NUMERICAL INTEGRATION USING V–UNIFORMLY
ERGODIC MARKOV CHAINS

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Abstract. We study numerical integration based on Markov chains. Focus is on error bounds uniformly on classes of integrands. Since on general state space the concept of uniform ergodicity is too restrictive to cover important cases we analyze the error of V-uniformly ergodic Markov chains. Emphasis is on the interplay between ergodicity properties of the transition kernel, the initial distributions and the classes of integrands. The analysis is based on arguments from interpolation theory.

1. Introduction

We study numerical integration based on Markov chains. Precisely, for a fixed probability \( \pi \) on \( X \) we aim at approximating \( \int_X f(x) \pi(dx) \) by means of a sample mean of random variables arising from an ergodic Markov chain with transition kernel \( K \), having \( \pi \) as its invariant distribution. Thus, if \( \nu \) is an initial distribution, then we use for a given \( f \) on \( X \) the sample mean

\[
\vartheta_N(f) := \frac{1}{N} \sum_{j=1}^{N} f(X_j),
\]

where \( X_1, \ldots, X_N \) are the consecutive steps of our Markov chain. Typically, the error is measured in mean square sense, i.e.,

\[
e(f, \vartheta_N, \nu, K) := \left( \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^{N} f(X_j) - \int_X f(x) \pi(dx) \right|^2 \right)^{1/2}
\]
denotes the individual error of the sample mean at function \( f \).

Typically \( e(f, \vartheta_N, \nu, K) \) will tend to 0 at a rate \( N^{-1/2} \). One problem to be discussed is to describe classes \( F \) of functions, for which the uniform error \( e(F, \vartheta_N, \nu, K) = \sup_{f \in F} e(f, \vartheta_N, \nu, K) \) retains this speed.

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of convergence. More precisely, which is the interaction between ergodicity properties of the kernel \( K \) and respective classes \( F \) and initial distributions \( \nu \)?

We extend previous analysis for uniformly ergodic Markov chains from \[4\] to \( V \)-uniformly ergodic Markov chains. When doing this the interplay between \( K, F \) and \( \nu \) will become transparent.

2. Some preliminary analysis

Our notation will be close to \[5\]. We assume that \((X, \mathcal{F})\) is a given (countably generated) measurable space and that the Markov chain \( X_n, n = 1, 2, \ldots \) is given through a family \( \{K(x, \cdot), x \in X\} \) of transition probabilities (Markov kernels) with the usual assumptions

(1) For each \( x \in X \) the mapping \( A \in \mathcal{F} \rightarrow K(x, A) \) induces a probability on \( X \),
(2) For each \( A \in \mathcal{F} \) the mapping \( x \in X \rightarrow K(x, A) \) is an \( \mathcal{F} \)-measurable real function.

By \( K^n(x, \cdot) \) we denote the \( n \)-step transition probabilities. We shall strengthen the assumptions below. Throughout we shall restrict our study to (aperiodic) and ergodic Harris chains, see \[3, 6\], such that there is a unique invariant probability, called \( \pi \), below, i.e.,

\[
\pi(A) = \int_X K(x, A) \pi(dx), \quad A \in \mathcal{F}.
\]

It will be convenient to use operator theoretic terms in our analysis. Therefore we introduce the following operators. Any family of transition probabilities induces a bounded operator \( P \) on \( L^\infty(X, \pi) \) by assigning

\[
(P f)(x) := \int_X f(y) K(x, dy).
\]

This operator has norm 1. In general, operators corresponding to some Markov chain in the above way are called Markov operators.

Remark 1. In the above reasoning the operator \( P \) maps uniformly bounded functions to uniformly bounded ones. Since \( P \) is \( \pi \)-invariant, this extends to the Banach spaces \( L^\infty(X, \pi) \) of equivalence classes, see e.g. \[6\] Chapt. 4, Prop. 1.4].

Also, the mathematical expectation induces a projection \( E \) onto the constant functions by assigning for every integrable function \( f \in L^1(X, \pi) \)

\[
E f := \int_X f(x) \pi(dx).
\]
3. $V$-uniformly ergodic Markov chains

In this section we recall some of the properties of $V$-uniformly ergodic Markov chains, as introduced in [5, Chap. 16]. Let $V \geq 1$ be a real valued function on $X$. By $L^\infty(V) := L^\infty(X, V, \pi)$ we denote the Banach space of all (equivalence classes of) functions $f$ on $X$, for which $\|f\|_V = \|f\|_{L^\infty(V)} := \text{ess-sup} |f(x)/V(x)| < \infty$. As in [5, Chap. 16] we introduce the following

**Definition 1.** The Harris Markov chain $K$ is called $V$-uniformly ergodic, if

$$\|P^n - E: L^\infty(V) \to L^\infty(V)\| \to 0.$$ 

**Remark 2.** There are several equivalent reformulation from ergodic theory. Since $E$ has rank 1, the operator $P$ must be quasi-compact in the sense, that there are $m \in \mathbb{N}$ and some compact operator $Q$ in $L^\infty(V)$ for which $\|P^m - Q : L^\infty(V) \to L^\infty(V)\| < 1$. Actually, quasi-compactness is equivalent to being $V$-uniformly ergodic, by virtue of the Yosida-Kakutani-Theorem, see [3, Thm. 2.8]. Moreover, it follows, that $\pi \to \langle \pi, f \rangle$ is continuous on $L^\infty(V)$, such that necessarily $\int V(x) \pi(dx) < \infty$. In [5, Thm. 14.0.1] there is a different proof for this fact, based on the Foster-Lyapunov drift condition, equivalently describing $V$-uniform ergodicity. Our reasoning will be based on an operator approach.

**Remark 3.** The origin to extending the notion of ergodicity lies in the fact, that on general state space most Markov chains will not be uniformly ergodic, but they are $V$–uniformly ergodic for appropriate choice of $V$. For more details we refer to [2]. In particular this has been shown for Markov chains on $\mathbb{R}^d$ with uniformly tight increments.

One might argue which properties of $V$ have impact on the ergodicity. The first observation is worth mentioning.

**Proposition 1.** Let $\varphi : X \to \mathbb{R}^+$ be any bounded function bounded away from 0. A Markov chain $K$ is $V$-uniformly ergodic for some function $V$ if and only if it is $\varphi V$-uniformly ergodic.

**Proof.** Suppose, it is $V$-uniformly ergodic. Then following estimates are valid.

$$\|P^n - E: L^\infty(\varphi V) \to L^\infty(\varphi V)\|$$

$$= \|\varphi^{-1}(P^n - E)\varphi : L^\infty(V) \to L^\infty(V)\|$$

$$\leq \|\varphi^{-1}\|_\infty \|\varphi\|_\infty \|P^n - E: L^\infty(V) \to L^\infty(V)\|,$$

whicn tends to 0 by assumption. The other implication follows by replacing $V$ by $\varphi V$ and $\varphi$ by $\varphi^{-1}$. \qed
This shows, that requiring $V \geq 1$ is merely a matter of taste. $V$ must be bounded away from 0 to ensure, that constant functions belong to $L^\infty(V)$, which is natural. As further consequence is

**Proposition 2.** A Markov chain $K$ is $V$-uniformly ergodic for some bounded function $V$ if and only if it is $1$-uniformly ergodic.

Thus we recourse to the original definition of uniform ergodicity as 1-uniform ergodicity in the sense of Definition 1.

Another consequence is less obvious. Nevertheless it has partly been observed in [5, Lemma 15.2.9].

**Proposition 3.** If a Markov chain $K$ is $V$-uniformly ergodic, then it is $V^\theta$-uniformly ergodic for all $0 < \theta < 1$. Moreover,

$$
\| P^n - E : L^\infty(V^\theta) \to L^\infty(V^\theta) \| \leq 2^{1-\theta} \| P^n - E : L^\infty(V) \to L^\infty(V) \|^{\theta}.
$$

**Proof.** The proof will follow from an interpolation argument. We first note, that $L^\infty(V)$ is dual to $L^1(V, \pi)$ with respect to the mapping $g \mapsto \int f(x)g(x) \pi(dx)$. The Stein-Weiss Interpolation Theorem [1] Thm. 5.4.1 applied with $p = 1$ implies that $L^1(V^\theta, \pi)$ is an interpolation space. The duality theorem [1] Thm. 3.7.1 yields, that $L^\infty(V^\theta, \pi)$ is interpolating between $L^\infty(X, \pi)$ and $L^\infty(V)$. Therefore

$$
\| P^n - E : L^\infty(V^\theta) \to L^\infty(V^\theta) \|
\| P^n - E : L^\infty(X) \to L^\infty(X) \|^{1-\theta} \| P^n - E : L^\infty(V) \to L^\infty(V) \|^{\theta}.
$$

Using the Markov property we have $\| P^n - E : L^\infty(X) \to L^\infty(X) \| \leq 2$, which implies (2).

So, among $V$, which are not bounded, significantly different behavior is for polynomial growth somewhere, in which case $V$ can be found growing less than any polynomial, and functions $V$, which are exponential or super-exponential. Then interpolation will still retain this type of growth.

We close this section with a further consequence, which will be crucial in our stability investigations.

**Proposition 4** (see e.g. [6, Thm. 3.10]).

$$
(I - P)L^\infty(V) = L^\infty(V)^0,
$$

where $L^\infty(V)^0$ denotes the set of all functions $f \in L^\infty(V)$ with $E f = 0$. As a consequence, the inverse $(I - P)^{-1}$ exists as a bounded operator on $L^\infty(V)^0$. 

Proof. First, since \( I - P^n = (I - P) \sum_{j=0}^{n-1} P^j \), and \( P \) is \( V \)-uniformly ergodic, thus \( P^n \to 0 \) on \( L^\infty(V)^0 \), the operator \( I - P \) must be invertible, now on \( L^\infty(V)^0 \). Hence \( (I - P)^{-1} \) exists on \( L^\infty(V)^0 \). \( \square \)

4. Stability for \( V \)-uniformly ergodic Markov chains

The implications of the concept of \( V \)-uniform ergodicity for numerical integration are established in the following result. We first note, that we cannot expect convergence for functions bounded by \( V \), since such need not even be square-integrable with respect to \( \pi \), a natural assumption for the error criterion under consideration. Also, the class of initial distributions \( \nu \), for which convergence can be expected is limited to those satisfying \( \int V(x) \nu(dx) < \infty \).

Let us introduce the following (quadratic) functional

\[
\Phi(f) := \langle \pi, [(I - P)^{-1}(I + P)(I - E)f][(I - E)f] \rangle.
\]

**Theorem 1.** Let the Markov chain \( K \) be \( V \)-uniformly ergodic. Then for every initial distribution \( \nu \), satisfying \( \int V(x) \nu(dx) < \infty \) and for all bounded sets \( F \subset L^\infty(X) \) the following asymptotics holds true.

\[
\lim_{N \to \infty} \sup_{f \in F} \left| Ne^2(f, \vartheta_N, \nu, K) - \Phi(f) \right| = 0.
\]

**Proof.** As in the previous paper [4] we rewrite the error and functional in operator form. We agree, given a function \( f \), to put \( g := f - Ef \in L^\infty(V)^0 \). Moreover, again given \( f \) we let \( h \in L^\infty(V)^0 \) be such that \( g = (I - P)h \), which is possible by (3) above. We obtain

\[
e^2(f, \vartheta_N, \nu, K)
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \langle \nu P^j, g^2 \rangle + \frac{2}{N} \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \langle \nu P^j, g P^{k-j} g \rangle
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \langle \nu P^j, g^2 \rangle + \frac{2}{N} \sum_{j=1}^{N-1} \langle \nu P^j, g P(I - P) \sum_{k=0}^{N-j-1} P^k h \rangle
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \langle \nu P^j, g^2 \rangle + \frac{2}{N} \sum_{j=1}^{N-1} \langle \nu P^j, g P(I - P^{N-j})h \rangle
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \langle \nu P^j, g^2 \rangle + \frac{2}{N} \sum_{j=1}^{N} \langle \nu P^j, g P h \rangle - \frac{2}{N} \sum_{j=1}^{N} \langle \nu P^j, g P^{N-j+1} h \rangle,
\]
where we extended the sums in (6) to \( N \) be adding and subtracting \( \frac{2}{N} \langle \nu P^N, g P h \rangle \), respectively. Since analogously

\[ \Phi(f) = \langle \pi, g^2 \rangle + 2 \langle \pi, g P h \rangle, \]

we can argue as follows. First note, that averages with respect to \( j \) tend to 0 if only the summands do, this is (the easy) part of Toeplitz’ Theorem. The third summand in (6) is further rewritten as

\[ (7) \quad \frac{2}{N} \sum_{j=1}^{N} \langle \nu P^j, g P^{N-j+1} h \rangle \]

\[ = \frac{2}{N} \sum_{j=1}^{N} \langle \nu P^j - \pi, g P^{N-j+1} h \rangle + \frac{2}{N} \sum_{j=1}^{N} \langle \pi, g(P^{N-j+1} - E) h \rangle, \]

since \( E h = 0 \). But

\[ (8) \quad \sup_{f \in F} \left| \frac{2}{N} \sum_{j=1}^{N} \langle \nu P^j - \pi, g P^{N-j+1} h \rangle \right| \leq \frac{2}{N} \sum_{j=1}^{N} \sup_{f \in F} \left| \langle \nu P^j - \pi, g P^{N-j+1} h \rangle \right|. \]

Therefore, if

\[ (9) \quad \sup_{f \in F} \left| \langle \nu P^l - \pi, g P^l h \rangle \right| \to 0, \]

uniformly for \( l = 0, 1, 2 \ldots \), then the first and second summand in (6) converge to 0, as well as the right hand side in (8). But \( g P^l h \) is in \( L^\infty(V) \) for \( f \in F \) and \( l \in \mathbb{N} \) by Proposition 4, such that convergence in (9) is an immediate consequence of \( V \)-uniform ergodicity.

It remains to investigate the second term on the right in (7) above.

\[ (10) \quad \sup_{f \in F} \left| \frac{2}{N} \sum_{j=1}^{N} \langle \pi, g(P^{N-j+1} - E) h \rangle \right| \leq \frac{2}{N} \sum_{j=1}^{N} \sup_{f \in F} \| g(P^j - E) h \|_V, \]

tending to 0, again by Toeplitz’ argument and \( V \)-uniform ergodicity. This completes the proof of the theorem.

\[ \square \]

Remark 4. The function \( h \) with \( g = (I - P) h \) is convenient for representing \( \Phi(f) \), indeed

\[ \Phi(f) = \langle \pi, (I - P) h (I + P) h \rangle = \| h \|_{L^2(X, \pi)}^2 - \| P h \|_{L^2(X, \pi)}^2. \]
We will now extend uniform convergence to larger classes $F$ of functions. The first question to be answered is: For which class of square-integrable functions does $(I - P)^{-1}$ exist? This question will be answered, again by an interpolation argument. Given any $V$, let $L^p(X, V, \pi)$ denote the Banach space of all functions which are $p$-integrable with respect to $\pi_V := V \pi$ and $L^p(X, V, \pi)^0$ the subspace of functions integrating to 0 with respect to $\pi$.

**Lemma 1.** The operator $(I - P)^{-1}$ exists on $L^2(X, V^{-1}, \pi)^0$.

**Proof.** We have to deal with weighted spaces and proceed as follows. Instead of the operators $P_n - E$ we consider $P_n := V^{-1}(P^n - E)V$. Since $P$ is $V$-uniformly ergodic we have convergence of the norms $\|P_n : L^\infty(X) \to L^\infty(X)\| \to 0$. Now, letting $L^1(X, V, \pi)$ denote the space of all functions integrable with respect to $\pi_V$, then $P_n$ is invariant with respect to $\pi_V$, thus extends as a bounded linear operator from $L^1(X, V, \pi)$ to $L^1(X, V, \pi)$. Riesz-Thorin interpolation, see [1, Thm. 1.1.1], provides $\|P_n : L^2(X, V, \pi) \to L^2(X, V, \pi)\| \to 0$. Rewriting this in terms of $P_n - E$ we obtain

$$(11) \quad \|P^n - E : L^2(X, V^{-1}, \pi) \to L^2(X, V^{-1}, \pi)\| \to 0,$$

which in turn implies the existence of $(I - P)^{-1}$ on $L^2(X, V^{-1}, \pi)^0$. □

Finally, to make the functional $\Phi(f)$ be finite, we need $f \in L^2(X, V, \pi)$, which suggests, that this is the proper class $F$ of functions to look at.

In [4] the extension to square-integrable functions was done at the expenses of restrictions on the initial distributions. Here we shall impose a condition on the operator to keep the class of initial distributions large. A careful inspection of the above proof allows to verify

**Corollary 1.** Suppose, $P^k : L^1(X, \pi) \to L^\infty(V)$ for some $k \in \mathbb{N}$. Then

$$(12) \quad \lim_{N \to \infty} \sup_{f \in F} |N e^2(f, \vartheta_N, \nu, K) - \Phi(f)| = 0,$$

for all initial distributions $\nu$ with $\int V(x) \nu(dx) < \infty$ and bounded sets $F \subset L^2(X, V, \pi)$.

**Proof.** In the proof of Theorem [1] Toeplitz-type arguments reduced the analysis to convergence to 0 of (9) and the left hand side in (10). For $j > k$, we can rewrite (9) as

$$\sup_{f \in F} \left| \langle \nu P^j - \pi, g P^j h \rangle \right| = \sup_{f \in F} \left| \langle \nu P^j - \pi, P^k (g P^j h) \rangle \right|.$$
Since for \( l \in \mathbb{N} \) and \( f \in F \subset L^2(X, V, \pi) \) the product \( gP^l h \) belongs to a uniformly bounded set in \( L^1(X, \pi) \), which in turn is mapped to \( L^\infty(V) \) by \( P^k \), we have again convergence to 0 in the present context.

Finally, since \( \pi = \pi P^k \), we can insert this power of \( P \) into the left hand side in (10) and need to show convergence

\[
\| P^k(g(P^j - E)h) \|_V \to 0,
\]

uniformly for \( f \in F \).

By (11) the norms of \( (P^j - E)h \) tend to 0 in \( L^2(X, V^{-1}, \pi) \), uniformly for \( f \) the unit ball of \( L^2(X, V^{-1}, \pi) \). Therefore \( \| g(P^j - E)h \|_{L^1(X, \pi)} \to 0 \) and, by our assumption on \( P^k \), this implies \( \| P^k(g(P^j - E)h) \|_V \to 0 \), completing the proof of the corollary. \( \square \)

**Remark 5.** A simple sufficient condition for \( P: L^1(X, \pi) \to L^\infty(V) \) can be given for kernels which are absolutely continuous with respect to \( \pi \). If \( k(x, dy) = k(x, y)\pi(dy) \), then \( \sup_{x,y} k(x,y) < \infty \) guarantees \( \| P: L^1(X, \pi) \to L^\infty(V) \| < \infty \).

If we cannot guarantee \( P^k: L^1(X, \pi) \to L^\infty(V) \), then additional assumptions on the initial distribution \( \nu \) have to be made, exactly those from Theorem 1 in [4], namely \( \nu \ll \pi \) (or at least \( \nu P^k \ll \pi \) for some power \( k \)) with uniformly bounded density.

If the underlying Markov chain \( K \) is reversible with respect to \( \pi \) then we can slightly relax the assumptions of Corollary 1.

**Definition 2** (see [7]). A Markov chain \( K \) is reversible with respect to \( \pi \), if for all \( A, B \in \mathcal{F} \) the balance

\[
\int_A K(x, B)\pi(dx) = \int_B K(x, A)\pi(dx)
\]

holds.

**Corollary 2.** Suppose, the Markov chain \( K \) is reversible and that \( P^k: L^1(X, V, \pi) \to L^\infty(X, V, \pi) \) for some \( k \in \mathbb{N} \). Then

\[
\lim_{N \to \infty} \sup_{f \in F} |N\epsilon^2(f, \theta_N, \nu, K) - \Phi(f)| = 0,
\]

for all initial distributions \( \nu \), satisfying \( \int V(x) \nu(dx) < \infty \) and bounded sets \( F \subset L^2(X, V, \pi) \).

**Proof.** The arguments are similar to those in previous proofs, so we sketch only the important point. Assumption (13) implies, that the operator \( P \) is self-adjoint in \( L^2(X, \pi) \). Then (11) yields

\[
\| P^n - E : L^2(X, V, \pi) \to L^2(X, V, \pi) \| \to 0,
\]
since $L^2(X, V, \pi)$ is the dual space of $L^2(X, V^{-1}, \pi)$ with respect to the duality $\langle f, g \rangle = \int f(x)g(x) \pi(dx)$.

Therefore $(I - P)^{-1}$ exists on $L^2(X, V, \pi)_0$, and the products $g P^j h$ are uniformly bounded in $L^1(X, V, \pi)$, if $f \in F$. Therefore

$$\sup_{f \in F} \left| \langle \nu P^j - \pi, P^k(g P^j h) \rangle \right| \to 0,$$

uniformly in $l$, provided $\int V(x) \nu(dx) < \infty$.

Similarly, $\|P^k(g(P^j - E))h\|_V \to 0$, since $\|P^n - E\| \to 0$ on $L^2(X, V, \pi)$. This allows to establish the

$$\square$$

**Remark 6.** Again, for integral kernels $K(x, dy) = k(x, y)\pi(dy)$, the assumption $P: L^1(X, V, \pi) \to L^\infty(V)$ is valid if $\sup_{x,y \in X} \frac{k(x,y)}{V(x)V(y)} < \infty$.

### 5. Extensions to other function classes

The preceding error criterion $e(f, \vartheta, \nu, K)$ substantially made use of square integrability of the underlying functions. If we replace the second absolute moment by the first one, then we may consider the error of integration of larger classes of functions.

To be precise, let

$$\bar{e}(f, \vartheta, \nu, K) := E \left| \frac{1}{N} \sum_{j=1}^{N} f(X_j) - \int_X f d\pi \right|$$

and correspondingly

$$\bar{e}(F, \vartheta, \nu, K) := \sup_{f \in F} \bar{e}(f, \vartheta, \nu, K),$$

the uniform error of some function class $F$. Evidently, $\bar{e}(f, \vartheta, \nu, K) \leq e(f, \vartheta, \nu, K)$. Therefore, the results from Section yield $\bar{e}(F, \vartheta, \nu, K) \leq CN^{-1/2}$ under the respective assumptions. On the other hand, for integrable functions no rate of convergence can be expected, in general. Instead we have

**Proposition 5.** Let $\nu$ be any initial distribution with $\int V(x) \nu(dx) < \infty$. If $P^k : L^1(X, \pi) \to L^\infty(V)$ for some $k$, then $\{\bar{e}(F, \vartheta, \nu, K), \ N \in \mathbb{N}\}$ is uniformly bounded.
Proof. A crude estimate yields, again letting $g := f - Ef$,

$$
\bar{e}(f, \vartheta_N, \nu, K) \leq \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}|g(X_j)| \leq \frac{1}{N} \sum_{j=1}^{N} \langle \nu, P^j |g| \rangle
$$

$$
\leq \frac{1}{N} \sum_{j=1}^{N} \langle \nu, (P^j - E) |g| \rangle + E |g|.
$$

The average on the right tends to 0 uniformly for $f \in F$, since for $j > k$ we can bound

$$
\sup_{f \in F} \langle \nu, (P^j - E) |g| \rangle \sup_{f \in F} \langle \nu, (P^{j-k} - E) P^k |g| \rangle
$$

$$
\leq C \| (P^{j-k} - E) : L^\infty(V) \to L^\infty(V) \|
$$

for a certain constant $C$. Therefore $\bar{e}(F, \vartheta_N, \nu, K)$ must be bounded, provided $F \in L^1(X, \pi)$ is bounded. □

The above crude bound allows to establish the main result in this section.

**Corollary 3.** Let $1 < p < 2$ and $\nu$ be any initial distribution with $\int V(x) \nu(dx) < \infty$ and assume $P^k : L^1(X, V, \pi) \to L^\infty(V)$ for some $k$. For every bounded set $F \in L^p(X, V, \pi)$ there is a constant $C$, such that

$$
\bar{e}(F, \vartheta_N, \nu, K) \leq CN^{-1+1/p}.
$$

**Proof.** Without loss of generality we assume $F$ to be the unit ball. The argument will again be based on interpolation. To this end let $\Omega := X^N$ the countable product of the state space $X$, endowed with product $\sigma$-algebra $\mathcal{F}^N$ and probability $\mathcal{P}$, the (canonical) distribution of the Markov chain $K$ with initial distribution $\nu$ on the path space. The sample mean $\vartheta_N$ induces an operator

$$
T_N : f \in F \to \vartheta_N(f) - Ef \in L^1(\Omega, \mathcal{F}^N, \mathcal{P}),
$$

for which $\bar{e}(f, \vartheta_N, \nu, K) = \|T_N f\|_{L^1(\Omega, \mathcal{F}, \mathcal{P})}$. Thus, by Proposition 5 its norm on $L^1(X, V, \pi)$ is

$$
\|T_N : L^1(X, V, \pi) \to L^1(\Omega, \mathcal{F}, \mathcal{P})\| \leq C.
$$

Now, from Corollary 1 from Section 4 we infer

$$
\|T_N : L^2(X, V, \pi) \to L^1(\Omega, \mathcal{F}^N, \mathcal{P})\| \leq CN^{-1/2}.
$$

Next we observe, that $L^p(X, V, \pi)$ is an interpolation space between $L^1(X, V, \pi)$ and $L^2(X, V, \pi)$, see Theorem 1.1.1. Therefore

$$
\bar{e}(F, \vartheta_N, \nu, K) = \|T_N : L^p(X, V, \pi) \to L^1(\Omega, \mathcal{F}^N, \mathcal{P})\| \leq CN^{-1+1/p},
$$

which completes the proof of the corollary. □
CONCLUSION

We analyze the asymptotic error of numerical integration based on Markov chains. Previous results in this direction were obtained for uniformly ergodic Markov chains. Since on general state spaces this assumption is too restrictive, the analysis is extended to Markov chains meeting the weaker condition of \(V\)-uniformly ergodicity. We establish error bounds uniformly on certain classes of integrands, related to the weight function \(V\). The analysis is based on interpolation in functional classes.

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