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Numerical Mathematics III – Partial Differential Equations

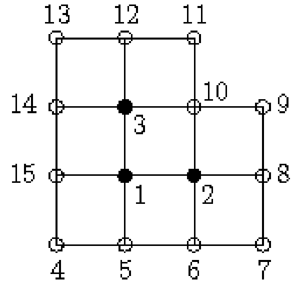
Exercise Problems 03

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Second order FD approximation of first derivative on non-equidistant grid.* Let $v(\mathbf{x})$ be a sufficiently smooth function. Derive a second order finite difference approximation of the first derivative in the node x_i for a non-equidistant grid with the points $x_i - h^-, x_i, x_i + h^+$. Is a third order approximation possible? Hint: For an equidistant grid, the central difference is of second order, which is just the arithmetic average of the forward and the backward difference. To solve this problem, one has to find a different linear combination of the forward and backward difference.

2. *Finite difference approximation of the second order derivative at a non-equidistant grid.* Consider the interval $[x - h_x^-, x + h_x^+]$ with $h_x^-, h_x^+ > 0$, $h_x^- \neq h_x^+$.
   i) Assume $u \in C^3\left([x - h_x^-, x + h_x^+]\right)$. Show the following consistency estimate
   $$\left| u''(x) - \frac{2}{h_x^+ + h_x^-}\left(\frac{u(x + h_x^+) - u(x)}{h_x^+} - \frac{u(x) - u(x - h_x^-)}{h_x^-}\right)\right| \le C\left(h_x^+ + h_x^-\right).$$
   ii) Prove that there is no other approximation which satisfies
   $$\left|u''(x) - \left(\alpha\, u\left(x - h_x^-\right) + \beta\, u(x) + \gamma\, u\left(x + h_x^+\right)\right)\right| \le C\left(h_x^+ + h_x^-\right)$$
   with $\alpha = \alpha\left(h_x^-, h_x^+\right)$, $\beta = \beta\left(h_x^-, h_x^+\right)$, $\gamma = \gamma\left(h_x^-, h_x^+\right) \in \mathbb{R}$, and the constant C does not explicitly depends on $u(\mathbf{x})$, $u'(\mathbf{x})$, and $u''(\mathbf{x})$.

3. *Five point stencil.* Consider the Dirichlet problem for the Poisson equation
   $$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$
   and the corresponding finite difference discretization with the five point stencil on the following grid:



Compute the matrices $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 12}$ for the finite difference equation

$$A\underline{u} = \underline{f} + B\underline{g}$$

with $\underline{u} = (u_1, u_2, u_3)^T$ and $\underline{g} = (u_4, \ldots, u_{15})^T$.

4. **This problem has to be solved until May 16, 2019.** *Code for five point stencil.* Write a MATLAB or Octave code for the numerical solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

The right-hand side and the boundary conditions should be chosen such that

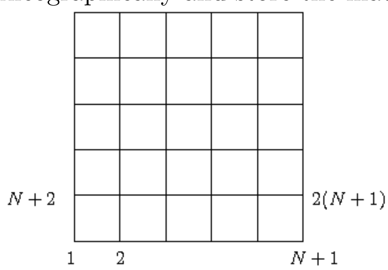
$$u(x,y) = x^4 y^5 - 17\sin(xy)$$

is the solution of the boundary value problem.

Use the five point stencil for the discretizing the partial differential equation. The mesh widths should be chosen to be

$$h_x = h_y = h = 2^{-n} \quad n = 2, 3, 4, \ldots, 8.$$

Order the nodes lexicographically and store the matrix in `sparse` format.



Compute the following errors

$$\|u - u_h\|_{l^\infty(\omega_h)}, \qquad \|u - u_h\|_{l^2(\omega_h)},$$

and the orders of convergence, based on the errors on the two finest meshes, where the second norm is the standard Euclidean vector norm.

The exercise problems should be solved in groups of two or three students. The written parts have to be submitted until **Thursday, May 09, 2019** to A. Jha. The executable codes have to be send by email to A. Jha.