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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 07

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Rational functions in Lebesgue spaces in a ball.* Solve the following problems.
   i) For which values of $a \in \mathbb{R}$ is the function $f : (-1,1) \to \mathbb{R}$
   $$f(x) = \begin{cases} |x|^a & x \neq 0, \\ 0 & x = 0 \end{cases}$$
   an element of $L^p((-1,1))$ with $p \in [1,\infty]$?
   ii) Let
   $$B_1(\mathbf{0}) = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 < 1\}$$
   be the d-dimensional unit ball, $d > 1$. Find the values $a \in \mathbb{R}$ for which the function $f : B_1(\mathbf{0}) \to \mathbb{R}$ with
   $$f(x) = \begin{cases} \|\boldsymbol{x}\|_2^a & \boldsymbol{x} \neq \mathbf{0}, \\ 0 & \boldsymbol{x} = \mathbf{0}, \end{cases}$$
   belongs to $L^p(B_1(\mathbf{0}))$ with $p \in [1,\infty]$!

2. *Interpolation inequality.* Let Ω be a bounded Lipschitz domain and let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Prove the interpolation inequality
   $$\|\nabla u\|_{L^2(\Omega)}^2 \leq \|u\|_{L^2(\Omega)} \|\Delta u\|_{L^2(\Omega)}.$$

3. *Poincaré–Friedrichs type inequality.* Prove the following inequality of Poincaré–Friedrichs type. Let Ω be a bounded domain with Lipschitz boundary and let $\Omega' \subset \Omega$ with $\mathrm{meas}_{\mathbb{R}^d}(\Omega') = \int_{\Omega'} d\mathbf{x} > 0$, then for all $u \in W^{1,p}(\Omega)$ it is
   $$\int_\Omega |u(\mathbf{x})|^p \ d\mathbf{x} \leq C \left(\left| \int_{\Omega'} u(\mathbf{x}) \ d\mathbf{x} \right|^p + \int_\Omega \|\nabla u(\mathbf{x})\|_2^p \ d\mathbf{x} \right).$$

4. *Integration by parts.* Prove Corollary 3.44: Let the conditions of Theorem 3.42 on the domain Ω be satisfied. Consider $w \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $p \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, it is
   $$\int_\Omega \partial_i w(\mathbf{x}) v(\mathbf{x}) \ d\mathbf{x} = \int_\Gamma w(\boldsymbol{s}) v(\boldsymbol{s}) \boldsymbol{n}_i(\boldsymbol{s}) \ d\boldsymbol{s} - \int_\Omega w(\mathbf{x}) \partial_i v(\mathbf{x}) \ d\mathbf{x}.$$

The exercise problems should be solved in groups of two or three students. The written parts have to be submitted until **Thursday, June 06, 2019**. The executable codes have to be send by email to A. Jha.