Chapter 4

Grid Transfer

Remark 4.1 *Contents of this chapter.* Consider a grid with grid size h and the corresponding linear system of equations

$$A^h \mathbf{u}^h = \mathbf{f}^h.$$

The summary given in Section 3.4 leads to the idea that there might be an iterative method for solving this system efficiently, which uses also coarser grids. In order to construct such a method, one needs mechanisms that transfer the information in an appropriate way between the grids. □

4.1 The Coarse Grid System and the Residual Equation

Remark 4.2 *Basic idea for obtaining a good initial iterate with a coarse grid solution.* One approach for improving the behavior of iterative methods, at least at the beginning of the iteration, consists in using a good initial iterate. For the model problem, one can try to find a good initial iterate, e.g., by solving the problem approximately on a coarse grid, using only a few iterations. The application of only a few iterations is called smoothing, and the iterative method itself smoother, since only the oscillating error modes (on the coarse grid) are damped. The solution from the coarse grid can be used as initial iterate on the fine grid. □

Remark 4.3 *Study of the discrete Fourier modes on different grids.* Given a grid Ω^{2h}. In practice, a uniform refinement step consists in dividing in halves all intervals of Ω^{2h}, leading to the grid Ω^h. Then, the nodes of Ω^{2h} are the nodes of Ω^h with even numbers, see Figure 4.1.

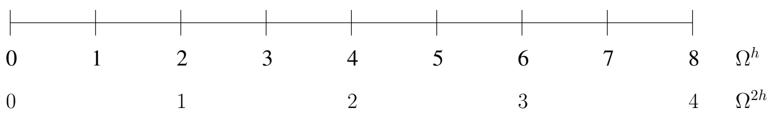

Figure 4.1: Coarse and fine grid.

Consider the k-th Fourier mode of the fine grid Ω^h. If $1 \le k \le N/2$, then it follows for the even nodes that

$$w_{k,2j}^h = \sin\left(\frac{2jk\pi}{N}\right) = \sin\left(\frac{jk\pi}{N/2}\right) = w_{k,j}^{2h}, \quad j = 1, \ldots, \frac{N}{2} - 1.$$

Hence, the k-th Fourier mode on Ω^h is the k-th Fourier mode on Ω^{2h}. From the definition of the smooth and oscillating modes, Remark 3.7, it follows that by going from the fine to the coarse grid, the k-th mode gets a higher frequency if $1 \le l \le N/2$. Note again that the notion of frequency depends on the grid size. The Fourier mode on Ω^h for $k = N/2$ is represented on Ω^{2h} by the zero vector.

For the transfer of the oscillating modes on Ω^h, i.e., for $N/2 < k < N$, one obtains a somewhat unexpected results. These modes are represented on Ω^{2h} as relatively smooth modes. The k-th mode on Ω^h becomes the negative of the $(N-k)$-th mode on Ω^{2h} (*exercise ?*):

$$\begin{aligned}
w_{k,2j}^h &= \sin\left(\frac{2jk\pi}{N}\right) = \sin\left(\frac{jk\pi}{N/2}\right), \\
-w_{N-k,2j}^{2h} &= -\sin\left(\frac{j(N-k)\pi}{N/2}\right) = -\sin\left(\frac{2j(N-k)\pi}{N}\right) \\
&= -\sin\left(\frac{2jN\pi}{N} - \frac{2jk\pi}{N}\right) \\
&= -\underbrace{\sin(2j\pi)}_{=0}\cos\left(\frac{2jk\pi}{N}\right) + \underbrace{\cos(2j\pi)}_{=1}\sin\left(\frac{2jk\pi}{N}\right) \\
&= \sin\left(\frac{2jk\pi}{N}\right),
\end{aligned}$$

i.e., $w_{k,2j}^h = -w_{N-k,2j}^{2h}$. This aspect shows that it is necessary to damp the oscillating error modes on Ω^h before a problem on Ω^{2h} is considered. Otherwise, one would get additional smooth error modes on the coarser grid. $\square$

Remark 4.4 *The residual equation.* An iterative method for the solution of $A\mathbf{u} = \mathbf{f}$ can be applied either directly to this equation or to an equation for the error, the so-called residual equation. Let $\mathbf{u}^{(m)}$ be an approximation of $\mathbf{u}$, then the error $\mathbf{e}^{(m)} = \mathbf{u} - \mathbf{u}^{(m)}$ satisfies the equation

$$A\mathbf{e}^{(m)} = \mathbf{f} - A\mathbf{u}^{(m)} =: \mathbf{r}^{(m)}. \tag{4.1}$$

$\square$

Remark 4.5 *Nested iteration.* This remark gives a first strategy for using coarse grid problems for the improvement of an iterative method for solving $A\mathbf{u}^h = \mathbf{f}^h$. This strategy is a generalization of the idea from Remark 4.2. It is called nested iteration:

- solve $A^{h_0}\mathbf{u}^{h_0} = \mathbf{f}^{h_0}$ on a very coarse grid approximately by applying a smoother,
- $\vdots$
- smooth $A^{2h}\mathbf{u}^{2h} = \mathbf{f}^{2h}$ on Ω^{2h},
- solve $A^h\mathbf{u}^h = \mathbf{f}^h$ on Ω^h by an iterative method with the initial iterate provided from the coarser grids.

However, there are some open questions with this strategy. How are the linear systems defined on the coarser grids? What can be done if there are still smooth error modes on the finest grid? In this case, the convergence of the last step will be slowly. $\square$

Remark 4.6 *Coarse grid correction, two-level method.* A second strategy uses also the residual equation (4.1):

- Smooth $A^h\mathbf{u}^h = \mathbf{f}^h$ on Ω^h. This step gives an approximation $\mathbf{v}^h$ of the solution which still has to be updated appropriately. Compute the residual $\mathbf{r}^h = \mathbf{f}^h - A^h\mathbf{v}^h$.

- Project (restrict) the residual to Ω^{2h}. The result is called $R(\mathbf{r}^h)$.
- Solve $A^{2h}\mathbf{e}^{2h} = R(\mathbf{r}^h)$ on Ω^{2h}. With this step, one obtains an approximation $\mathbf{e}^{2h}$ of the error.
- Project (prolongate) $\mathbf{e}^{2h}$ to Ω^h. The result is denoted by $P(\mathbf{e}^{2h})$.
- Update the approximation of the solution on Ω^h by $\mathbf{v}^h := \mathbf{v}^h + P(\mathbf{e}^{2h})$.

This approach is called coarse grid correction or two-level method. With this approach, one computes on Ω^{2h} an approximation of the error. However, also for this approach one has to answer some questions. How to define the system on the coarse grid? How to restrict the residual to the coarse grid and how to prolongate the correction to the fine grid? $\square$

4.2 Prolongation or Interpolation

Remark 4.7 *General remarks.* The transfer from the coarse to the fine grid is called prolongation or interpolation. In many situations, one can use the simplest approach, which is the linear interpolation. For this reason, this section will only consider this approach. $\square$

Example 4.8 *Linear interpolation for finite difference methods.* For finite difference methods, the prolongation operator is defined by a local averaging. Let Ω^{2h} be divided into $N/2$ intervals and Ω^h into N intervals. The node j on Ω^{2h} corresponds to the node $2j$ on Ω^h, $0 \le j \le N/2$, see Figure 4.1. Let $\mathbf{v}^{2h}$ be given on Ω^{2h}. Then, the linear interpolation

$$I_{2h}^h \;:\; \mathbb{R}^{N/2-1} \to \mathbb{R}^{N-1}, \quad \mathbf{v}^h = I_{2h}^h \mathbf{v}^{2h}$$

is given by

$$\begin{array}{rcll} v_{2j}^h &=& v_j^{2h}, & j = 1,\ldots,N/2-1, \\ v_{2j+1}^h &=& \dfrac{1}{2}\left(v_j^{2h} + v_{j+1}^{2h}\right), & j = 0,\ldots,N/2-1, \end{array} \tag{4.2}$$

see Figure 4.2. For even nodes of Ω^h, one takes directly the value of the corresponding node of Ω^{2h}. For odd nodes of Ω^h, the arithmetic mean of the values of the neighbor nodes is computed.

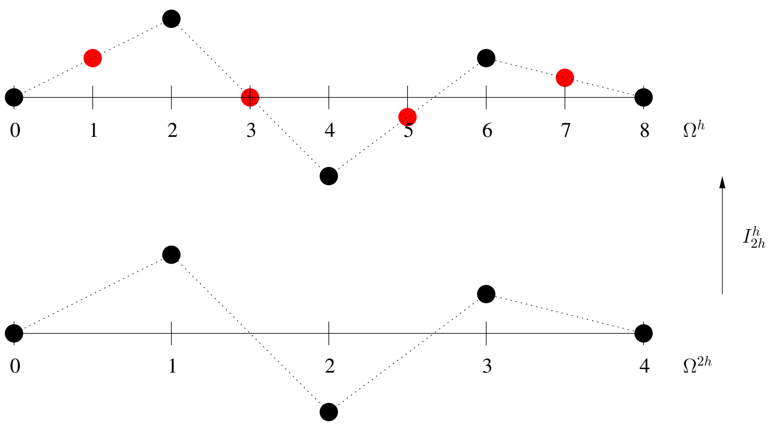

Figure 4.2: Linear interpolation for finite difference methods.

The linear prolongation is a linear operator, see below Lemma 4.10, between two finite-dimensional spaces. Hence, it can be represented as a matrix. Using the standard basis of $\mathbb{R}^{N/2-1}$ and $\mathbb{R}^{N-1}$, then

$$I_{2h}^h = \frac{1}{2}\begin{pmatrix} 1 & & & & \\ 2 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & 1 & 1 & & \\ & & 2 & \ddots & \\ & & 1 & \ddots & \\ & & & & 1 \\ & & & & 2 \\ & & & & 1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N/2-1)}. \tag{4.3}$$

□

Example 4.9 *Canonical prolongation for finite element methods.* Consider conforming finite element methods and denote the spaces on Ω^{2h} and Ω^h with V^{2h} and V^h, respectively. Because Ω^h is a uniform refinement of Ω^{2h}, it follows that $V^{2h} \subset V^h$. Hence, each finite element function defined on Ω^{2h} is contained in the space V^h. This aspect defines a canonical prolongation

$$\mathcal{I}_{2h}^h \;:\; V^{2h} \to V^h, \quad v^{2h} \mapsto v^{2h}.$$

The canonical prolongation will be discussed in detail for P_1 finite elements. Let $\{\varphi_i^{2h}\}_{i=1}^{N/2-1}$ be the local basis of V^{2h} and $\{\varphi_i^h\}_{i=1}^{N-1}$ be the local basis of V^h. Each function $v^{2h} \in V^{2h}$ has a representation of the form

$$v^{2h}(x) = \sum_{i=1}^{N/2-1} v_i^{2h}\varphi_i^{2h}(x), \quad v_i^{2h} \in \mathbb{R},\, i = 1,\ldots,N/2-1.$$

There is a bijection between V^{2h} and $\mathbb{R}^{N/2-1}$.

Let $j = 2i$ be the corresponding index on Ω^h to the index i on Ω^{2h}. From the property of the local basis, it follows that

$$\varphi_i^{2h} = \frac{1}{2}\varphi_{j-1}^h + \varphi_j^h + \frac{1}{2}\varphi_{j+1}^h.$$

Inserting this representation gives

$$\begin{aligned} v^{2h}(x) &= \sum_{i=1}^{N/2-1} v_i^{2h}\left(\frac{1}{2}\varphi_{2i-1}^h + \varphi_{2i}^h + \frac{1}{2}\varphi_{2i+1}^h\right) \\ &= v_1^{2h}\left(\frac{1}{2}\varphi_1^h + \varphi_2^h + \frac{1}{2}\varphi_3^h\right) \\ &\quad +v_2^{2h}\left(\frac{1}{2}\varphi_3^h + \varphi_4^h + \frac{1}{2}\varphi_5^h\right) \\ &\quad +v_3^{2h}\left(\frac{1}{2}\varphi_5^h + \varphi_6^h + \frac{1}{2}\varphi_7^h\right) \\ &\quad +\ldots. \end{aligned}$$

From this formula, one can see that the representation in the basis of V^h is of the following form. For basis functions that correspond to nodes which are already on Ω^{2h} (even indices on the fine grid), the coefficient is the same as for the basis function on the coarser grids. For basis functions that correspond to new nodes, the coefficient is the arithmetic mean of the coefficients of the neighbor basis functions. Hence, if local bases are used, the coefficients for the prolongated finite element function can be computed by multiplying the coefficients of the coarse grid finite element function with the matrix (4.3). □

Lemma 4.10 Properties of the linear interpolation operator. *The operator $I_{2h}^h : \mathbb{R}^{N/2-1} \to \mathbb{R}^{N-1}$ defined in (4.2) is a linear operator. It has full rank and only the trivial kernel.*

Proof: *i) I_{2h}^h is a linear operator.* The operator is homogeneous, since for $\alpha \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^{N/2-1}$ it is

$$\begin{aligned} v_{2j}^h &= (\alpha \mathbf{v})_j = \alpha v_j, \\ v_{2j+1}^h &= \frac{1}{2}\left((\alpha\mathbf{v})_j + (\alpha\mathbf{v})_{j+1}\right) = \alpha\frac{1}{2}\left(v_j + v_{j+1}\right). \end{aligned}$$

The operator is additive. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{N/2-1}$, then

$$\begin{aligned} \left(I_{2h}^h(\mathbf{v}+\mathbf{w})\right)_{2j} &= (\mathbf{v}+\mathbf{w})_j = v_j + w_j = \left(I_{2h}^h(\mathbf{v})\right)_{2j} + \left(I_{2h}^h(\mathbf{w})\right)_{2j}, \\ \left(I_{2h}^h(\mathbf{v}+\mathbf{w})\right)_{2j+1} &= \frac{1}{2}\left((\mathbf{v}+\mathbf{w})_j + (\mathbf{v}+\mathbf{w})_{j+1}\right) = \frac{1}{2}\left(v_j + v_{j+1}\right) + \frac{1}{2}\left(w_j + w_{j+1}\right) \\ &= \left(I_{2h}^h(\mathbf{v})\right)_{2j+1} + \left(I_{2h}^h(\mathbf{w})\right)_{2j+1}. \end{aligned}$$

An homogeneous and additive operator is linear.

ii) I_{2h}^h has full rank and trivial kernel. Since $N/2-1 < N-1$, both properties are equivalent. Let $\mathbf{0} = \mathbf{v}^h = I_{2h}^h(\mathbf{v}^{2h})$. From (4.2) it follows from the vanishing of the even indices of $\mathbf{v}^h$ immediately that $v_j^{2h} = 0$, $j = 1, \ldots, N/2-1$, i.e., $\mathbf{v}^{2h} = \mathbf{0}$. Hence, the only element in the kernel of I_{2h}^h is the zero vector. ■

Remark 4.11 *Effect of the prolongation on different error modes.* Assume that the error, which is of course unknown is a smooth function on the fine grid Ω^h. In addition, the coarse grid approximation on Ω^{2h} is computed and it should be exact in the nodes of the coarse grid. The interpolation of this coarse grid approximation is a smooth function on the fine grid (there are no new oscillations). For this reason, one can expect a rather good approximation of the smooth error on the fine grid.

If the error on the fine grid is oscillating, then each interpolation of a coarse grid approximation to the fine grid is a smooth function and one cannot expect that the error on the fine grid is approximated well, see Figure 4.3.

Altogether, the prolongation gives the best results, if the error on the fine grid is smooth. Hence, the prolongation is an appropriate complement to the smoother, which works most efficiently if the error is oscillating. □

4.3 Restriction

Remark 4.12 *General remarks.* For the two-level method, one has to transfer the residual from Ω^h to Ω^{2h} before the coarse grid equation can be solved. This transfer is called restriction. □

Example 4.13 *Injection for finite difference schemes.* The simplest restriction is the injection. It is defined by

$$I_h^{2h} : \mathbb{R}^{N-1} \to \mathbb{R}^{N/2-1}, \quad \mathbf{v}^{2h} = I_h^{2h}\mathbf{v}^h, \quad v_j^{2h} = v_{2j}^h, \quad j = 1, \ldots, \frac{N}{2} - 1,$$

see Figure 4.4. For this restriction, one takes for each node on the coarse grid simply the value of the grid function at the corresponding node on the fine grid.

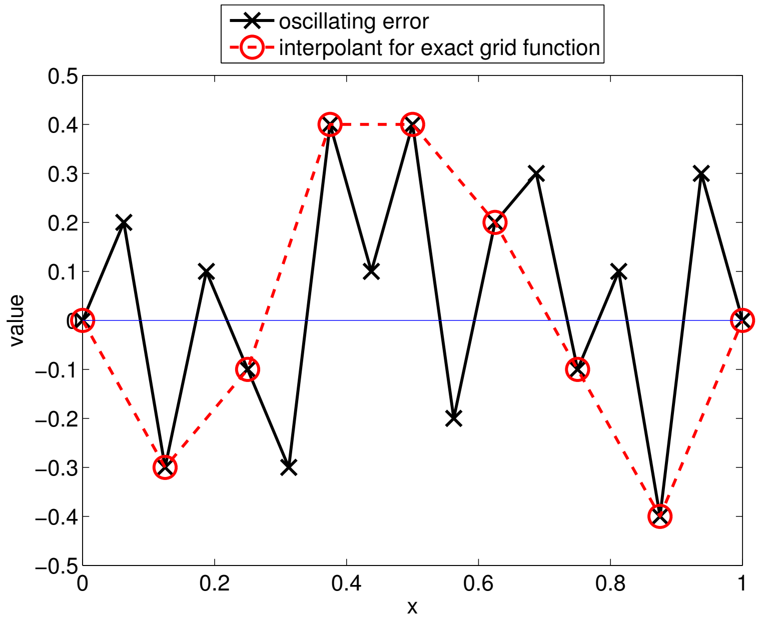

Figure 4.3: Oscillating error and interpolation.

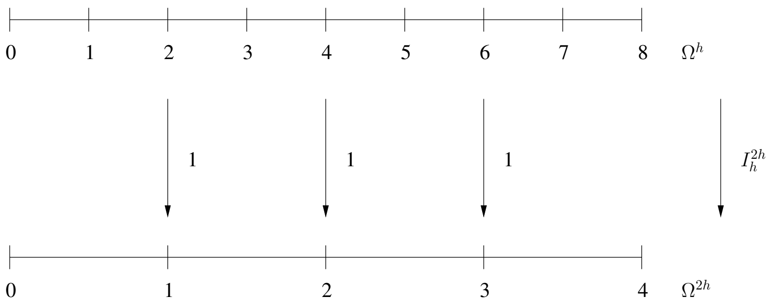

Figure 4.4: Injection.

It turns out that the injection does not lead to an efficient method. If one ignores every other node on Ω^h, then the values of the residual in these nodes, and with that also the error in these nodes, do not possess any impact on the system on the coarse grid. Consequently, these errors will generally not be corrected. □

Example 4.14 *Weighted restriction for finite difference schemes.* The weighted restriction uses all nodes on the fine grid. It is defined by an appropriate averaging

$$\begin{aligned} I_h^{2h} &: \quad \mathbb{R}^{N-1} \to \mathbb{R}^{N/2-1}, \\ \mathbf{v}^{2h} &= I_h^{2h}\mathbf{v}^h, \quad v_j^{2h} = \frac{1}{4}\left(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h\right), \quad j = 1, \ldots, \frac{N}{2} - 1, \end{aligned} \tag{4.4}$$

see Figure 4.5. For finite difference schemes, only the weighted restriction will be considered in the following.

If the spaces $\mathbb{R}^{N-1}$ and $\mathbb{R}^{N/2-1}$ are equipped with the standard bases, the matrix

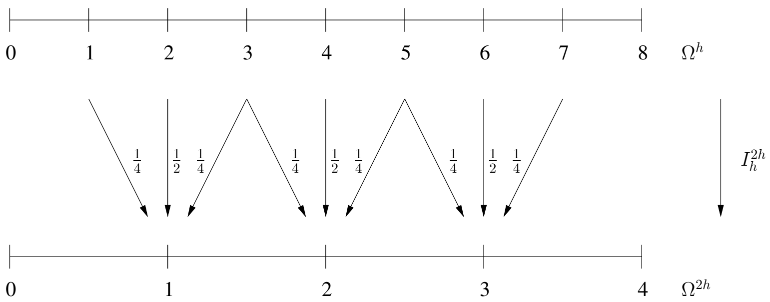

Figure 4.5: Weighted restriction.

representation of the weighted restriction operator has the form

$$I_h^{2h} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & 2 & 1 \end{pmatrix} \in \mathbb{R}^{(N/2-1)\times(N-1)}. \qquad (4.5)$$

With this representation, one can see an important connection between weighted restriction I_h^{2h} and interpolation I_{2h}^h:

$$I_{2h}^h = 2\left(I_h^{2h}\right)^T.$$

□

Lemma 4.15 Properties of the weighted restriction operator. *Let the restriction operator I_h^{2h} given by* (4.4)*. This operator is linear. The rank of this operator is $N/2-1$ and the kernel has dimension $N/2$.*

Proof: *i) Linearity. exercise.*

ii) Rank and kernel. From linear algebra, it is known that the sum of the dimension of the kernel and the rank is $N-1$. The rank of I_h^{2h} is equal to the dimension of its range (row rank). The range of I_h^{2h} is equal to $\mathbb{R}^{N/2-1}$, since every vector from $\mathbb{R}^{N/2-1}$ might be the image in the space of grid functions of Ω^{2h} of a vector corresponding to grid functions of Ω^h. Hence, the rank is $N/2-1$ and consequently, the dimension of the kernel is $N-1-(N/2-1) = N/2$. ■

Example 4.16 *Canonical restriction for finite element schemes.* Whereas for finite difference methods, one works only with vectors of real numbers, finite element methods are imbedded into the Hilbert space setting. In this setting, a finite element function is, e.g, from the space V^h, but the residual, which is the right-hand side minus the finite element operator applied to a finite element function (current iterate) is from the dual space $\left(V^h\right)^*$ of V^h. In this setting, it makes a difference if one restricts an element from V^h or from its dual space.

For restricting a finite element function from V^h to V^{2h}, one can take the analogon of the weighted restriction. If local bases are used, then the coefficients of the finite element function from V^h are multiplied with the matrix (4.5) to get the coefficients of the finite element function in V^{2h}.

In the two-level method, one has to restrict the residual, i.e., one needs a restriction from $\left(V^h\right)^*$ to $\left(V^{2h}\right)^*$. In this situation, a natural choice consists in using the dual prolongation operator, i.e.,

$$\mathcal{I}_h^{2h} \ : \ \left(V^h\right)^* \to \left(V^{2h}\right)^*, \quad \mathcal{I}_h^{2h} = \left(\mathcal{I}_{2h}^h\right)^*.$$

The dual operator is defined by

$$\left\langle \mathcal{I}_{2h}^{h} v^{2h}, r^h \right\rangle_{V^h,(V^h)^*} = \left\langle v^{2h}, \mathcal{I}_h^{2h} r^h \right\rangle_{V^{2h},(V^{2h})^*} \quad \forall\, v^{2h} \in V^{2h}, r^h \in \left(V^h\right)^* .$$

Thus, if local bases and the bijection between finite element spaces and the Euclidean spaces are used, then the restriction of the residual can be represented by the transposed of the matrix (4.3). This issue makes a difference of a factor of 2 compared with the matrix for the weighted restriction. □