\2&-2&4\end{pmatrix}$$

Gradient system $([0,\infty[^3,\mathcal{E},\mathbb{K})$: $\mathcal{E}$ indep. of ε but $\mathbb{K}_\varepsilon(\boldsymbol{c})$

Evol.Γ-convergence proved in "Disser, Liero, Zinsl 2016 WIAS preprint 2227".

Overview

3. Motivating examples

$\boldsymbol{X}$ sep./refl. Banach space and functionals $\mathcal{J}_\varepsilon : \boldsymbol{X} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$

Definition (Weak/strong Γ-convergence and Mosco convergence)

Weak Γ-convergence: $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}$ if (G1w) and (G2w) hold:

(G1w) $u_\varepsilon \rightharpoonup u \implies \mathcal{J}(u) \leq \liminf_{\varepsilon \to 0} \mathcal{J}_\varepsilon(u_\varepsilon)$ (liminf estimate)

(G2w) $\forall \widehat{u}\ \exists (\widehat{u}_\varepsilon)_\varepsilon$: $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ and $\mathcal{J}(\widehat{u}) = \lim_{\varepsilon \to 0} \mathcal{J}_\varepsilon(\widehat{u}_\varepsilon)$ (ex. recovery seq.)

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Strong Γ-convergence $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ if (G1s) and (G2s) hold:

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Mosco convergence $\mathcal{J}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{J}$ if $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ and $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}$ hold

(or simply (G1w) and (G2s))

The (primal) dissipation potentials $\mathcal{R}(u,\dot u)$ is always **convex in $\dot u$**.

The dual dissipation potential $\mathcal{R}^*$ is always **convex in ξ**.

$$\mathcal{R}^*(u,\xi) := \sup\{\, \langle \xi, v\rangle - \mathcal{R}(u,v) \mid v \in \boldsymbol{X}\,\}$$

Theorem (Attouch 1984)

Let $\boldsymbol{X}$ be a reflexive Banach space and assume that all $\mathcal{F}_n : \boldsymbol{X} \to \mathbb{R}_\infty$ are proper, convex, equicoercive and that $(\mathcal{F}_n^*)^*$. Then,

$$\mathcal{F}_n \xrightharpoonup{\Gamma} \mathcal{F} \qquad \Longleftrightarrow \qquad \mathcal{F}_n^* \xrightarrow{\Gamma} \mathcal{F}^*\,.$$

(HU specialist are F. Bethke and N. Farchmin)

In particular, we have $\mathcal{F}_n \xrightarrow{\mathrm{M}} \mathcal{F} \quad \Leftrightarrow \quad \mathcal{F}_n^* \xrightarrow{\mathrm{M}} \mathcal{F}^*$.

Easy to remember via the well-known convergence result of linear functional analysis:

$v_n \rightharpoonup v$ and $\xi_n \to \xi$ implies $\langle \xi_n, v_n\rangle \to \langle \xi, v\rangle$

Overview

$\boldsymbol{X} = \mathbb{R}^2$

$\mathcal{E}_\varepsilon(u) = \frac{1}{2}u_1^2 + \frac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \frac{1}{2}u \cdot A_\varepsilon u$ with $A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$

$\mathcal{R}_\varepsilon(\dot u) = \frac{1}{2}\dot u_1^2 + \frac{1}{2\varepsilon^\beta}\dot u_2^2 = \frac{1}{2}\dot u \cdot \mathbb{G}_\varepsilon \dot u$ with $\mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$

ODE reads $\mathbb{G}_\varepsilon \dot u_\varepsilon = -A_\varepsilon u_\varepsilon$ with $u_\varepsilon(0) = \binom{1}{0}$

Explicit solutions can be calculated for all $\varepsilon > 0$. We find, for all $t \geq 0$,

$\beta \in [0, 2[\,: \quad u_\varepsilon(t) \to \binom{\mathrm{e}^{-t}}{0}$ as $\varepsilon \to 0$

$\beta = 2: \quad u_\varepsilon(t) \to \binom{w(t)}{0}$ as $\varepsilon \to 0$

$\beta > 2: \quad u_\varepsilon(t) \to \binom{\mathrm{e}^{-2t}}{0}$ as $\varepsilon \to 0$

where $w(t) = \frac{1}{2\sqrt{5}}\big((\sqrt{5}+1)\mathrm{e}^{-\mu_1 t} + (\sqrt{5}-1)\mathrm{e}^{-\mu_2 t}\big)$ with $\mu_{1,2} = (3 \pm \sqrt{5})/2$

3. Motivating examples

$$\mathcal{E}_\varepsilon(u) = \tfrac{1}{2}u_1^2 + \tfrac{1}{2\varepsilon^2}\big(u_2 - \varepsilon u_1\big)^2 = \tfrac{1}{2}u\cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

$$\mathcal{R}_\varepsilon(\dot u) = \tfrac{1}{2}\dot u_1^2 + \tfrac{1}{2\varepsilon^\beta}\dot u_2^2 = \tfrac{1}{2}\dot u\cdot \mathbb{G}_\varepsilon \dot u \text{ with } \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

What are the limits of the functionals?

3. Motivating examples

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What are the limits of the functionals?

$$\mathcal{E}_\varepsilon \xrightarrow{\text{pointwise}} \mathcal{E}_{\text{pw}} : u \mapsto \begin{cases} (\frac{1}{2}{+}\frac{1}{2})u_1^2 & \text{for } u_2 = 0, \\ \infty & \text{otherwise} \end{cases}$$

$$\mathcal{E}_\varepsilon \xrightarrow{\text{M}} \mathcal{E}_0 : u \mapsto \begin{cases} \frac{1}{2}u_1^2 & \text{for } u_2 = 0, \\ \infty & \text{otherwise} \end{cases} \qquad \mathcal{R}_\varepsilon \xrightarrow{\text{M}} \mathcal{R}_0 : v \mapsto \begin{cases} \frac{1}{2}v_1^2 & \text{for } v_2 = 0, \\ \infty & \text{otherwise} \end{cases}$$

$$\mathcal{E}_\varepsilon(u) = \frac12 u_1^2 + \frac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \frac12 u\cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

$$\mathcal{R}_\varepsilon(\dot u) = \frac12 \dot u_1^2 + \frac{1}{2\varepsilon^\beta}\dot u_2^2 = \frac12 \dot u \cdot \mathbb{G}_\varepsilon \dot u \text{ with } \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

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$(\mathbb{R}^2, \mathcal{E}, \mathcal{R}_0)$ gives $u(t) = \binom{\mathrm{e}^{-t}}{0}$ and $(\mathbb{R}^2, \mathcal{E}_{\mathrm{pw}}, \mathcal{R}_0)$ gives $u(t) = \binom{\mathrm{e}^{-2t}}{0}$.

$\boxed{\beta < 2}$ $(\mathbb{R}^2, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}^2, \mathcal{E}, \mathcal{R}_0)$

$\boxed{\beta = 2}$ **no** evolutionary Γ convergence

$\boxed{\beta > 2}$ $(\mathbb{R}^2, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}^2, \mathcal{E}_{\mathrm{pw}}, \mathcal{R}_0)$

3. Motivating examples

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2} u \cdot A_\varepsilon u, \ \ \mathcal{R}_\varepsilon(\dot u) = \frac{1}{2} \dot u \cdot \mathbb{G}_\varepsilon \dot u \quad A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}, \ \mathbb{G}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$$

Reason for non-convergence is seen via the energy-dissipation relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(u_\varepsilon(t)) = \langle \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon), \dot u_\varepsilon \rangle = -\langle \mathbb{G}_\varepsilon \dot u_\varepsilon, \dot u_\varepsilon \rangle = -\big((\dot u_{1,\varepsilon})^2 + (\dot u_{2,\varepsilon})^2/\varepsilon^\beta \big)$$

$\rightsquigarrow \quad \mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T 2\mathcal{R}_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0)) \ = \ 1$ (finite, indep. of ε)

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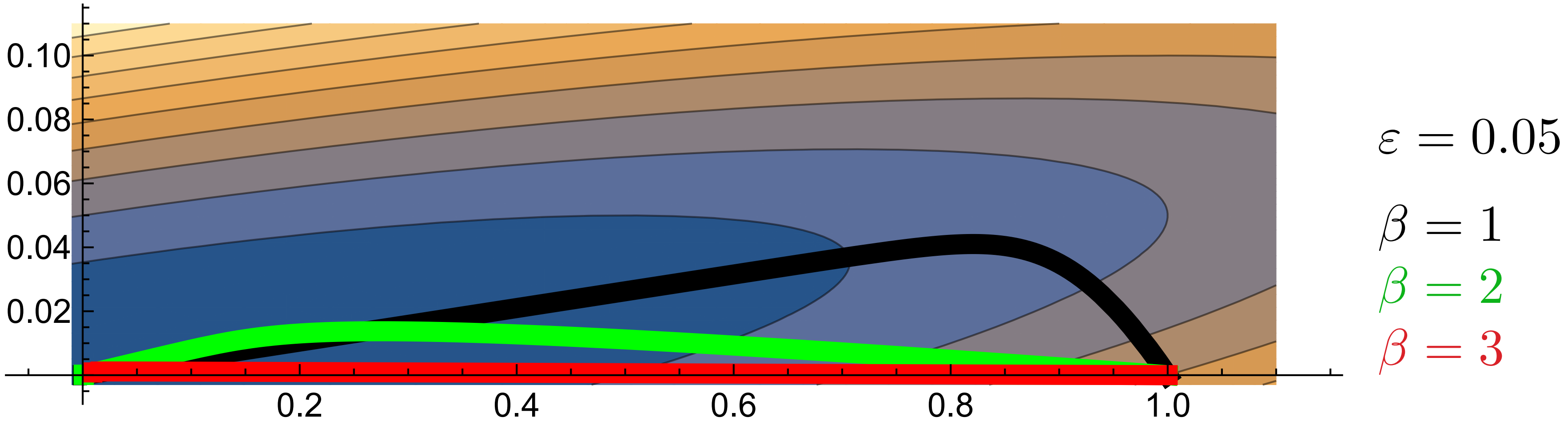
$$\leadsto \quad \mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T 2\mathcal{R}_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0)) \ = \ 1 \text{ (finite, indep. of } \varepsilon)$$

- Energy landspace $\mathcal{E}_\varepsilon(u)$ wants to have $u_2 \approx \varepsilon u_1 \approx \varepsilon \mathrm{e}^{-\lambda t}$
- Dissipation $1 \geq \int_0^T 2\mathcal{R}_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t \geq \int_0^T (\dot u_{2,\varepsilon}(t))^2/\varepsilon^\beta\,\mathrm{d}t.$

$\beta > 2$: "dissipation" doesn't allow solutions to move away from $u_2 \equiv 0$.



Overview

We consider **one-dimensional homogenization** of a parabolic equation on $x \in \Omega =]0,\ell[$ for $t > 0$:

$$\boxed{c(\tfrac{x}{\varepsilon})\dot u(t,x) = \big(a(\tfrac{x}{\varepsilon})u_x(t,x)\big)_x - b(\tfrac{x}{\varepsilon})u(t,x) \qquad u_x(t,0) = 0 = u_x(t,\ell)}$$

where $a,b,c \in \mathrm{L}^\infty(\mathbb{R})$ are 1-periodic and are $\geq c_0 > 0$.

Family of gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac12 \int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2 \,\mathrm{d}x, \qquad \mathcal{R}_\varepsilon(v) = \frac12 \int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2 \,\mathrm{d}x$$

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Family of gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2\,\mathrm{d}x, \qquad \mathcal{R}_\varepsilon(v) = \frac{1}{2}\int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2\,\mathrm{d}x$$

Aim: Find $\mathcal{E}_{\text{eff}}$ and $\mathcal{R}_{\text{eff}}$ in the form

$$\mathcal{E}_{\text{eff}}(u) = \tfrac{1}{2}\int_\Omega a_{\text{eff}}\, u_x(x)^2 + b_{\text{eff}}\, u^2\,\mathrm{d}x, \qquad \mathcal{R}_{\text{eff}}(v) = \tfrac{1}{2}\int_\Omega c_{\text{eff}}\, v^2\,\mathrm{d}x$$

$a_{\text{eff}} = ?$

$b_{\text{eff}} = ?$

$c_{\text{eff}} = ?$

Quadratic functionals:

$$\Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega v(x)\cdot G_\varepsilon(x)v(x)\,\mathrm{d}x \iff \Psi^*_\varepsilon(x) = \frac{1}{2}\int_\Omega \xi(x)\cdot G_\varepsilon(x)^{-1}\xi(x)\,\mathrm{d}x$$

Lemma (One-dimensional homogenization)
Let $G_\varepsilon(x) = \mathbb{G}(x/\varepsilon)$ with $0 < c_0 \leq \mathbb{G}(y) \leq C_1$ and $\mathbb{G}$ 1-periodic.
In $\mathrm{L}^2(]x_1, x_2[)$ we have

weak-Γ: $\Psi_\varepsilon \overset{\Gamma}{\rightharpoonup} \Psi_{\text{harm}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\text{harm}}v\,\mathrm{d}x$

strong-Γ: $\Psi_\varepsilon \overset{\Gamma}{\longrightarrow} \Psi_{\text{arith}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\text{arithm}}v\,\mathrm{d}x$

with $G_{\text{harm}} = \Big(\int_0^1 \mathbb{G}(y)^{-1}\,\mathrm{d}y\Big)^{-1} \leq G_{\text{arithm}} = \int_0^1 \mathbb{G}(y)\,\mathrm{d}y$.

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with $G_{\text{harm}} = \Big(\int_0^1 \mathbb{G}(y)^{-1}\,\mathrm{d}y\Big)^{-1} \le G_{\text{arithm}} = \int_0^1 \mathbb{G}(y)\,\mathrm{d}y$.

Proof of weak-Γ: Assume $v_\varepsilon \rightharpoonup v$ in $\mathrm{L}^2(]a,b[)$.

$\Psi_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{x_1}^{x_2} v_\varepsilon\cdot G_\varepsilon v_\varepsilon\,\mathrm{d}x =$

$$\frac{1}{2}\int_{x_1}^{x_2} \underbrace{(\mathbb{G}_\varepsilon v_\varepsilon - G_{\text{ha}} v)\cdot\mathbb{G}_\varepsilon^{-1}(G_\varepsilon v_\varepsilon - G_{\text{ha}} v)}_{\ge 0} + 2\underbrace{\mathbb{G}_\varepsilon v_\varepsilon\cdot\mathbb{G}_\varepsilon^{-1}}_{=v_\varepsilon \rightharpoonup v} G_{\text{ha}} v - G_{\text{ha}} v\cdot \underbrace{G_\varepsilon^{-1}}_{\overset{*}{\rightharpoonup} G_{\text{ha}}^{-1}} G_{\text{ha}} v\,\mathrm{d}x$$

Quadratic functionals:

$$\Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega v(x)\cdot G_\varepsilon(x)v(x)\,\mathrm{d}x \iff \Psi^*_\varepsilon(x) = \frac{1}{2}\int_\Omega \xi(x)\cdot G_\varepsilon(x)^{-1}\xi(x)\,\mathrm{d}x$$

Lemma (One-dimensional homogenization)
Let $G_\varepsilon(x) = \mathbb{G}(x/\varepsilon)$ with $0 < c_0 \leq \mathbb{G}(y) \leq C_1$ and $\mathbb{G}$ 1-periodic.
In $\mathrm{L}^2(]x_1, x_2[)$ we have

weak-Γ: $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\text{harm}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\text{harm}} v\,\mathrm{d}x$

strong-Γ: $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\text{arith}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\text{arithm}} v\,\mathrm{d}x$

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Hence, $\liminf_{\varepsilon\to 0}\Psi_\varepsilon(v_\varepsilon) \geq \frac{1}{2}\int_{x_1}^{x_2} 0 + 2v\cdot G_{\text{ha}} v - v\cdot G_{\text{ha}} v\,\mathrm{d}x = \Psi_{\text{harm}}(v)$

Quadratic functionals:

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Given $\widehat{v}$ choose the recovery sequence $\widehat{v}_\varepsilon = G_\varepsilon^{-1} G_{\text{ha}}\widehat{v} \rightharpoonup \widehat{v}$ and first term $= 0$.
Hence, $\Psi_\varepsilon(\widehat{v}_\varepsilon) = \int_{x_1}^{x_2} 0 + G_{\text{ha}}\widehat{v}\cdot G_\varepsilon^{-1} G_{\text{ha}}\widehat{v}\,\mathrm{d}x \to \Psi_{\text{harm}}(\widehat{v})$

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Proof of strong-Γ is much simpler:
If $v_\varepsilon \to v$ in $\mathrm{L}^2(]a,b[)$, then
$$\Psi_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{x_1}^{x_2} v\cdot \underbrace{G_\varepsilon v}_{\rightharpoonup G_{\mathsf{ar}} v} - 2v\cdot G_\varepsilon \underbrace{(v-v_\varepsilon)}_{\to 0} + \underbrace{(v-v_\varepsilon)}_{\to 0}\cdot G_\varepsilon (v-v_\varepsilon)\,\mathrm{d}x \;\to\; \Psi_{\mathsf{ar}}(v)$$

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Result is compatible with Attouch's theorem: $\boxed{\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\text{harm}} \iff \Psi^*_\varepsilon \xrightarrow{\Gamma} \Psi^*_{\text{harm}}}$

For this, simply use $\text{arith}(\mathbb{G}^{-1}) = \text{harm}(\mathbb{G})^{-1}$.

One-dimensional homogenization for parabolic equation on $x \in \Omega =]0, \ell[$:

$$\boxed{c(\tfrac{x}{\varepsilon})\dot u(t,x) = \big(a(\tfrac{x}{\varepsilon})u_x(t,x)\big)_x - b(\tfrac{x}{\varepsilon})u(t,x) \qquad u_x(t,0) = 0 = u_x(t,\ell)}$$

where $a, b, c \in \mathrm{L}^\infty(\mathbb{R})$ are 1-periodic and are $\geq c_0 > 0$.

Gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \Psi_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2\,\mathrm{d}x, \qquad \Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2\,\mathrm{d}x$$

- $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\text{harm}}$ or $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\text{arith}}$ in the **dynamic space** $\mathrm{L}^2(\Omega)$
- Analogously the energy satisfies in the **energy space** $\mathrm{H}^1(\Omega) \Subset \mathrm{L}^2(\Omega)$

$$\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_{\text{ha}} : u \mapsto \frac{1}{2}\int_\Omega a_{\text{harm}}u_x^2 + b_{\text{arith}}u^2\,\mathrm{d}x \qquad \text{(weakly in } \mathrm{H}^1(\Omega))$$

$$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_{\text{ar}} : u \mapsto \frac{1}{2}\int_\Omega a_{\text{arith}}u_x^2 + b_{\text{arith}}u^2\,\mathrm{d}x \qquad \text{(strongly in } \mathrm{H}^1(\Omega))$$

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We will use later: $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_{\text{ha}}$ (Mosco in $\mathrm{L}^2(\Omega)$)

$\rightsquigarrow$ expected limit eqn $c_{\text{eff}}\, u_t = a_{\text{harm}} u_{xx} - b_{\text{arith}} u$ with $c_{\text{eff}} \in \{c_{\text{harm}}, c_{\text{arith}}\}$

Overview

1. Introduction

2. Gradient systems

3. Motivating examples

4. **Energy-dissipation formulations**

5. Evolutionary variational inequality (EVI)

6. Rate-independent systems (RIS)

Overview

Legendre-Fenchel theory for a reflexive Banach space

$\Psi : \boldsymbol{X} \to \mathbb{R}_\infty$ proper, convex, lower semicontinuous

Legendre transform $\Psi^* = \mathcal{L}\Psi : \boldsymbol{X}^* \to \mathbb{R}_\infty$ with

$\Psi^*(\xi) := \sup\{\, \langle \xi, v\rangle - \Psi(v) \mid v \in \boldsymbol{X}\,\}$

Basic properties:

- $\mathcal{L}(\mathcal{L}\Psi) = \Psi$ or $\Psi^{**} = \Psi$
- Young-Fenchel estimate: $\forall v \in \boldsymbol{X}\ \forall \xi \in \boldsymbol{X}^* :\ \ \Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$
- $\Psi(v) = \frac{1}{2}\langle Gv, v\rangle \implies \Psi^*(\xi) = \frac{1}{2}\langle \xi, G^{-1}\xi\rangle$
- $\Psi(v) = \frac{1}{p}\|v\|_{\boldsymbol{X}}^p \implies \Psi^*(\xi) = \frac{1}{p^*}\|\xi\|_{\boldsymbol{X}^*}^{p^*}$ for $1 < p < \infty$, $p^* = p/(p{-}1)$

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$\Psi^*(\xi) := \sup\{ \langle \xi, v\rangle - \Psi(v) \,|\, v \in \boldsymbol{X} \}$

$\Psi^{**} = \Psi$ and $\Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$

Subdifferential of convex Ψ

$\partial\Psi(v) = \{ \eta \in \boldsymbol{X}^* \,|\, \forall\, w \in \boldsymbol{X} : \; \Psi(w) \geq \Psi(v) + \langle \eta, w{-}v\rangle \} \subset \boldsymbol{X}^*$

If $\Psi \in \mathrm{C}^1(\boldsymbol{X};\mathbb{R})$ and convex, then $\partial\Psi(v) = \{\mathrm{D}\Psi(v)\}$.

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Theorem (Fenchel equivalence)

(i) $\xi \in \partial\Psi(v) \iff$ *(ii)* $v \in \partial\Psi^*(\xi) \iff$ *(iii)* $\Psi(v) + \Psi^*(\xi) \leq \langle \xi, v \rangle$

4. Energy-dissipation formulations

(i) $\xi \in \partial\Psi(v) \iff$ (ii) $v \in \partial\Psi^*(\xi) \iff$ (iii) $\Psi(v) + \Psi^*(\xi) \leq \langle \xi, v \rangle$

Generalized gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$

Energy funct. $\mathcal{E} : [0,T]\times\boldsymbol{X} \to \mathbb{R}_\infty$, dissipation pot. $\mathcal{R}(u,\cdot) : \boldsymbol{X} \to [0,\infty]$

$$\boxed{\text{(i)} \quad 0 \in \partial_{\dot{u}}\mathcal{R}(u(t), \dot{u}(t)) + \mathrm{D}\mathcal{E}(t, u(t)) \; \in \boldsymbol{X}^* \text{ for a.a. } t \in [0,T]}$$

force balance in $\boldsymbol{X}^*$ Biot's equation 1954

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force balance in $\boldsymbol{X}^*$ Biot's equation 1954

Dual dissipation potential $\mathcal{R}^*(u,\xi) = \mathcal{L}(\mathcal{R}(u,\cdot))(\xi)$

$$\boxed{\text{(ii)} \quad \dot u(t) \in \partial_\xi \mathcal{R}^*(u(t), -\mathrm{D}\mathcal{E}(t,u(t))) \ \in \boldsymbol{X} \text{ for a.a. } t \in [0,T]}$$

rate equation in $\boldsymbol{X}$ Onsager's equation 1931

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(iii) $\mathcal{R}(u(t),\dot u(t))+\mathcal{R}^*(u(t),-\mathrm{D}\mathcal{E}(t,u(t))) \leq \langle -\mathrm{D}\mathcal{E}(t,u(t)),\dot u(t)\rangle$

power balance in $\mathbb{R}$ (equivalent to equality by Young-Fenchel)

De Giorgi's (Ψ,Ψ^*) formulation 1980

(i) $0 \in \partial_{\dot u}\mathcal{R}(u,\dot u) + \mathrm{D}\mathcal{E}(t,u)$ (ii) $\dot u \in \partial_\xi \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(t,u))$

Theorem (Energy-Dissipation Principle (EDP), De Giorgi'80)

Assume that $\mathcal{E}$ satisfies the ***chain rule on $\boldsymbol{X}$****, then $u \in \mathrm{W}^{1,1}([0,T];\boldsymbol{X})$ solves (i) or (ii) if and only if* ***(EDE)*** *holds:*

(EDE)
$$\begin{aligned}\mathcal{E}(T,u(T)) + \int_0^T \mathcal{R}(u(t),\dot u(t)) + \mathcal{R}^*(u(t),-\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t \\ \leq \mathcal{E}(0,u(0)) + \int_0^T \partial_s\mathcal{E}(s,u(s))\,\mathrm{d}s\end{aligned}$$

Final energy + dissipated energy = initial energy + external work

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Proof: $\int_0^T -\langle \mathrm{D}\mathcal{E}(t,u),\dot u\rangle\,\mathrm{d}t \overset{\text{YF}}{\le} \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E})\,\mathrm{d}t$

$$\overset{\text{(EDE)}}{\le} \mathcal{E}(0,u(0)) + \int_0^T \partial_t\mathcal{E}\,\mathrm{d}t - \mathcal{E}(T,u(T)) \overset{\text{Ch.Rule}}{=} \int_0^T -\langle \mathrm{D}\mathcal{E}(t,u),\dot u\rangle\,\mathrm{d}t$$

$\Rightarrow$ all estimates are equalities $\Rightarrow$ Young-Fenchel estimate is equality a.e. QED

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Fundamental and more general tool **Chain-Rule Estimate (CR)**

$\mathcal{E} : \boldsymbol{X} \to \mathbb{R}_\infty$ satisfies **CRE**, if

$$\left.\begin{array}{c} u \in \mathrm{W}^{1,p}([0,T];\boldsymbol{X}),\ \xi \in \mathrm{L}^{p'}([0,T];\boldsymbol{X}^*) \\ \xi(t) \in \partial\mathcal{E}(u(t)) \end{array}\right\} \implies \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) \ge \langle \xi(t), \dot u(t)\rangle$$

(e.g. always true for lsc and convex $\mathcal{E}(\cdot)$)

Overview

$0 \in \partial_{\dot u}\mathcal{R}_\varepsilon(u,\dot u) + \mathrm{D}\mathcal{E}_\varepsilon(u) \overset{\text{Fenchel}}{\Longleftrightarrow}$ (EDE) = Energy-Dissipation Estimate

$$\boxed{\text{(EDE)}\quad \mathcal{E}_\varepsilon(u^\varepsilon(t)) + \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon,\dot u^\varepsilon) + \mathcal{R}^*_\varepsilon(u^\varepsilon,-\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t \le \mathcal{E}_\varepsilon(u^\varepsilon(0))}$$

Evolutionary Γ convergence based on (EDP)

- Sandier-Serfaty'04 (general approach)
- here: improved version of M-Rossi-Savare'12 (CVPDE) $\mathcal{R}(u,v) = \Psi(v)$

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Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ reflexive, $\exists\, c, C, \lambda_c > 0$, $p > 1$ such that $\mathcal{E}_\varepsilon(\cdot) + \lambda_c\|\cdot\|^2_{\boldsymbol{X}}$ is convex,
$\Psi_\varepsilon(v) \ge c\|v\|^p_{\boldsymbol{X}} - C$, $\Psi^*_\varepsilon(\xi) \ge c\|\xi\|^p_{\boldsymbol{X}^*} - C$, $\mathcal{E}_\varepsilon(u) \ge c\|u\|_{\boldsymbol{Z}} - C$ with $\boldsymbol{Z} \Subset \boldsymbol{X}$

$$\boxed{\Big(\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0 \ \&\ \Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \text{ in } \boldsymbol{X}\Big) \quad \Longrightarrow \quad (\boldsymbol{X}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}_0, \Psi_0)}$$

Compatibility: $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ and $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0$ in SAME topology $\boldsymbol{X}$

Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ *reflexive,* $\exists\, c, C, \lambda_{\rm c} > 0,\ p > 1$ *such that* $\mathcal{E}_\varepsilon(\cdot) + \lambda_{\rm c}\|\cdot\|^2_{\boldsymbol{X}}$ *is convex,*
$\Psi_\varepsilon(v) \geq c\|v\|^p_{\boldsymbol{X}} - C,\ \Psi^*_\varepsilon(\xi) \geq c\|\xi\|^p_{\boldsymbol{X}^*} - C,\ \mathcal{E}_\varepsilon(u) \geq c\|u\|_{\boldsymbol{Z}} - C$ *with* $\boldsymbol{Z} \Subset \boldsymbol{X}$

$$\boxed{\Big(\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0 \ \&\ \Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \ \textit{in}\ \boldsymbol{X}\Big) \quad \Longrightarrow \quad (\boldsymbol{X}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\textit{evol}} (\boldsymbol{X}, \mathcal{E}_0, \Psi_0)}$$

Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ reflexive, $\exists\, c, C, \lambda_{\rm c} > 0,\ p > 1$ such that $\mathcal{E}_\varepsilon(\cdot) + \lambda_{\rm c}\|\cdot\|^2_{\boldsymbol{X}}$ is convex,
$\Psi_\varepsilon(v) \geq c\|v\|^p_{\boldsymbol{X}} - C,\ \Psi^*_\varepsilon(\xi) \geq c\|\xi\|^p_{\boldsymbol{X}^*} - C,\ \mathcal{E}_\varepsilon(u) \geq c\|u\|_{\boldsymbol{Z}} - C$ with $\boldsymbol{Z} \Subset \boldsymbol{X}$

$$\boxed{\Big(\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0 \ \&\ \Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \text{ in } \boldsymbol{X}\Big) \quad\Longrightarrow\quad (\boldsymbol{X}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{evol} (\boldsymbol{X}, \mathcal{E}_0, \Psi_0)}$$

Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

- M-Rossi-Savare'12 (CVPDE) uses the much stronger statement $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi_0$ (but allows state dependence via $\mathcal{R}_\varepsilon(u, \dot{u})$)

- Sina Reichelt 2014 (not contained in my survey article, see Reichelt-Liero SIMA'16 and excercise in tutorial)

$$\boxed{\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \text{ is sufficient}}$$

Theorem (Mosco convergence implies evolutionary Γ-convergence)

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

ODE model on $\boldsymbol{X} = \mathbb{R}^2$

We always have $\mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}$ and $\mathcal{R}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{R}$.

$\mathcal{R}(v) = \frac{1}{2}(v_1^2 + v_2^2/\varepsilon^\beta)$ and $\mathcal{R}^*(\xi) = \frac{1}{2}(\xi_1^2 + \varepsilon^\beta \xi_2^2)$

Theorem is applicable for $\beta = 0$ only,
because of needed equicoercivity of for Ψ^*_ε.

Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ reflexive, $\exists\, c, C, \lambda_c > 0$, $p > 1$ such that $\mathcal{E}_\varepsilon(\cdot) + \lambda_c\|\cdot\|^2_{\boldsymbol{X}}$ is convex,
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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

Homogenization: $c\|v\|^2_{L^2} \leq \Psi_\varepsilon(v) \leq C\|v\|^2_{L^2} \quad \Longrightarrow \quad \boldsymbol{X} = L^2(0,\ell)$.

$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_0^\ell a_\varepsilon u_x^2 + b_\varepsilon u^2 \,dx$: $\qquad \mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ in $\boldsymbol{X} = L^2(0,\ell)$ $\qquad \oplus$

$\Psi_\varepsilon(v) = \frac{1}{2}\int_0^\ell c(x/\varepsilon)v(x)^2 \,dx$: $\qquad$ not $\xrightarrow{M}$, but $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\text{strong}}$ $\qquad \oplus$

Theorem is applicable and gives $c_{\text{eff}} = c_{\text{arith}} \geq c_{\text{harm}}$.

$$\boxed{c_\varepsilon u_t = (a_\varepsilon u_x)_x - b_\varepsilon u \quad \xrightarrow{\text{evol}} \quad c_{\text{arith}} u_t = (a_{\text{harm}} u_x)_x - b_{\text{arith}} u}$$

Sketch of proof of theorem: u_ε are solutions of (i) = $(\text{EDB})_\varepsilon$:

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon)+\Psi^*_\varepsilon(\xi_\varepsilon)\big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0)) \text{ where } -\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$$

- Uniform coercivity of $\mathcal{E}_\varepsilon$, Ψ_ε. and Ψ^*_ε yield uniform a priori bounds

$$\|u_\varepsilon\|_{\mathrm{L}^\infty([0,T];\boldsymbol{Z})} + \|u_\varepsilon\|_{\mathrm{W}^{1,p}([0,T];\boldsymbol{X})} + \|\xi_\varepsilon\|_{\mathrm{L}^p([0,T];\boldsymbol{X}^*)} \le C$$

- We find convergent subsequences (still $\varepsilon = \varepsilon_k$)

$$u_\varepsilon(t) \to u(t) \text{ in } \boldsymbol{X},\ u_\varepsilon \rightharpoonup u \text{ in } \mathrm{W}^{1,p}([0,T];\boldsymbol{X}),\ \xi_\varepsilon \rightharpoonup \xi \text{ in } \mathrm{L}^p([0,T];\boldsymbol{X}^*)$$

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- Lower semicontinuity of the dual dissipation (use $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0 \iff \Psi^*_\varepsilon \xrightharpoonup{\Gamma} \Psi^*_0$)

Ioffe's lsc result: $\int_0^T \Psi^*_0(\xi)\,\mathrm{d}t \leq \liminf_{\varepsilon\to0} \int_0^T \Psi^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}t$

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Ioffe's lsc result: $\displaystyle \int_0^T \Psi^*_0(\xi)\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^T \Psi^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}t$

(Sebastian Hensel is specialist for versions of Ioffe's theorem!)

The integrand $(\varepsilon,\xi) \mapsto \Psi^*_\varepsilon(\xi)$ for the functional $\mathcal{J}(\varepsilon,\xi) := \int_0^T \Psi^*_\varepsilon(\xi)\,\mathrm{d}t$

- is seq. weakly lower semicontinuous
- and convex in $\xi \in \boldsymbol{X}^*$

The convergence $\varepsilon_\varepsilon := \varepsilon \to 0$ is strong, while the convergence $\xi_\varepsilon \rightharpoonup \xi$ is weak.

$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon)+\Psi^*_\varepsilon(\xi_\varepsilon)\big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ where $-\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$

Sketch of proof of theorem (continued):

- $\int_0^T \Psi_0(\dot u(t))\,\mathrm{d}t \leq \liminf_{\varepsilon\to0} \int_0^T \Psi_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t.$

Ioffe's theorem doesn't apply as $\dot u_\varepsilon \rightharpoonup \dot u$ in $\mathrm{W}^{1,p}(0,T;\boldsymbol{X})$ and $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_0$.

Reichelt's lemma (see tutorial): It still holds because of $u_\varepsilon(t) \xrightarrow{\boldsymbol{X}} u(t)$.

- Taking $\varepsilon \to 0$ in (EDE)$_\varepsilon$ and using well-prepared initial cond. (i.c.) gives

$$\mathcal{E}_0(u(T)) + \int_0^T \Big(\Psi_0(\dot u)+\Psi^*_0(\xi)\Big)\,\mathrm{d}t \ \leq\ \lim_{\varepsilon\to0} \mathcal{E}_\varepsilon(u_\varepsilon(0)) \overset{\text{(i.c.)}}{=} \mathcal{E}_0(u(0))$$

- We would be finished if we knew $-\xi(t) = \mathrm{D}\mathcal{E}_0(u(t))$ for a.a. $t \in [0,T]$.

$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon) + \Psi_\varepsilon^*(\xi_\varepsilon)\big)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ where $-\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$

Sketch of proof of theorem (continued):

- $\int_0^T \Psi_0(\dot u(t))\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \Psi_\varepsilon(\dot u_\varepsilon(t))\,\mathrm{d}t.$

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- Taking $\varepsilon \to 0$ in (EDE)$_\varepsilon$ and using well-prepared initial cond. (i.c.) gives

$$\mathcal{E}_0(u(T)) + \int_0^T \Big(\Psi_0(\dot u) + \Psi_0^*(\xi)\Big)\,\mathrm{d}t \;\leq\; \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(0)) \overset{\text{(i.c.)}}{=} \mathcal{E}_0(u(0))$$

- We would be finished if we knew $-\xi(t) = \mathrm{D}\mathcal{E}_0(u(t))$ for a.a. $t \in [0,T]$.

Strong-weak closedness of $\mathrm{D}\mathcal{E}_\varepsilon$ if $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ & $\mathcal{E}_\varepsilon$ λ_c-convex (cf. Attouch'84)

$$u_\varepsilon \to u \text{ in } \boldsymbol{X} \text{ and } -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon) \ni \xi_\varepsilon \rightharpoonup \xi \text{ in } \boldsymbol{X}^* \;\Rightarrow\; -\xi \in \mathrm{D}\mathcal{E}_0(\mathrm{u})$$

- Now the Energy-Dissipation Principle shows
  - that u is a solution and
  - that $\mathcal{E}_\varepsilon(t) \to \mathcal{E}_0(u(t))$ for all $t \in [0,T]$. QED

4. Energy-dissipation formulations

Main tools is Strong-Weak Closedness of the graph of $(\mathrm{D}\mathcal{E}_\varepsilon)_{\varepsilon\in]0,1[}$

$$\boxed{\text{(SWC)}\quad u_\varepsilon \to u \text{ in } \boldsymbol{X} \text{ and } \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon) \ni \xi_\varepsilon \rightharpoonup \xi \text{ in } \boldsymbol{X}^* \;\Rightarrow\; \xi \in \mathrm{D}\mathcal{E}(\mathrm{u})}$$

This is a consequence of Γ-convergence and convexity!

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This is a consequence of Γ-convergence and convexity!

Theorem (Convexity and $\xrightarrow{\Gamma}$ imply (SWC), cf. Attouch 1984)

If all $\mathcal{E}_\varepsilon$ are lsc and convex, then $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ implies (SWC).

Proof: Assume $u_\varepsilon \to u$, $\xi_\varepsilon \rightharpoonup \xi$, and $\mathcal{E}_\varepsilon(u_\varepsilon) \to e_*$

Then convexity gives $\quad (\text{Cvx})_\varepsilon \quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u_\varepsilon) + \langle \xi_\varepsilon, w{-}u_\varepsilon\rangle$

For given $\widehat{u}$ the Γ_s-convergence gives a rec. seq. $\widehat{u}_\varepsilon$ with $\widehat{u}_\varepsilon \to \widehat{u}$, $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(\widehat{u})$

Hence, setting $w = \widehat{u}_\varepsilon$ in $(\text{Cvx})_\varepsilon$ gives $\underbrace{\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon)}_{\to\mathcal{E}_0(\widehat{u})} \geq \underbrace{\mathcal{E}_\varepsilon(u_\varepsilon)}_{\to e_*} + \langle \underbrace{\xi_\varepsilon}_{\rightharpoonup\xi}, \underbrace{\widehat{u}_\varepsilon{-}u_\varepsilon}_{\to\widehat{u}-u}\rangle$

Taking the limit $\varepsilon \to 0$ we obtain the relation $\mathcal{E}_0(\widehat{u}) \geq e_* + \langle \xi, \widehat{u}{-}u\rangle$

Choose $\widehat{u} = u$ we see that $\mathcal{E}_0(u) \geq e_*$ but Γ_s-liminf gives $e_* \geq \mathcal{E}_0(u)$. Thus, $e_* = \mathcal{E}_0(u)$ and we conclude $\xi \in \partial\mathcal{E}_0(u)$ as desired. $\square$

4. Energy-dissipation formulations

(EDE) $\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \int_0^T \big(\Psi_\varepsilon(\dot u_\varepsilon) + \Psi_\varepsilon^*(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\big)\,\mathrm{d}t \leq \mathcal{E}_\varepsilon(u_\varepsilon(0))$

The **Sandier-Serfaty [2004]** approach is more general.

They do assume

neither Strong-Weak Closedness of $(\partial\mathcal{E}_\varepsilon)_{\varepsilon\in[0,1]}$

nor the Mosco convergence of $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi$

Instead they assume

(i) $v_\varepsilon \rightharpoonup v \implies \liminf_{\varepsilon\to} \Psi_\varepsilon(v_\varepsilon) \geq \Psi_0(v)$ (w-Γ-liminf)

(ii) $u_\varepsilon \to u \implies \liminf_{\varepsilon\to} \Psi_\varepsilon^*(\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)) \geq \Psi_0^*(\mathrm{D}\mathcal{E}_0(u))$ (dual w-Γ-liminf)

Clearly, (SWC) & $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi$ imply (i) and (ii) but not vice-versa.