Chapter 5

Finite Element Methods

5.1 Finite Element Spaces

Remark 5.1 *Mesh cells, faces, edges, vertices.* A mesh cell K is a compact polyhedron in $\mathbb{R}^d$, $d \in \{2,3\}$, whose interior is not empty. The boundary ∂K of K consists of m-dimensional linear manifolds (points, pieces of straight lines, pieces of planes), $0 \le m \le d-1$, which are called m-faces. The 0-faces are the vertices of the mesh cell, the 1-faces are the edges, and the $(d-1)$-faces are just called faces. □

Remark 5.2 *Finite dimensional spaces defined on K.* Let $s \in \mathbb{N}$. Finite element methods use finite dimensional spaces $P(K) \subset C^s(K)$ which are defined on K. In general, $P(K)$ consists of polynomials. The dimension of $P(K)$ will be denoted by $\dim P(K) = N_K$. □

Example 5.3 *The space $P(K) = P_1(K)$.*The space consisting of linear polynomials on a mesh cell K is denoted by $P_1(K)$:

$$P_1(K) = \left\{ a_0 + \sum_{i=1}^{d} a_i x_i \ : \ \mathbf{x} = (x_1, \ldots, x_d)^T \in K \right\}.$$

There are $d+1$ unknown coefficients a_i, $i = 0, \ldots, d$, such that $\dim P_1(K) = N_K = d+1$. □

Remark 5.4 *Linear functionals defined on $P(K)$.* For the definition of finite elements, linear functional which are defined on $P(K)$ are of importance.

Consider linear and continuous functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K} \ : C^s(K) \to \mathbb{R}$ which are linearly independent. There are different types of functionals which can be utilized in finite element methods:

- point values: $\Phi(v) = v(\mathbf{x})$, $\mathbf{x} \in K$,
- point values of a first partial derivative: $\Phi(v) = \partial_i v(\mathbf{x})$, $\mathbf{x} \in K$,
- point values of the normal derivative on a face E of K: $\Phi(v) = \nabla v(\mathbf{x}) \cdot \mathbf{n}_E$, $\mathbf{n}_E$ is the outward pointing unit normal vector on E,
- integral mean values on K: $\Phi(v) = \frac{1}{|K|} \int_K v(\mathbf{x}) \, d\mathbf{x}$,
- integral mean values on faces E: $\Phi(v) = \frac{1}{|E|} \int_E v(\mathbf{s}) \, d\mathbf{s}$.

The smoothness parameter s has to be chosen in such a way that the functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$ are continuous. If, e.g., a functional requires the evaluation of a partial derivative or a normal derivative, then one has to choose at least $s = 1$. For the other functionals given above, $s = 0$ is sufficient. □

Definition 5.5 Unisolvence of $P(K)$ with respect to the functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$. The space $P(K)$ is called unisolvent with respect to the functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$ if there is for each $\mathbf{a} \in \mathbb{R}^{N_K}$, $\mathbf{a} = (a_1, \ldots, a_{N_K})^T$, exactly one $p \in P(K)$ with

$$\Phi_{K,i}(p) = a_i, \quad 1 \le i \le N_K.$$

□

Remark 5.6 *Local basis.* Unisolvence means that for each vector $\mathbf{a} \in \mathbb{R}^{N_K}$, $\mathbf{a} = (a_1, \ldots, a_{N_K})^T$, there is exactly one element in $P(K)$ such that a_i is the image of the i-th functional, $i = 1, \ldots, N_K$.

Choosing in particular the Cartesian unit vectors for $\mathbf{a}$, then it follows from the unisolvence that a set $\{\phi_{K,i}\}_{i=1}^{N_K}$ exists with $\phi_{K,i} \in P(K)$ and

$$\Phi_{K,i}(\phi_{K,j}) = \delta_{ij}, \quad i, j = 1, \ldots, N_K.$$

Consequently, the set $\{\phi_{K,i}\}_{i=1}^{N_K}$ forms a basis of $P(K)$. This basis is called local basis. □

Remark 5.7 *Transform of an arbitrary basis to the local basis.* If an arbitrary basis $\{p_i\}_{i=1}^{N_K}$ of $P(K)$ is known, then the local basis can be computed by solving a linear system of equations. To this end, represent the local basis in terms of the known basis

$$\phi_{K,j} = \sum_{k=1}^{N_K} c_{jk} p_k, \quad c_{jk} \in \mathbb{R},\ j = 1, \ldots, N_K,$$

with unknown coefficients c_{jk}. Applying the definition of the local basis leads to the linear system of equations

$$\Phi_{K,i}(\phi_{K,j}) = \sum_{k=1}^{N_K} c_{jk} a_{ik} = \delta_{ij}, \quad i, j = 1, \ldots, N_K, \quad a_{ik} = \Phi_{K,i}(p_k).$$

Because of the unisolvence, the matrix $A = (a_{ij})$ is non-singular and the coefficients c_{jk} are determined uniquely. □

Example 5.8 *Local basis for the space of linear functions on the reference triangle.* Consider the reference triangle $\hat{K}$ with the vertices $(0,0)$, $(1,0)$, and $(0,1)$. A linear space on $\hat{K}$ is spanned by the functions $1, \hat{x}, \hat{y}$. Let the functionals be defined by the values of the functions in the vertices of the reference triangle. Then, the given basis is not a local basis because the function 1 does not vanish at the vertices.

Consider first the vertex $(0,0)$. A linear basis function $a\hat{x} + b\hat{y} + c$ which has the value 1 in $(0,0)$ and which vanishes in the other vertices has to satisfy the following set of equations

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is $a = -1, b = -1, c = 1$. The two other basis functions of the local basis are $\hat{x}$ and $\hat{y}$, such that the local basis has the form $\{1 - \hat{x} - \hat{y}, \hat{x}, \hat{y}\}$. □

Remark 5.9 *Triangulation, grid, mesh, grid cell.* For the definition of global finite element spaces, a decomposition of the domain Ω into polyhedrons K is needed. This decomposition is called triangulation $\mathcal{T}^h$ and the polyhedrons K are called mesh cells. The union of the polyhedrons is called grid or mesh.

A triangulation is called regular, see the definition in Ciarlet Ciarlet (1978), if:

- It holds $\overline{\Omega} = \cup_{K \in \mathcal{T}^h} K$.

- Each mesh cell $K \in \mathcal{T}^h$ is closed and the interior $\mathring{K}$ is non-empty.
- For distinct mesh cells K_1 and K_2 there holds $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$.
- For each $K \in \mathcal{T}^h$, the boundary ∂K is Lipschitz-continuous.
- The intersection of two mesh cells is either empty or a common m-face, $m \in \{0, \ldots, d-1\}$.

□

Remark 5.10 *Global and local functionals.* Let $\Phi_1, \ldots, \Phi_N : C^s(\overline{\Omega}) \to \mathbb{R}$ continuous linear functionals of the same types as given in Remark 5.4. The restriction of the functionals to $C^s(K)$ defines local functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$, where it is assumed that the local functionals are unisolvent on $P(K)$. The union of all mesh cells K_j, for which there is a $p \in P(K_j)$ with $\Phi_i(p) \neq 0$, will be denoted by ω_i. □

Example 5.11 *On subdomains ω_i.* Consider the two-dimensional case and let Φ_i be defined as nodal value of a function in $\mathbf{x} \in K$. If $\mathbf{x} \in \mathring{K}$, then $\omega_i = K$. In the case that $\mathbf{x}$ is on a face of K but not in a vertex, then ω_i is the union of K and the other mesh cell whose boundary contains this face. Last, if $\mathbf{x}$ is a vertex of K, then ω_i is the union of all mesh cells which possess this vertex, see Figure 5.1. □

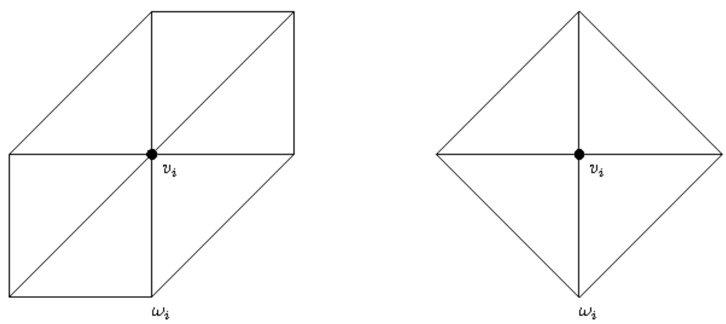

Figure 5.1: Subdomains ω_i.

Definition 5.12 Finite element space, global basis. A function $v(\mathbf{x})$ defined on Ω with $v|_K \in P(K)$ for all $K \in \mathcal{T}^h$ is called continuous with respect to the functional $\Phi_i : \Omega \to \mathbb{R}$ if

$$\Phi_i(v|_{K_1}) = \Phi_i(v|_{K_2}), \quad \forall\, K_1, K_2 \in \omega_i.$$

The space

$$S = \Big\{ v \in L^\infty(\Omega) \ :\ v|_K \in P(K) \text{ and } v \text{ is continuous with respect to } \Phi_i, i = 1, \ldots, N \Big\}$$

is called finite element space.

The global basis $\{\phi_j\}_{j=1}^N$ of S is defined by the condition

$$\phi_j \in S, \quad \Phi_i(\phi_j) = \delta_{ij}, \quad i, j = 1, \ldots, N.$$

□

Example 5.13 *Piecewise linear global basis function.* Figure 5.2 shows a piecewise linear global basis function in two dimensions. Because of its form, such a function is called hat function. □

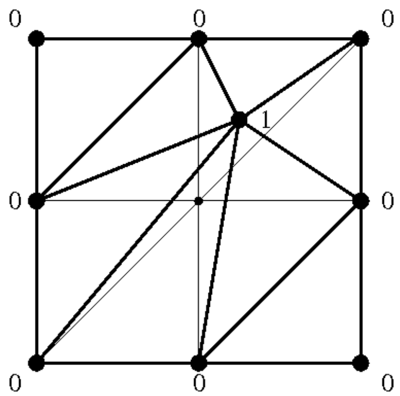

Figure 5.2: Piecewise linear global basis function (boldface lines), hat function.

Remark 5.14 *On global basis functions.* A global basis function coincides on each mesh cell with a local basis function. This property implies the uniqueness of the global basis functions.

For many finite element spaces it follows from the continuity with respect to $\{\Phi_i\}_{i=1}^N$, the continuity of the finite element functions themselves. Only in this case, one can speak of values of finite element functions on m-faces with $m < d$. □

Definition 5.15 Parametric finite elements. Let $\hat{K}$ be a reference mesh cell with the local space $P(\hat{K})$, the local functionals $\hat{\Phi}_1, \ldots, \hat{\Phi}_{\hat{N}}$, and a class of bijective mappings $\{F_K \ : \hat{K} \to K\}$. A finite element space is called a parametric finite element space if:

- The images $\{K\}$ of $\{F_K\}$ form the set of mesh cells.
- The local spaces are given by

$$P(K) = \left\{ p \ : \ p = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P}(\hat{K}) \right\}. \tag{5.1}$$

- The local functionals are defined by

$$\Phi_{K,i}(v(\mathbf{x})) = \hat{\Phi}_i \left(v(F_K(\hat{\mathbf{x}})) \right), \tag{5.2}$$

  where $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_d)^T$ are the coordinates of the reference mesh cell and it holds $\mathbf{x} = F_K(\hat{\mathbf{x}})$.

□

Remark 5.16 *Motivations for using parametric finite elements.* Definition 5.12 of finite elements spaces is very general. For instance, different types of mesh cells are allowed. However, as well the finite element theory as the implementation of finite element methods become much simpler if only parametric finite elements are considered. □

5.2 Finite Elements on Simplices

Definition 5.17 d-simplex. **A d-simplex $K \subset \mathbb{R}^d$ is the convex hull of $(d+1)$ points $\mathbf{a}_1, \ldots, \mathbf{a}_{d+1} \in \mathbb{R}^d$ which form the vertices of K.** □

Remark 5.18 *On d-simplices.* It will be always assumed that the simplex is not degenerated, i.e., its d-dimensional measure is positive. This property is equivalent to the non-singularity of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,d+1} \\ a_{21} & a_{22} & \dots & a_{2,d+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \dots & a_{d,d+1} \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

where $\mathbf{a}_i = (a_{1i}, a_{2i}, \ldots, a_{di})^T$, $i = 1, \ldots, d+1$.

For $d = 2$, the simplices are the triangles and for $d = 3$ they are the tetrahedrons. □

Definition 5.19 Barycentric coordinates. Since K is the convex hull of the points $\{\mathbf{a}_i\}_{i=1}^{d+1}$, the parametrization of K with a convex combination of the vertices reads as follows

$$K = \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{x} = \sum_{i=1}^{d+1} \lambda_i \mathbf{a}_i, \ 0 \le \lambda_i \le 1, \ \sum_{i=1}^{d+1} \lambda_i = 1 \right\}.$$

The coefficients $\lambda_1, \ldots, \lambda_{d+1}$ are called barycentric coordinates of $\mathbf{x} \in K$. □

Remark 5.20 *On barycentric coordinates.* From the definition it follows that the barycentric coordinates are the solution of the linear system of equations

$$\sum_{i=1}^{d+1} a_{ji}\lambda_i = x_j, \quad 1 \le j \le d, \quad \sum_{i=1}^{d+1} \lambda_i = 1.$$

Since the system matrix is non-singular, see Remark 5.18, the barycentric coordinates are determined uniquely.

The barycentric coordinates of the vertex $\mathbf{a}_i$, $i = 1, \ldots, d+1$, of the simplex is $\lambda_i = 1$ and $\lambda_j = 0$ if $i \ne j$. Since $\lambda_i(\mathbf{a}_j) = \delta_{ij}$, the barycentric coordinate λ_i can be identified with the linear function which has the value 1 in the vertex $\mathbf{a}_i$ and which vanishes in all other vertices $\mathbf{a}_j$ with $j \ne i$.

The barycenter of the simplex is given by

$$S_K = \frac{1}{d+1} \sum_{i=1}^{d+1} \mathbf{a}_i = \sum_{i=1}^{d+1} \frac{1}{d+1} \mathbf{a}_i.$$

Hence, its barycentric coordinates are $\lambda_i = 1/(d+1)$, $i = 1, \ldots, d+1$. □

Remark 5.21 *Simplicial reference mesh cells.* A commonly used reference mesh cell for triangles and tetrahedrons is the unit simplex

$$\hat{K} = \left\{ \hat{\mathbf{x}} \in \mathbb{R}^d \ : \ \sum_{i=1}^{d} \hat{x}_i \le 1, \ \hat{x}_i \ge 0, \ i = 1, \ldots, d \right\},$$

see Figure 5.3. The class $\{F_K\}$ of admissible mappings are the bijective affine mappings

$$F_K \hat{\mathbf{x}} = B\hat{\mathbf{x}} + \mathbf{b}, \qquad B \in \mathbb{R}^{d \times d}, \ \det(B) \ne 0, \ \mathbf{b} \in \mathbb{R}^d.$$

The images of these mappings generate the set of the non-degenerated simplices $\{K\} \subset \mathbb{R}^d$. □

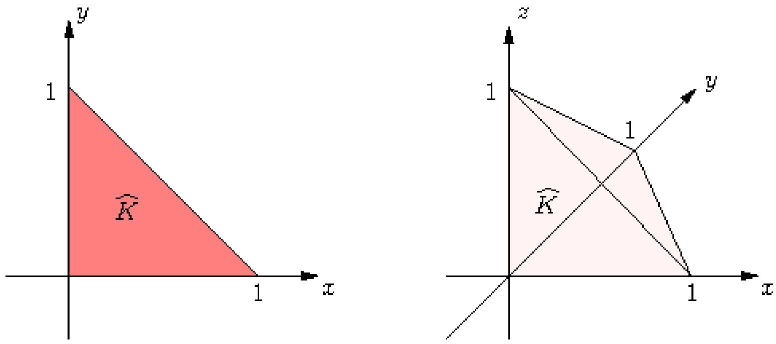

Figure 5.3: The unit simplices in two and three dimensions.

Definition 5.22 Affine family of simplicial finite elements. Given a simplicial reference mesh cell $\hat{K}$, affine mappings $\{F_K\}$, and an unisolvent set of functionals on $\hat{K}$. Using (5.1) and (5.2), one obtains a local finite element space on each non-degenerated simplex. The set of these local spaces is called affine family of simplicial finite elements. □

Definition 5.23 Polynomial space P_k. Let $\mathbf{x} = (x_1, \ldots, x_d)^T$, $k \in \mathbb{N} \cup \{0\}$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)^T$. Then, the polynomial space P_k is given by

$$P_k = \operatorname{span}\left\{\prod_{i=1}^{d} x_i^{\alpha_i} = \mathbf{x}^{\boldsymbol{\alpha}} \ : \ \alpha_i \in \mathbb{N} \cup \{0\} \ \text{ for } \ i = 1, \ldots, d, \ \sum_{i=1}^{d} \alpha_i \le k\right\}.$$

□

Remark 5.24 *Lagrangian finite elements.* In all examples given below, the linear functionals on the reference mesh cell $\hat{K}$ are the values of the polynomials with the same barycentric coordinates as on the general mesh cell K. Finite elements whose linear functionals are values of the polynomials on certain points in K are called Lagrangian finite elements. □

Example 5.25 P_0 : *piecewise constant finite element.* The piecewise constant finite element space consists of discontinuous functions. The linear functional is the value of the polynomial in the barycenter of the mesh cell, see Figure 5.4. It is $\dim P_0(K) = 1$. □

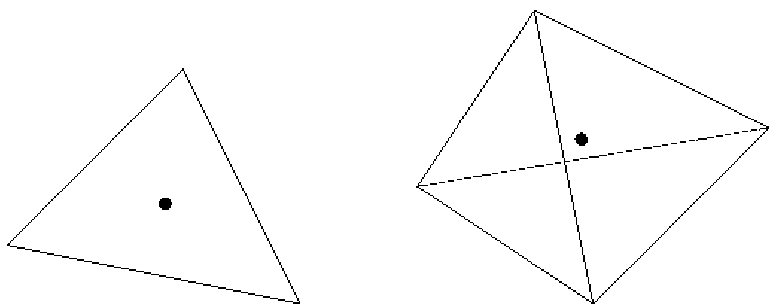

Figure 5.4: The finite element $P_0(K)$.

Example 5.26 P_1 : *conforming piecewise linear finite element.* This finite element space is a subspace of $C(\overline{\Omega})$. The linear functionals are the values of the function in the vertices of the mesh cells, see Figure 5.5. It follows that $\dim P_1(K) = d + 1$.

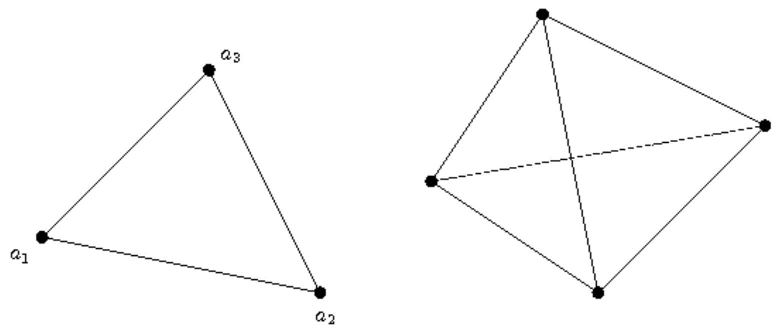

Figure 5.5: The finite element $P_1(K)$.

The local basis for the functionals $\{\Phi_i(v) = v(\mathbf{a}_i),\ i = 1, \ldots, d+1\}$, is $\{\lambda_i\}_{i=1}^{d+1}$ since $\Phi_i(\lambda_j) = \delta_{ij}$, see Remark 5.20. Since a local basis exists, the functionals are unisolvent with respect to the polynomial space $P_1(K)$.

Now, it will be shown that the corresponding finite element space consists of continuous functions. Let K_1, K_2 be two mesh cells with the common face E and let $v \in P_1(= S)$. The restriction of v_{K_1} on E is a linear function on E as well as the restriction of v_{K_2} on E. It has to be shown that both linear functions are identical. A linear function on the $(d-1)$-dimensional face E is uniquely determined with d linearly independent functionals which are defined on E. These functionals can be chosen to be the values of the function in the d vertices of E. The functionals in S are continuous, by the definition of S. Thus, it must hold that both restrictions on E have the same values in the vertices of E. Hence, it is $v_{K_1}|_E = v_{K_2}|_E$ and the functions from P_1 are continuous. □

Example 5.27 P_2 : *conforming piecewise quadratic finite element.* This finite element space is also a subspace of $C(\overline{\Omega})$. It consists of piecewise quadratic functions. The functionals are the values of the functions in the $d+1$ vertices of the mesh cell and the values of the functions in the centers of the edges, see Figure 5.6. Since each vertex is connected to each other vertex, there are $\sum_{i=1}^{d} i = d(d+1)/2$ edges. Hence, it follows that $\dim P_2(K) = (d+1)(d+2)/2$.

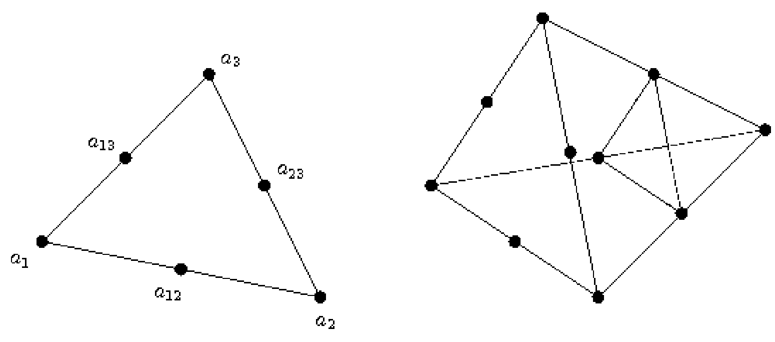

Figure 5.6: The finite element $P_2(K)$.

The part of the local basis which belongs to the functionals $\{\Phi_i(v) = v(\mathbf{a}_i),$ $i = 1, \ldots, d+1\}$, is given by

$$\{\phi_i(\lambda) = \lambda_i(2\lambda_i - 1), \quad i = 1, \ldots, d+1\}.$$

Denote the center of the edges between the vertices $\mathbf{a}_i$ and $\mathbf{a}_j$ by $\mathbf{a}_{ij}$. The corresponding part of the local basis is given by

$$\{\phi_{ij} = 4\lambda_i\lambda_j, \quad i,j = 1,\ldots,d+1,\ i<j\}.$$

The unisolvence follows from the fact that there exists a local basis. The continuity of the corresponding finite element space is shown in the same way as for the P_1 finite element. The restriction of a quadratic function in a mesh cell to a face E is a quadratic function on that face. Hence, the function on E is determined uniquely with $d(d+1)/2$ linearly independent functionals on E.

The functions ϕ_{ij} are called in two dimensions edge bubble functions. □

Example 5.28 *P_3 : conforming piecewise cubic finite element.* This finite element space consists of continuous piecewise cubic functions. It is a subspace of $C(\overline{\Omega})$. The functionals in a mesh cell K are defined to be the values in the vertices ($(d+1)$ values), two values on each edge (dividing the edge in three parts of equal length) ($2\sum_{i=1}^{d} i = d(d+1)$ values), and the values in the barycenter of the 2-faces of K, see Figure 5.7. Each 2-face of K is defined by three vertices. If one considers for each vertex all possible pairs with other vertices, then each 2-face is counted three times. Hence, there are $(d+1)(d-1)d/6$ 2-faces. The dimension of $P_3(K)$ is given by

$$\dim P_3(K) = (d+1) + d(d+1) + \frac{(d-1)d(d+1)}{6} = \frac{(d+1)(d+2)(d+3)}{6}.$$

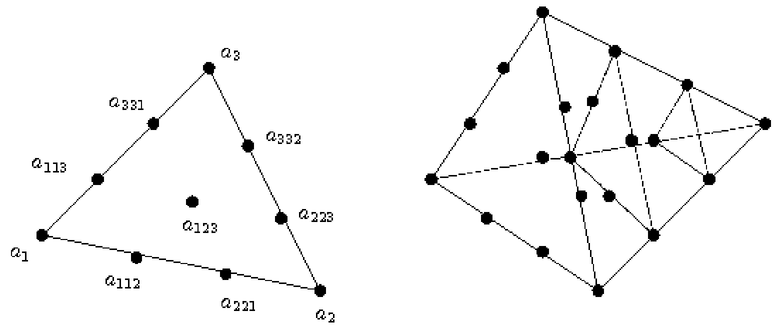

Figure 5.7: The finite element $P_3(K)$.

For the functionals

$$\left\{\begin{array}{rcll} \Phi_i(v) & = & v(\mathbf{a}_i),\ i=1,\ldots,d+1, & \text{(vertex)}, \\ \Phi_{iij}(v) & = & v(\mathbf{a}_{iij}),\ i,j=1,\ldots,d+1, i\neq j, & \text{(point on edge)}, \\ \Phi_{ijk}(v) & = & v(\mathbf{a}_{ijk}),\ i=1,\ldots,d+1, i<j<k & \text{(point on 2-face)} \end{array}\right\},$$

the local basis is given by

$$\left\{\begin{array}{rcl} \phi_i(\lambda) & = & \frac{1}{2}\lambda_i(3\lambda_i-1)(3\lambda_i-2), \\ \phi_{iij}(\lambda) & = & \frac{9}{2}\lambda_i\lambda_j(3\lambda_i-1), \\ \phi_{ijk}(\lambda) & = & 27\lambda_i\lambda_j\lambda_k \end{array}\right\}.$$

In two dimensions, the function $\phi_{ijk}(\lambda)$ is called cell bubble function. □

Example 5.29 *Cubic Hermite element.* The finite element space is a subspace of $C(\overline{\Omega})$, its dimension is $(d+1)(d+2)(d+3)/6$ and the functionals are the values of the function in the vertices of the mesh cell ($(d+1)$ values), the value of the barycenter at the 2-faces of K ($(d+1)(d-1)d/6$ values), and the partial derivatives at the vertices ($d(d+1)$ values), see Figure 5.8. The dimension is the same as for the P_3 element. Hence, the local polynomials can be defined to be cubic.

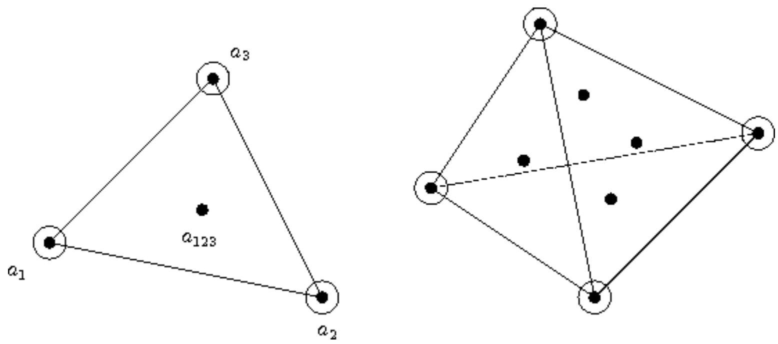

Figure 5.8: The cubic Hermite element.

This finite element does not define an affine family in the strict sense, because the functionals for the partial derivatives $\hat{\Phi}_i(\hat{v}) = \partial_i \hat{v}(\mathbf{0})$ on the reference cell are mapped to the functionals $\Phi_i(v) = \partial_{\mathbf{t}_i} v(\mathbf{a})$, where $\mathbf{a} = F_K(\mathbf{0})$ and $\mathbf{t}_i$ are the directions of edges which are adjacent to $\mathbf{a}$, i.e., $\mathbf{a}$ is an end point of this edge. This property suffices to control all first derivatives. On has to take care of this property in the implementation of this finite element.

Because of this property, one can use the derivatives in the direction of the edges as functionals

$$\begin{array}{rcll} \Phi_i(v) &=& v(\mathbf{a}_i), & \text{(vertices)} \\ \Phi_{ij}(v) &=& \nabla v(\mathbf{a}_i)\cdot(\mathbf{a}_j - \mathbf{a}_i),\ i,j = 1,\ldots,d-1, i \neq j, & \text{(directional derivative)} \\ \Phi_{ijk}(v) &=& v(\mathbf{a}_{ijk}),\ i<j<k, & \text{(2-faces)} \end{array}$$

with the corresponding local basis

$$\begin{array}{rcl} \phi_i(\lambda) &=& -2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \sum_{j<k, j\neq i, k\neq i} \lambda_j \lambda_k, \\ \phi_{ij}(\lambda) &=& \lambda_i \lambda_j (2\lambda_i - \lambda_j - 1), \\ \phi_{ijk}(\lambda) &=& 27\lambda_i\lambda_j\lambda_k. \end{array}$$

The proof of the unisolvence can be found in the literature.

Here, the continuity of the functions will be shown only for $d = 2$. Let K_1, K_2 be two mesh cells with the common edge E and the unit tangential vector $\mathbf{t}$. Let V_1, V_2 be the end points of E. The restriction $v|_{K_1}, v|_{K_2}$ to E satisfy four conditions

$$v|_{K_1}(V_i) = v|_{K_2}(V_i), \quad \partial_{\mathbf{t}} v|_{K_1}(V_i) = \partial_{\mathbf{t}} v|_{K_2}(V_i),\ i = 1,2.$$

Since both restrictions are cubic polynomials and four conditions have to be satisfied, their values coincide on E.

The cubic Hermite finite element possesses an advantage in comparison with the P_3 finite element. For $d = 2$, it holds for a regular triangulation $\mathcal{T}_h$ that

$$\#(K) \approx 2\#(V), \quad \#(E) \approx 2\#(V),$$

where $\#(\cdot)$ denotes the number of triangles, nodes, and edges, respectively. Hence, the dimension of P_3 is approximately $7\#(V)$, whereas the dimension of the cubic

Hermite element is approximately $5\#(V)$. This difference comes from the fact that both spaces are different. The elements of both spaces are continuous functions, but for the functions of the cubic Hermite finite element, in addition, the first derivatives are continuous at the nodes. That means, these two spaces are different finite element spaces whose degree of the local polynomial space is the same (cubic). One can see at this example the importance of the functionals for the definition of the global finite element space. □

Example 5.30 P_1^{nc} *: nonconforming linear finite element, Crouzeix–Raviart finite element Crouzeix and Raviart (1973).* This finite element consists of piecewise linear but discontinuous functions. The functionals are given by the values of the functions in the barycenters of the faces such that $\dim P_1^{\text{nc}}(K) = (d+1)$. It follows from the definition of the finite element space, Definition 5.12, that the functions from P_1^{nc} are continuous in the barycenter of the faces

$$P_1^{\text{nc}} = \big\{v \in L^2(\Omega) \;:\; v|_K \in P_1(K),\ v(\mathbf{x}) \text{ is continuous at the barycenter of all faces}\big\}. \tag{5.3}$$

Equivalently, the functionals can be defined to be the integral mean values on the faces and then the global space is defined to be

$$P_1^{\text{nc}} = \left\{v \in L^2(\Omega) \;:\; v|_K \in P_1(K), \int_E v|_K \, d\mathbf{s} = \int_E v|_{K'} \, d\mathbf{s} \;\forall\, E \in \mathcal{E}(K) \cap \mathcal{E}(K')\right\}, \tag{5.4}$$

where $\mathcal{E}(K)$ is the set of all $(d-1)$ dimensional faces of K.

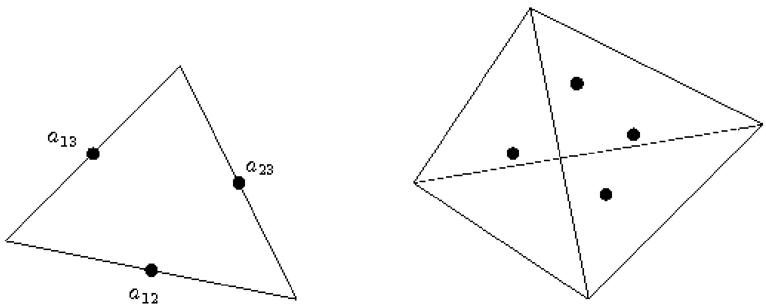

Figure 5.9: The finite element P_1^{nc}.

For the description of this finite element, one defines the functionals by

$$\Phi_i(v) = v(\mathbf{a}_{i-1,i+1}) \text{ for } d = 2, \quad \Phi_i(v) = v(\mathbf{a}_{i-2,i-1,i+1}) \text{ for } d = 3,$$

where the points are the barycenters of the faces with the vertices that correspond to the indices. This system is unisolvent with the local basis

$$\phi_i(\lambda) = 1 - d\lambda_i, \quad i = 1, \ldots, d+1.$$

□

5.3 Finite Elements on Parallelepipeds

Remark 5.31 *Reference mesh cells, reference map.* On can find in the literature two reference cells: the unit cube $[0,1]^d$ and the large unit cube $[-1,1]^d$. It does not matter which reference cell is chosen. Here, the large unit cube will be used: $\hat{K} = [-1,1]^d$. The class of admissible reference maps $\{F_K\}$ consists of bijective affine mappings of the form

$$F_K\hat{\mathbf{x}} = B\hat{\mathbf{x}} + \mathbf{b}, \quad B \in \mathbb{R}^{d\times d},\ \mathbf{b} \in \mathbb{R}^d.$$

If B is a diagonal matrix, then $\hat{K}$ is mapped to d-rectangles.

The class of mesh cells which are obtained in this way is not sufficient to triangulate general domains. If one wants to use more general mesh cells than parallelepipeds, then the class of admissible reference maps has to be enlarged, see Section 5.4. □

Definition 5.32 Polynomial space Q_k. Let $\mathbf{x} = (x_1,\ldots,x_d)^T$ and denote by $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d)^T$ a multi-index. Then, the polynomial space Q_k is given by

$$Q_k = \operatorname{span}\left\{\prod_{i=1}^d x_i^{\alpha_i} = \mathbf{x}^{\boldsymbol{\alpha}} \ :\ 0 \le \alpha_i \le k \ \text{ for } \ i = 1,\ldots,d\right\}.$$

□

Example 5.33 Q_1 *vs.* P_1*.* The space Q_1 consists of all polynomials which are d-linear. Let $d = 2$, then it is

$$Q_1 = \operatorname{span}\{1, x, y, xy\},$$

whereas

$$P_1 = \operatorname{span}\{1, x, y\}.$$

□

Remark 5.34 *Finite elements on d-rectangles.* For simplicity of presentation, the examples below consider d-rectangles. In this case, the finite elements are just tensor products of one-dimensional finite elements. In particular, the basis functions can be written as products of one-dimensional basis functions. □

Example 5.35 Q_0 *: piecewise constant finite element.* Similarly to the P_0 space, the space Q_0 consists of piecewise constant, discontinuous functions. The functional is the value of the function in the barycenter of the mesh cell K and it holds $\dim Q_0(K) = 1$. □

Example 5.36 Q_1 *: conforming piecewise d-linear finite element.* This finite element space is a subspace of $C(\overline{\Omega})$. The functionals are the values of the function in the vertices of the mesh cell, see Figure 5.10. Hence, it is $\dim Q_1(K) = 2^d$.

The one-dimensional local basis functions, which will be used for the tensor product, are given by

$$\hat{\phi}_1(\hat{x}) = \frac{1}{2}(1-\hat{x}), \quad \hat{\phi}_2(\hat{x}) = \frac{1}{2}(1+\hat{x}).$$

With these functions, e.g., the basis functions in two dimensions are computed by

$$\hat{\phi}_1(\hat{x})\hat{\phi}_1(\hat{y}),\ \hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{y}),\ \hat{\phi}_2(\hat{x})\hat{\phi}_1(\hat{y}),\ \hat{\phi}_2(\hat{x})\hat{\phi}_2(\hat{y}).$$

The continuity of the functions of the finite element space Q_1 is proved in the same way as for simplicial finite elements. It is used that the restriction of a function from $Q_k(K)$ to a face E is a function from the space $Q_k(E)$, $k \ge 1$. □

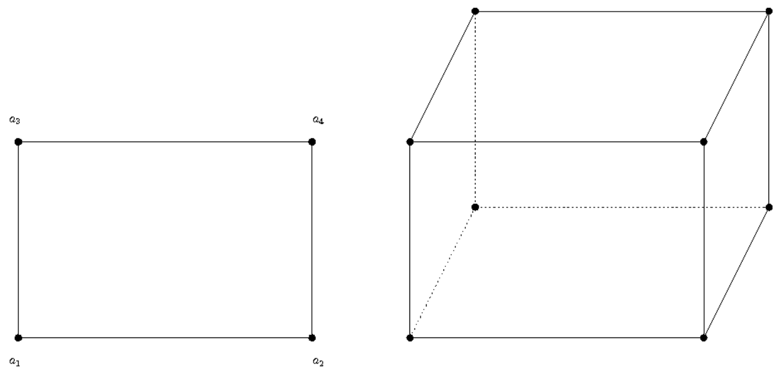

Figure 5.10: The finite element Q_1.

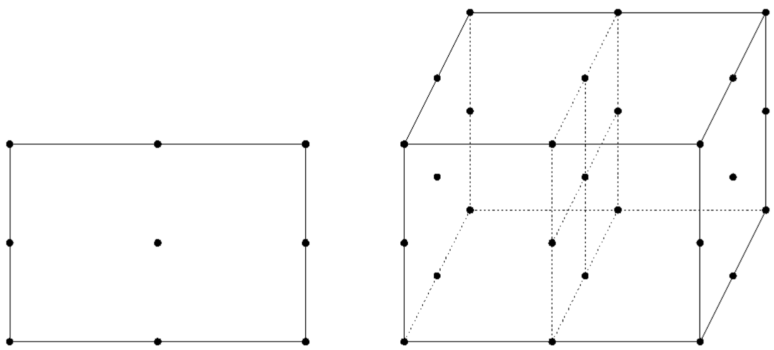

Figure 5.11: The finite element Q_2.

Example 5.37 Q_2 *: conforming piecewise d-quadratic finite element.* It holds that $Q_2 \subset C(\overline{\Omega})$. The functionals in one dimension are the values of the function at both ends of the interval and in the center of the interval, see Figure 5.11. In d dimensions, they are the corresponding values of the tensor product of the intervals. It follows that $\dim Q_2(K) = 3^d$.

The one-dimensional basis function on the reference interval are defined by

$$\hat{\phi}_1(\hat{x}) = -\frac{1}{2}\hat{x}(1-\hat{x}), \quad \hat{\phi}_2(\hat{x}) = (1-\hat{x})(1+\hat{x}), \quad \hat{\phi}_3(\hat{x}) = \frac{1}{2}(1+\hat{x})\hat{x}.$$

The basis function $\prod_{i=1}^d \hat{\phi}_2(\hat{x}_i)$ is called cell bubble function. □

Example 5.38 Q_3 *: conforming piecewise d-quadratic finite element.* This finite element space is a subspace of $C(\overline{\Omega})$. The functionals on the reference interval are given by the values at the end of the interval and the values at the points $\hat{x} = -1/3$, $\hat{x} = 1/3$. In multiple dimensions, it is the corresponding tensor product, see Figure 5.12. The dimension of the local space is $\dim Q_3(K) = 4^d$.

The one-dimensional basis functions in the reference interval are given by

$$\begin{aligned}
\hat{\phi}_1(\hat{x}) &= -\frac{1}{16}(3\hat{x}+1)(3\hat{x}-1)(\hat{x}-1), \\
\hat{\phi}_2(\hat{x}) &= \frac{9}{16}(\hat{x}+1)(3\hat{x}-1)(\hat{x}-1), \\
\hat{\phi}_3(\hat{x}) &= -\frac{9}{16}(\hat{x}+1)(3\hat{x}+1)(\hat{x}-1), \\
\hat{\phi}_4(\hat{x}) &= \frac{1}{16}(3\hat{x}+1)(3\hat{x}-1)(\hat{x}+1).
\end{aligned}$$

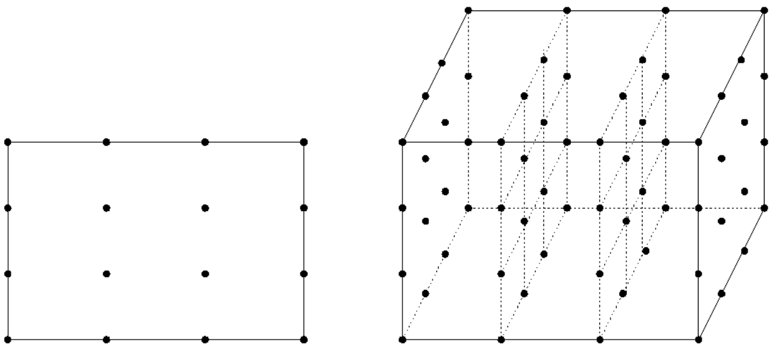

Figure 5.12: The finite element Q_3.

□

Example 5.39 Q_1^{rot} *: rotated nonconforming element of lowest order, Rannacher–Turek element Rannacher and Turek (1992):* This finite element space is a generalization of the P_1^{nc} finite element to quadrilateral and hexahedral mesh cells. It consists of discontinuous functions which are continuous at the barycenter of the faces. The dimension of the local finite element space is $\dim Q_1^{\text{rot}}(K) = 2d$. The space on the reference mesh cell is defined by

$$\begin{aligned} Q_1^{\text{rot}}\left(\hat{K}\right) &= \left\{\hat{p} \,:\, \hat{p} \in \operatorname{span}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}\right\} && \text{for } d = 2, \\ Q_1^{\text{rot}}\left(\hat{K}\right) &= \left\{\hat{p} \,:\, \hat{p} \in \operatorname{span}\{1, \hat{x}, \hat{y}, \hat{z}, \hat{x}^2 - \hat{y}^2, \hat{y}^2 - \hat{z}^2\}\right\} && \text{for } d = 3. \end{aligned}$$

Note that the transformed space

$$Q_1^{\text{rot}}(K) = \{p = \hat{p} \circ F_K^{-1}, \hat{p} \in Q_1^{\text{rot}}(\hat{K})\}$$

contains polynomials of the form $ax^2 - by^2$, where a, b depend on F_K.

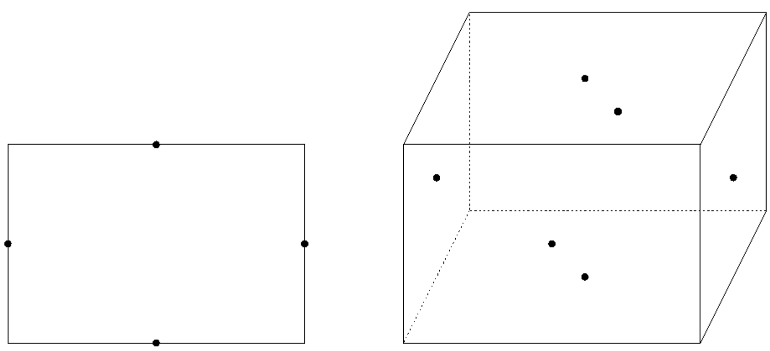

Figure 5.13: The finite element Q_1^{rot}.

For $d = 2$, the local basis on the reference cell is given by

$$\begin{aligned} \phi_1(\hat{x}, \hat{y}) &= -\frac{3}{8}(\hat{x}^2 - \hat{y}^2) - \frac{1}{2}\hat{y} + \frac{1}{4}, \\ \phi_2(\hat{x}, \hat{y}) &= \frac{3}{8}(\hat{x}^2 - \hat{y}^2) + \frac{1}{2}\hat{x} + \frac{1}{4}, \\ \phi_3(\hat{x}, \hat{y}) &= -\frac{3}{8}(\hat{x}^2 - \hat{y}^2) + \frac{1}{2}\hat{y} + \frac{1}{4}, \\ \phi_4(\hat{x}, \hat{y}) &= \frac{3}{8}(\hat{x}^2 - \hat{y}^2) - \frac{1}{2}\hat{x} + \frac{1}{4}. \end{aligned}$$

Analogously to the Crouzeix–Raviart finite element, the functionals can be defined as point values of the functions in the barycenters of the faces, see Figure 5.13, or as integral mean values of the functions at the faces. Consequently, the finite element spaces are defined in the same way as (5.3) or (5.4), with $P_1^{\mathrm{nc}}(K)$ replaced by $Q_1^{\mathrm{rot}}(K)$.

In the code MooNMD John and Matthies (2004), the mean value oriented Q_1^{rot} finite element space is implemented fro two dimensions and the point value oriented Q_1^{rot} finite element space for three dimensions. For $d = 3$, the integrals on the faces of mesh cells, whose equality is required in the mean value oriented Q_1^{rot} finite element space, involve a weighting function which depends on the particular mesh cell K. The computation of these weighting functions for all mesh cells is an additional computational overhead. For this reason, Schieweck (Schieweck, 1997, p. 21) suggested to use for $d = 3$ the simpler point value oriented form of the Q_1^{rot} finite element. □

5.4 Parametric Finite Elements on General d-Dimensional Quadrilaterals

Remark 5.40 *Parametric mappings.* The image of an affine mapping of the reference mesh cell $\hat{K} = [-1, 1]^d$, $d \in \{2, 3\}$, is a parallelepiped. If one wants to consider finite elements on general q-quadrilaterals, then the class of admissible reference maps has to be enlarged.

The simplest parametric finite element on quadrilaterals in two dimensions uses bilinear mappings. Let $\hat{K} = [-1, 1]^2$ and let

$$F_K(\hat{\mathbf{x}}) = \begin{pmatrix} F_K^1(\hat{\mathbf{x}}) \\ F_K^2(\hat{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\hat{x} + a_{13}\hat{y} + a_{14}\hat{x}\hat{y} \\ a_{21} + a_{22}\hat{x} + a_{23}\hat{y} + a_{24}\hat{x}\hat{y} \end{pmatrix}, F_K^i \in Q_1, \ i = 1, 2,$$

be a bilinear mapping from $\hat{K}$ on the class of admissible quadrilaterals. A quadrilateral K is called admissible if

- the length of all edges of K is larger than zero,
- the interior angles of K are smaller than π, i.e. K is convex.

This class contains, e.g., trapezoids and rhombi. □

Remark 5.41 *Parametric finite element functions.* The functions of the local space $P(K)$ on the mesh cell K are defined by $p = \hat{p} \circ F_K^{-1}$. These functions are in general rational functions. However, using d-linear mappings, then the restriction of F_K on an edge of $\hat{K}$ is an affine map. For instance, in the case of the Q_1 finite element, the functions on K are linear functions on each edge of K for this reason. It follows that the functions of the corresponding finite element space are continuous, see Example 5.26. □

5.5 Transform of Integrals

Remark 5.42 *Motivation.* The transform of integrals from the reference mesh cell to mesh cells of the grid and vice versa is used as well for analysis as for the implementation of finite element methods. This section provides an overview of the most important formulae for transforms.

Let $\hat{K} \subset \mathbb{R}^d$ be the reference mesh cell, K be an arbitrary mesh cell, and $F_K : \hat{K} \to K$ with $\mathbf{x} = F_K(\hat{\mathbf{x}})$ be the reference map. It is assumed that the reference map is a continuous differentiable one-to-one map. The inverse map is denoted by $F_K^{-1} : K \to \hat{K}$. For the integral transforms, the derivatives (Jacobians) of F_K and F_K^{-1} are needed

$$DF_K(\hat{\mathbf{x}})_{ij} = \frac{\partial x_i}{\partial \hat{x}_j}, \quad DF_K^{-1}(\mathbf{x})_{ij} = \frac{\partial \hat{x}_i}{\partial x_j}, \quad i,j = 1,\ldots,d.$$

□

Remark 5.43 *Integral with a function without derivatives.* This integral transforms with the standard rule of integral transforms

$$\int_K v(\mathbf{x})\ d\mathbf{x} = \int_{\hat{K}} \hat{v}(\hat{\mathbf{x}})\,|\det DF_K(\hat{\mathbf{x}})|\ d\hat{\mathbf{x}}, \tag{5.5}$$

where $\hat{v}(\hat{\mathbf{x}}) = v(F_K(\hat{\mathbf{x}}))$. □

Remark 5.44 *Transform of derivatives.* Using the chain rule, one obtains

$$\begin{aligned}
\frac{\partial v}{\partial x_i}(\mathbf{x}) &= \sum_{j=1}^{d} \frac{\partial \hat{v}}{\partial \hat{x}_j}(\hat{\mathbf{x}})\frac{\partial \hat{x}_j}{\partial x_i} = \nabla_{\hat{\mathbf{x}}}\hat{v}(\hat{\mathbf{x}}) \cdot \left(\left(DF_K^{-1}(\mathbf{x})\right)^T\right)_i \\
&= \nabla_{\hat{\mathbf{x}}}\hat{v}(\hat{\mathbf{x}}) \cdot \left(\left(DF_K^{-1}(F_K(\hat{\mathbf{x}}))\right)^T\right)_i, \qquad (5.6)\\
\frac{\partial \hat{v}}{\partial \hat{x}}(\hat{\mathbf{x}}) &= \sum_{j=1}^{d} \frac{\partial v}{\partial x_j}(\mathbf{x})\frac{\partial x_j}{\partial \hat{x}_i} = \nabla v(\mathbf{x}) \cdot \left(\left(DF_K(\hat{\mathbf{x}})\right)^T\right)_i \\
&= \nabla v(\mathbf{x}) \cdot \left(\left(DF_K(F_K^{-1}(\mathbf{x}))\right)^T\right)_i. \qquad (5.7)
\end{aligned}$$

The index i denotes the i-th row of a matrix. Derivatives on the reference mesh cell are marked with a symbol on the operator. □

Remark 5.45 *Integrals with a gradients.* Using the rule for transforming integrals and (5.6) gives

$$\begin{aligned}
&\int_K \mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x})\ d\mathbf{x} \\
&= \int_{\hat{K}} \mathbf{b}\left(F_K(\hat{\mathbf{x}})\right) \cdot \left[\left(DF_K^{-1}\right)^T\left(F_K(\hat{\mathbf{x}})\right)\right] \nabla_{\hat{\mathbf{x}}}\hat{v}(\hat{\mathbf{x}})\,|\det DF_K(\hat{\mathbf{x}})|\ d\hat{\mathbf{x}}. \qquad (5.8)
\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}
&\int_K \nabla v(\mathbf{x}) \cdot \nabla w(\mathbf{x})\ d\mathbf{x} \\
&= \int_{\hat{K}} \left[\left(DF_K^{-1}\right)^T\left(F_K(\hat{\mathbf{x}})\right)\right] \nabla_{\hat{\mathbf{x}}}\hat{v}(\hat{\mathbf{x}}) \cdot \left[\left(DF_K^{-1}\right)^T\left(F_K(\hat{\mathbf{x}})\right)\right] \nabla_{\hat{\mathbf{x}}}\hat{w}(\hat{\mathbf{x}}) \\
&\quad \times |\det DF_K(\hat{\mathbf{x}})|\ d\hat{\mathbf{x}}. \qquad (5.9)
\end{aligned}$$

□

Remark 5.46 *Integral with the divergence.* Integrals of the following type are important for the Navier–Stokes equations

$$\begin{aligned}
&\int_K \nabla \cdot v(\mathbf{x}) q(\mathbf{x})\ d\mathbf{x} = \int_K \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}(\mathbf{x}) q(\mathbf{x})\ d\mathbf{x} \\
&= \int_{\hat{K}} \sum_{i=1}^{d} \left[\left(\left(DF_K^{-1}(F_K(\hat{\mathbf{x}}))\right)^T\right)_i \cdot \nabla_{\hat{\mathbf{x}}}\hat{v}_i(\hat{\mathbf{x}})\right] \hat{q}(\hat{\mathbf{x}})\,|\det DF_K(\hat{\mathbf{x}})|\ d\hat{\mathbf{x}} \\
&= \int_{\hat{K}} \left[\left(DF_K^{-1}(F_K(\hat{\mathbf{x}}))\right)^T : D_{\hat{\mathbf{x}}}\mathbf{v}(\hat{\mathbf{x}})\right] \hat{q}(\hat{\mathbf{x}})\,|\det DF_K(\hat{\mathbf{x}})|\ d\hat{\mathbf{x}}. \qquad (5.10)
\end{aligned}$$

In the derivation, (5.6) was used. □

Example 5.47 *Affine transform.* The most important class of reference maps are affine transforms

$$\mathbf{x} = B\hat{\mathbf{x}} + \mathbf{b}, \quad B \in \mathbb{R}^{d\times d}, \mathbf{b} \in \mathbb{R}^d,$$

where the invertible matrix B and the vector $\mathbf{b}$ are constants. It follows that

$$\hat{\mathbf{x}} = B^{-1}(\mathbf{x} - \mathbf{b}) = B^{-1}\mathbf{x} - B^{-1}\mathbf{b}.$$

In this case, there are

$$DF_K = B, \quad DF_K^{-1} = B^{-1}, \quad \det DF_K = \det(B).$$

One obtains for the integral transforms from (5.5), (5.8), (5.9), and (5.10)

$$\int_K v(\mathbf{x})\, d\mathbf{x} = |\det(B)| \int_{\hat{K}} \hat{v}(\hat{\mathbf{x}})\, d\hat{\mathbf{x}}, \tag{5.11}$$

$$\int_K \mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, d\mathbf{x} = |\det(B)| \int_{\hat{K}} \mathbf{b}\left(F_K(\hat{\mathbf{x}})\right) \cdot B^{-T}\nabla_{\hat{\mathbf{x}}}\hat{v}(\hat{\mathbf{x}})\, d\hat{\mathbf{x}}, \tag{5.12}$$

$$\int_K \nabla v(\mathbf{x}) \cdot \nabla w(\mathbf{x})\, d\mathbf{x} = |\det(B)| \int_{\hat{K}} B^{-T}\nabla_{\hat{\mathbf{x}}}\hat{v}(\hat{\mathbf{x}}) \cdot B^{-T}\nabla_{\hat{\mathbf{x}}}\hat{w}(\hat{\mathbf{x}})\, d\hat{\mathbf{x}}, \tag{5.13}$$

$$\int_K \nabla \cdot v(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x} = |\det(B)| \int_{\hat{K}} \left[B^{-T} : D_{\hat{\mathbf{x}}}\mathbf{v}(\hat{\mathbf{x}})\right] \hat{q}(\hat{\mathbf{x}})\, d\hat{\mathbf{x}}. \tag{5.14}$$

□