Lösungen zum 32. Aufgabenblatt für MfI 3

1. Aufgabe :
   Für $\mathbf{v}$ mit $\|\mathbf{v}\| = 1$ gilt:

$$\begin{aligned}
\frac{\partial f}{\partial \mathbf{v}}(0,0) &= \lim_{h\to 0} \frac{f(h\mathbf{v}) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{hv_1h^3v_2^2}{h^2v_1^2 + h^2v_2^2}\right) \\
&= \lim_{h\to 0} h(v_1v_2^2) \\
&= 0
\end{aligned}$$

   Die partiellen Ableitungen sind:

$$\frac{\partial f}{\partial x} = \begin{cases} -\dfrac{y^3\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

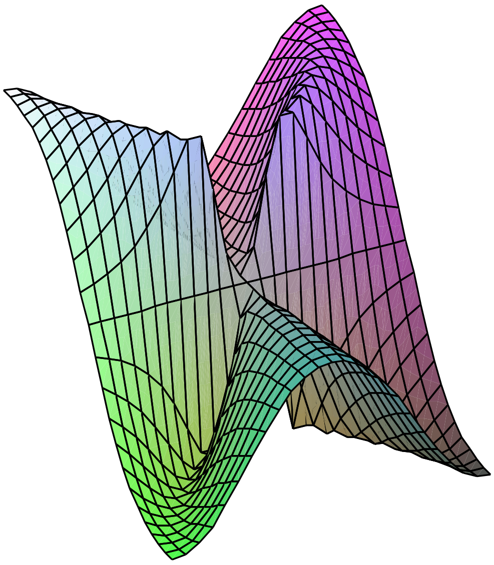
$$\frac{\partial f}{\partial y} = \begin{cases} \dfrac{xy^2\left(3\,x^2+y^2\right)}{\left(x^2+y^2\right)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   Stetigkeit der partiellen Ableitungen (in Polarkoordinaten):

$$\begin{aligned}
\lim_{r\to 0} f_x(r,\varphi) &= \lim_{r\to 0} \frac{r^3\sin^3(\varphi)\left(r^2(\cos^2(\varphi) - \sin^2(\varphi))\right)}{r^4} \\
&= \lim_{r\to 0} r\left(\sin^3(\varphi)\cos^2(\varphi) - \sin^5(\varphi)\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{r\to 0} f_y(r,\varphi) &= \lim_{r\to 0} \frac{r^3\cos(\varphi)\sin^2(\varphi)\left(3r^2\cos^2(\varphi) + r^2\sin^2(\varphi)\right)}{r^4} \\
&= r\left(3\cos^3(\varphi)\sin^2(\varphi) + \cos(\varphi)\sin^4(\varphi)\right) \\
&= 0
\end{aligned}$$

2. Aufgabe :



(a) Unstetigkeit:

$$\text{Wähle}: \quad x = 0 \longrightarrow f(0,y) = \frac{0}{y^4} = 0$$
$$\lim_{y\to 0} f(0,y) = 0$$

$$\text{Wähle}: \quad x = y^2 \longrightarrow f(y^2,y) = \frac{y^4}{2y^4} = \frac{1}{2}$$
$$\frac{1}{2}\lim_{y\to 0} f(y^2,y) = \frac{1}{2}$$

(b) Richtungsableitung in Richtung $\mathbf{a}$ mit $\|\mathbf{a}\| \neq 0$:

$$\begin{aligned}
\partial_{\mathbf{a}} f((0,0)) &= \lim_{t\to 0, t\neq 0} \frac{f((0,0)+t\mathbf{a}) - f((0,0))}{t} \\
&= \lim_{t\to 0, t\neq 0} \frac{f(t\mathbf{a})}{t} \\
&= \lim_{t\to 0, t\neq 0} \frac{1}{t}\,\frac{ta_1(ta_2)^2}{(ta_1)^2+(ta_2)^4} \\
&= \lim_{t\to 0} \frac{t^3 a_1 a_2^2}{t^3(a_1^2+t^2a_2^4)} \\
&= \lim_{t\to 0} \frac{a_1 a_2^2}{a_1^2+t^2a_2^4}
\end{aligned}$$

1. Fall : $a_1 \neq 0$

$$\Longrightarrow \quad \partial_{\mathbf{a}} f((0,0)) = \frac{a_1 a_2^2}{a_1^2}$$

$$= \frac{a_2^2}{a_1}$$

2. Fall : $a_1 = 0$

$$\Longrightarrow \quad \partial_{\mathbf{a}} f\left((0,0)\right) = \lim_{t \to 0} \frac{0}{t} a_2^4 = 0$$

3. Aufgabe :

$$\begin{aligned}
RT &= \left(P + \frac{a}{V_m^2}\right)(V_m - b) \\
T &= \frac{1}{R}\left(P + \frac{a}{V_m^2}\right)(V_m - b) \\
\frac{\partial T}{\partial P} &= \frac{\partial}{\partial P}\frac{1}{R}\left(P + \frac{a}{V_m^2}\right)(V_m - b) \\
&= \frac{1}{R}(V_m - b)
\end{aligned}$$

$\frac{\partial P}{\partial V_m}$:

$$\begin{aligned}
0 &= \left(\frac{\partial P}{\partial V_m} - \frac{2aV_m}{V_m^3}\right)(V_m - b) + \left(P + \frac{a}{V_m^2}\right) \\
\frac{\partial P}{\partial V_m} &= \frac{1}{V_m - b}\left(-P - \frac{a}{V_m^2}\right) + \frac{2a}{V_m^3} \\
&= \frac{-PV_m^3 - aV_m + 2a(V_m - b)}{V_m^3 (V_m - b)}
\end{aligned}$$

$\frac{\partial V_m}{\partial T}$:

$$\begin{aligned}
R &= \frac{-2a\frac{\partial V_m}{\partial T}}{V_m^3}(V_m - b) + \left(P + \frac{a}{V_m^2}\right)\frac{\partial V_m}{\partial T} \\
R &= \frac{\partial V_m}{\partial T}\left(\frac{-2a}{V_m^3}(V_m - b) + P + \frac{a}{V_m^2}\right) \\
\frac{\partial V_m}{\partial T} &= \frac{RV_m^3}{-2a(V_m - b) + PV_m^3 + aV_m} \\
\frac{\partial V_m}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V_m} &= \frac{RV_m^3}{-2a(V_m - b) + PV_m^3 + aV_m} \cdot \frac{1}{R}(V_m - b) \cdot \frac{-PV_m^3 - aV_m + 2a(V_m - b)}{V_m^3 (V_m - b)} \\
&= -1
\end{aligned}$$