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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 07

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Boundedness of a bilinear form.* Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ denote the bilinear form

$$a(u,v) = \int_\Omega \nabla u(\mathbf{x})^T A(\mathbf{x}) \nabla v(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x}) \ d\mathbf{x},$$

with $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,\ldots,d$, $c \in L^\infty(\Omega)$, $c \geq 0$. Show that this bilinear form is bounded, i.e., there is a constant C such that

$$|a(u,v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall\ u,v \in H^1(\Omega).$$

2 points

2. *Connection of properties of matrices and bilinear forms.* Let

$$A = (a_{ij}) = a(\phi_j, \phi_i),$$

where $\{\phi_i\}_{i=1}^k$ is the basis of a finite-dimensional space V_k. Show that

i)

$$A = A^T \quad \Longleftrightarrow \quad a(v,w) = a(w,v) \quad \forall\ v,w \in V_k,$$

ii)

$$\underline{v}^T A \underline{v} > 0 \quad \forall\ \underline{v} \in \mathbb{R}^k,\ \underline{v} \neq \underline{0} \quad \Longleftrightarrow \quad a(v,v) > 0 \quad \forall\ v \in V_k,\ v \neq 0.$$

2 points

3. *Stability estimate for the Poisson problem.* Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary and let $f \in L^2(\Omega)$. Consider the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

Show the stability estimate

$$\|u\|_{H^1(\Omega)} \leq C \, \|f\|_{L^2(\Omega)} \, .$$

2 points

4. *Weak formulation for Robin boundary conditions.* Consider the partial differential equation

$$-\nabla \cdot (A(\mathbf{x}) \nabla u(\mathbf{x})) + c(\mathbf{x}) u(\mathbf{x}) = f(\mathbf{x}) \quad \text{in } \Omega$$

with so-called Robin boundary conditions

$$(A(\mathbf{x})\nabla u(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) + a(\mathbf{x})u(\mathbf{x}) = g(\mathbf{x}) \quad \text{on } \partial\Omega.$$

All functions are assumed to be sufficiently regular with $A(\mathbf{x}) = A^T(\mathbf{x})$. Derive a weak or variational formulation of this problem. **1 point**

The exercise problems should be solved in groups of four to five students. The solutions have to be submitted until **Monday, June 19th, 2023, 10:00 a.m.** via the whiteboard.