Chapter 7
Finite Element Methods for Second Order Elliptic Problems

7.1 General Convergence Theorems

Remark 7.1. Motivation. In Section 5.1, non-conforming finite element spaces were introduced, i.e., methods where the finite element space V^h is not a subspace of V, which is the space in the definition of the continuous variational problem. The property $V^h \not\subset V$ is given for the Crouzeix–Raviart and the Rannacher–Turek element. Another case of non-conformity is given if the domain does not possess a polyhedral boundary and one has to apply some approximation of the boundary.

For non-conforming methods, the finite element approach is not longer a Ritz method. Hence, the convergence proof from Theorem 4.14 cannot be applied in this case. In addition, in practice, one is interested also in the order of convergence in other norms than $\|\cdot\|_V$ or one has to take into account that the values of the bilinear or linear form need to be approximated numerically. The abstract convergence theorem, which will be proved in this section, allows the numerical analysis of complex finite element methods. □

Remark 7.2. Notations, Assumptions. Let $\{h > 0\}$ be a set of mesh widths and let S^h, V^h normed spaces of functions which are defined on domains $\{\Omega^h \subset \mathbb{R}^d\}$. It will be assumed that the space S^h has a finite dimension and that S^h and V^h possess a common norm $\|\cdot\|_h$. In the application of the abstract theory, S^h will be a finite element space and V^h is defined such that the restriction and/or extension of the solution of the continuous problem to Ω^h is contained in V^h. The index h indicates that V^h might depend on h but not that V^h is finite-dimensional. Strictly speaking, the modified solution of the continuous problem does not solve the given problem any longer. Hence, it is consequent that the continuous problem does not appear explicitly in the abstract theory.

Given the bilinear forms

$$a^h : S^h \times S^h \to \mathbb{R},$$

$$\tilde{a}^h : (S^h + V^h) \times (S^h + V^h) \to \mathbb{R}.$$

Let the bilinear form a^h be regular in the sense that there is a constant $m > 0$, which is independent of h, such that for each $v^h \in S^h$ there is a $w^h \in S^h$ with $\|w^h\|_h = 1$ such that[1]

$$m \|v^h\|_h \le a^h(v^h, w^h). \tag{7.1}$$

This condition is equivalent to the requirement that the stiffness matrix A with the entries $a_{ij} = a^h(\phi_j, \phi_i)$, where $\{\phi_i\}$ is a basis of S^h, is uniformly non-singular, i.e., its non-singularity is independent of h (eigenvalues are bounded away from zero uniformly with respect to h). For the second bilinear form, only its boundedness will be assumed

$$\tilde{a}^h(u, v) \le M \|u\|_h \|v\|_h \quad \forall\ u, v \in S^h + V^h. \tag{7.2}$$

Let the linear functionals $\{f^h(\cdot)\}\ :\ S^h \to \mathbb{R}$ be given. Then, the following discrete problems will be considered: Find $u^h \in S^h$ with

$$a^h(u^h, v^h) = f^h(v^h) \quad \forall\, v^h \in S^h. \tag{7.3}$$

Because the stiffness matrix is assumed to be non-singular, there is a unique solution of (7.3).

Note the similarities of the whole setup with the assumptions for the Theorem of Lax–Milgram. In fact, the current setup can be considered as a generalization of the Lax–Milgram theory. □

Theorem 7.3. Abstract finite element error estimate. *Let the conditions* (7.1) *and* (7.2) *be satisfied and let u^h be the solution of* (7.3)*. Then, the following error estimate holds for each $\tilde{u} \in V^h$*

$$\|\tilde{u} - u^h\|_h \le C \inf_{v^h \in S^h} \left\{ \|\tilde{u} - v^h\|_h + \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\|w^h\|_h} \right\} + C \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\|w^h\|_h} \tag{7.4}$$

with $C = C(m, M)$.

Proof. Because of (7.1), there is for each $v^h \in S^h$ a $w^h \in S^h$ with $\|w^h\|_h = 1$ and

$$m \|u^h - v^h\|_h \le a^h(u^h - v^h, w^h).$$

[1] note that this condition can be formulated as an inf-sup condition:

$$0 < m \le \inf_{v^h \in S^h} \sup_{w^h \in S^h} \frac{a^h(v^h, w^h)}{\|v^h\|_h \|w^h\|_h}$$

Using the definition of u^h from (7.3), one obtains

$$m \left\| u^h - v^h \right\|_h \leq f^h(w^h) - a^h(v^h, w^h) + \tilde{a}^h(v^h, w^h) + \tilde{a}^h(\tilde{u} - v^h, w^h) - \tilde{a}^h(\tilde{u}, w^h).$$

From (7.2) and $\left\| w^h \right\|_h = 1$, it follows that

$$\tilde{a}^h(\tilde{u} - v^h, w^h) \leq M \left\| \tilde{u} - v^h \right\|_h .$$

Rearranging the terms appropriately and using $\left\| w^h / \left\| w^h \right\|_h \right\|_h = 1$ yields

$$\begin{aligned} m \left\| u^h - v^h \right\|_h \leq M \left\| \tilde{u} - v^h \right\|_h &+ \sup_{w^h \in S^h} \frac{\left| \tilde{a}^h(v^h, w^h) - a^h(v^h, w^h) \right|}{\left\| w^h \right\|_h} \\ &+ \sup_{w^h \in S^h} \frac{\left| \tilde{a}^h(\tilde{u}, w^h) - f^h(w^h) \right|}{\left\| w^h \right\|_h}. \end{aligned} \tag{7.5}$$

Applying the triangle inequality

$$\left\| \tilde{u} - u^h \right\|_h \leq \left\| \tilde{u} - v^h \right\|_h + \left\| u^h - v^h \right\|_h$$

and inserting the estimate (7.5) gives (7.4). ■

Remark 7.4. To Theorem 7.3.

- An important special case of this theorem is the case that the stiffness matrix is uniformly positive definite, i.e., the condition

$$m \left\| v^h \right\|_h^2 \leq a^h(v^h, v^h) \quad \forall\, v^h \in S^h \tag{7.6}$$

  is satisfied. Dividing (7.6) by $\left\| v^h \right\|_h$ reveals that condition (7.1) is implied by (7.6).
- If the continuous problem is also defined with the bilinear form $\tilde{a}^h(\cdot, \cdot)$, then

$$\sup_{w^h \in S^h} \frac{\left| \tilde{a}^h(v^h, w^h) - a^h(v^h, w^h) \right|}{\left\| w^h \right\|_h}$$

  can be considered as consistency error of the bilinear forms and the term

$$\sup_{w^h \in S^h} \frac{\left| \tilde{a}^h(\tilde{u}, w^h) - f^h(w^h) \right|}{\left\| w^h \right\|_h}$$

  as consistency error of the right-hand sides.

□

Theorem 7.5. First Strang[2] lemma *Let S^h be a conforming finite element space, i.e., $S^h \subset V$, with $\|\cdot\|_h = \|\cdot\|_V$ and let the space V^h be independent of h. Consider a continuous problem of the form*

$$\tilde{a}^h(u, v) = f(v) \quad \forall\, v \in V,$$

[2] Gilbert Strang, born 1934

then the following error estimate holds

$$\left\|u-u^h\right\|_V \le C \inf_{v^h\in S^h}\left\{\left\|u-v^h\right\|_V + \sup_{w^h\in S^h}\frac{\left|\tilde{a}^h(v^h,w^h)-a^h(v^h,w^h)\right|}{\|w^h\|_V}\right\}$$
$$+C\sup_{w^h\in S^h}\frac{\left|f(w^h)-f^h(w^h)\right|}{\|w^h\|_V}.$$

Proof. The statement of this theorem follows directly from Theorem 7.3. ■

7.2 Finite Element Method with the Non-conforming Crouzeix–Raviart Element

Remark 7.6. The continuous problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary. Let

$$Lu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{7.7}$$

where the operator is given by

$$Lu = -\nabla\cdot(A\nabla u)$$

with $A = A^T$ and

$$A(\boldsymbol{x}) = (a_{ij}(\boldsymbol{x}))_{i,j=1}^d, \quad a_{ij} \in W^{1,p}(\Omega), p > d. \tag{7.8}$$

It will be assumed that there are two positive real numbers m, M such that

$$m\|\boldsymbol{\xi}\|_2^2 \le \boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \le M\|\boldsymbol{\xi}\|_2^2 \quad \forall\, \boldsymbol{\xi}\in\mathbb{R}^d, \boldsymbol{x}\in\overline{\Omega}. \tag{7.9}$$

From the Sobolev inequality, Theorem 3.51, it follows that $a_{ij} \in L^\infty(\Omega)$. With

$$a(u,v) = \int_\Omega (A(\boldsymbol{x})\nabla u(\boldsymbol{x}))\cdot\nabla v(\boldsymbol{x})\; d\boldsymbol{x}$$

and the Cauchy–Schwarz inequality, one obtains

$$|a(u,v)| \le \|A\|_{L^\infty(\Omega)}\int_\Omega |\nabla u(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})|\; d\boldsymbol{x} \le C\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$$

for all $u, v \in H_0^1(\Omega)$. In addition, it follows from (7.9) that

$$m\|\nabla u\|_{L^2(\Omega)}^2 \le a(u,u) \quad \forall\, u\in H_0^1(\Omega).$$

Hence, the bilinear form is bounded and elliptic. Using the Theorem of Lax–Milgram, Theorem 4.5, it follows that for given $f \in H^{-1}(\Omega)$ there es a unique

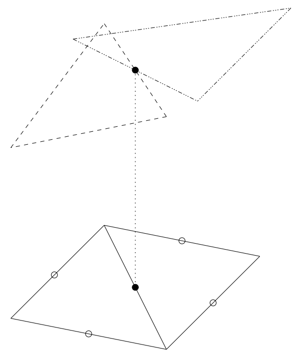

Fig. 7.1 Function from P_1^{nc}.

weak solution $u \in H_0^1(\Omega)$ of

$$a(u,v) = f(v) \quad \forall\, v \in H_0^1(\Omega). \tag{7.10}$$

□

Remark 7.7. Assumptions and the discrete problem. The non-conforming Crouzeix–Raviart finite element P_1^{nc} was introduced in Example 5.30. To simplify the presentation, it will be restricted here on the two-dimensional case. In addition, to avoid the estimate of the error coming from approximating the domain, it will be assumed that Ω is a convex domain with polygonal boundary. It can be shown that in this case the boundary is Lipschitz. In addition, it is assumed that $f \in L^2(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$.

Let $\mathcal{T}^h$ be a regular triangulation of Ω with triangles. Let P_1^{nc} (nc – non-conforming) denote the finite element space of piecewise linear functions that are continuous at the midpoints of the edges. This space is non-conforming if it is applied for the discretization of a second order elliptic equation since the continuous problem is given in $H_0^1(\Omega)$ and the functions of $H_0^1(\Omega)$ do not possess jumps. The functions of P_1^{nc} have generally jumps, see Figure 7.1, and they are not weakly differentiable. In addition, the space is also non-conforming with respect to the boundary condition, which is not satisfied exactly. The functions from P_1^{nc} that will be sought as an approximation of the solution of the boundary value problem (7.7) vanish in the midpoint of the edges at the boundary. However, in the other points at the boundary, their value is generally not equal to zero.

The bilinear form

$$a(u,v) = \int_\Omega (A(\boldsymbol{x})\nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x}) \; d\boldsymbol{x}$$

will be extended to $H_0^1(\Omega) + P_1^{\mathrm{nc}}$ by

$$a^h(u,v) = \sum_{K\in\mathcal{T}^h} \int_K (A(\boldsymbol{x})\nabla u(\boldsymbol{x})) \cdot \nabla v(\boldsymbol{x})\ d\boldsymbol{x} \quad \forall\ u,v \in H_0^1(\Omega) + P_1^{\mathrm{nc}}.$$

Then, the non-conforming finite element method is given by: Find $u^h \in P_1^{\mathrm{nc}}$ with

$$a^h(u^h,v^h) = (f,v^h) \quad \forall\ v^h \in P_1^{\mathrm{nc}}.$$

The goal of this section consists in proving the linear convergence with respect to h in the energy norm $\|\cdot\|_h = \left(a^h(\cdot,\cdot)\right)^{1/2}$. It can be proved that the solution of the continuous problem (7.10) is smooth, i.e., that $u \in H^2(\Omega)$, since $f \in L^2(\Omega)$, the coefficients $a_{ij}(\boldsymbol{x})$ are weakly differentiable with bounded derivatives, and Ω is a convex domain with polygonal boundary. □

Remark 7.8. The error equation. The first step of proving an error estimate consists in deriving an equation for the error. To this end, multiply the continuous problem (7.7) with a test function from $v^h \in P_1^{\mathrm{nc}}$, integrate the product on Ω, and apply integration by parts on each triangle. This approach gives

$$\begin{aligned}
(f,v^h) &= -\sum_{K\in\mathcal{T}^h} \int_K \nabla\cdot(A(\boldsymbol{x})\nabla u(\boldsymbol{x}))\, v^h(\boldsymbol{x})\ d\boldsymbol{x} \\
&= \sum_{K\in\mathcal{T}^h} \int_K (A(\boldsymbol{x})\nabla u(\boldsymbol{x}))\cdot\nabla v^h(\boldsymbol{x})\ d\boldsymbol{x} \\
&\quad - \sum_{K\in\mathcal{T}^h} \int_{\partial K} (A(s)\nabla u(s))\cdot \boldsymbol{n}_K(s) v^h(s)\ ds \\
&= a^h(u,v^h) - \sum_{K\in\mathcal{T}^h} \int_{\partial K} (A(s)\nabla u(s))\cdot \boldsymbol{n}_K(s) v^h(s)\ ds,
\end{aligned}$$

where $\boldsymbol{n}_K$ is the unit outer normal at the edges of the triangles. Subtracting the finite element equation, one obtains

$$a^h(u-u^h,v^h) = \sum_{K\in\mathcal{T}^h} \int_{\partial K} (A(s)\nabla u(s))\cdot \boldsymbol{n}_K(s) v^h(s)\ ds \quad \forall\ v^h \in P_1^{\mathrm{nc}}. \tag{7.11}$$

□

Lemma 7.9. Estimate of the right-hand side of the error equation (7.11)**.** *Assume that $u \in H^2(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$, $i,j=1,2$, then it is*

$$\left| \sum_{K\in\mathcal{T}^h} \int_{\partial K} A(s)\nabla u(s)\cdot \boldsymbol{n}_K(s) v^h(s)\ ds \right| \le Ch\, \|u\|_{H^2(\Omega)} \left\|v^h\right\|_h.$$

Proof. Every edge of the triangulation that is in Ω appears exactly twice in the boundary integrals on ∂K. The corresponding unit normals possess opposite signs. One can choose for each edge one fixed unit normal and then one can write the integrals in the form

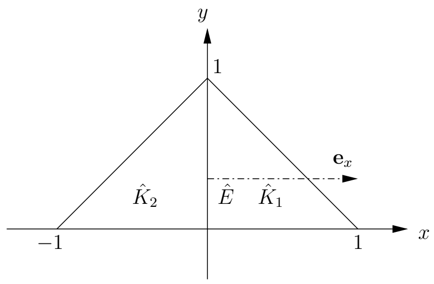

Fig. 7.2 Reference configuration.

$$\sum_E \int_E \left[\left|(A(s)\nabla u(s)) \cdot \boldsymbol{n}_E(s) v^h(s)\right|\right]_E \, ds = \sum_E \int_E (A(s)\nabla u(s)) \cdot \boldsymbol{n}_E(s) \left[\left|v^h\right|\right]_E(s) \, ds,$$

where the sum is taken over all edges $\{E\}$. Here, $\left[\left|v^h\right|\right]_E$ denotes the jump of v^h

$$\left[\left|v^h\right|\right]_E(s) = \begin{cases} v^h|_{K_1}(s) - v^h|_{K_2}(s) & s \in E \subset \Omega, \\ v^h(s) & s \in E \subset \partial\Omega, \end{cases}$$

where $\boldsymbol{n}_E$ is directed from K_1 to K_2 or it is the outer normal on $\partial\Omega$. For writing the integrals in this form, it was used that $\nabla u(s)$, $A(s)$, and $\boldsymbol{n}_E(s)$ are almost everywhere continuous, so that these functions can be written as factor in front of the jumps.

Because of the continuity condition for the functions from P_1^{nc} and the homogeneous Dirichlet boundary condition, it is for all $v^h \in P_1^{\text{nc}}$ that $\left[\left|v^h\right|\right]_E(P) = 0$ for the midpoints P of all edges. From the linearity of the functions on the edges, it follows that

$$\int_E \left[\left|v^h\right|\right]_E(s) \, ds = 0 \quad \forall\, E. \tag{7.12}$$

Let E be an arbitrary edge in Ω that belongs to the triangles K_1 and K_2. The next goal consists in proving the estimate

$$\begin{aligned} &\left|\int_E (A(s)\nabla u(s)) \cdot \boldsymbol{n}_E(s) \left[\left|v^h\right|\right]_E(s) \, ds\right| \\ &\le Ch \left\|u\right\|_{H^2(K_1)} \left(\left\|\nabla v^h\right\|_{L^2(K_1)} + \left\|\nabla v^h\right\|_{L^2(K_2)}\right). \end{aligned} \tag{7.13}$$

To this end, one uses a reference configuration $\left(\hat{K}_1, \hat{K}_2, \hat{E}\right)$, where $\hat{K}_1$ is the unit triangle and $\hat{K}_2$ is the triangle that is obtained by reflecting the unit triangle at the y-axis. The common edge $\hat{E}$ is the interval $(0,1)$ on the y-axis. The unit normal on $\hat{E}$ will be chosen to be the Cartesian unit vector $\boldsymbol{e}_x$, see Figure 7.2. This choice is the other way around than in the definition of the jump, but it is just for simplicity of notation and it does not influence the estimate. The reference configuration can be transformed to (K_1, K_2, E) by a map that is continuous and on both triangles $\hat{K}_i$ affine. For this map, one can prove the same properties for the transform as proved in Chapter 6.

Using (7.12), the Cauchy–Schwarz inequality, and the trace theorem, one obtains for an arbitrary constant $\alpha \in \mathbb{R}$

$$\int_{\hat{E}} \left(\hat{A}(\hat{s})\nabla \hat{u}(\hat{s})\right) \cdot \boldsymbol{e}_x \left[\left|\hat{v}^h\right|\right]_{\hat{E}} \, d\hat{s} = \int_{\hat{E}} \left(\left(\hat{A}(\hat{s})\nabla \hat{u}(\hat{s})\right) \cdot \boldsymbol{e}_x - \alpha\right) \left[\left|\hat{v}^h\right|\right]_{\hat{E}} \, d\hat{s}$$

$$
\begin{aligned}
&\leq \left\| \left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x - \alpha \right\|_{L^2(\hat{E})} \left\| \left[\left|\hat{v}^h\right|\right]_{\hat{E}} \right\|_{L^2(\hat{E})} \\
&\leq C \left\| \left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x - \alpha \right\|_{H^1(\hat{K}_1)} \left\| \left[\left|\hat{v}^h\right|\right]_{\hat{E}} \right\|_{L^2(\hat{E})}
\end{aligned} \qquad (7.14)
$$

In particular, one can choose α such that

$$\int_{\hat{K}_1} \left(\left(\hat{A}(\hat{\boldsymbol{x}})\nabla\hat{u}(\hat{\boldsymbol{x}})\right)\cdot \boldsymbol{e}_x - \alpha\right)\, d\hat{\boldsymbol{x}} = 0.$$

Using first that $(a^2+b^2)^{1/2} \leq a+b$ for $a,b\geq 0$, then the $L^2(\Omega)$ term in the first factor of the right-hand side of (7.14) can be bounded using the estimate from Lemma 6.4 for $k=0$ and $l=1$ and the choice of α

$$
\begin{aligned}
&\left\| \left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x - \alpha \right\|_{H^1(\hat{K}_1)} \\
&\leq \left\| \left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x - \alpha \right\|_{L^2(\hat{K}_1)} + \left\| \nabla\left(\left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x - \alpha\right) \right\|_{L^2(\hat{K}_1)} \\
&\leq C \left\| \nabla\left(\left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x - \alpha\right) \right\|_{L^2(\hat{K}_1)} \\
&= C \left\| \nabla\left(\left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x\right) \right\|_{L^2(\hat{K}_1)}.
\end{aligned}
$$

To estimate the second factor, the trace theorem and the equivalence of norms in finite-dimensional spaces are applied

$$
\begin{aligned}
\left\| \left[\left|\hat{v}^h\right|\right]_{\hat{E}} \right\|_{L^2(\hat{E})} &\leq C\left(\left\|\hat{v}^h\right\|_{H^1(\hat{K}_1)} + \left\|\hat{v}^h\right\|_{H^1(\hat{K}_2)} \right) \\
&\leq C\left(\left\|\nabla\hat{v}^h\right\|_{L^2(\hat{K}_1)} + \left\|\nabla\hat{v}^h\right\|_{L^2(\hat{K}_2)} \right).
\end{aligned} \qquad (7.15)
$$

To apply the norm equivalence, one has to prove that the terms in the last line are in fact norms. Let the terms in the last line be zero, then it follows that $\hat{v}^h = c_1$ in $\hat{K}_1$ and $\hat{v}^h = c_2$ in $\hat{K}_2$. Because $\hat{v}^h$ is continuous in the midpoint of $\hat{E}$, one finds that $c_1 = c_2$ and consequently that $\left[\left|\hat{v}^h\right|\right]_{\hat{E}} = 0$. Hence, also the left-hand side of the estimate is zero and (7.15) is true. It follows that the right-hand side of estimate (7.15) defines a norm in the quotient space of P_1^{nc} with respect to the constraint $\left[\left|\hat{v}^h\right|\right]_{\hat{E}} = 0$.

Altogether, one obtains for the reference configuration

$$
\begin{aligned}
&\left| \int_{\hat{E}} \left(\hat{A}(\hat{s})\nabla\hat{u}(\hat{s})\right)\cdot \boldsymbol{e}_x \left[\left|\hat{v}^h\right|\right]_{\hat{E}}\, d\hat{s} \right| \\
&\leq C \left\| \nabla\left(\left(\hat{A}\nabla\hat{u}\right)\cdot \boldsymbol{e}_x\right) \right\|_{L^2(\hat{K}_1)} \left(\left\|\nabla\hat{v}^h\right\|_{L^2(\hat{K}_1)} + \left\|\nabla\hat{v}^h\right\|_{L^2(\hat{K}_2)} \right).
\end{aligned}
$$

This estimate has to be transformed to the triple (K_1, K_2, E). In this step, one gets for the integral on the edge the factor C (Ch for ∇ and Ch^{-1} for $d\hat{s}$). For the product of the norms on the right-hand side, one obtains the factor Ch (Ch for the first factor and C for the second factor). In addition, one uses that $A(s)$ and all first order derivatives of $A(s)$ are bounded to estimated the first term on the right-hand side (*exercise*). In summary, (7.13) is proved.

The statement of the lemma follows by summing over all edges and by applying on the right-hand side the Cauchy–Schwarz inequality for sums. ■

Theorem 7.10. Finite element error estimate. *Let the assumptions of Lemma 7.9 be satisfied, then it holds the following error estimate*

$$\left\|u-u^h\right\|_h^2 \leq Ch\left\|u\right\|_{H^2(\Omega)}\left\|u-u^h\right\|_h + Ch^2\left\|u\right\|_{H^2(\Omega)}^2. \qquad (7.16)$$

Table 7.1 Example 7.12. Number of degrees of freedom, including nodes at the Dirichlet boundary.

level	P_1^{nc}
1	56
2	208
3	800
4	3136
5	12416
6	49408
7	197120
8	787456
9	3147776

Proof. Applying Lemma 7.9, it follows from the error equation (7.11) that

$$\left|a^h(u-u^h, v^h)\right| \le Ch\,\|u\|_{H^2(\Omega)}\,\left\|v^h\right\|_h \quad \forall\, v^h \in P_1^{\text{nc}}.$$

Let $I^h : H_0^1(\Omega) \to P_1^{\text{nc}}$ be an interpolation operator with optimal interpolation order in $\|\cdot\|_h$. Then, one obtains with the Cauchy–Schwarz inequality, the triangle inequality, and the interpolation estimate

$$\begin{aligned}
\left\|u-u^h\right\|_h^2 &= a^h(u-u^h, u-u^h) = a^h(u-u^h, u-I^h u) + a^h(u-u^h, I^h u - u^h)\\
&\le \left|a^h(u-u^h, u-I^h u)\right| + Ch\,\|u\|_{H^2(\Omega)}\left\|I^h u - u^h\right\|_h\\
&\le \left\|u-u^h\right\|_h \left\|u-I^h u\right\|_h + Ch\,\|u\|_{H^2(\Omega)}\left(\left\|I^h u - u\right\|_h + \left\|u-u^h\right\|_h\right)\\
&\le Ch\left\|u-u^h\right\|_h \|u\|_{H^2(\Omega)} + Ch\,\|u\|_{H^2(\Omega)}\left(h\,\|u\|_{H^2(\Omega)} + \left\|u-u^h\right\|_h\right).
\end{aligned}$$

■

Remark 7.11. To the error estimate. Let the first term on the error bound (7.16) dominate the second term, i.e., $Ch\,\|u\|_{H^2(\Omega)}\left\|u-u^h\right\|_h \ge Ch^2\,\|u\|_{H^2(\Omega)}^2$. Then, it follows that

$$\left\|u-u^h\right\|_h^2 \le Ch\,\|u\|_{H^2(\Omega)}\left\|u-u^h\right\|_h \quad \Longleftrightarrow \quad \left\|u-u^h\right\|_h \le Ch\,\|u\|_{H^2(\Omega)}.$$

If the second term dominates, i.e., $Ch\,\|u\|_{H^2(\Omega)}\left\|u-u^h\right\|_h < Ch^2\,\|u\|_{H^2(\Omega)}^2$, one finds that

$$\left\|u-u^h\right\|_h^2 \le Ch^2\,\|u\|_{H^2(\Omega)}^2 \quad \Longleftrightarrow \quad \left\|u-u^h\right\|_h \le Ch\,\|u\|_{H^2(\Omega)}.$$

In both cases, one obtains that the method converges of first order. □

Example 7.12. Numerical study that supports the finite element error estimate. The same problem as in Example 6.18 is considered. The number of degrees of freedom are given in Table 7.1. One can see that on the same grid, the Crouzeix–Raviart finite element has more degrees of freedom than the P_1 finite element. The order of convergence for $\left\|u-u^h\right\|_h$ is displayed in Figure 7.3. These results support the first order error estimate. □

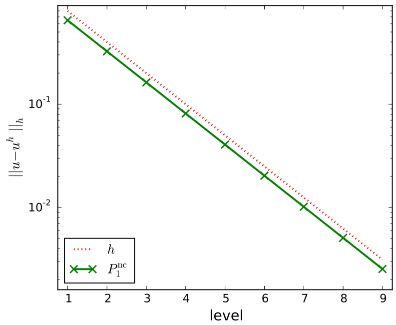

Fig. 7.3 Example 7.12. Convergence of $\left\|u - u^h\right\|_h$ for P_1^{nc}.

7.3 $L^2(\Omega)$ Error Estimate

Remark 7.13. Motivation. A method is called quasi-optimal in a given norm, if the order of the method is the same as the optimal approximation order. Already for one dimension, one can show that at most linear convergence in $H^1(\Omega)$ can be achieved for the best approximation in P_1. This statement can be already verified with the function $v(x) = x^2$. Hence, all considered methods so far are quasi-optimal in the energy norm.

However, the best approximation error in $L^2(\Omega)$ is of one order higher than the best approximation error in $H^1(\Omega)$. A natural question is whether finite element methods converge also of higher order with respect to the error in $L^2(\Omega)$ than with respect to the error in the energy norm.

In this section, it will be shown that one can obtain for finite element methods a higher order of convergence in $L^2(\Omega)$ than in $H^1(\Omega)$. However, there are more restrictive assumptions to prove this property in comparison with the convergence proof for the energy norm. □

Remark 7.14. Model problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a convex polyhedral domain with Lipschitz boundary. The model problem has the form

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{7.17}$$

For proving an error estimate in $L^2(\Omega)$, the regularity of the solution of (7.17) plays an essential role. □

Definition 7.15. m-regular differential operator. Let L be a second order differential operator. This operator is called m-regular, $m \ge 2$, if for all $f \in H^{m-2}(\Omega)$ the solutions of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$, are in the space $H^m(\Omega)$ and the following estimate holds

$$\|u\|_{H^m(\Omega)} \le C \|f\|_{H^{m-2}(\Omega)} + C \|u\|_{H^1(\Omega)} . \tag{7.18}$$

□

Remark 7.16. On the m-regularity.

- The definition is formulated in a way that it can be applied also if the solution of the problem is not unique.
- For the Laplacian, the term $\|u\|_{H^1(\Omega)}$ can be estimated by $\|f\|_{L^2(\Omega)}$ such that with (7.18) one obtains (*exercise*)

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

- Many regularity results can be found in the literature. Loosely speaking, they say that regularity is given if the data of the problem (coefficients of the operator, boundary of the domain) are sufficiently regular. For instance, an elliptic operator in divergence form ($\Delta = \nabla \cdot (A\nabla)$) is 2-regular if the coefficients are from $W^{1,p}(\Omega)$, $p \geq 1$, and if $\partial\Omega$ is a C^2 boundary. Another important result is the 2-regularity of the Laplacian on a convex domain. A comprehensive overview on regularity results can be found in Grisvard (1985).

□

Remark 7.17. Variational form and finite element formulation of the model problem. The variational form of (7.17) is: Find $u \in H_0^1(\Omega)$ with

$$(\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H_0^1(\Omega).$$

The P_1 finite element space, with zero boundary conditions, will be used for the discretization. Then, the finite element problem reads as follows: Find $u^h \in P_1$ such that

$$(\nabla u^h, \nabla v^h) = (f, v^h) \quad \forall\, v^h \in P_1. \tag{7.19}$$

□

Theorem 7.18. Finite element error estimates. *Let $u(\boldsymbol{x})$ be the solution of (7.17), let (7.17) be 2-regular, and let $u^h(\boldsymbol{x})$ be the solution of (7.19). Then, the following error estimates hold*

$$\begin{aligned} \left\|\nabla(u - u^h)\right\|_{L^2(\Omega)} &\leq Ch \|f\|_{L^2(\Omega)}, \\ \left\|u - u^h\right\|_{L^2(\Omega)} &\leq Ch^2 \|f\|_{L^2(\Omega)}. \end{aligned}$$

Proof. With the error estimate in $H^1(\Omega)$, Corollary 6.16, and the 2-regularity, one obtains

$$\left\|\nabla(u - u^h)\right\|_{L^2(\Omega)} \leq Ch \|u\|_{H^2(\Omega)} \leq Ch \|f\|_{L^2(\Omega)}. \tag{7.20}$$

For proving the $L^2(\Omega)$ error estimate, let $w \in H_0^1(\Omega)$ be the unique solution of the so-called dual problem

$$(\nabla v, \nabla w) = (u - u^h, v) \quad \forall\, v \in H_0^1(\Omega).$$

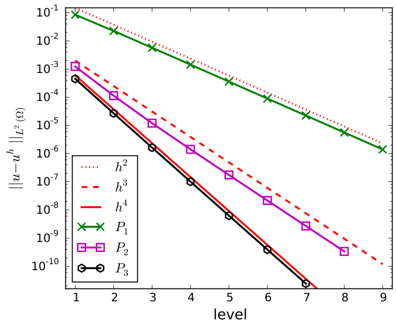

Fig. 7.4 Example 7.19. Convergence of $\|u-u^h\|_{L^2(\Omega)}$ for different finite elements.

For a symmetric differential operator, the dual problem has the same form like the original (primal) problem. Hence, the dual problem is also 2-regular and it holds the estimate

$$\|w\|_{H^2(\Omega)} \le C \left\|u-u^h\right\|_{L^2(\Omega)}.$$

For performing the error estimate, the Galerkin orthogonality of the error is utilized

$$(\nabla(u-u^h), \nabla v^h) = (\nabla u, \nabla v^h) - (\nabla u^h, \nabla v^h) = (f, v^h) - (f, v^h) = 0$$

for all $v^h \in P_1$. Now, the error $u-u^h$ is used as test function v in the dual problem. Let $I^h w$ be the interpolant of w in P_1. Using the Galerkin orthogonality, the interpolation estimate, and the regularity of w, one obtains

$$\begin{aligned}
\left\|u-u^h\right\|_{L^2(\Omega)}^2 &= (\nabla(u-u^h), \nabla w) = (\nabla(u-u^h), \nabla(w - I^h w)) \\
&\le \left\|\nabla(u-u^h)\right\|_{L^2(\Omega)} \left\|\nabla(w-I^h w)\right\|_{L^2(\Omega)} \\
&\le Ch \|w\|_{H^2(\Omega)} \left\|\nabla(u-u^h)\right\|_{L^2(\Omega)} \\
&\le Ch \left\|u-u^h\right\|_{L^2(\Omega)} \left\|\nabla(u-u^h)\right\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, division by $\left\|u-u^h\right\|_{L^2(\Omega)}$ and the application of the already known error estimate (7.20) for $\left\|\nabla(u-u^h)\right\|_{L^2(\Omega)}$ are used for completing the proof of the theorem. ■

Example 7.19. Numerical study that supports the finite element error estimate. The same problem as in Example 6.18 is considered. Numerical results concerning the convergence of the error $\left\|u-u^h\right\|_{L^2(\Omega)}$ are presented in Figure 7.4. These results support the error estimate given in Theorem 7.18. □

7.4 Outlook

Remark 7.20. Outlook to forthcoming classes. This class provided an introduction to numerical methods for solving partial differential equations and the numerical analysis of these methods. There are many further aspects that might be covered in forthcoming classes.

Further aspects for elliptic problems.

- Adaptive methods and a posteriori error estimators. It will be shown how it is possible to estimate the error of the computed solution only using known quantities and in this way one can decide where it makes sense to refine the mesh and where not.
- Multigrid methods. Multigrid methods are for certain classes of problems optimal solvers.
- Numerical analysis of problems with other boundary conditions or taking into account quadrature rules.

Time-dependent problems. As mentioned in Remark 1.7, standard approaches for the numerical solution of time-dependent problems are based on solving stationary problems in each discrete time.

- The numerical analysis of discretizations of time-dependent problems has some new aspects, but also many tools from the analysis of steady-state problems are used.

Convection-diffusion equations. Convection-diffusion equations are of importance in many applications. Generally, the convection (first order differential operator) dominates the diffusion (second order differential operator).

- In the convection-dominated regime, the Galerkin method as presented in this class does not work. One needs new ideas for discretizations and these new discretizations create new challenges for the numerical analysis.

Problems with more than one unknown function. The fundamental equation of fluid dynamics, the Navier–Stokes equations, Section 1.3, belong to this class.

- It will turn out that the discretization of the Navier–Stokes equations requires special care in the choice of the finite element spaces. The numerical analysis becomes rather involved.

$\square$