MEAN-FIELD INTERACTION OF
BROWNIAN OCCUPATION MEASURES, I:
UNIFORM TUBE PROPERTY OF THE COULOMB FUNCTIONAL

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Abstract: We study the transformed path measure arising from the self-interaction of a three-dimensional Brownian motion via an exponential tilt with the Coulomb energy of the occupation measures of the motion by time t. The logarithmic asymptotics of the partition function were identified in the 1980s by Donsker and Varadhan [DV83-P] in terms of a variational formula. Recently [MV14] a new technique for studying the path measure itself was introduced, which allows for proving that the normalized occupation measure asymptotically concentrates around the set of all maximizers of the formula. In the present paper, we show that likewise the Coulomb functional of the occupation measure concentrates around the set of corresponding Coulomb functionals of the maximizers in the uniform topology. This is a decisive step on the way to a rigorous proof of the convergence of the normalized occupation measures towards an explicit mixture of the maximizers, which will be carried out elsewhere. Our methods rely on deriving Hölder-continuity of the Coulomb functional of the occupation measure with exponentially small deviation probabilities and invoking the large-deviation theory developed in [MV14] to a certain shift-invariant functional of the occupation measures.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study a transformed path measure that arises from a mean-field type interaction of a three dimensional Brownian motion in a Coulomb potential. Under the influence of such a transformed measure, the large-t behavior of the normalized occupation measures, denoted by L_t, is of high interest. This is intimately connected to the well-known polaron problem from statistical mechanics and a full understanding of the behavior of L_t under the aforementioned transformation is crucial for the analysis of the polaron path measure under 'strong coupling', its effective mass and justification of mean-field approximations. For physical relevance of this model, we refer to the article of Spohn (see [S86]). Some mathematically rigorous research in this direction began in the 1980s with the analysis of the partition function of Donsker and Varadhan ([DV83-P]), but it was not until recently that a new technique was developed [MV14] for handling the actual path measures, which promises to make amenable a deeper analysis and a full identification of the limiting distribution of L_t. The present paper makes decisive steps towards this goal, which are also interesting on their own.

We start with developing the mathematical layout of the model in Section 1.1, remind on earlier results in Section 1.2, present our new progress in Section 1.3, report on the achievements of [MV14]

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in Section 1.4 and give in Section 1.5 a quick outlook on how the results of this paper will be utilized in future work.

1.1 The transformed path measure.

We start with the Wiener measure $\mathbb{P}$ on $\Omega = C([0,\infty), \mathbb{R}^3)$ corresponding to a 3-dimensional Brownian motion $W = (W_t)_{t \geq 0}$ starting from the origin. We are interested in the transformed path measure

$$\widehat{\mathbb{P}}_t(\mathrm{d}\omega) = \frac{1}{Z_t} \exp\left\{\frac{1}{t} \int_0^t \int_0^t \mathrm{d}\sigma \mathrm{d}s \, \frac{1}{|\omega_\sigma - \omega_s|}\right\} \mathbb{P}(\mathrm{d}\omega) \qquad \omega \in \Omega, \tag{1.1}$$

with the normalizing constant, the *partition function*,

$$Z_t = \mathbb{E}\left[\exp\left\{\frac{1}{t} \int_0^t \int_0^t \mathrm{d}\sigma \mathrm{d}s \, \frac{1}{|W_\sigma - W_s|}\right\}\right]. \tag{1.2}$$

We remark that the asymptotic behavior of $\widehat{\mathbb{P}}_t$ is determined by those influential paths which make $|W_\sigma - W_s|$ small, i.e., the interaction is *self-attractive*.

Let

$$L_t = \frac{1}{t} \int_0^t \mathrm{d}s \, \delta_{W_s} \tag{1.3}$$

be the normalized occupation measure of W until time t. This is a random element of $\mathcal{M}_1(\mathbb{R}^3)$, the space of probability measures on $\mathbb{R}^3$. Then the path measure $\widehat{\mathbb{P}}_t$ can be written as

$$\widehat{\mathbb{P}}_t(A) = \frac{1}{Z_t} \mathbb{E}\left[\mathbb{1}_A \exp\left\{tH(L_t)\right\}\right] \qquad A \subset \Omega,$$

where

$$H(\mu) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}x)\,\mu(\mathrm{d}y)}{|x - y|}, \qquad \mu \in \mathcal{M}_1(\mathbb{R}^3), \tag{1.4}$$

denotes the *Coulomb potential energy functional* of μ. Hence, $\widehat{\mathbb{P}}_t$ is an exponential tilt of the Coulomb energy function of L_t with parameter t. It is the goal of this paper to make a contribution to a rigorous understanding of the behavior of L_t under $\widehat{\mathbb{P}}_t$.

For any $\mu \in \mathcal{M}_1(\mathbb{R}^3)$, we define the function

$$(\Lambda\mu)(x) = \left(\mu \star \frac{1}{|\cdot|}\right)(x) = \int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}y)}{|x - y|},$$

which is also sometimes called its *Coulomb potential energy functional*. In order to avoid misunderstandings, we will call $H(\mu)$ the *Coulomb energy* and $\Lambda(\mu)$ the *Coulomb functional* of μ. Note that $H(\mu) = \langle \mu, \Lambda\mu \rangle = \int (\Lambda\mu)(x)\,\mu(\mathrm{d}x)$. We remark that the Coulomb energy of the Brownian occupation measure,

$$\Lambda_t(x) = (\Lambda L_t)(x) = \int_{\mathbb{R}^3} \frac{L_t(\mathrm{d}y)}{|x - y|} = \frac{1}{t} \int_0^t \frac{\mathrm{d}s}{|W_s - x|}, \tag{1.5}$$

is almost surely finite in $\mathbb{R}^3$.

1.2 Existing results.

Donsker and Varadhan [DV83-P] studied the asymptotic behavior of Z_t resulting in the variational formula

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \log Z_t &= \sup_{\mu\in\mathcal{M}_1(\mathbb{R}^3)} \left\{ H(\mu) - I(\mu) \right\} \\
&= \sup_{\substack{\psi\in H^1(\mathbb{R}^3) \\ \|\psi\|_2=1}} \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{d}x\mathrm{d}y \, \frac{\psi^2(x)\psi^2(y)}{|x-y|} - \frac{1}{2}\|\nabla\psi\|_2^2 \right\} = \rho,
\end{aligned}
\tag{1.6}
$$

with $H^1(\mathbb{R}^3)$ denoting the usual Sobolev space of square integrable functions with square integrable gradient. Furthermore, we put

$$
I(\mu) = \frac{1}{2}\|\nabla\psi\|_2^2
\tag{1.7}
$$

if μ has a density ψ^2 with $\psi \in H^1(\mathbb{R}^3)$, and $I(\mu) = \infty$ otherwise. Note that both H and I are shift-invariant functionals, i.e., $H(\mu) = H(\mu \star \delta_x)$ and $I(\mu) = I(\mu \star \delta_x)$ for any $x \in \mathbb{R}^3$.

The above result is a consequence of a *large deviation principle* (LDP) for L_t under $\mathbb{P}$ in $\mathcal{M}_1(\mathbb{R}^3)$, developed by Donsker and Varadhan ([DV75-83]). This means, when $\mathcal{M}_1(\mathbb{R}^3)$ is equipped with the usual weak topology, for every open set $G \subset \mathcal{M}_1(\mathbb{R}^3)$,

$$
\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(L_t \in G\big) \geq - \inf_{\mu\in G} I(\mu),
\tag{1.8}
$$

and for any compact set $K \subset \mathcal{M}_1(\mathbb{R}^3)$,

$$
\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(L_t \in K\big) \leq - \inf_{\mu\in K} I(\mu).
\tag{1.9}
$$

The above statement is also called a *weak large deviation principle* since the upper bound (1.9) holds only for compact subsets. We say that a family of probability distributions satisfies a *strong large deviation principle* if, along with the lower bound (1.8), the upper bound (1.9) holds also for all closed sets.

The variational formula (1.6) has been analyzed by Lieb ([L76]). It turns out that there is a smooth, rotationally symmetric and centered maximizer ψ_0 which is unique except for spatial translations. In other words, if $\mathfrak{m}$ denotes the set of maximizing densities, then

$$
\mathfrak{m} = \big\{ \mu_0 \star \delta_x \colon x \in \mathbb{R}^3 \big\},
\tag{1.10}
$$

where μ_0 is a probability measure with a density ψ_0^2 so that ψ_0 maximizes the variational problem (1.6). We will often write $\mu_x = \mu_0 \star \delta_x$ and write ψ_x^2 for its density.

Given (1.6) and (1.10), we expect the distribution of L_t under the transformed measure $\widehat{\mathbb{P}}_t$ to concentrate around $\mathfrak{m}$ and, even more, to converge towards a mixture of spatial shifts of μ_0. Such a precise analysis was carried out by Bolthausen and Schmock [BS97] for a spatially discrete version of $\widehat{\mathbb{P}}_t$, i.e., for the continuous-time simple random walk on $\mathbb{Z}^d$ instead of Brownian motion and an interaction potential $v \colon \mathbb{Z}^d \to [0,\infty)$ with finite support instead of the singular Coulomb potential $x \mapsto 1/|x|$. A first key step in [BS97] was to show that, under the transformed measure, the probability of the local times falling outside any neighborhood of the maximizers decays exponentially. For its proof, the lack of a strong LDP for the local times was handled by an extended version of a standard periodization procedure by folding the random walk into some large torus. Combined with this, an explicit tightness property of the distributions of the local times led to an identification of the limiting distribution.

However, in the context of the continuous setting with a singular Coulomb interaction, the afore-mentioned periodization technique or any standard compactification procedure does not work well to circumvent the lack of a strong LDP. An investigation of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$, the distribution of L_t under $\widehat{\mathbb{P}}_t$, remained open until a recent result [MV14] rigorously justified the above heuristics, leading to the statement

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \{L_t \notin U(\mathfrak{m})\} < 0, \tag{1.11}$$

where $U(\mathfrak{m})$ is any neighborhood of $\mathfrak{m}$ in the weak topology induced by the Prohorov metric, the metric that is induced by all the integrals against continuous bounded test functions. Hence, (1.11) implies that the distribution of L_t under $\widehat{\mathbb{P}}_t$ is asymptotically concentrated around $\mathfrak{m}$. Since a one-dimensional picture of $\mathfrak{m}$ is an infinite line, its neighborhood resembles an infinite tube. Therefore, assertions similar to (1.11) are sometimes called a *tube property*.

It is worth pointing out that although (1.11) requires only the weak topology in the statement, its proof is crucially based on a robust theory of *compactification* $\widetilde{\mathcal{X}}$ of the quotient space

$$\widetilde{\mathcal{M}}_1(\mathbb{R}^d) \hookrightarrow \widetilde{\mathcal{X}}$$

of orbits $\widetilde{\mu} = \{\mu \star \delta_x \colon x \in \mathbb{R}^3\}$ of probability measures μ on $\mathbb{R}^d$ under translations and a full LDP for the distributions of $\widetilde{L}_t \in \widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ embedded in the compactification. In particular, this is based on a topology induced by a different metric in the compactfication $\widetilde{\mathcal{X}}$, see Section 1.4 for details and its consequences in the present context.

1.3 Our results: uniform tube property and regularity of $\Lambda(L_t)$

Let us turn to our main results. Roughly speaking, we will show that $\Lambda(L_t)$ converges to $\Lambda(\mathfrak{m})$ in the uniform metric under $\widehat{\mathbb{P}}_t$ and that $\Lambda(L_t)$ possesses a certain uniform Hölder continuity property with exponential error bounds. These results will make determinant contribution to the the full identification of the limiting distribution of L_t under $\widehat{\mathbb{P}}_t$. We refer the reader to Section 1.5 for a heuristic explanation as to why the present results are crucial in this respect and turn to the statements of our main results.

Let us write $\Lambda(\psi^2)(x) = \int \mathrm{d}y \, \frac{\psi^2(y)}{|x-y|}$ for functions ψ^2, and recall that $\psi_w^2 = \psi_0^2 \star \delta_w$ denotes the shift of the maximizer ψ_0^2 of the second variational formula (1.6) by $w \in \mathbb{R}^3$. Here is the statement of our first main result.

Theorem 1.1. *For any $\varepsilon > 0$,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \inf_{w \in \mathbb{R}^3} \left\| \Lambda_t - \Lambda \psi_w^2 \right\|_\infty > \varepsilon \right\} < 0. \tag{1.12}$$

This is a tube property for Λ_t in the uniform metric, since the ε-neighbourhood of $\Lambda(\mathfrak{m}) = \{\Lambda(\psi_w^2) \colon w \in \mathbb{R}^3\}$ can be visualized as a tube around the 'line' $\mathfrak{m}$. The proof of Theorem 1.1 is given in Section 3.

As a consequence of Theorem 1.1, the Hamiltonian $H(L_t) = \langle L_t, \Lambda L_t \rangle$ converges in distribution towards the common Coulomb energy of any member of $\mathfrak{m}$ and we state this fact as

Corollary 1.2. *Under $\widehat{\mathbb{P}}_t$, the distributions of $H(L_t)$ converge weakly to the Dirac measure at*

$$H(\psi_0^2) = \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\psi_0^2(x)\psi_0^2(y)}{|x-y|} \, \mathrm{d}x\mathrm{d}y.$$

Let us highlight the core of the proof of Theorem 1.1. An important technical hindrance in the proof of Theorem 1.1 stems from the singularity of the Coulomb potential $x \mapsto 1/|x|$, which does not fit within the set up of standard large deviation theory. This problem was encountered also in [MV14] for deriving (1.11). As it concerns L_t, this turned out to be a mild technical issue. Indeed, a simple truncation argument with replacing $1/|x|$ by its regularized version $1/\sqrt{|x|^2 + \delta^2}$ sufficed to carry over the theory developed in [MV14] to this singular potential. However, as it now concerns $\Lambda(L_t)$ in the uniform metric, the singularity of $1/|\cdot|$ turns out be a more serious problem, since a standard contraction principle combined with the truncation argument does not work well here. We need a strategy that extracts some approximate cancellation in the difference of terms $\Lambda_t(x_1)$ and $\Lambda_t(x_2)$ when two Coulomb singularities x_1 and x_2 come close to each other, and we need this cancellation 'exponentially fast' in t. In other words, we need some 'exponential regularity' of the random map $x \mapsto \Lambda_t(x)$ and inspired by an earlier work of Donsker and Varadhan for one dimensional Brownian local times [DV77], we are led to proving the following super exponential estimate and this is our second main result.

Theorem 1.3. *For every $b > 0$,*

$$\lim_{\delta \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left\{ \sup_{x_1, x_2 \in \mathbb{R}^3 \,:\, |x_1 - x_2| \leq \delta} \left| \Lambda_t(x_1) - \Lambda_t(x_2) \right| \geq b \right\} = -\infty. \tag{1.13}$$

In Section 2, we prove Theorem 1.3. It is a crucial step in the proof of Theorem 1.1, but also rather interesting on its own sake. Note that Theorem 1.3 implies in particular some Hölder continuity property of Λ_t, which we do not explore further here. In fact, in the course of the proof of Theorem 1.3, we identify a polynomial gauge function for the supremum in (1.13).

Let us state the following useful corollary to Theorem 1.3, which is also of independent interest.

Corollary 1.4. *For any $b > 0$,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left\{ \|\Lambda_t\|_\infty > b \right\} < 0.$$

The proof of this corollary is deferred to Section 3.

1.4 Review: compactness and large deviations for Λ_t

Let us turn to the second main ingredient in the proof of Theorem 1.1 besides Theorem 1.3. This was derived in [MV14], and we provide and explain it here for future reference.

Let $\widetilde{\mathcal{M}}_1(\mathbb{R}^d) = \{\widetilde{\mu} \colon \mu \in \mathcal{M}_1(\mathbb{R}^d)\}$ be the quotient space of orbits $\widetilde{\mu} = \{\mu \star \delta_x \colon x \in \mathbb{R}^d\}$ of $\mathcal{M}_1(\mathbb{R}^d)$ under translations. When endowed with the weak topology, $\mathcal{M}_1(\mathbb{R}^d)$ as well as $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ fail to be compact. Let

$$\widetilde{\mathcal{X}} = \left\{ \xi = (\widetilde{\alpha}_j)_{j \in J} \colon J \text{ at most countable}, \alpha_j \in \mathcal{M}_{\leq 1}(\mathbb{R}^d) \, \forall j \in J \right\}$$

be the space of collections of orbits of sub-probability measures. Then we have a natural embedding

$$\widetilde{\mathcal{M}}_1(\mathbb{R}^d) \hookrightarrow \widetilde{\mathcal{X}}.$$

There is a metric $\mathbf{D}$ on $\widetilde{\mathcal{X}}$ so that $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ is dense in $(\widetilde{\mathcal{X}}, \mathbf{D})$ and any sequence $(\widetilde{\mu}_n)_n$ in $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ finds a subsequence which converges in the metric $\mathbf{D}$ to some element $\xi \in \widetilde{\mathcal{X}}$. In other words, $\widetilde{\mathcal{X}}$ is the compactification of $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ and also the completion under the metric $\mathbf{D}$ of the totally bounded space $\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$. Furthermore, the distribution of the orbits $\widetilde{L}_t$ embedded in $\widetilde{\mathcal{X}}$ satisfy a strong LDP in

the compact metric space $\widetilde{\mathcal{X}}$ with the rate function

$$\widetilde{J}(\xi) = \sum_{j \in J} I(\widetilde{\alpha}_j) = \sum_{j \in J} I(\alpha_j), \qquad \xi = (\widetilde{\alpha}_j)_{j \in J} \in \widetilde{\mathcal{X}},$$

where we recall that $I(\cdot)$ is defined in (1.7) and is shift-invariant and for any $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$, $I(\alpha)$ is a function only of the orbit $\widetilde{\alpha}$, which we call $I(\widetilde{\alpha})$. Furthermore, by Varadhan's lemma, for any continuous functional $\widetilde{\mathcal{H}} \colon \widetilde{\mathcal{X}} \to \mathbb{R}$,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left\{ e^{t \widetilde{\mathcal{H}}(\widetilde{L}_t)} \right\} = \sup_{\xi \in \widetilde{\mathcal{X}}} \left\{ \widetilde{\mathcal{H}}(\xi) - \widetilde{J}(\xi) \right\}.$$

For applications, the underlying novelty of the above theory lies in the wide range of choice for functionals $\widetilde{\mathcal{H}}$ which are inherently shift-invariant. For deriving (1.11), we plugged in (ignoring the singularity which was tamed down by the aforementioned truncation argument)

$$H(L_t) = H(L_t \star \delta_x) = \widetilde{\mathcal{H}}(\widetilde{L}_t).$$

In the present context of the functional $\|\Lambda_t\|_\infty$, we take note of the following simple fact: For any $x \in \mathbb{R}^3$,

$$\|\Lambda_t\|_\infty = \sup_{y \in \mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{L_t(\mathrm{d}z)}{|z-y|} \right) = \sup_{y \in \mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{(L_t \star \delta_x)(\mathrm{d}z)}{|z-y|} \right) = \|\Lambda_t \star \delta_x\|_\infty.$$

This simple shift-invariance of the norm allows a free passage to the orbits $\widetilde{\Lambda}_t = \{\Lambda_t \star \delta_x \colon x \in \mathbb{R}^3\}$ enabling us to invoke the strong large deviation principle for $\widetilde{L}_t$ in $\widetilde{\mathcal{X}}$ and forms the second main ingredient for the proof of Theorem 1.1, see the proof of (3.9) in Section 3.

1.5 Outlook: convergence of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$

We explain here how the novel result of the present paper, Theorem 1.1, will be instrumental in proving convergence of the distribution of L_t under $\widehat{\mathbb{P}}_t$. The statement in (1.11) implies that with high $\widehat{\mathbb{P}}_t$-probability, L_t stays in a neighborhood of $\mathfrak{m}$. This suggests that under $\widehat{\mathbb{P}}_t$, L_t should look like some shift μ_{X_t} of the maximizer μ_0 of the variational formula in (1.6), where X_t is some random t-dependent location in $\mathbb{R}^3$. The next decisive step must be to prove that X_t is tight as $t \to \infty$. This implies tightness of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$ as $t \to \infty$. The last task to be done is then to identify a limiting distribution of X_t and therefore establish convergence of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$, but we will not elaborate on this here.

For showing tightness of X_t, we need to justify that L_t can not build up its mass over a long time around some maximizer $\mu_x \in \mathfrak{m}$ if x is far away from the origin. To show this, we emulate a strategy similar to [BS97] and show that the ratio

$$\frac{\widehat{\mathbb{P}}_t(L_t \approx \mu_x)}{\widehat{\mathbb{P}}_t(L_t \approx \mu_0)} = \frac{\mathbb{E}\left[\exp\{tH(L_t)\} \mathbb{1}_{\{L_t \approx \mu_x\}} \right]}{\mathbb{E}\left[\exp\{tH(L_t)\} \mathbb{1}_{\{L_t \approx \mu_0\}} \right]} \tag{1.14}$$

is small for large t if $|x|$ gets large. It is reasonable to argue that if $|x|$ is large, for $\{L_t \approx \mu_x\}$ to happen, the path starting from the origin has to reach a neighborhood of x relatively quickly, say after time $t_0 \ll t$ and concentrate in that neighborhood for the remaining time $t - t_0$. Then we can split the occupation measure

$$L_t = \frac{t_0}{t} L_{t_0} + \frac{t - t_0}{t} L_{t_0, t}$$

where $L_{t_0, t}$ denotes the normalized occupation measure of the path from t_0 to t. Then, with high probability, on $\{L_t \approx \mu_x\}$, we expect that $L_{t_0, t} \approx \mu_x$, and L_{t_0} is the normalized occupation times of a path that runs relatively quickly from the origin to x.

Using that the Hamiltonian is quadratic, we accordingly obtain

$$tH(L_t) = \frac{t_0^2}{t} H(L_{t_0}) + \frac{(t-t_0)^2}{t} H(L_{t_0,t}) + 2\frac{t_0(t-t_0)}{t}\langle L_{t_0}, \Lambda_{t_0,t}\rangle, \tag{1.15}$$

where $\Lambda_{t_0,t}(x) = \int_{\mathbb{R}^3} \frac{L_{t_0,t}(\mathrm{d}y)}{|x-y|}$ is the Coulomb functional of $L_{t_0,t}$. Since $t_0 \ll t$, the first term on the right-hand side can be shown to be negligible. In the second term, we note that the main bulk of the path $L_{t_0,t}$ stays close to x. Hence to compare this part to a path which typically stays close to 0 after time t_0 instead of x, we can just shift the path close to x by $-x$. Note that this shifting does not cost anything to the bulk of the path since $H(L_{t_0,t})$ is shift-invariant. Finally, the third term in (1.15) makes the difference between numerator (large $|x|$) and denominator ($x = 0$) on the event $\{L_t \approx \mu_x\}$. Indeed, thanks to Theorem 1.1, we will see that $\Lambda_{t_0,t} \approx \Lambda(\mu_x)$, which is concentrated around x and therefore will have only vanishing interaction with L_{t_0} in the numerator, as $|x|$ is large. On the contrary, for $x = 0$, we expect that, for large t_0, we will have $L_{t_0} \approx \mu_0$ and therefore, $\Lambda_{t_0,t}$ (which is also $\approx \mu_0$) will have a non-trivial interaction with L_{t_0}.

Summarizing, the ratio (1.14) gets small for large t, as $|x|$ gets large and implies that under $\widehat{\mathbb{P}}_t$, L_t must have its main weight, up to a small factor, close to its starting point. Justifying the above heuristics goes beyond the scope of the present article and will appear in a forthcoming paper ([BKM15]).

2. Super-exponential estimate: Proof of Theorem 1.3

We will prove Theorem 1.3 in a succession of five lemmas. For any $x \in \mathbb{R}^3$ we will denote by $\mathbb{P}_x$ the Wiener measure for the Brownian motion $W = (W_t)_{t\geq 0}$ starting at x and by $\mathbb{E}_x$ the corresponding expectation and we continue to write $\mathbb{P}_0 = \mathbb{P}$ and $\mathbb{E}_0 = \mathbb{E}$.

For any $x_1, x_2 \in \mathbb{R}^3$ satisfying $|x_1 - x_2| \leq 1$ and for $y \in \mathbb{R}^3$, let us define the function

$$V(y) = V_{x_1,x_2}(y) = \left(\frac{1}{|y-x_1|} - \frac{1}{|y-x_2|}\right) \mathbb{1}_{\left\{\left|y - \frac{x_1+x_2}{2}\right|\leq h\right\}}, \qquad \text{where } h = |x_1 - x_2|^{1/2}. \tag{2.1}$$

We will later approximate $\Lambda_1(x_1) - \Lambda_1(x_2)$ by $\int_0^1 V(W_s)\,\mathrm{d}s$. Therefore, we need to control exponential moments of $\int_0^1 V(W_s)\,\mathrm{d}s$. We begin with controlling high polynomial moments.

Lemma 2.1. *For every $\varepsilon \in (0, \frac{1}{2})$ there exists a constant $C = C_\varepsilon > 0$ such that, for any $k \in \mathbb{N}$ and any $x_1, x_2 \in \mathbb{R}^3$ satisfying $|x_1 - x_2| \leq 1$,*

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3:\\|x_1-x_2|\leq 1}} \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left\{\left(\int_0^1 V(W_s)\,\mathrm{d}s\right)^{2k}\right\} \leq C^k\, h^{4k(1-2\varepsilon)} \frac{(2k)!}{\Gamma(1+2k\varepsilon)}. \tag{2.2}$$

where $\Gamma(t) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x} x^{t-1}$ denotes the usual Gamma function.

Lemma 2.1 has an important consequence in the present context which we formulate as

Lemma 2.2. *For any $\varepsilon \in (0, \frac{1}{2})$, abbreviate $a = 1 - 2\varepsilon$ and $\rho = \frac{1}{1-\varepsilon} > 1$. Then, for some $\beta_1 > 0$,*

$$K_1 = \sup_{x_1,x_2\in\mathbb{R}^3:\,|x_1-x_2|\leq 1} \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left\{\mathrm{e}^{\beta_1\left|h^{-2a}\int_0^1 V(W_s)\,\mathrm{d}s\right|^\rho}\right\} < \infty.$$

Proof. We can estimate the left hand side as

$$\mathbb{E}_x\left\{e^{\beta_1\left|h^{-2a}\int_0^1 V(W_s)\,ds\right|^\rho}\right\} = \sum_{k=0}^\infty \frac{\beta_1^k}{k!} h^{-2ak\rho}\,\mathbb{E}_x\left\{\left|\int_0^1 V(W_s)\,ds\right|^{\rho k}\right\}$$

$$\leq \sum_{k=0}^\infty \frac{\beta_1^k}{k!} h^{-2ak\rho}\,\mathbb{E}_x\left\{\left(\int_0^1 V(W_s)\,ds\right)^{2\lceil\rho k/2\rceil}\right\}^{\rho k/2\lceil\rho k/2\rceil}$$

$$\leq \sum_{k=0}^\infty \frac{\beta_1^k}{k!} h^{-2ak\rho}\,C^k h^{4a\rho k}\left(\frac{(2\lceil\rho k/2\rceil)!}{\Gamma(1+2\lceil\rho k/2\rceil\varepsilon)}\right)^{\rho k/2\lceil\rho k/2\rceil},$$

where we used Jensens inequality in the first estimate and (2.2) in the second (with a possibly different value of C). Since, by Stirlings formula, up to a factor of $e^{O(k)}$,

$$\frac{1}{k!}\left(\frac{(2\lceil\rho k/2\rceil)!}{\Gamma(1+2\lceil\rho k/2\rceil\varepsilon)}\right)^{\rho k/2\lceil\rho k/2\rceil} \approx \frac{1}{k!}\left(\frac{(2\lceil\rho k/2\rceil)!}{(2\varepsilon\lceil\rho k/2\rceil)!}\right)^{\rho k/2\lceil\rho k/2\rceil} \approx \frac{k^{k\rho}}{k^k\,k^{k\varepsilon\rho}} \approx k^{k(-1+\rho-\rho\varepsilon)},$$

and $-1+\rho-\rho\varepsilon = 0$, we can choose $\beta_1 > 0$ small enough to make the above sum convergent. □

We turn to the proof of Lemma 2.1.

Proof of Lemma 2.1. We compute the moments in the left hand side of (2.2) as

$$\mathbb{E}_x\left\{\left(\int_0^1 ds\,V(W_s)\,ds\right)^{2k}\right\}$$

$$= (2k)!\int_{(\mathbb{R}^3)^{2k}}\prod_{j=1}^{2k}\left(dy_j V(y_j)\right)\int_{0\leq s_1<\cdots<s_{2k}\leq 1}ds_1\ldots ds_{2k}\prod_{j=1}^{2k}p_{s_j-s_{j-1}}(y_{j-1},y_j),$$

(2.3)

where $p_t(x,y)$ is the standard Gaussian kernel with variance t, and we put $y_0 = x$. We estimate this kernel as follows: For any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that, for any $t > 0$ and any $y \in \mathbb{R}^3$,

$$p_t(0,y) = \frac{1}{(2\pi)^{3/2}}\frac{1}{t^{1-\varepsilon}}\left\{\frac{|y|^{1+2\varepsilon}}{t^{\frac{1+2\varepsilon}{2}}}\exp\left(-\frac{|y|^2}{2t}\right)\right\}\frac{1}{|y|^{1+2\varepsilon}} \leq C_\varepsilon\frac{1}{t^{1-\varepsilon}}\frac{1}{|y|^{1+2\varepsilon}},$$

since the map $(0,\infty) \ni x \mapsto x^{\frac{1}{2}+\varepsilon}e^{-x}$ is bounded. We use this simple bound to estimate the right hand side of (2.3) and conclude

$$\mathbb{E}_x\left\{\left(\int_0^1 ds\,V(W_s)\,ds\right)^{2k}\right\} \leq (2k)!C_\varepsilon^{2k}\int_{0\leq s_1<\cdots<s_{2k}\leq 1}\prod_{j=1}^{2k}\left(ds_j\frac{1}{(s_j-s_{j-1})^{1-\varepsilon}}\right)$$

$$\times \int_{(\mathbb{R}^3)^{2k}}\prod_{j=1}^{2k}\left(dy_j\frac{|V(y_j)|}{|y_j-y_{j-1}|^{1+2\varepsilon}}\right).$$

A simple computation using iterated Euler beta integral and its identification in terms of the Gamma function shows that

$$\int_{0\leq s_1<\cdots<s_{2k}\leq 1}\prod_{j=1}^{2k}\left(ds_j\frac{1}{(s_j-s_{j-1})^{1-\varepsilon}}\right) = \prod_{j=1}^{2k}\frac{\Gamma(\varepsilon)\Gamma(1+(j-1)\varepsilon)}{\Gamma(1+j\varepsilon)} = \frac{\Gamma(\varepsilon)^{2k}}{\Gamma(1+2k\varepsilon)}.$$

Let us abbreviate $I(z) = \int dy\,|V(y)|\,|y-z|^{-1-2\varepsilon}$, then Lemma 2.1 will follow from the estimate $I(z) \leq \widetilde{C}h^{2-4\varepsilon}$ for any $z \in \mathbb{R}^3$ and any $x_1, x_2 \in \mathbb{R}^3$ satisfying $h = \sqrt{|x_1-x_2|} \leq 1$, for some suitable

constant $\widetilde{C}$. Let us prove that. We recall the definition of $V = V_{x_1, x_2}$ from (2.1). Then the reversed triangle inequality implies that

$$|V(y)| = |V_{x_1, x_2}(y)| \le h^2 \frac{1}{|y - x_1|} \frac{1}{|y - x_2|} \mathbb{1}_{\left\{ \left| y - \frac{x_1 + x_2}{2} \right| \le h \right\}}.$$

Hence

$$I(z) \le h^2 \int_{B_h\left(\frac{x_1 + x_2}{2}\right)} dy \, \frac{1}{|y - x_1|} \frac{1}{|y - x_2|} \frac{1}{|y - z|^{1+2\varepsilon}}. \tag{2.4}$$

Depending on the location of the point z in the ball $B_h(\frac{x_1 + x_2}{2})$, we distinguish two cases.

Case 1: Suppose $z \in B_{h^2/4}(x_1)$ or $z \in B_{h^2/4}(x_2)$. By symmetry, it is sufficient to handle only $z \in B_{h^2/4}(x_1)$. We decompose

$$B_h\left(\frac{x_1 + x_2}{2}\right) = B_{h^2/2}(x_1) \cup B_{h^2/2}(x_2) \cup \left(B_h\left(\frac{x_1 + x_2}{2}\right) \setminus \left(B_{h^2/2}(x_1) \cup B_{h^2/2}(x_2) \right) \right). \tag{2.5}$$

For $y \in B_{h^2/2}(x_1)$, we can estimate $|y - x_2| \ge |x_1 - x_2| - h^2/2 = h^2/2$ and therefore

$$\int_{B_{h^2/2}(x_1)} dy \, \frac{1}{|y - x_1|} \frac{1}{|y - x_2|} \frac{1}{|y - z|^{1+2\varepsilon}}$$

$$\le 2h^{-2} \int_{B_{h^2/2}(x_1)} dy \, \frac{1}{|y - x_1|} \frac{1}{|y - z|^{1+2\varepsilon}}$$

$$\le 2h^{-2} \left(\int_{B_{h^2/2}(x_1)} dy \, \frac{1}{|y - x_1|^p} \right)^{1/p} \left(\int_{B_{h^2/2}(x_1)} dy \, \frac{1}{|y - z|^{q(1+2\varepsilon)}} \right)^{1/q},$$

by Hölder's inequality for any $p, q \ge 1$ and $1/p + 1/q = 1$. Since $\varepsilon < \frac{1}{2}$, it is possible to choose p and q in such a way that $p < 3$ and $q < 3/(1+2\varepsilon)$. Then both integrals on the right-hand side are finite and can be easily and explicitly calculated using the shift and rotation invariance of the integrand. This gives, for some constant $C = C(p, q, \varepsilon)$,

$$\int_{B_{h^2/2}(x_1)} dy \, \frac{1}{|y - x_1|} \frac{1}{|y - x_2|} \frac{1}{|y - z|^{1+2\varepsilon}} \le Ch^{-2} \times h^{\frac{2(3-p)}{p}} \times h^{\frac{2(3-q(1+2\varepsilon))}{q}} = Ch^{-4\varepsilon}. \tag{2.6}$$

Similarly,

$$\int_{B_{h^2/2}(x_2)} dy \, \frac{1}{|y - x_1|} \frac{1}{|y - x_2|} \frac{1}{|y - z|^{1+2\varepsilon}} \le Ch^{-4\varepsilon}. \tag{2.7}$$

Also, the remaining integral is estimated by Hölder's inequality as

$$\int_{B_h\left(\frac{x_1 + x_2}{2}\right) \setminus \left(B_{h^2/2}(x_1) \cup B_{h^2/2}(x_2) \right)} dy \, \frac{1}{|y - x_1|} \frac{1}{|y - x_2|} \frac{1}{|y - z|^{1+2\varepsilon}}$$

$$\le \left\{ \int_{B_h\left(\frac{x_1 + x_2}{2}\right) \setminus B_{h^2/2}(x_1)} dy \, \frac{1}{|y - x_1|^{2+\delta}} \right\}^{1/(2+\delta)} \times \left\{ \int_{B_h\left(\frac{x_1 + x_2}{2}\right) \setminus B_{h^2/2}(x_2)} dy \, \frac{1}{|y - x_2|^{2+\delta}} \right\}^{1/(2+\delta)}$$

$$\times \left\{ \int_{B_h\left(\frac{x_1 + x_2}{2}\right) \setminus B_{h^2/4}(z)} dy \, \frac{1}{|y - z|^{(1+2\varepsilon)\left(\frac{2+\delta}{\delta}\right)}} \right\}^{\delta/(2+\delta)},$$

for any $\delta > 0$, where we recall that $z \in B_{h^2/4}(x_1)$. If we choose $\delta \in (1, \frac{1+2\varepsilon}{1-\varepsilon})$, then a similar computation as above shows that, for some $C = C(\varepsilon, \delta)$,

$$\int_{B_h\left(\frac{x_1+x_2}{2}\right)\setminus\left(B_{h^2/2}(x_1)\cup B_{h^2/2}(x_2)\right)} \mathrm{d}y \, \frac{1}{|y-x_1|} \frac{1}{|y-x_2|} \frac{1}{|y-z|^{1+2\varepsilon}} \le Ch^{-4\varepsilon}. \tag{2.8}$$

We combine (2.6), (2.7) and (2.8) with the decomposition (2.5) and (2.4) to conclude that or goal $I(z) \le Ch^{2-4\varepsilon}$ holds for any $z \in B_{h^2/4}(x_1)$, by symmetry also for any $z \in B_{h^2/4}(x_2)$.

Case 2: Suppose $z \in B_h(\frac{x_1+x_2}{2}) \setminus \left(B_{h^2/4}(x_1) \cup B_{h^2/4}(x_2)\right)$.

In this case, we decompose the integration area into

$$B_h\left(\frac{x_1+x_2}{2}\right) = B_{h^2/8}(x_1) \cup B_{h^2/8}(x_2) \cup \left(B_h\left(\frac{x_1+x_2}{2}\right) \setminus \left(B_{h^2/8}(x_1) \cup B_{h^2/8}(x_2)\right)\right)$$

and repeat the estimates of Case 1 to conclude that $I(z) \le Ch^{2-4\varepsilon}$ holds true in this case too. To avoid repetitions we drop the details. This proves Lemma 2.1. $\qquad\square$

Lemma 2.3. *Let $\varepsilon \in [1 - 1/\sqrt{2}, \frac{1}{2})$, and abbreviate $a = 1 - 2\varepsilon$ and $\rho = \frac{1}{1-\varepsilon}$ as in Lemma 2.2. Then there is a constant $\beta_2 = \beta_2(\varepsilon) > 0$ such that*

$$K_2 = \sup_{\substack{x_1,x_2\in\mathbb{R}^3 \\ |x_1-x_2|\le 1}} \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left[\exp\left\{\beta_2\left(\frac{|\Lambda_1(x_1)-\Lambda_1(x_2)|}{|x_1-x_2|^a}\right)^\rho\right\}\right] < \infty.$$

Proof. We approach the difference quotient of Λ_1 with $h^{-2a}\int_0^1 V(W_s)\,\mathrm{d}s$ with V as in (2.1) and decompose the expectation in the left hand side as

$$\mathbb{E}_x\left[\exp\left\{\beta_2\left(\frac{|\Lambda_1(x_1)-\Lambda_1(x_2)|}{|x_1-x_2|^a}\right)^\rho\right\}\right]$$
$$\le \mathbb{E}_x\left[\exp\left\{\beta_2 2^\rho\left\{\left|\frac{1}{h^{2a}}\int_0^1 V(W_s)\,\mathrm{d}s\right|^\rho + \left|\frac{\Lambda_1(x_1)-\Lambda_1(x_2)}{|x_1-x_2|^a} - \frac{\int_0^1 V(W_s)\mathrm{d}s}{h^{2a}}\right|^\rho\right\}\right\}\right]. \tag{2.9}$$

We claim that for some $\beta_3 > 0$,

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^3: \\ |x_1-x_2|\le 1}} \sup_{x\in\mathbb{R}^3} \mathbb{E}_x\left[\exp\left\{\beta_3\left|\frac{\Lambda_1(x_1)-\Lambda_1(x_2)}{|x_1-x_2|^a} - \frac{\int_0^1 V(W_s)\mathrm{d}s}{h^{2a}}\right|^\rho\right\}\right] < \infty. \tag{2.10}$$

Then Hölder's inequality, applied to the right-hand side of (2.9), together with Lemma 2.2 imply Lemma 2.3. We turn to the proof of (2.10). Recall that $B_h = B_h\left(\frac{x_1+x_2}{2}\right)$ denotes the ball of radius $h = |x_1-x_2|^{1/2}$ around $(x_1+x_2)/2$. Note that

$$\left|\frac{\Lambda_1(x_1)-\Lambda_1(x_2)}{|x_1-x_2|^a} - \frac{\int_0^1 V(W_s)\,\mathrm{d}s}{h^{2a}}\right|^\rho \le h^{-2a\rho}\left|\int_0^1 \mathrm{d}s\left|\frac{1}{|W_s-x_1|} - \frac{1}{|W_s-x_2|}\right|\mathbb{1}_{\left\{|W_s-\frac{x_1+x_2}{2}|>h\right\}}\right|^\rho$$
$$\le h^{-2a\rho}\int_{B_h^c} L_1(\mathrm{d}z)\left|f\left(z-\frac{x_1-x_2}{2}\right) - f\left(z-\frac{x_1+x_2}{2}\right)\right|^\rho, \tag{2.11}$$

where the second inequality above follows from Jensen's inequality applied to the probability measure $L_1(\mathrm{d}z) = \int_0^1 \mathrm{d}s\,\delta_{W_s}(\mathrm{d}z)$, and we wrote $f(z) = |z - \frac{x_1+x_2}{2}|^{-1}$.

We choose a ball $B_r = B_r(\frac{x_1+x_2}{2})$ of radius $r = |x_1-x_2|^\eta$ with some $\eta \in (0, 1/2)$. Then $B_r^c \subset B_h^c$ and we decompose the integral on the right-hand side of (2.11) into the integral over B_r^c and $B_h^c \setminus B_r^c$.

Note that $|\nabla f(z)| = |z - \frac{x_1 + x_2}{2}|^{-2}$. By the mean value theorem,

$$\left| f\left(z - \frac{x_1 - x_2}{2}\right) - f\left(z - \frac{x_1 + x_2}{2}\right) \right| \leq |x_1 - x_2| \, |\nabla f(\xi)| = |x_1 - x_2| \, \left| \xi - \frac{x_1 + x_2}{2} \right|^{-2},$$

for some $\xi = \xi_z \in \mathbb{R}^3$ lying on the straight line between $x - \frac{x_1 - x_2}{2}$ and $x + \frac{x_1 + x_2}{2}$, i.e.,

$$\xi_z = \lambda\left(z - \frac{x_1 - x_2}{2}\right) + (1 - \lambda)\left(z - \frac{x_1 + x_2}{2}\right) \qquad \text{for some } \lambda \in [0, 1]. \tag{2.12}$$

Let us now estimate the integral in the right hand side of (2.11) over B_r^c. A simple geometric argument shows that

$$\inf_{z \in B_r^c} \left| \xi_z - \frac{x_1 + x_2}{2} \right| \geq \inf_{z : \, |z - \frac{x_1 + x_2}{2}| > r} |z - x_2| \geq r - \frac{|x_1 - x_2|}{2}.$$

Since $r = |x_1 - x_2|^\eta \geq |x_1 - x_2|$ for $|x_1 - x_2| \leq 1$, the integral over B_r in the right hand side of (2.11) can be estimated as

$$\int_{B_r^c} L_1(\mathrm{d}z) \left| f\left(z - \frac{x_1 - x_2}{2}\right) - f\left(z - \frac{x_1 + x_2}{2}\right) \right|^\rho \leq \left[|x_1 - x_2| \left(r - \frac{|x_1 - x_2|}{2} \right)^{-2} \right]^\rho L_1(B_r^c)$$
$$\leq C |x_1 - x_2|^{\rho(1 - 2\eta)}, \tag{2.13}$$

for some $C > 0$, not depending on $|x_1 - x_2|$. Similarly, the integral over $B_h^c \setminus B_r^c = B_r \setminus B_h$ on the right-hand side of (2.11) can be estimated as (recall that $h = |x_1 - x_2|^{1/2} \geq |x_1 - x_2|$),

$$\int_{B_r \setminus B_h} L_1(\mathrm{d}z) \left| f\left(z - \frac{x_1 - x_2}{2}\right) - f\left(z - \frac{x_1 + x_2}{2}\right) \right|^\rho \leq \left[|x_1 - x_2| \left(h - \frac{|x_1 - x_2|}{2} \right)^{-2} \right]^\rho L_1(B_r \setminus B_h)$$
$$\leq C L_1(B_r), \tag{2.14}$$

for some constant $C > 0$, not depending on h. The last two estimates, combined with (2.11), imply that, for any x, x_1, x_2 such that $|x_1 - x_2| \leq 1$,

$$\mathbb{E}_x \left[\exp\left\{ \beta_3 \left| \frac{\Lambda_1(x_1) - \Lambda_1(x_2)}{|x_1 - x_2|^a} - \frac{\int_0^1 V(W_s) \, \mathrm{d}s}{h^{2a}} \right| \right\} \right] \leq \mathrm{e}^{\beta_3 |x_1 - x_2|^{\rho(1 - a - 2\eta)}} \, \mathbb{E}_x \left\{ \mathrm{e}^{C\beta_3 |x_1 - x_2|^{-a\rho} L_1(B_r)} \right\}. \tag{2.15}$$

Let us handle the expectation in the right-hand side above as follows. Note that it is maximal for x equal to the centre point, $(x_1 + x_2)/2$, of B_r. Hence, it suffices to handle only the case $x = x_1 + x_2 = 0$. Here we have, for some constants c, c_2 that depend only on the dimension,

$$\mathbb{E}\left(L_1(B_r)\right) = \int_0^1 \mathbb{P}(W_s \in B_r) \, \mathrm{d}s \leq c \int_{B_r} \frac{\mathrm{d}z}{|z|} = c_2 r^2 = c_2 |x_1 - x_2|^{2\eta}.$$

We now assume that $2\eta \geq a\rho$. Then, for $\beta_3 > 0$ small enough and all $x_1, x_2 \in \mathbb{R}^3$ such that $|x_1 - x_2| \leq 1$,

$$\mathbb{E}\left\{ C\beta_3 |x_1 - x_2|^{-a\rho} L_1(B_r) \right\} \leq c_2 C \beta_3 |x_1 - x_2|^{-a\rho + 2\eta} \leq \frac{1}{2}.$$

Hence by Portenko's lemma (see [P76]),

$$\mathbb{E}\left\{ \mathrm{e}^{C\beta_3 |x_1 - x_2|^{-a\rho} L_1(B_r)} \right\} \leq \frac{1}{1 - c_2 C \beta_3 |x_1 - x_2|^{-a\rho + 2\eta}} \leq 2. \tag{2.16}$$

Then (2.15) and (2.16) imply that, for the choice

$$\eta \geq \frac{a\rho}{2} \qquad \text{and} \qquad \rho(1 - a - 2\eta) \geq 0$$

i.e, for $\eta \in [\frac{1-2\varepsilon}{2(1-\varepsilon)}, \varepsilon]$, (2.10) holds. This interval is non-empty if the condition $1 - 2\varepsilon \leq 2\varepsilon(1 - 2\varepsilon)$ is satisfied, which is equivalent to $2\varepsilon^2 - 4\varepsilon + 1 \leq 0$, i.e., to

$$\varepsilon \in \left[1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right].$$

Recalling that $\varepsilon < 1/2$, this is the requirement of Lemma 2.3, which is proved now. □

Lemma 2.4. *Fix* $\varepsilon \in (\frac{1}{3}, \frac{1}{2})$ *and let* $a = 1 - 2\varepsilon$ *and* $\rho = \frac{1}{1-\varepsilon}$ *as in Lemma 2.3. Then there exists a constant* $\beta_4 = \beta_4(\varepsilon) > 0$ *such that the random variable*

$$M = \int_{\mathbb{R}^3} \mathrm{d}x_1 \int_{\mathbb{R}^3} \mathrm{d}x_2 \, \mathbb{1}\{|x_1 - x_2| \leq 1\} \left[\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1 \right] \qquad (2.17)$$

has a finite expectation under $\mathbb{P}_0$.

Proof. We note that the interval $(\frac{1}{3}, \frac{1}{2})$ is contained in $[1 - 1/\sqrt{2}, \frac{1}{2}]$ and hence we can apply Lemma 2.3. By Fubini's theorem, it suffices to show that

$$\int\int_{|x_1 - x_2| \leq 1} \mathrm{d}x_1 \mathrm{d}x_2 \, \mathbb{E}\left[\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1 \right] < \infty. \qquad (2.18)$$

We decompose $\mathbb{R}^3 \subset \bigcup_{n=0}^\infty \{x \in \mathbb{R}^3 \colon n \leq |x| < n+1\}$ and put $\tau_n = \inf\{t > 0 \colon |W_t| > n - n^\alpha\}$ for some $\alpha \in (0, 1)$. Then

$$\int\int_{|x_1 - x_2| \leq 1} \mathrm{d}x_1 \mathrm{d}x_2 \, \mathbb{E}\left[\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1 \right]$$

$$\leq \sum_{n=0}^\infty \int_{|x_1| \in [n, n+1)} \mathrm{d}x_1 \int_{B_1(x_1)} \mathrm{d}x_2 \left[\mathbb{E}\left\{\mathbb{1}_{\{\tau_n > 1\}}\left(\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right)\right\} \qquad (2.19)$$

$$+ \mathbb{E}\left\{\mathbb{1}_{\{\tau_n \leq 1\}}\left(\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right)\right\}\right].$$

The first expectation inside the integrals is handled as follows. We note that, with $|x_1| \in [n, n+1)$ and $x_2 \in B_1(x_1)$, if $\tau_n > 1$, then $|W_s - x_1| > n^\alpha$ and $|W_s - x_2| > n^\alpha - 1$ for any $s \in [0, 1]$. Hence, for any $n \in \mathbb{N}$, on the event $\{\tau_n > 1\}$,

$$\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a} \leq \frac{|x_1 - x_2|}{|x_1 - x_2|^{1-2\varepsilon}} \int_0^1 \frac{\mathrm{d}s}{|W_s - x_1||W_s - x_2|} \leq c_1|x_1 - x_2|^{2\varepsilon} n^{-2\alpha} \leq c_1 n^{-2\alpha}.$$

Hence,

$$\sum_{n=0}^\infty \int_{|x_1| \in [n, n+1)} \mathrm{d}x_1 \int_{B_1(x_1)} \mathrm{d}x_2 \, \mathbb{E}\left\{\mathbb{1}_{\{\tau_n > 1\}}\left(\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right)\right\}$$

$$\leq \sum_{n=0}^\infty \left(e^{\beta_4 c_1^\rho n^{-2\alpha\rho}} - 1\right) \mathrm{Leb}\left\{x_1 \in \mathbb{R}^3 \colon |x_1| \in [n, n+1)\right\} \mathrm{Leb}(B_1(0)). \qquad (2.20)$$

Since the first term is of size $O(n^{-2\alpha\rho})$ and the first Lebesgue measure is of size $O(n^2)$, the above sum is finite for $\alpha > \frac{3}{2\rho}$. Since we chose $\varepsilon > \frac{1}{3}$ and hence $\rho = \frac{1}{1-\varepsilon} > \frac{3}{2}$, we can choose some $\alpha \in (0, 1)$ so that $\alpha > \frac{3}{2\rho}$, as desired.

Let us now handle the second expectation in (2.19). By the Cauchy-Schwarz inequality and Lemma 2.3, if β_4 is small enough, for any $x_1, x_2 \in \mathbb{R}^3$ such that $|x_1 - x_2| \leq 1$,

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_n \leq 1\}}\left\{\exp\left\{\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\} - 1\right\}\right]$$

$$\leq \mathbb{P}(\tau_n \leq 1)^{\frac{1}{2}} \, \mathbb{E}\left[\exp\left\{2\beta_4\left(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1 - x_2|^a}\right)^\rho\right\}\right]^{\frac{1}{2}}$$

$$\leq C\mathbb{P}\left(\max_{[0,1]} W > n - n^\alpha\right)^{\frac{1}{2}},$$

where C does not depend on x_1, x_2. Since the last probability is of order e^{-cn^2}, the second sum on n in (2.19) is obviously finite. This, combined with the finiteness of the sum in (2.20), proves (2.18) and hence finishes the proof of Lemma 2.4. $\qquad\square$

For the proof of Theorem 1.3 we will use the following (multidimensional) estimate of Garsia-Rodemich-Rumsey [SV79, p. 60].

Lemma 2.5. *Let $p(\cdot)$ and $\Psi(\cdot)$ be strictly increasing continuous functions on $[0, \infty)$ so that $p(0) = \Psi(0) = 0$ and $\lim_{t \uparrow \infty} \Psi(t) = \infty$. If $f \colon \mathbb{R}^d \to \mathbb{R}$ is continuous on the closure of the ball $B_{2r}(z)$ for some $z \in \mathbb{R}^d$ and $r > 0$, then the bound*

$$\int_{B_r(z)} \mathrm{d}x \int_{B_r(z)} \mathrm{d}y \, \Psi\left(\frac{|f(x) - f(y)|}{p(|x - y|)}\right) \leq M < \infty, \tag{2.21}$$

implies that

$$\left|f(x) - f(y)\right| \leq 8 \int_0^{2|x-y|} \Psi^{-1}\left(\frac{M}{\gamma u^{2d}}\right) p(\mathrm{d}u), \qquad x, y \in B_r(z), \tag{2.22}$$

for some constant γ that depends only on d.

Finally we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The Brownian scaling property implies that

$$\Lambda_t(x) = \frac{1}{t}\int_0^t \frac{1}{|W_s - x|} \, \mathrm{d}s = \int_0^1 \frac{1}{|W(ts) - x|} \, \mathrm{d}s \overset{\mathcal{D}}{=} \int_0^1 \frac{1}{|\sqrt{t}W(s) - x|} \, \mathrm{d}s = t^{-1/2}\Lambda_1(xt^{-1/2}),$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution. Hence, the claim of Theorem 1.3 is equivalent to

$$\lim_{\delta \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left\{\sup_{x_1, x_2 \in \mathbb{R}^3 \colon |x_1 - x_2| \leq \delta t^{-1/2}} \left|\Lambda_1(x_1) - \Lambda_1(x_2)\right| \geq bt^{1/2}\right\} = -\infty, \qquad b > 0. \tag{2.23}$$

Now we would like to apply Lemma 2.5. We pick $\varepsilon \in (\frac{1}{3}, \frac{1}{2})$ and $a = 1 - 2\varepsilon$ and $\rho = \frac{1}{1-\varepsilon}$ and $\beta = \beta_4$ as in Lemma 2.4 and choose

$$\Psi(x) = \mathrm{e}^{\beta|x|^\rho} - 1, \qquad p(x) = |x|^a = |x|^{1-2\varepsilon}, \qquad f(x) = \Lambda_1(x). \tag{2.24}$$

Then $\Psi(\cdot)$, $p(\cdot)$ and $f(\cdot)$ all satisfy the requirements of Lemma 2.5. Furthermore, Lemma 2.4 implies that hypothesis (2.21) is satisfied if $|x_1 - x_2| \leq \delta$ and $\delta > 0$ is chosen small enough, where the random variable M is given in (2.17). Hence, (2.22) implies that for $|x_1 - x_2| \leq \delta t^{-1/2}$ and all $t \geq 1$,

$$\left|\Lambda_1(x_1) - \Lambda_1(x_2)\right| \leq 8 \int_0^{\delta t^{-1/2}} \Psi^{-1}\left(\frac{M}{\gamma u^6}\right) p(\mathrm{d}u) = 8\frac{1 - 2\varepsilon}{\beta^{1/\rho}} \int_0^{\delta t^{-1/2}} \log\left(1 + \frac{M}{\gamma u^6}\right)^{1/\rho} u^{-2\varepsilon} \, \mathrm{d}u. \tag{2.25}$$

For $u \in (0, \delta t^{-1/2}]$ and all sufficiently large t, we estimate

$$8 \frac{1 - 2\varepsilon}{\beta^{1/\rho}} \log\left(1 + \frac{M}{\gamma u^6}\right)^{1/\rho} \leq C\left((\log(M \vee 1))^{1/\rho} + (\log \tfrac{1}{u})^{1/\rho}\right),$$

for some constant C that does not depend on t if t is sufficiently large. Hence, the right-hand side of (2.25) is not larger than

$$C_\delta (\log(M \vee 1))^{1/\rho} t^{\varepsilon - 1/2} + C_\delta (\log t)^c t^{\varepsilon - 1/2}$$

for some C_δ, c, not depending on t. Substituting this in (2.25) and recalling that $\rho = \frac{1}{1-\varepsilon}$, we obtain

$$\mathbb{P}\left\{ \sup_{x_1, x_2 \in \mathbb{R}^3 : \, |x_1 - x_2| \leq \delta t^{-1/2}} \left|\Lambda_1(x_1) - \Lambda_1(x_2)\right| \geq b t^{1/2} \right\} \leq \mathbb{P}\left\{ (\log(M \vee 1))^{1/\rho} + (\log t)^c \geq \frac{b}{C_\delta} t^{1-\varepsilon} \right\}$$

$$\leq \mathbb{P}\left\{ \log(M \vee 1) \geq C_1 b^\rho t - C_2 (\log t)^{c\rho} \right\} \leq \mathbb{E}(M \vee 1) e^{-C_1 b^\rho t + C_2 (\log t)^{c\rho}}.$$

$$(2.26)$$

Recall that by Lemma 2.4, $\mathbb{E}(M \vee 1) < \infty$. Hence, the above estimate now implies (2.23) and therefore Theorem 1.3. $\qquad \square$

Corollary 2.6. *For any $b > 0$,*

$$\lim_{\delta \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \sup_{x_1, x_2 \in \mathbb{R}^3 : \, |x_1 - x_2| \leq \delta} \left|\Lambda_t(x_1) - \Lambda_t(x_2)\right| \geq b \right\} = -\infty.$$

Proof. Let us denote by $A_{t,\delta}$ the above event inside the probability. Then the Cauchy-Schwarz inequality gives that

$$\frac{1}{t} \log \widehat{\mathbb{P}}_t \{A_{t,\delta}\} = \frac{1}{2t} \log \mathbb{E}\left\{ e^{2tH(L_t)} \right\} - \frac{1}{t} \log \mathbb{E}\left\{ e^{tH(L_t)} \right\} + \frac{1}{2} \frac{1}{t} \log \mathbb{P}\{A_{t,\delta}\}.$$

While the first two terms have finite large-t limits, by Theorem 1.3 the large-t limit of the third term tends to $-\infty$ as $\delta \to 0$. This proves the corollary. $\qquad \square$

3. LDP for Λ_t in the uniform metric: Proof of Theorem 1.1

Recall that we need to show, for any $\varepsilon > 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \inf_{w \in \mathbb{R}^3} \left\| \Lambda_t - \Lambda \psi_w^2 \right\|_\infty \geq \varepsilon \right\} < 0. \qquad (3.1)$$

We approximate the sup-norm inside the probability via a coarse graining argument as follows. For any $\delta \in (0, 1)$, we can estimate

$$\inf_{w \in \mathbb{R}^3} \left\| \Lambda_t - \Lambda \psi_w^2 \right\|_\infty = \inf_{w \in \mathbb{R}^3} \sup_{x \in \mathbb{R}^3} \left| \Lambda_t(x) - \left(\Lambda \psi_w^2\right)(x) \right|$$

$$\leq \sup_{x_1, x_2 \in \mathbb{R}^3 : \, |x_1 - x_2| \leq \delta} \left| \Lambda_t(x_1) - \Lambda_t(x_2) \right| \qquad (3.2)$$

$$+ \inf_{w \in \mathbb{R}^3} \sup_{z \in \delta \mathbb{Z}^3} \left[\left| \Lambda_t(z) - \left(\Lambda \psi_w^2\right)(z) \right| + \sup_{\tilde{z} \in B_\delta(z)} \left| \left(\Lambda \psi_w^2\right)(\tilde{z}) - \left(\Lambda \psi_w^2\right)(z) \right| \right].$$

Note that, for any $w \in \mathbb{R}^3$ the deterministic function $\Lambda \psi_w^2$ is uniformly continuous on $\mathbb{R}^3$ and hence

$$\lim_{\delta \downarrow 0} \sup_{z \in \delta \mathbb{Z}^3} \sup_{\tilde{z} \in B_\delta(z)} \left| \left(\Lambda \psi_w^2\right)(\tilde{z}) - \left(\Lambda \psi_w^2\right)(z) \right| = 0.$$

Since $\varepsilon > 0$ is arbitrary, the above fact and Corollary 2.6 imply that, to deduce (3.1), it suffices to prove, for any $\varepsilon, \delta > 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \inf_{w \in \mathbb{R}^3} \sup_{z \in \delta \mathbb{Z}^3} \left| \Lambda_t(z) - (\Lambda \psi_w^2)(z) \right| \geq \varepsilon \right\} < 0. \tag{3.3}$$

For any $z \in \delta \mathbb{Z}^3$, $w \in \mathbb{R}^3$ and any $\eta > 0$, we can estimate

$$\left| \Lambda_t(z) - (\Lambda \psi_w^2)(z) \right| \leq \int_{B_\eta(z)} \frac{\psi_w^2(y)}{|y - z|} \, \mathrm{d}y + \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} + \left| \int_{\mathbb{R}^3} \frac{\mathbb{1}\{|y - z| \geq \eta\}}{|y - z|} \left(L_t(\mathrm{d}y) - \psi_w^2(y)\mathrm{d}y \right) \right|. \tag{3.4}$$

The first term can be handled easily. Note that, for any $w \in \mathbb{R}^3$, ψ_w is radially symmetric and $\|\psi_w\|_2 = 1$. Hence using polar coordinates and invoking the dominated convergence theorem we can argue that

$$\lim_{\eta \to 0} \sup_{z \in \delta \mathbb{Z}^3} \int_{B_\eta(z)} \frac{\psi_w^2(y)}{|y - z|} \mathrm{d}y = 0. \tag{3.5}$$

Let us turn to the second term in (3.4). We claim that, for any $\delta > 0$ and $\eta > 0$ small enough,

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \left\{ \sup_{z \in \delta \mathbb{Z}^3} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} \geq \varepsilon \right\} < 0. \tag{3.6}$$

Let us first handle the above event with the Wiener measure $\mathbb{P}$ replacing $\widehat{\mathbb{P}}_t$. Then we can estimate

$$\mathbb{P} \left\{ \sup_{z \in \delta \mathbb{Z}^3} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} > \varepsilon \right\} \leq \sum_{\substack{z \in \delta \mathbb{Z}^3 \\ |z| \leq t^2}} \mathbb{P} \left\{ \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} \geq \varepsilon/2 \right\} + \mathbb{P} \left\{ \sup_{\substack{z \in \delta \mathbb{Z}^3 \\ |z| > t^2}} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} \geq \varepsilon/2 \right\}. \tag{3.7}$$

The second term can be estimated by the probability that the Brownian path, starting at origin, travels a distance $t^2 - \varepsilon$ by time t. This probability is of order $\exp\{-ct^3\}$ and can be ignored. For the first term we note that a box of size t^2 in $\mathbb{R}^3$ can be covered by $O(t^6)$ sub-boxes of side length δ and that the probability is maximal for $z = 0$. Hence, we can estimate, with the help of Markov's inequality, for any $\beta > 0$,

$$\sum_{\substack{z \in \delta \mathbb{Z}^3 \\ |z| \leq t^2}} \mathbb{P} \left\{ \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} > \varepsilon/2 \right\} \leq Ct^6 \, \mathbb{P} \left\{ \beta \int_0^t V_\eta(W_s)\mathrm{d}s > t\beta\varepsilon/2 \right\} \leq Ct^6 \mathrm{e}^{-\frac{\varepsilon}{2}t\beta} \, \mathbb{E} \left\{ \mathrm{e}^{\beta \int_0^t V_\eta(W_s)\, \mathrm{d}s} \right\}, \tag{3.8}$$

where $V_\eta(x) = \mathbb{1}_{\{|x| \leq \eta\}} \frac{1}{|x|}$. Note that, for any $\beta > 0$ and some constants c_1, c_2 independent of η,

$$\sup_{y \in \mathbb{R}^3} \mathbb{E}_y \left\{ \beta \int_0^1 V_\eta(W_s) \, \mathrm{d}s \right\} \leq \beta \int_{B_\eta(0)} \frac{\mathrm{d}x}{|x|} \int_0^1 p_s(0, x) \, \mathrm{d}s \leq \beta c_1 \int_{B_\eta(0)} \frac{\mathrm{d}x}{|x|^2} = c_2 \eta \beta.$$

For any fixed $\beta > 0$ and η small enough, this is not larger than $1/2$, and by Portenko's lemma ([P76]), successive conditioning and the Markov property,

$$\mathbb{E} \left\{ \mathrm{e}^{\beta \int_0^t V_\eta(W_s)\mathrm{d}s} \right\} \leq 2^{\lceil t \rceil}.$$

Then (3.8) and (3.7) imply, for any $\beta > 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left\{ \sup_{z \in \delta \mathbb{Z}^3} \int_{B_\eta(z)} \frac{L_t(\mathrm{d}y)}{|y - z|} \geq \varepsilon \right\} \leq -\varepsilon\beta/2 + \log 2.$$

From this we can deduce (3.6) by choosing $\beta > 0$ large enough and invoking Hölder's inequality as in the proof of Corollary 2.6. We drop the details to avoid repetition.

Let us turn to the third term on the right hand side of (3.4). Then by (3.5) and (3.6), it suffices to prove that, for every $\eta, \varepsilon > 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \{ L_t \in F_\eta \} < 0, \tag{3.9}$$

where

$$F_\eta = \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^3) \colon \ \forall w \in \mathbb{R}^3 \ \sup_{z \in \mathbb{R}^3} \left| \int \left(\frac{1}{|y-z|} \wedge \frac{1}{\eta} \right) \left(\mu(\mathrm{d}y) - \psi_w^2(y)\mathrm{d}y \right) \right| \geq \varepsilon \right\}.$$

We claim that for each $\eta > 0$, F_η is a closed set in the weak topology in $\mathcal{M}_1(\mathbb{R}^3)$. Indeed, for each $\eta > 0$, we set

$$f_{z,\eta}(y) = \left(\frac{1}{|y-z|} \wedge \frac{1}{\eta} \right) \mathbb{1}_{\{|z-y| \geq \eta\}}.$$

Then the family $\mathcal{A}_\eta = \{ f_{z,\eta} \colon z \in \mathbb{R}^d \}$ is equicontinuous and uniformly bounded. Hence, for any $\eta > 0$, the set

$$G_{\eta,w} = \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^3) \colon \ \sup_{f \in \mathcal{A}_\eta} \left| \langle f, \mu - \psi_w^2 \rangle \right| < \varepsilon \right\}$$

is weakly open and hence

$$F_\eta = \bigcap_{w \in \mathbb{R}^d} G_{\eta,w}^{\mathrm{c}}$$

is weakly closed.

Furthermore, we note that F_η is shift-invariant, i.e., if $\mu \in F_\eta$, then $\mu \star \delta_x \in F_\eta$ for any $x \in \mathbb{R}^3$. In other words,

$$\widehat{\mathbb{P}}_t \{ L_t \in F_\eta \} = \widehat{\mathbb{P}}_t \{ \widetilde{L}_t \in \widetilde{F}_\eta \}$$

where $\widetilde{F}_\eta = \{ \widetilde{\mu} \colon \mu \in F_\eta \}$, the set of orbits $\widetilde{\mu} = \{ \mu \star \delta_x \colon x \in \mathbb{R}^3 \}$ of members of F_η, is a closed set in $\widetilde{\mathcal{M}}_1(\mathbb{R}^3) \hookrightarrow \widetilde{\mathcal{X}}$, recall Section 1.4. Then [MV14, Theorem 5.3] implies

$$\limsup_{t \to \infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \{ \widetilde{L}_t \in \widetilde{F}_\eta \} \leq - \inf_{\xi \in \widetilde{F}_\eta} \widetilde{J}(\xi),$$

and [MV14, Lemma 5.4] implies

$$\inf_{\xi \in \widetilde{F}_\eta} \widetilde{J}(\xi) > 0.$$

These two facts imply (3.9) and hence Theorem 1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We end this section with the proof of Corollary 1.4.

Proof of Corollary 1.4. The proof is straightforward and similar to the last line of arguments. Indeed, we note that for any $\delta > 0$,

$$\mathbb{P}\{ \|\Lambda_t\|_\infty > b \} \leq \mathbb{P}\left\{ \sup_{|x_1 - x_2| \leq \delta} \left| \Lambda_t(x_1) - \Lambda_t(x_2) \right| \geq b/2 \right\} + \mathbb{P}\left\{ \sup_{x \in \delta\mathbb{Z}^3} \Lambda_t(x) \geq b/2 \right\}$$

$$\leq \mathbb{P}\left\{ \sup_{|x_1 - x_2| \leq \delta} \left| \Lambda_t(x_1) - \Lambda_t(x_2) \right| \geq b/2 \right\} + \mathbb{P}\left\{ \sup_{\substack{x \in \delta\mathbb{Z}^3 \\ |x| \leq t^2}} \Lambda_t(x) \geq b/2 \right\}$$

$$+ \mathbb{P}\left\{ \sup_{\substack{x \in \delta\mathbb{Z}^3 \\ |x| > t^2}} \Lambda_t(x) \geq b/2 \right\}$$

By Theorem 1.3, the first term has a strictly negative exponential rate. The third term can again be neglected since this is of order $\exp\{-ct^3\}$. Also for the second term, the box of size t^2 can be covered by $O(t^6)$ sub-boxes of side length δ. Therefore,

$$\mathbb{P}\bigg\{\sup_{\substack{x\in\delta\mathbb{Z}^3\\|x|\le t^2}}\Lambda_t(x)\ge b/2\bigg\}\le Ct^6\mathbb{P}\Big\{\Lambda_t(0)>b/2\Big\}.$$

For any $\kappa>0$,

$$\mathbb{P}\Big\{\Lambda_t(0)>b/2\Big\}\le \mathrm{e}^{-\kappa bt/2}\,\mathbb{E}\bigg\{\exp\Big\{\kappa\int_0^t\frac{\mathrm{d}s}{|W_s|}\Big\}\bigg\}.$$

We choose $t>u\gg 1$ and $\kappa>0$ small enough so that $\sqrt{u}\kappa\ll 1$ and

$$\alpha=\sup_{x\in\mathbb{R}^3}\mathbb{E}_x\bigg\{\kappa\int_0^u\frac{\mathrm{d}s}{|W_s|}\bigg\}=\mathbb{E}_0\bigg\{\kappa\int_0^u\frac{\mathrm{d}s}{|W_s|}\bigg\}=2\kappa\sqrt{u}\,\mathbb{E}\bigg(\frac{1}{|W_1|}\bigg)\ll 1.$$

Then by Portenko's lemma [P76],

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}_x\bigg\{\exp\Big\{\kappa\int_0^u\frac{\mathrm{d}s}{|W_s|}\Big\}\bigg\}\le\frac{1}{1-\alpha},$$

and by successive conditioning and the Markov property,

$$\mathbb{E}\bigg\{\exp\Big\{\kappa\int_0^t\frac{\mathrm{d}s}{|W_s|}\Big\}\bigg\}\le\bigg(\frac{1}{1-\alpha}\bigg)^{t/u}.$$

Since $\log(1+\alpha)\approx\alpha$ as $\alpha\to 0$, for any $b>0$ and $\kappa>0$ suitably chosen and u large enough,

$$\mathbb{P}\Big\{\Lambda_t(0)>b/2\Big\}\le\exp\Big\{-\frac{\kappa bt}{2}+\frac{t}{u}\log(1-\alpha)\Big\}$$

$$\le\exp\Big[-t\kappa\Big\{\frac{b}{2}-\frac{1}{\sqrt{u}}c\Big\}\Big]$$

$$\le\exp\big\{-t\kappa\widetilde{C}\big\}$$

for some $\widetilde{C}=\widetilde{C}(u,a,c)>0$. This proves the corollary. $\qquad\square$

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