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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 02

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Basic properties of finite difference approximations.* Solve the following problems.

   i) Show that
   $$v_{\mathring{x},i} = \frac{1}{2}\left(v_{x,i} + v_{\bar{x},i}\right), \quad v_{\bar{x}x,i} = \left(v_{\bar{x},i}\right)_{x,i}.$$
   1 point

   ii) Consider a function $v(x)$ at x_i and show the following consistency estimates
   $$v_{\mathring{x},i} = v'\left(x_i\right) + \mathcal{O}\left(h^2\right), \quad v_{\bar{x}x,i} = v''\left(x_i\right) + \mathcal{O}\left(h^2\right).$$
   1 point

   iii) Compute the order of consistency of the following finite difference approximation
   $$u''\left(x\right) \sim \frac{1}{12h^2}\Big(-u\left(x+2h\right)+16u\left(x+h\right)-30u\left(x\right)+16u\left(x-h\right)-u\left(x-2h\right)\Big).$$
   2 points

2. *Finite difference approximation of the second order derivative for non-constant diffusion.* Consider the differential operator $Lu = \frac{\partial}{\partial x}\left(k\left(x\right)\frac{\partial u}{\partial x}\right)$ and its finite difference approximation
   $$\left(L_h u_h\right)_i = \frac{1}{h}\left(a_{i+1}\frac{u_{i+1}-u_i}{h} - a_i\frac{u_i - u_{i-1}}{h}\right).$$
   Show that $a_i = \frac{k_i + k_{i-1}}{2}$ and $a_i = k\left(x_i - \frac{h}{2}\right)$ satisfy the conditions for second order consistency
   $$\frac{a_{i+1}-a_i}{h} = k'\left(x_i\right) + \mathcal{O}\left(h^2\right), \qquad \frac{a_{i+1}+a_i}{2} = k\left(x_i\right) + \mathcal{O}\left(h^2\right),$$
   which were derived in the lecture. **4 points**

3. *Finite difference approximation of the second order derivative at a non-equidistant grid.* Consider the interval $[x-h_x^-, x+h_x^+]$ with $h_x^-, h_x^+ > 0$, $h_x^- \neq h_x^+$.

   i) Assume $u \in C^3\left(\left[x-h_x^-, x+h_x^+\right]\right)$. Show the following consistency estimate
   $$\left|u''\left(x\right) - \frac{2}{h_x^+ + h_x^-}\left(\frac{u\left(x+h_x^+\right)-u\left(x\right)}{h_x^+} - \frac{u\left(x\right)-u\left(x-h_x^-\right)}{h_x^-}\right)\right| \le C\left(h_x^+ + h_x^-\right).$$
   3 points

ii) Prove that there is no other approximation which satisfies

$$\left|u''(x) - \left(\alpha\, u\left(x - h_x^-\right) + \beta\, u(x) + \gamma\, u\left(x + h_x^+\right)\right)\right| \le C\left(h_x^+ + h_x^-\right),$$

where the weights depend only on the mesh width, i.e., $\alpha = \alpha\left(h_x^-, h_x^+\right)$, $\beta = \beta\left(h_x^-, h_x^+\right)$, $\gamma = \gamma\left(h_x^-, h_x^+\right) \in \mathbb{R}$, and the constant C does not explicitly depend on $u(\boldsymbol{x})$, $u'(\boldsymbol{x})$, and $u''(\boldsymbol{x})$. **2 points**

4. This problem has to be solved until May 15th, 2023!
Code for five point stencil. Write a code for the numerical solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

The right-hand side and the boundary conditions should be chosen such that

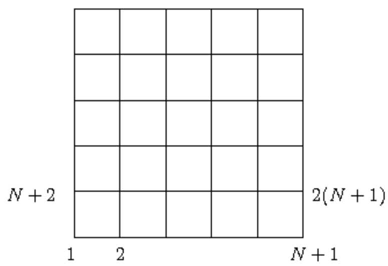
$$u(x,y) = x^4 y^5 - 17 \sin(xy)$$

is the solution of the boundary value problem.

Use the five point stencil for the discretizing the partial differential equation. The mesh widths should be chosen to be

$$h_x = h_y = h = 2^{-n} \quad n = 2, 3, 4, \ldots, 8.$$

Order the nodes lexicographically and store the matrix in `sparse` format. **6 points**



Compute the following errors

$$\|u - u_h\|_{l^\infty(\omega_h)}, \qquad \|u - u_h\|_{l^2(\omega_h)},$$

and the orders of convergence, based on the errors on the two finest meshes, where the second norm is the standard Euclidean vector norm. **2 points**

The exercise problems should be solved in groups of four to five students. The solutions have to be submitted until **Monday, May 08th, 2023, 10:00 a.m.** via the whiteboard.