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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 10

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

Solving the following problems is voluntary. The solutions will not be graded.

1. *Properties of the interpolation operator.* Show that the interpolation operator $I_{\hat{K}} : C^s(\hat{K}) \to \hat{P}(\hat{K})$, which was defined in the lecture, is a linear and continuous operator. The spaces $C^s(\hat{K})$ and $P(\hat{K}) \subset C^s(\hat{K})$ are equipped with the norm of $C^s(\hat{K})$. In addition, show that the restriction of $I_{\hat{K}}$ to $P(\hat{K})$ is the identity operator.

2. Let the domain $\Omega = (0,1)^2$ be triangulated with the two triangles with the vertices $(0,0),(1,1),(0,1)$ and $(0,0),(1,0),(1,1)$. Compute the error between the function
$$v(x,y) = \sin(2x+y) + 6x^3y^2 \in C^\infty(\Omega)$$
and its interpolant Iv:
   - in the finite element space P_0, the functional is the value of the function in the barycenter of the triangle, the error should be given in the $L^2(\Omega)$ norm,
   - in the finite element space P_1, the functionals are the values of the function in the vertices of the triangles, the error should be given in the $L^2(\Omega)$ norm and in the $H^1(\Omega)$ semi norm.

3. *Estimates in the proof of the interpolation estimate of the Clément operator.*
   i) Show the estimate
   $$|p_i(V_i)| \le ch^{-d/q}\|v\|_{L^q(\omega_i)}.$$
   ii) Prove that one can find a polynomial $p \in P_1(\omega_K)$ with
   $$\left\|D^j(v-p)\right\|_{L^q(\omega_K)} \le Ch^{l-j}\left\|D^l v\right\|_{L^q(\omega_K)}, \quad 0 \le j \le l \le 2.$$

4. Consider the function
$$v(x) = \begin{cases} 0.5 & x \in (0, 0.25) \\ 1 & x \in (0.25, 0.5) \\ 0 & x \in (0.5, 0.75) \\ -0.5 & x \in (0.75, 1). \end{cases}$$
Compute the Clément interpolant of this function in the finite element space P_1, which is defined on the domain $\Omega = (0,1)$ and the grid with the nodes $\{0, 0.25, 0.5, 0.75, 1\}$.
Hint: the integrals can be computed with Matematica etc.