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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 01

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Basic properties of the nabla operator.* The following operators are defined for a scalar function u a and a vector-valued function $\mathbf{v} = (v_1, v_2, v_3)^T$ with the help of the nabla operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)^T = (\partial_{x_1}, \cdots, \partial_{x_d}) :$$

- $\mathrm{grad} u = \nabla u = (\partial_x u, \partial_y u, \partial_z u)^T$,
- $\mathrm{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3$,
- $\mathrm{rot} \mathbf{v} = \nabla \times \mathbf{v} = \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix}$.

Assuming that all considered functions are sufficiently smooth (sufficiently often differentiable), show the following identities:

i) $\nabla \cdot \nabla u = \Delta u = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u$,

ii) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$,

iii) $\nabla \times (\nabla u) = 0$,

iv) $\nabla \times (u\mathbf{v}) = u\,(\nabla \times \mathbf{v}) - (\mathbf{v} \times \nabla u)$,

v) $\nabla \cdot (u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\,\nabla \cdot \mathbf{v}$.

4 points

2. *Analytic solution of a one-dimensional heat equation.* Show that

$$u\,(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0\,(z)\, \mathrm{e}^{\frac{-(x-z)^2}{4t}}\, dz$$

is a solution of the one-dimensional heat equation

$$\begin{array}{rcll} \partial_t u - \partial_{xx} u & = & 0, & x \in \mathbb{R}, t > 0, \\ u\,(0, x) & = & u_0\,(x)\,, & x \in \mathbb{R}. \end{array}$$

It shall be assumed that $u_0(x)$ is sufficiently smooth.
Hint. To check the initial condition, assume that $u_0(x)$ can be expanded in a uniformly convergent Fourier series

$$u_0(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos\left(n\omega x\right) + \beta_n \sin\left(n\omega x\right)\right),$$

with $\omega \in \mathbb{R}$, and use the following identities

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(nz)\, \mathrm{e}^{\frac{-(x-z)^2}{4t}}\, dz &= \sqrt{4\pi t}\mathrm{e}^{-n^2 t} \sin(nx)\,,\\ \int_{-\infty}^{\infty} \cos(nz)\, \mathrm{e}^{\frac{-(x-z)^2}{4t}}\, dz &= \sqrt{4\pi t}\mathrm{e}^{-n^2 t} \cos(nx)\,, \quad n \in \mathbb{R}.\end{aligned}$$

4 points

3. *Classification of second order partial differential equations.* Classify the following partial differential equations

$$\begin{aligned}\text{i)}\quad \partial_{xx}u + 2\partial_{xy}u + 2\partial_{yy}u + 4\partial_{yz}u + 5\partial_{zz}u + \partial_x u + \partial_y u &= 0, \quad (3d)\,,\\ \text{ii)}\quad e^z\partial_{xy}u - \partial_{xx}u - \log\left(x^2 + y^2 + z^2\right) &= 0, \quad (3d)\,,\\ \text{iii)}\quad \partial_{xx}u + 4\partial_{xy}u + 3\partial_{yy}u + 3\partial_x u - \partial_y u + 2u &= 0, \quad (2d)\,,\\ \text{iv)}\quad a\partial_{xx}u + 2a\partial_{xy}u + a\partial_{yy}u + b\partial_x u + c\partial_y u + u &= 0, \quad (2d)\,,\end{aligned}$$

with $a \neq 0$ in iv) and $2d$ – in two dimensions, $3d$ – in three dimensions.

4 points

The exercise problems should be solved in groups of four to five students. The solutions have to be submitted until **Monday, May 01st, 2023, 10:00 a.m.** via the whiteboard.