2. *Local basis of* $P_2(\hat{K})$. Consider the reference triangle $\widehat{K}$ with the vertices $(0,0)$, $(1,0)$, and $(0,1)$. The space of polynomials of degree two is spanned by

$$1,\ \widehat{x},\ \widehat{y},\ \widehat{x}\widehat{y},\ \widehat{x}^2,\ \widehat{y}^2.$$

Use as functionals the values of the functions in the vertices and the barycenters of the edges. Compute the local basis with respect to these functionals.

Solution:

Denote the basis given in the problem by $\{p_k\}$, $k = 1, \ldots 6$. For transforming this basis, one uses the ansatz

$$\phi_{\widehat{K},j} = \sum_{k=1}^{6} c_{jk} p_k \quad j = 1, \ldots, 6.$$

From the condition for the local basis, one gets

$$\Phi_{\widehat{K},i}(\phi_{\widehat{K},j}) = \delta_{ij} = \sum_{k=1}^{6} c_{jk}\, \Phi_{\widehat{K},i}(p_k) \quad i, j = 1, \ldots 6. \tag{5.1}$$

The functions of $P_2(\hat{K})$ are the values in the nodes $A_1 = (0,0)$, $A_2 = (1/2,0)$, $A_3 = (1,0)$, $A_4 = (0,1/2)$, $A_5 = (1/2,1/2)$, and $A_6 = (0,1)$. With this numbering, equation (5.1) leads to a linear system of equations whose matrix can be computed by evaluating $\Phi_{\widehat{K},i}(p_k)$, where the i-th row corresponds to the i-th functional and the k-th column to the k-th basis function (not vanishing at the k-th node)

$$\Phi_{\widehat{K},i}(p_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

One obtains, for $j = 1$, i.e., for the node A_1,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \\ c_{16} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad c_{1j} = \begin{pmatrix} 1 \\ -3 \\ -3 \\ 4 \\ 2 \\ 2 \end{pmatrix}.$$

For the linear systems of equations for the other nodes, one gets

$$\begin{aligned} c_{2j} &= (0,4,0,-4,-4,0)^T, \\ c_{3j} &= (0,-1,0,0,2,0)^T, \\ c_{4j} &= (0,0,4,-4,0,-4)^T, \\ c_{5j} &= (0,0,0,4,0,0)^T, \\ c_{6j} &= (0,0,-1,0,0,2)^T. \end{aligned}$$

Collecting the results, one finds that the local basis has the form

$$
\begin{aligned}
\phi_{\widehat{K},1} &= 1 - 3\widehat{x} - 3\widehat{y} + 4\widehat{x}\widehat{y} + 2\widehat{x}^2 + 2\widehat{y}^2, \\
\phi_{\widehat{K},2} &= 4\widehat{x} - 4\widehat{x}\widehat{y} - 4\widehat{x}^2, \\
\phi_{\widehat{K},3} &= -\widehat{x} + 2\widehat{x}^2, \\
\phi_{\widehat{K},4} &= 4\widehat{y} - 4\widehat{x}\widehat{y} - 4\widehat{y}^2, \\
\phi_{\widehat{K},5} &= 4\widehat{x}\widehat{y}, \\
\phi_{\widehat{K},6} &= -\widehat{y} + 2\widehat{y}^2.
\end{aligned}
$$

□

3. *Local basis of* $Q_1^{\text{rot}}(\hat{K})$. Consider the reference square $\widehat{K} = [-1,1]^2$ and the space that is spanned by the functions

$$1,\ \widehat{x},\ \widehat{y},\ \widehat{x}^2 - \widehat{y}^2.$$

Let the integral mean values on the edges E_i, $i = 1, \ldots, 4$, of $\widehat{K}$ be the functionals of the finite element

$$\Phi(v) = \frac{1}{\text{meas}(E_i)} \int_{E_i} v(s)\, ds \quad i = 1, \ldots, 4.$$

Compute the local basis with respect to these functionals.

Solution:

The way to solve this problem is analogous to the solution of Problem 2. Numerate the the edges as follows: E_1 at $y = -1$, E_2 at $x = 1$, E_3 at $y = 1$, and E_4 at $x = -1$. The matrix for the linear systems of equations is given by

$$\Phi_{\widehat{K},i}(p_k) = \begin{pmatrix} 1 & 0 & -1 & -\frac{2}{3} \\ 1 & 1 & 0 & \frac{2}{3} \\ 1 & 0 & 1 & -\frac{2}{3} \\ 1 & -1 & 0 & \frac{2}{3} \end{pmatrix}$$

The solutions are

$$\begin{aligned} c_{1j} &= \left(\frac{1}{4}, 0, -\frac{1}{2}, -\frac{3}{8}\right)^T, \\ c_{2j} &= \left(\frac{1}{4}, \frac{1}{2}, 0, \frac{3}{8}\right)^T, \\ c_{3j} &= \left(\frac{1}{4}, 0, \frac{1}{2}, -\frac{3}{8}\right)^T, \\ c_{4j} &= \left(\frac{1}{4}, -\frac{1}{2}, 0, \frac{3}{8}\right)^T, \end{aligned}$$

and the local basis has the form

$$\begin{aligned} \phi_{\widehat{K},1} &= \frac{1}{8}\left(2 - 4\widehat{y} - 3\widehat{x}^2 + 3\widehat{y}^2\right), \\ \phi_{\widehat{K},2} &= \frac{1}{8}\left(2 + 4\widehat{x} + 3\widehat{x}^2 - 3\widehat{y}^2\right), \\ \phi_{\widehat{K},3} &= \frac{1}{8}\left(2 + 4\widehat{y} - 3\widehat{x}^2 + 3\widehat{y}^2\right), \\ \phi_{\widehat{K},4} &= \frac{1}{8}\left(2 - 4\widehat{x} + 3\widehat{x}^2 - 3\widehat{y}^2\right). \end{aligned}$$

□