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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 05

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Weak derivative in one dimension.* Solve the following problems.

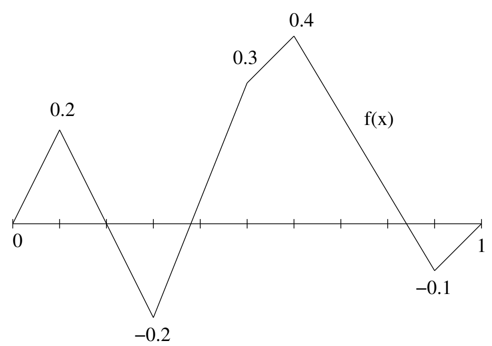
   i) Let $f(x) = 1$ in Ω. Investigate whether or not $f \in L^1(\Omega)$, $f \in L^1_{\text{loc}}(\Omega)$ for $\Omega = (0,1)$ and $\Omega = \mathbb{R}$. **2 points**

   ii) Show with the help of the definition that

   $$f'(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0, \end{cases}$$

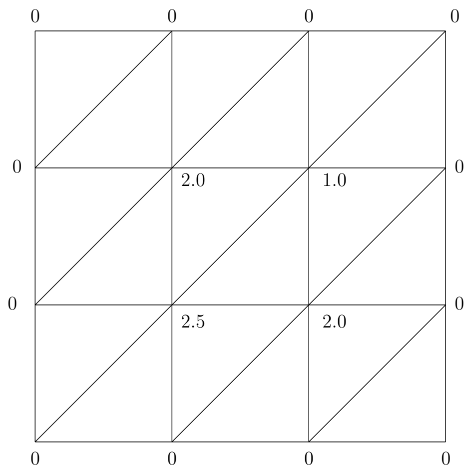
   is the weak derivative of $f(x) = |x|$. **2 points**

   iii) Compute the weak derivative of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)$.



1 point

2. *Weak derivative in two dimensions.* Compute the first weak derivatives of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)^2$, which is continuous and piecewise (with respect to the grid) linear and which is therefore completely determined by the values in the nodes.



2 points

3. *Hölder's inequality.*

   i) Let $r \in [1, \infty)$, $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, $u \in L^{rp}(\Omega)$, $v \in L^{rq}(\Omega)$. Show that
   $$\|uv\|_{L^r(\Omega)} \le \|u\|_{L^{rp}(\Omega)} \|v\|_{L^{rq}(\Omega)}.$$
   1 point

   ii) Show that for $p \in (2, \infty)$
   $$\|uv\|_{L^2(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^{2p/(p-2)}(\Omega)}$$
   holds. **1 point**

4. *Rational functions in Lebesgue spaces in a ball.* Solve the following problems.

   i) For which values of $a \in \mathbb{R}$ is the function $f : (-1, 1) \to \mathbb{R}$
   $$f(x) = \begin{cases} |x|^a & x \neq 0, \\ 0 & x = 0 \end{cases}$$
   an element of $L^p((-1, 1))$ with $p \in [1, \infty]$? **2 points**

   ii) Let
   $$B_1(\mathbf{0}) = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 < 1\}$$
   be the d-dimensional unit ball, $d > 1$. Find the values $a \in \mathbb{R}$ for which the function $f : B_1(\mathbf{0}) \to \mathbb{R}$ with
   $$f(x) = \begin{cases} \|\boldsymbol{x}\|_2^a & \boldsymbol{x} \neq \mathbf{0}, \\ 0 & \boldsymbol{x} = \mathbf{0}, \end{cases}$$
   belongs to $L^p(B_1(\mathbf{0}))$ with $p \in [1, \infty]$! **2 points**

The exercise problems should be solved in groups of four to five students. The solutions have to be submitted until **Thursday, June 01st, 2023, 10:00 a.m.** via the whiteboard.