Applied Numerical Mathematics 37 (2001) 503–518

APPLIED NUMERICAL MATHEMATICS

www.elsevier.nl/locate/apnum

Residual a posteriori error estimates for two-level finite element methods for the Navier–Stokes equations ☆

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Abstract

Residual based a posteriori error estimates for conforming finite element solutions of the incompressible Navier–Stokes equations which are computed with four recently proposed two-level methods are derived. The a posteriori error estimates contain additional terms in comparison to the estimates for the solution obtained by the standard one-level method. The importance of these additional terms in the error estimates is investigated by studying their asymptotic behaviour. For optimally scaled meshes, these bounds are not of higher order than the order of convergence of the discrete solution. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

Keywords: Navier–Stokes equations; Residual a posteriori error estimators; Two-level methods; Finite element methods

1. Introduction

Modeling of processes in physics and engineering leads often to nonlinear partial differential equations. An example are the incompressible Navier–Stokes equations which model incompressible fluid flows. For standard conforming finite element discretizations of these equations, which fulfill the Babuška–Brezzi stability condition, optimal a priori error estimates can be proven, provided there is a sufficiently smooth nonsingular solution.

In practice, the numerical solution of the nonlinear system of equations arising in the discretization of the Navier–Stokes equations may be very time consuming. Two-level methods aim to compute a discrete approximation of the solution of the nonlinear partial differential equation with less computational work and to preserve the optimal order of convergence. The basis of two-level methods are a coarse grid and a fine grid. First, the given problem is solved on the coarse grid, which is in general inexpensive. The second step is to spend some numerical work in doing very few (e.g., one) steps of an iterative method for solving the problem on the fine mesh. It turns out that the crucial point for asymptotically optimal

☆ This work was supported by the Deutsche Akademische Austauschdienst (DAAD).

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PII: S0168-9274(00)00058-1

a priori error estimates is an appropriate scaling between the coarse and fine mesh. Recently investigated two-level methods contain as a third step a defect correction on the coarse mesh.

Two-level methods have gained some attraction in the last couple of years. They have been investigated for semilinear elliptic equations and scalar nonlinear partial differential equations by Xu [23,24]. Layton [14], Layton and Lenferink [15], and Layton and Tobiska [17] have studied four different two-level methods for the incompressible Navier–Stokes equations. The methods differ in the (linear) equation which is solved on the fine mesh and whether or not a correction step on the coarse mesh is applied. Asymptotically optimal a priori error estimates in two norms have been proven. The numerical efficiency of two-level methods has been demonstrated by Wu and Allen [21] for nonlinear reaction–diffusion equations.

In this paper, residual based a posteriori error estimates for the four two-level methods given in [14, 15,17] (Algorithms 2–5) are proved. In contrast to a priori estimates, the error is estimated by known and computed quantities, like the right hand side and the discrete solution. This enables an error control during the computation. For each two-level method, a posteriori error estimates in two norms are proved. The basic ideas to obtain these estimates go back to Verfürth [19] and Eriksson et al. [8]. They are sketched here in the proofs of Propositions 1 and 2. It turns out that the estimates for the two-level methods contain additional terms in comparison to a posteriori error estimates for standard solution techniques. This is due to the fact that the discrete solutions obtained by the two-level methods fulfill only an approximate or violated (as called by Angermann [1]) Galerkin orthogonality. We analyze the asymptotic behaviour of these additional terms. In this way, we get useful information on the effect of solving different equations on the fine mesh, on the gain of applying a final coarse mesh correction step, and on the necessity of computing the additional terms for obtaining an asymptotically optimal a posteriori error estimate. We show that for optimally scaled meshes, the bounds of the asymptotic behaviour of the additional terms are in general not of higher order than the standard terms in the residual based a posteriori error estimators. Thus, the computation of these terms is unavoidable in practice.

2. Notations and mathematical preliminaries

We consider a finite element approximation of the stationary Navier–Stokes equations in primitive variables

$$\begin{aligned} -\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p&=\boldsymbol{f} && \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u}&=0 && \text{in } \Omega,\\ \boldsymbol{u}&=\boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, Ω is a polyhedral domain in $\mathbb{R}^d$, $d=2,3$, with boundary $\partial\Omega$, $\boldsymbol{u}:\Omega\to\mathbb{R}^d$ is the flow field, $p:\Omega\to\mathbb{R}$ is the pressure which is normalized by $\int_\Omega p\,\mathrm{d}x=0$, $\boldsymbol{f}:\Omega\to\mathbb{R}^d$ a body force, and the parameter $\nu>0$ is the viscosity. Throughout this paper, we consider laminar fluid flows and assume that there is a nonsingular solution $(\boldsymbol{u},p)$ of (1). Hence, the viscosity ν has to be sufficiently large. A weak formulation of (1) reads as: Find $(\boldsymbol{u},p)\in V\times Q$ satisfying

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})+(q,\nabla\cdot\boldsymbol{u})-(p,\nabla\cdot\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})\quad\forall(\boldsymbol{v},q)\in V\times Q \tag{2}$$

with

$$V = \left(H_0^1(\Omega)\right)^d = \left\{\boldsymbol{v} \in \left(H^1(\Omega)\right)^d\colon\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\right\},$$
$$Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega)\colon \int_\Omega q\,\mathrm{d}x = 0\right\},$$
$$a(\boldsymbol{u}, \boldsymbol{v}) = (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \qquad b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \left((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}\right)$$

and $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and in its vector valued versions. The norm in $(L^2(\Omega))^d$ is denoted by $\|\cdot\|_0$, the seminorm in the standard Sobolev space $(H^k(\Omega))^d$ by $|\cdot|_k$, and the norm in $(H^k(\Omega))^d$ by $\|\cdot\|_k$. The space V is equipped with the norm $|\cdot|_1$ which is possible as a consequence of Poincaré's inequality. Bilinear forms, trilinear forms, and norms in subdomains $\omega \subset \Omega$ are marked by an additional index, e.g., $b_\omega(\cdot,\cdot,\cdot)$ or $\|\cdot\|_{0,\omega}$. The product space $V \times Q$ is equipped with the norm

$$|||(\boldsymbol{u}, p)||| = \left(|\boldsymbol{u}|_1^2 + \|p\|_0^2\right)^{1/2} \quad \text{for } (\boldsymbol{u}, p) \in V \times Q.$$

Let $\mathcal{T}_h$ denote a decomposition of Ω into d-simplices. We denote by h_T the diameter of a simplex T, by h_E the diameter of a face E, and we set $h = \max_{T\in\mathcal{T}_h}\{h_T\}$. Each family of triangulations $\{\mathcal{T}_h\}_h$ is assumed to be admissible and shape regular in the usual sense, e.g., see [5].

Given the decomposition $\mathcal{T}_h$, we can construct conforming velocity–pressure finite element spaces $V_h \times Q_h$ with $V_h \subset V$ and $Q_h \subset Q$. These spaces are assumed to satisfy the inf–sup condition, i.e., there exists a constant $\beta > 0$ independent of the triangulation such that

$$\sup_{\boldsymbol{v}_h \in V_h} \frac{(\nabla \cdot \boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_1} \geqslant \beta \|q_h\|_0 \quad \forall q_h \in Q_h. \tag{3}$$

Examples of spaces V_h, Q_h satisfying (3) can be found, e.g., in [2,10]. The standard finite element approximation to (1) is to find $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ satisfying for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h). \tag{4}$$

We assume that the solution of (2) and the finite element spaces satisfy

$$\begin{cases} \boldsymbol{u} \in (H^{k+1}(\Omega))^d \cap V, \quad p \in H^k(\Omega) \cap Q, \qquad k \geqslant 1, \\ V_h \text{ contains piecewise polynomials of degree } k, \\ Q_h \text{ contains piecewise polynomials of degree } k-1. \end{cases} \tag{5}$$

Furthermore, the a priori error estimates

$$\begin{cases} |||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant ch^k\left(|\boldsymbol{u}|_{k+1} + |p|_k\right), \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant ch^{k+1}\left(|\boldsymbol{u}|_{k+1} + |p|_k\right), \end{cases} \tag{6}$$

are assumed to hold, where c denotes throughout this paper a generic constant independent of $\mathcal{T}_h$. The properties (5) are important to prove (6) for a wide variety of velocity–pressure finite element spaces satisfying (3), see, e.g., [10].

Let $(\boldsymbol{u}, p)$ be the solution of (2). We consider a dual linearized Navier–Stokes problem of finding $(\boldsymbol{z}, s) \in V \times Q$ satisfying for all $(\boldsymbol{w}, t) \in V \times Q$

$$a(\boldsymbol{w}, \boldsymbol{z}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z}) + b(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{z}) - (t, \nabla \cdot \boldsymbol{z}) + (s, \nabla \cdot \boldsymbol{w}) = (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}). \tag{7}$$

We assume that (7) is $H^2 \times H^1$-regular, namely (z, s) exists uniquely and satisfies

$$\|z\|_2 + \|s\|_1 \leqslant C_s(\boldsymbol{u}, \boldsymbol{u}_h, \nu)\|\boldsymbol{u} - \boldsymbol{u}_h\|_0. \tag{8}$$

As indicated by Johnson [12], the worst case behaviour of the stability constant is $C_\mathrm{s} \sim \mathrm{O}(\exp(\nu^{-1}))$. If this behaviour reflects the actual stability property of a flow, every error estimate involving C_s would be worthless in practice even for relatively large viscosity parameters. However, the behaviour of C_s for laminar flows is substantially better in many situations. For some model problems, Johnson and Rannacher [13] could prove $C_\mathrm{s} \sim \mathrm{O}(\nu^{-1})$ and numerical results by Johnson [12] show $C_\mathrm{s} \leqslant 1.6$ for the driven cavity flow in three dimensions and $\nu \geqslant \frac{1}{700}$.

In the following, we need some estimates of the trilinear term, see [7,17,18]:

$$\begin{aligned}
&\left|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\right| \leqslant c(\varepsilon)\|\boldsymbol{u}\|_0^{1-\varepsilon}|\boldsymbol{u}|_1^{\varepsilon}|\boldsymbol{v}|_1|\boldsymbol{w}|_1 \qquad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V,\\
&\quad \varepsilon \in (0,1) \text{ if } d = 2, \quad \varepsilon = \tfrac{1}{2} \text{ if } d = 3,
\end{aligned} \tag{9}$$

$$\left|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\right| \leqslant c\|\boldsymbol{u}\|_0|\boldsymbol{v}|_1\|\boldsymbol{w}\|_2 \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,\ \boldsymbol{w} \in \left(H^2(\Omega)\right)^d, \tag{10}$$

$$\left|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\right| \leqslant c\|\boldsymbol{u}\|_0|\boldsymbol{v}|_{1,\infty}\|\boldsymbol{w}\|_0 \qquad \forall\, \boldsymbol{u}, \boldsymbol{w} \in V, \boldsymbol{v} \in \left(W^{1,\infty}(\Omega)\right)^d. \tag{11}$$

Let R_{V_h} and R_{Q_h} denote interpolation operators of Clément [6] into V_h and Q_h, respectively. These operators are based on local L^2-projections and satisfy for $0 \leqslant l \leqslant m \leqslant 1$ and an interpolation constant C_i

$$\begin{cases}
\left|\boldsymbol{v} - R_{V_h}(\boldsymbol{v})\right|_{l,T} \leqslant C_\mathrm{i} h_T^{m-l+1}|\boldsymbol{v}|_{m+1,\widetilde{\omega}(T)} & \forall \boldsymbol{v} \in V \cap \left(H^{m+1}(\Omega)\right)^d,\\
\left\|\boldsymbol{v} - R_{V_h}(\boldsymbol{v})\right\|_{0,E} \leqslant C_\mathrm{i} h_E^{m+1/2}|\boldsymbol{v}|_{m+1,\widetilde{\omega}(E)} & \forall \boldsymbol{v} \in V \cap \left(H^{m+1}(\Omega)\right)^d,\\
\left\|q - R_{Q_h}(q)\right\|_{0,T} \leqslant C_\mathrm{i} h_T|q|_{1,\widetilde{\omega}(T)} & \forall q \in Q \cap H^1(\Omega).
\end{cases} \tag{12}$$

Here, $\widetilde{\omega}(T)$ denotes the union of mesh cells which contains T and all mesh cells whose closure has a point with $\overline{T}$ in common. Similarly, $\widetilde{\omega}(E)$ denotes the union of all mesh cells whose closure has a common point with the closure of the face E. The interpolation operator of Clément satisfies, see also [5, formulae (17.10) and (17.11)],

$$\begin{cases}
\left\|q - R_{Q_h}(q)\right\|_0 \leqslant c\|q\|_0 & \forall q \in L^2(\Omega),\\
\left|\boldsymbol{v} - R_{V_h}(\boldsymbol{v})\right|_1 \leqslant c|\boldsymbol{v}|_1 & \forall \boldsymbol{v} \in V.
\end{cases} \tag{13}$$

From (13) and Poincaré's inequality, we get

$$\begin{cases}
\left\|R_{Q_h}(q)\right\|_0 \leqslant c\|q\|_0 & \forall q \in L^2(\Omega),\\
\left\|R_{V_h}(\boldsymbol{v})\right\|_1 \leqslant c|R_{V_h}(\boldsymbol{v})|_1 \leqslant c|\boldsymbol{v}|_1 \leqslant c\|\boldsymbol{v}\|_1 & \forall \boldsymbol{v} \in V.
\end{cases} \tag{14}$$

Analytical as well as numerical results indicate that the constant C_i in (12) are of moderate size, see Carstensen and Funken [3,4].

The jump $[|\boldsymbol{v}_h|]_E$ of a function $\boldsymbol{v}_h$ across a face E is defined by

$$[|\boldsymbol{v}_h|]_E(x) := \begin{cases}
\lim\limits_{t\to+0}\left\{\boldsymbol{v}_h(x + tn_E) - \boldsymbol{v}_h(x - tn_E)\right\}, & x \in E \not\subset \partial\Omega,\\
\lim\limits_{t\to+0}\left\{-\boldsymbol{v}_h(x - tn_E)\right\}, & x \in E \subset \partial\Omega,
\end{cases}$$

where n_E is a normal unit vector on E. If $E \subset \partial\Omega$, n_E denotes the outer normal, otherwise n_E has an arbitrary but fixed orientation. With that, every face E which separates two neighboring mesh cells T_1

and T_2 is associated with a uniquely oriented normal (for definiteness from T_1 to T_2). If $\boldsymbol{v} \in V$, then we know from the trace theorem $\boldsymbol{v}|_E \in (H^{1/2}(E))^d$ and therefore $[|\boldsymbol{v}|]_E = \mathbf{0}$ for almost every $x \in E$.

3. On the analysis of residual based a posteriori error estimators

Let $\|\cdot\|$ be a prescribed norm in $V \times Q$, η_T the error estimate on the mesh cell T and $\eta = (\sum_{T\in\mathcal{T}_h} \eta_T^2)^{1/2}$ the global error estimate.

The local error estimate $\eta_T \leqslant c_T \|\boldsymbol{e}_h\|_{U(T)}$ serves to control an adaptive grid refinement. Here, $\boldsymbol{e}_h = (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)$ is the error, $U(T)$ is a neighbourhood of T and c_T is a constant which should be approximately the same for all mesh cells. Proofs of local lower estimates for residual based a posteriori error estimators use suitable cut-off functions as given in Verfürth [19,20] or in [11]. It turns out that they can be proven for the error $(\boldsymbol{u} - \boldsymbol{v}_h, p - q_h)$ with an arbitrary pair of discrete functions $(\boldsymbol{v}_h, q_h)$, in particular, for the solution obtained by each two-level method which will be described in this paper. Thus, an adaptive grid refinement for these methods can be controlled by the local estimators (16) or (19) given below. However, it is not clear how to combine two-level methods with adaptive grid refinement and to our knowledge there is no literature on this subject available.

Information on the global error are obtained from an upper estimate

$$\|\boldsymbol{e}_h\| \leqslant c\eta.$$

This estimate serves as a stopping criterion for the solution process. Given a required accuracy *tol*, the discrete solution is sufficiently accurate if $c\eta \leqslant tol$. Therefore, the constant c must be known at least approximately. An important property for proving global upper estimates is the so-called Galerkin orthogonality of the discretization. That means, the error of the discrete solution is in some sense orthogonal to the finite element space, see (17) for the standard finite element discretization of the Navier–Stokes equations. The Galerkin orthogonality allows to introduce an interpolant of a suitable function into the error estimate such that the asymptotic correctness of the a posteriori error estimates can be proved by applying interpolation estimates of type (12). However, there are many situations where a Galerkin orthogonality might not hold, e.g., if

- only an approximation of the discrete solution is computed, e.g., by an iterative solver or a two-level method,
- quadrature errors occur,
- the discrete problem differs from the standard finite element discretization,
- nonconforming finite element discretizations are used.

In these cases, only an approximate Galerkin orthogonality holds and a global a posteriori error estimate of the form

$$\|\boldsymbol{e}_h\| \leqslant c\eta + E(\boldsymbol{u}_h, p_h) \tag{15}$$

with the additional term $E(\boldsymbol{u}_h, p_h)$ can be obtained. Only if $E(\boldsymbol{u}_h, p_h)$ is of higher order than η, this term can be neglected in computations.

The aim of this paper is to derive estimates of the form (15) for discrete solutions of the Navier–Stokes equations in the case of several two-level methods. The asymptotic behaviour of the extra terms will be bounded and it turns out that these bounds are in general not of higher order for optimally scaled meshes.

4. The standard one-level method

We start by recalling the a posteriori error estimates in the $|||\cdot|||$-norm and the L^2-norm for the one-level conforming finite element method (4).

Algorithm 1. Standard one-level algorithm
Step 1. Compute $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that $\forall(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h).$$

Proposition 1. *If h is sufficiently small, then for $(\boldsymbol{u}_h, p_h)$ computed with Algorithm* 1, *the a posteriori error estimate*

$$|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leqslant c\eta_1((\boldsymbol{u}_h, p_h)) = c\left(\sum_T \eta^2_{T,|||\cdot|||}\right)^{1/2}$$

with

$$\eta^2_{T,|||\cdot|||} := h_T^2\big\|\boldsymbol{f} + \nu\Delta\boldsymbol{u}_h - (\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h - \nabla p_h\big\|^2_{0,T} + \|\nabla\cdot\boldsymbol{u}_h\|^2_{0,T} + \sum_{E\subset\partial T, E\not\subset\partial\Omega} h_E\big\|[|\nu\nabla\boldsymbol{u}_h\cdot n_E - p_h n_E|]_E\big\|^2_{0,E} \tag{16}$$

holds true, where c depends on $(\boldsymbol{u}, p)$ and C_{i}.

Proof. A proof of the $|||\cdot|||$-norm a posteriori error estimate can be found in Verfürth [19]. For completeness and to show the rôle of the Galerkin orthogonality, we will sketch the proof here.

If h is sufficiently small, then Proposition 7.1 in [19] gives

$$|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leqslant 2\big\|\mathrm{D}F(\boldsymbol{u}, p)^{-1}\big\|_{\mathcal{L}((V\times Q)^*,(V\times Q))}\|F(\boldsymbol{u}_h, p_h)\|_{(V\times Q)^*},$$

where $F(\boldsymbol{u}_h, p_h): V\times Q \to \mathbb{R}$ is the residual

$$F(\boldsymbol{u}_h, p_h) = a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) - (p_h, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_h) - (\boldsymbol{f}, \boldsymbol{v}),$$

$\mathrm{D}F(\boldsymbol{u}, p)$ is the Fréchet derivative of F at $(\boldsymbol{u}, p)$, and $(V\times Q)^*$ is the dual space of $V\times Q$. Since $(\boldsymbol{u}, p)$ is assumed to be a nonsingular solution of (1), $\|\mathrm{D}F(\boldsymbol{u}, p)^{-1}\|_{\mathcal{L}((V\times Q)^*,(V\times Q))} < \infty$. Because of $F(\boldsymbol{u}, p) = 0$, we have

$$\begin{aligned}\big\|F(\boldsymbol{u}_h, p_h)\big\|_{(V\times Q)^*} &= \big\|F(\boldsymbol{u}, p) - F(\boldsymbol{u}_h, p_h)\big\|_{(V\times Q)^*}\\ &= \sup_{(\boldsymbol{0},0)\neq(\boldsymbol{v},q)\in V\times Q}\big\{|||(\boldsymbol{v}, q)|||^{-1}\big|a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})\\ &\qquad - b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) - (p-p_h, \nabla\cdot\boldsymbol{v}) + \big(q, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)\big|\big\}.\end{aligned}$$

The second factor will be estimated. From (2) and (4), we get the Galerkin orthogonality of the discretization: $\forall(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$0 = -a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + (p-p_h, \nabla\cdot\boldsymbol{v}_h) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - \big(q_h, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big). \tag{17}$$

We set $\boldsymbol{r} = \boldsymbol{f} + \nu\Delta\boldsymbol{u}_h - (\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h - \nabla p_h$. The Galerkin orthogonality (17) with $\boldsymbol{v}_h = R_{V_h}(\boldsymbol{v})$, $q_h = R_{Q_h}(q)$, integration by parts, (2), $\nabla\cdot\boldsymbol{u} = 0$, the Cauchy–Schwarz inequality, the interpolation estimates (12), the shape regularity of the mesh, and the definition of the $|||\cdot|||$-norm imply

$$\begin{aligned}&\big|a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v})-(p-p_h,\nabla\cdot\boldsymbol{v})+\big(q,\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)\big|\\&\quad\leqslant c\left[\left(\sum_T h_T^2\|\boldsymbol{r}\|_{0,T}^2+\sum_E h_E\big\|\big[|\nu\nabla\boldsymbol{u}_h\cdot n_E-p_hn_E|\big]_E\big\|_{0,E}^2+\sum_T\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}^2\right)^{1/2}\right]|||(\boldsymbol{v},q)|||,\end{aligned}$$

which concludes the proof. □

Proposition 2. *With the assumption* (8) *on the stability of the linearized dual problem* (7)*, the a posteriori error estimate*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0\leqslant c\eta_2((\boldsymbol{u}_h,p_h))=c\left(\sum_T\eta_{T,L^2}^2\right)^{1/2}\tag{18}$$

with

$$\begin{aligned}\eta_{T,L^2}^2:=\;&h_T^4\big\|\boldsymbol{f}+\nu\Delta\boldsymbol{u}_h-(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h-\nabla p_h\big\|_{0,T}^2+h_T^2\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}^2\\&+\sum_{E\subset\partial T,E\not\subset\partial\Omega}h_E^3\big\|\big[|\nu\nabla\boldsymbol{u}_h\cdot n_E-p_hn|\big]_E\big\|_{0,E}^2\end{aligned}\tag{19}$$

is valid for $(\boldsymbol{u}_h,p_h)$ *computed with Algorithm* 1*, where* c *depends on* $C_\mathrm{s}C_\mathrm{i}$.

Proof. The proof, which is the basis of proving a posteriori L^2-error estimates below, follows the framework developed by Eriksson et al. [8].

1. *Error representation using the dual linearized problem.* With $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{u}_h$, $t=p-p_h$, we obtain from (7)

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2=\;&a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z})+b(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{z})+b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u},\boldsymbol{z})\\&+\big(s,\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)-(p-p_h,\nabla\cdot\boldsymbol{z}).\end{aligned}\tag{20}$$

2. *Galerkin orthogonality of the discrete problem* (17).

3. *Interpolation estimates for the solution of the dual linearized problem.* We set $\boldsymbol{v}_h=R_{V_h}(\boldsymbol{z})=:\boldsymbol{z}_h$, $q_h=R_{Q_h}(s)=:s_h$ in (17), add (20), note $\nabla\cdot\boldsymbol{u}=0$ and set $\boldsymbol{r}=\boldsymbol{f}+\nu\Delta\boldsymbol{u}_h-(\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h-\nabla p_h$. Integration by parts and the interpolation estimates (12) give

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2\leqslant\;&C_\mathrm{i}\sum_T\big[h_T^2\|\boldsymbol{r}\|_{0,T}\|\boldsymbol{z}\|_{2,\widetilde{\omega}(T)}+h_T\|\nabla\cdot\boldsymbol{u}_h\|_{0,T}\|s\|_{1,\widetilde{\omega}(T)}\big]\\&+C_\mathrm{i}\sum_E h_E^{3/2}\big\|[|\nu\nabla\boldsymbol{u}_h\cdot n_E-p_hn_E|]_E\big\|_{0,E}\|\boldsymbol{z}\|_{2,\tilde{\omega}(E)}.\end{aligned}$$

4. *Strong stability of the dual problem.* With assumption (8) and the shape regularity of the mesh, we obtain (18) after dividing by $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$. □

It is well known that the a posteriori error estimates are asymptotically optimal, i.e., $\eta_1((\boldsymbol{u}_h,p_h))=\mathrm{O}(h^k)$ and $\eta_2((\boldsymbol{u}_h,p_h))=\mathrm{O}(h^{k+1})$.

5. A two-level method with one Newton step on the fine mesh

We now consider a two-level method which works on a fine mesh $\mathcal{T}_h$ and on a coarse mesh $\mathcal{T}_H$ where $h<H$ is assumed throughout this paper. The coarse spaces V_H and Q_H are assumed to have the properties of the finite element spaces stated above.

Algorithm 2. Two-level algorithm, coarse mesh solve followed by a fine mesh Newton step
Step 1. Compute $(\boldsymbol{u}_H, p_H) \in V_H \times Q_H$ such that $\forall(\boldsymbol{v}_H, q_H) \in V_H \times Q_H$

$$a(\boldsymbol{u}_H, \boldsymbol{v}_H) + b(\boldsymbol{u}_H, \boldsymbol{u}_H, \boldsymbol{v}_H) + (q_H, \nabla \cdot \boldsymbol{u}_H) - (p_H, \nabla \cdot \boldsymbol{v}_H) = (\boldsymbol{f}, \boldsymbol{v}_H).$$

Step 2. Compute $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that $\forall(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$\begin{aligned} &a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_H, \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) \\ &\quad = (\boldsymbol{f}, \boldsymbol{v}_h) + b(\boldsymbol{u}_H, \boldsymbol{u}_H, \boldsymbol{v}_h). \end{aligned}$$

In Algorithm 2, only one step of a Newton iteration is performed on the fine mesh $\mathcal{T}_h$, i.e., one has to solve only one linear system on the fine mesh. It has been proposed by Layton [14] and further analyzed by Layton and Lenferink [16]. Let the assumption (5) be fulfilled. Then, the a priori error estimate

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c(\boldsymbol{u}, p)\bigl(h^k + H^{2k+1-\varepsilon}\bigr) \tag{21}$$

holds true with ε defined in (9). Thus, for an asymptotically optimal error estimate in the $|||\cdot|||$-norm, the optimal scaling is $h = \mathrm{O}(H^{2+(1-\varepsilon)/k})$. The L^2-error of the velocity can be estimated in the form

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c(\boldsymbol{u}, p)\bigl(h^{k+1} + H^{2k+1}\bigr), \tag{22}$$

which leads to an optimal mesh scaling of $h = \mathrm{O}(H^{2-1/(k+1)})$.

An a posteriori error estimate for the $|||\cdot|||$-norm of the solution computed with Algorithm 2 has been proven by Ervin et al. [9]. The proof uses the same basic ideas as the proof of Proposition 1. In addition, results of numerical tests with Algorithm 2 are presented in [9].

Proposition 3. *If H is sufficiently small, then for $(\boldsymbol{u}_h, p_h)$ computed with Algorithm* 2, *the a posteriori error estimate*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c\left[\left(\sum_T \eta^2_{T,|||\cdot|||}\right)^{1/2} + \|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{1+\varepsilon}\right] \tag{23}$$

holds true with c depending on $(\boldsymbol{u}, p)$ and C_{i}.

Proposition 4. *Under the assumptions of Proposition* 2 *the a posteriori error estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c\left[\left(\sum_T \eta^2_{T,L^2}\right)^{1/2} + h\|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{1+\varepsilon} + \|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0|\boldsymbol{u}_h - \boldsymbol{u}_H|_1\right] \tag{24}$$

is valid for $(\boldsymbol{u}_h, p_h)$ computed with Algorithm 2.

Proof. To apply the techniques of proving Proposition 2, the crucial point is to find an approximate Galerkin orthogonality similar to (17). Step 2 of Algorithm 2, Eq. (2) and the identity

$$b(\boldsymbol{u}_H, \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h) - b(\boldsymbol{u}_H, \boldsymbol{u}_H, \boldsymbol{v}_h) = b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{v}_h)$$

give for all $(\boldsymbol{v}_h, s_h) \in V_h \times Q_h$

$$\begin{aligned} b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{v}_h) = {} & -a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + (p - p_h, \nabla \cdot \boldsymbol{v}_h) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\ & + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - \bigl(s_h, \nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\bigr), \end{aligned}$$

see [9]. Now, following the proof of Proposition 2, we have to estimate the additional term $|b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, z_h)|$. Its estimation starts with the splitting

$$\left|b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, z_h)\right| = \left|b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, z_h - z) + b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, z)\right|.$$

Using now (9), (10), the interpolation estimate (12), and the strong stability assumption (8), we obtain the estimate (24). □

The asymptotic order of convergence of the fine grid solution $(\boldsymbol{u}_h, p_h)$ is at least as good as that of the coarse grid solution, see (21), (22). Thus, the following asymptotic behaviour of the extra term in (23) can be derived, using the a priori estimate (6) for $\boldsymbol{u} - \boldsymbol{u}_H$:

$$\begin{aligned}\|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{1+\varepsilon} &\leqslant \left(\|\boldsymbol{u} - \boldsymbol{u}_H\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_0\right)^{1-\varepsilon}\left(|\boldsymbol{u} - \boldsymbol{u}_H|_1 + |\boldsymbol{u} - \boldsymbol{u}_h|_1\right)^{1+\varepsilon} \\ &\leqslant \left(\left(H^{k+1} + H^{k+1}\right)c(\boldsymbol{u}, p)\right)^{1-\varepsilon}\left(\left(H^k + H^k\right)c(\boldsymbol{u}, p)\right)^{1+\varepsilon} \\ &= \mathrm{O}\left(H^{2k+1-\varepsilon}\right).\end{aligned}$$

In this bound, the asymptotic behaviour of the additional term does not depend on the fine mesh size h. If the fine mesh size is chosen asymptotically coarser than given by the optimal mesh scaling, $H^{2k+1-\varepsilon} = \mathrm{o}(h^k)$, we conclude from the a priori error estimate (21)

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c(\boldsymbol{u}, p)h^k.$$

In this case, the asymptotic behaviour of the extra term is of higher order. If h is chosen to be optimal or asymptotically smaller, the asymptotic behaviour of the additional term and the discrete solution coincide. Now, the computation of the additional term becomes important for an asymptotically optimal a posteriori error estimation. The extra term measures the difference of the coarse and fine grid solution and it becomes the more important the greater this difference becomes. Analogously, we find that the additional terms in the a posteriori L^2-error estimate behave like $\mathrm{O}(H^{2k+1})$.

6. A two-level method with one Oseen step on the fine mesh

The two-level method analyzed in this section is a simplification of Algorithm 2. The Newton step of Algorithm 2 is replaced by one step of a fixed point iteration.

Algorithm 3. Two-level algorithm, coarse mesh solve followed by a fine mesh Oseen step
Step 1. Same as Step 1 of Algorithm 2.
Step 2. Compute $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_H, \boldsymbol{u}_h, \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h).$$

Algorithm 3 is called modified Picard method and the equation in Step 2 is called Oseen equation. It has been proposed by Layton and Lenferink [15]. If the assumption (5) is fulfilled, the a priori error estimates

$$\begin{cases} |||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c(\boldsymbol{u}, p)\left(h^k + H^{k+1}\right), \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c(\boldsymbol{u}, p)\left(h^{k+1} + H^{k+1}\right), \end{cases} \tag{25}$$

are valid, [15], which results in optimal mesh scalings $h = \mathrm{O}(H^{1+1/k})$ for the $|||\cdot|||$-norm and $h = \mathrm{O}(H)$ for the L^2-norm of the velocity.

In Algorithm 3 as well as in Algorithm 2, only one linear system on the fine mesh has to be solved. But the term $b(\boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h)$ on the left-hand side of Step 2 in Algorithm 2 may lead to certain computational disadvantages, e.g., to bad matrix properties and large memory requirements. For this reason, the Newton iteration is replaced often by a fixed point iteration, where the term $b(\boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h)$ is put on the right-hand side and $\boldsymbol{u}_h$ is replaced by the current approximation of $\boldsymbol{u}_h$. In this fixed point iteration, a number of Oseen equations has to be solved. Algorithm 3 is obtained if only one step of this fixed point iteration is performed with the initial approximation $\boldsymbol{u}_h \approx \boldsymbol{u}_H$ in $b(\boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h)$.

Proposition 5. *Let $(\boldsymbol{u}_h, p_h)$ be computed with Algorithm* 3*. With the assumptions of Proposition* 3*, the a posteriori error estimate*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c\left[\left(\sum_T \eta^2_{T,|||\cdot|||}\right)^{1/2} + \|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{\varepsilon}|\boldsymbol{u}_h|_1\right] \tag{26}$$

holds true. With the assumption of Proposition 4*, the a posteriori error estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c\left[\left(\sum_T \eta^2_{T,L^2}\right)^{1/2} + h\|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{\varepsilon}|\boldsymbol{u}_h|_1 + \|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0|\boldsymbol{u}_h|_1\right] \tag{27}$$

is valid.

Proof. For the first estimate, we follow the proof of Proposition 1. Step 2 of Algorithm 3 and (2) yield the approximate Galerkin orthogonality for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$\begin{aligned} -b(\boldsymbol{u}_H - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = {} & -a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + (p - p_h, \nabla\cdot\boldsymbol{v}_h) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\ & + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - \big(q_h, \nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_h)\big). \end{aligned} \tag{28}$$

In addition to the proof of Proposition 1, we have to estimate the extra term $|b(\boldsymbol{u}_H - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)|$. To obtain (26), we use (9), (14), and the definition of the $|||\cdot|||$-norm.

The L^2-error estimate of the velocity can be proved analogously to Proposition 4 using the approximate Galerkin orthogonality (28). □

With (5) and (6), we can establish a $\mathrm{O}(H^{k+1-\varepsilon})$ asymptotic behaviour of the extra term in (26) and a $\mathrm{O}(H^{k+1})$ asymptotic behaviour of the additional terms in (27).

7. The modified Picard method with a correction step

The next algorithm extends the modified Picard method, Algorithm 3, by the solution of a defect correction equation on the coarse mesh.

Algorithm 4. Two-level algorithm, coarse mesh solve followed by a fine mesh Oseen step and a coarse mesh correction

Step 1. Same as Step 1 of Algorithms 2 and 3.
Step 2. Same as Step 2 of Algorithm 3, the solution is denoted by $(\boldsymbol{u}_*, p_*)$.
Step 3. Compute $(\boldsymbol{e}_H, \epsilon_H) \in V_H \times Q_H$ such that for all $(\boldsymbol{v}_H, q_H) \in V_H \times Q_H$

$$a(\boldsymbol{e}_H, \boldsymbol{v}_H) + b(\boldsymbol{u}_H, \boldsymbol{e}_H, \boldsymbol{v}_H) + b(\boldsymbol{e}_H, \boldsymbol{u}_H, \boldsymbol{v}_H) - (\epsilon_H, \nabla\cdot\boldsymbol{v}_H) + (q_H, \nabla\cdot\boldsymbol{e}_H) \\ = b(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{v}_H)$$

and set $\boldsymbol{u}_h = \boldsymbol{u}_* + \boldsymbol{e}_H,\ p_h = p_* + \epsilon_H$.

A two-level method with an additional coarse mesh correction was studied first by Xu [23] for semilinear elliptic equations. Algorithm 4 has been proposed and a priori error estimates have been given by Layton and Tobiska [17]:

$$\begin{cases} |||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c(\boldsymbol{u}, p)\big(h^k + H^{2k+1-\varepsilon}\big), \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c(\boldsymbol{u}, p)\big(h^{k+1} + h^k H^2 + hH^{k+1} + H^{\min(k+3,2k+1)}\big), \end{cases} \tag{29}$$

with ε defined in (9). Thus, scalings which ensure optimal order of convergence (6) are

$$\begin{array}{lll} |||\cdot|||\text{-norm}: & \text{for } k=1:\ h = \mathrm{O}\big(H^{3-\varepsilon}\big), & \text{for } k\geqslant 2:\ h = \mathrm{O}\big(H^{1+2/k}\big), \\ \|\cdot\|_0\text{-norm}: & \text{for } k=1:\ h = \mathrm{O}\big(H^{3/2}\big), & \text{for } k\geqslant 2:\ h = \mathrm{O}\big(H^{1+1/k}\big). \end{array}$$

The Galerkin projection $(R_V, R_Q): V\times Q \to V_H \times Q_H$ with respect to the dual linearized problem of Step 3 in Algorithm 4 is defined in the following way: for all $(\boldsymbol{w}, t) \in V_h \times Q_h$, $(\boldsymbol{v}, q) \in V \times Q$

$$0 = a\big(\boldsymbol{w}, \boldsymbol{v} - R_V(\boldsymbol{v}, q)\big) + b\big(\boldsymbol{u}_H, \boldsymbol{w}, \boldsymbol{v} - R_V(\boldsymbol{v}, q)\big) + b\big(\boldsymbol{w}, \boldsymbol{u}_H, \boldsymbol{v} - R_V(\boldsymbol{v}, q)\big) \\ - \big(t, \nabla\cdot\big(\boldsymbol{v} - R_V(\boldsymbol{v}, q)\big)\big) + \big(q - R_Q(\boldsymbol{v}, q), \nabla\cdot\boldsymbol{w}\big).$$

In [17], the approximate Galerkin orthogonality for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$b(\boldsymbol{u} - \boldsymbol{u}_H, \boldsymbol{u} - \boldsymbol{u}_H, \boldsymbol{v}_h) - b\big(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{v}_h - R_V(\boldsymbol{v}_h, q_h)\big) \\ = -a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) + (p - p_h, \nabla\cdot\boldsymbol{v}_h) - b(\boldsymbol{u}_H, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) \\ - b(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h) - \big(q_h, \nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_h)\big), \tag{30}$$

and the estimates

$$|||(\boldsymbol{v} - R_V(\boldsymbol{v}, q), q - R_Q(\boldsymbol{v}, q))||| \leqslant c|||(\boldsymbol{v}, q)|||, \quad \forall\,(\boldsymbol{v}, q) \in V\times Q, \tag{31}$$

$$\|\boldsymbol{v}_h - R_V(\boldsymbol{v}_h, q_h)\|_\theta \leqslant cH^{1-\theta}|||(\boldsymbol{v}_h, q_h)|||, \quad \forall(\boldsymbol{v}_h, q_h) \in V_h\times Q_h,\ \theta\in[0,1], \tag{32}$$

$$\|\boldsymbol{v} - R_V(\boldsymbol{v}, q)\|_0 + H|\boldsymbol{v} - R_V(\boldsymbol{v}, q)|_1 \leqslant cH^2\big(\|\boldsymbol{v}\|_2 + \|q\|_1\big) \\ \forall(\boldsymbol{v}, q) \in (V\times Q)\cap\big(H^2(\Omega)^d \times H^1(\Omega)\big) \tag{33}$$

have been proved.

Proposition 6. *Let $(\boldsymbol{u}_h, p_h)$ be computed with Algorithm* 4. *Then, with the assumptions of Proposition* 3, *the a posteriori error estimate*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \\ \leqslant c\left[\left(\sum_T \eta^2_{T,|||\cdot|||}\right)^{1/2} + \|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{1+\varepsilon} + \|\boldsymbol{u}_* - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_* - \boldsymbol{u}_H|_1^{\varepsilon}|\boldsymbol{u}_H|_1\right] \tag{34}$$

and with the assumption of Proposition 4, *the a posteriori error estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c\left[\left(\sum_T \eta^2_{T,L^2}\right)^{1/2} + h\|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h - \boldsymbol{u}_H|_1^{1+\varepsilon} + \|\boldsymbol{u}_h - \boldsymbol{u}_H\|_0|\boldsymbol{u}_h - \boldsymbol{u}_H|_1 \right.\\ \left. + H\|\boldsymbol{u}_* - \boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_* - \boldsymbol{u}_H|_1^{\varepsilon}|\boldsymbol{u}_H|_1\right] \tag{35}$$

are valid.

Proof. The proof of the first estimate starts like that of Proposition 1. Using the approximate Galerkin orthogonality (30) and reordering some terms, we obtain for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$\begin{aligned}
&\big|a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) - (p - p_h, \nabla\cdot\boldsymbol{v}) + \big(q, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)\big| \\
&\quad = \big|(\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{v}_h) - \big[a(\boldsymbol{u}_h, \boldsymbol{v}-\boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}-\boldsymbol{v}_h) + (q - q_h, \nabla\cdot\boldsymbol{u}_h) - \big(p_h, \nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_h)\big)\big] \\
&\qquad - b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{v}_h) + b\big(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{v}_h - R_V(\boldsymbol{v}_h, q_h)\big)\big|.
\end{aligned}$$

The first terms are estimated in the same way as in Proposition 1. For estimating the last two terms, we use estimate (9) for the trilinear form, estimate (32) for the Galerkin projection, (14), and the definition of the $|||\cdot|||$-norm.

For the second estimate, we follow the proof of Proposition 2, now using the approximate Galerkin orthogonality (30). In this way, we have to estimate the additional terms $|b(\boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{u}_h - \boldsymbol{u}_H, \boldsymbol{z}_h)|$ and $|b(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{z}_h - R_V(\boldsymbol{z}_h, s_h))|$. The first one was estimated already in Proposition 4. Only the second term needs to be estimated in a different way. Applying (9), we obtain

$$\big|b\big(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{z}_h - R_V(\boldsymbol{z}_h, s_h)\big)\big| \leqslant c\|\boldsymbol{u}_H - \boldsymbol{u}_*\|_0^{1-\varepsilon} |\boldsymbol{u}_H - \boldsymbol{u}_*|_1^{\varepsilon} |\boldsymbol{u}_H|_1 \big|\boldsymbol{z}_h - R_V(\boldsymbol{z}_h, s_h)\big|_1.$$

Further, we get with (31), (12) and (33)

$$\begin{aligned}
\big|\boldsymbol{z}_h - R_V(\boldsymbol{z}_h, s_h)\big|_1 &\leqslant \big|R_V(\boldsymbol{z}, s) - R_V(\boldsymbol{z}_h, s_h)\big|_1 + |\boldsymbol{z} - \boldsymbol{z}_h|_1 + \big|\boldsymbol{z} - R_V(\boldsymbol{z}, s)\big|_1 \\
&\leqslant c\big(|\boldsymbol{z} - \boldsymbol{z}_h|_1 + \|s - s_h\|_0\big) + \big|\boldsymbol{z} - R_V(\boldsymbol{z}, s)\big|_1 \\
&\leqslant cH\big(\|\boldsymbol{z}\|_2 + \|s\|_1\big).
\end{aligned}$$

Applying (8) completes the proof. □

The dominating extra term in (34) is $\|\boldsymbol{u}_* - \boldsymbol{u}_H\|_0^{1-\varepsilon} |\boldsymbol{u}_* - \boldsymbol{u}_H|_1^{\varepsilon} |\boldsymbol{u}_H|_1$. Since the asymptotic behaviour of $\boldsymbol{u} - \boldsymbol{u}_*$ is not worse than of $\boldsymbol{u} - \boldsymbol{u}_H$, see (25), the triangle inequality together with the a priori error estimate (6) for $\boldsymbol{u} - \boldsymbol{u}_H$ and the assumption (5) give a $\mathrm{O}(H^{k+1-\varepsilon})$ behaviour of the extra terms. For the L^2-error estimate, the asymptotic convergence of the extra terms in (35) can be estimated with the a priori error estimates (6) and (29) for $\boldsymbol{u} - \boldsymbol{u}_H$ and $\boldsymbol{u} - \boldsymbol{u}_h$, respectively. The result is a $\mathrm{O}(H^{k+2-\varepsilon})$ asymptotic behaviour of these terms.

The asymptotic behaviour of the extra terms in (34) and (35) differs from the asymptotic orders of convergence derived from (29). This comes from an alternative estimation technique of the trilinear term which has been used in [17]. However, this technique requires additional regularity of $\boldsymbol{u}$ and leads to the fact that the constants in (29) depend on other norms of $\boldsymbol{u}$ than, e.g., the constants in (21), (22), and (25). We show in Remark 7 that this technique leads to essentially the same asymptotic behaviour of the extra terms as given above after an appropriate modification of the constants.

Remark 7 (Alternative estimate of the trilinear term). If the additional assumption $|\boldsymbol{u}_H|_{1,\infty} \leqslant c$ with c independent of H holds true, the trilinear term in the proof of Proposition 6 can be estimated in a different way. Using (11), (32), and (14), we obtain

$$\sup_{(\boldsymbol{0},0)\neq(\boldsymbol{v},q)\in V} \frac{|b(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{v}_h - R_V(\boldsymbol{v}_h, q_h))|}{|||(\boldsymbol{v}, q)|||} \leqslant cH\|\boldsymbol{u}_H - \boldsymbol{u}_*\|_0 |\boldsymbol{u}_H|_{1,\infty}. \tag{36}$$

This techniques is used in [17] which results in the fact that the constants in the a priori error estimates (29) depend on $|\boldsymbol{u}|_{1,\infty}$. Formally, the behaviour of the right hand side in (36) is $\mathrm{O}(H^{2k+1})$.

Using (36), one can prove an asymptotic behaviour of the extra terms in the a posteriori error estimates which coincides with the asymptotic orders of convergence given in (29). But in order to have a fair comparison, e.g., to (34), $|\boldsymbol{u}_H|_{1,\infty}$ should be replaced in (36) by $|\boldsymbol{u}_H|_1$. From [22, formula (5.5)], we know the discrete Sobolev inequality

$$\|\boldsymbol{v}_H\|_{0,\infty} \leqslant c(d,H)|\boldsymbol{v}_H|_1 \quad \forall \boldsymbol{v}_H \in V_H$$

with $c(d,H) = c|\ln H|^{1/2}$ if $d = 2$ and $c = cH^{-1/2}$ if $d = 3$. Besides that, it holds $|\boldsymbol{u}_H|_{1,\infty} \leqslant cH^{-1}\|\boldsymbol{u}_H\|_{0,\infty}$, see [5, formula (17.22)]. Applying these estimates to $|\boldsymbol{u}_H|_{1,\infty}$ in (36) gives

$$\sup_{0\neq(\boldsymbol{v},q)\in V} \frac{|b(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_H, \boldsymbol{v}_h - R_V(\boldsymbol{v}_h, q_h))|}{|||(\boldsymbol{v},q)|||} \leqslant c(d,H)\|\boldsymbol{u}_H - \boldsymbol{u}_*\|_0 |\boldsymbol{u}_H|_1.$$

This is of order $\mathrm{O}(H^{k+1/2})$ for $d = 3$ and therefore no improvement. For $d = 2$, the extra term is now of order $\mathrm{O}(H^{k+1}|\ln H|^{1/2})$ which is a slight improvement compared to $\mathrm{O}(H^{k+1-\varepsilon})$ for $H \to 0$.

8. Algorithm 2 with a correction step

In the previous section, we have seen that the correction step improves the a posteriori error estimate in the L^2-norm of Algorithm 4 in comparison to Algorithm 3. In this section, we will study if there holds a similar result for Algorithm 2 with an appropriate correction step. The following algorithm has been proposed by Layton and Tobiska [17].

Algorithm 5. Two-level algorithm, coarse mesh solve followed by a fine mesh Newton step and a coarse mesh correction

Step 1. Same as Step 1 of Algorithms 2.
Step 2. Same as Step 2 of Algorithm 2, the solution is denoted by $(\boldsymbol{u}_*, p_*)$.
Step 3. Compute $(\boldsymbol{e}_H, \epsilon_H) \in V_H \times Q_H$ such that for all $(\boldsymbol{v}_H, q_H) \in V_H \times Q_H$

$$\begin{aligned} &a(\boldsymbol{e}_H, \boldsymbol{v}_H) + b(\boldsymbol{u}_H, \boldsymbol{e}_H, \boldsymbol{v}_H) + b(\boldsymbol{e}_H, \boldsymbol{u}_H, \boldsymbol{v}_H) + (q_H, \nabla\cdot\boldsymbol{e}_H) - (\epsilon_H, \nabla\cdot\boldsymbol{v}_H) \\ &\quad = b(\boldsymbol{u}_H - \boldsymbol{u}_*, \boldsymbol{u}_* - \boldsymbol{u}_H, \boldsymbol{v}_H) \end{aligned}$$

and set $\boldsymbol{u}_h = \boldsymbol{u}_* + \boldsymbol{e}_H,\ p_h = p_* + \epsilon_H$.

The a priori error estimates

$$\begin{cases} |||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leqslant c(\boldsymbol{u},p)\big(h^k + H^{2k+1-\varepsilon}\big), \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leqslant c(\boldsymbol{u},p)\big(h^{k+1} + H^{2k+3/2} + H^{k+1}h^k\big), \end{cases} \tag{37}$$

with ε defined in (9), have been proven in [17]. Thus, the optimal mesh scaling is $h = \mathrm{O}(H^{2+(1-\varepsilon)/k})$ for the $|||\cdot|||$-norm and $h = \mathrm{O}(H^{2-1/(2k+2)})$ for the L^2-norm of the velocity. The scaling with respect to the $|||\cdot|||$-norm is not better than for Algorithm 2, compare (21). For Algorithm 5, the approximate Galerkin orthogonality for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$

$$
\begin{aligned}
&-b(\boldsymbol{u}-\boldsymbol{u}_*, \boldsymbol{u}-\boldsymbol{u}_*, \boldsymbol{v}_h)+b(\boldsymbol{u}-\boldsymbol{u}_H, \boldsymbol{u}-\boldsymbol{u}_*, \boldsymbol{v}_h)+b(\boldsymbol{u}-\boldsymbol{u}_*, \boldsymbol{u}-\boldsymbol{u}_H, \boldsymbol{v}_h) \\
&\quad -b\big(\boldsymbol{u}_*-\boldsymbol{u}_H, \boldsymbol{u}_*-\boldsymbol{u}_H, R_V(\boldsymbol{v}_h, q_h)-\boldsymbol{v}_h\big) \\
&\quad = -a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h)+(p-p_h, \nabla\cdot\boldsymbol{v}_h)-b(\boldsymbol{u}_H, \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) \\
&\quad -b(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}_H, \boldsymbol{v}_h)-\big(q_h, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)
\end{aligned}
\tag{38}
$$

holds.

Proposition 8. *Let $(\boldsymbol{u}_h, p_h)$ be computed with Algorithm* 5. *With the assumptions of Proposition* 3*, the a posteriori error estimate*

$$
\begin{aligned}
|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leqslant c\Bigg[&\bigg(\sum_T \eta^2_{T,|||\cdot|||}\bigg)^{1/2}+\|\boldsymbol{u}_h-\boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h-\boldsymbol{u}_H|_1^{\varepsilon}|\boldsymbol{u}_*-\boldsymbol{u}_h|_1 \\
&+\|\boldsymbol{u}_h-\boldsymbol{u}_*\|_0^{1-\varepsilon}|\boldsymbol{u}_h-\boldsymbol{u}_*|_1^{\varepsilon}|\boldsymbol{u}_*-\boldsymbol{u}_H|_1 \\
&+\|\boldsymbol{u}_*-\boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_*-\boldsymbol{u}_H|_1^{1+\varepsilon}\Bigg]
\end{aligned}
\tag{39}
$$

and with the assumptions of Proposition 4*, the a posteriori error estimate*

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \leqslant c\Bigg[&\bigg(\sum_T \eta^2_{T,L^2}\bigg)^{1/2}+h\|\boldsymbol{u}_h-\boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_h-\boldsymbol{u}_H|_1^{\varepsilon}|\boldsymbol{u}_*-\boldsymbol{u}_h|_1 \\
&+\|\boldsymbol{u}_h-\boldsymbol{u}_H\|_0|\boldsymbol{u}_*-\boldsymbol{u}_h|_1+h\|\boldsymbol{u}_h-\boldsymbol{u}_*\|_0^{1-\varepsilon}|\boldsymbol{u}_h-\boldsymbol{u}_*|_1^{\varepsilon}|\boldsymbol{u}_*-\boldsymbol{u}_H|_1 \\
&+\|\boldsymbol{u}_h-\boldsymbol{u}_*\|_0|\boldsymbol{u}_*-\boldsymbol{u}_H|_1+H\|\boldsymbol{u}_*-\boldsymbol{u}_H\|_0^{1-\varepsilon}|\boldsymbol{u}_*-\boldsymbol{u}_H|_1^{1+\varepsilon}\Bigg]
\end{aligned}
\tag{40}
$$

hold true.

Proof. The proof of the first estimate is similar to those of Proposition 1 and the first part of Proposition 6. With the approximate Galerkin orthogonality (38), we obtain for all $(\boldsymbol{v}_h, q_h)\in V_h\times Q_h$

$$
\begin{aligned}
&\big|a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v})+b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})-b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v})-(p-p_h, \nabla\cdot\boldsymbol{v})+\big(q, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)\big| \\
&\quad =\big|(\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{v}_h)-\big[a(\boldsymbol{u}_h, \boldsymbol{v}-\boldsymbol{v}_h)+b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}-\boldsymbol{v}_h)+(q-q_h, \nabla\cdot\boldsymbol{u}_h)-\big(p_h, \nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_h)\big)\big] \\
&\quad +b(\boldsymbol{u}_h-\boldsymbol{u}_H, \boldsymbol{u}_*-\boldsymbol{u}_h, \boldsymbol{v}_h)-b(\boldsymbol{u}_h-\boldsymbol{u}_*, \boldsymbol{u}_H-\boldsymbol{u}_*, \boldsymbol{v}_h) \\
&\quad +b(\boldsymbol{u}_*-\boldsymbol{u}_H, \boldsymbol{u}_*-\boldsymbol{u}_H, R_V(\boldsymbol{v}_h, q_h)-\boldsymbol{v}_h)\big|.
\end{aligned}
$$

The proof continues like in the propositions mentioned above.

The proof of the second estimate is similar to those of Propositions 2 and 4. With the approximate Galerkin orthogonality (38) follows

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2 = &\big[a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{z}-\boldsymbol{z}_h)-\big(p-p_h, \nabla\cdot(\boldsymbol{z}-\boldsymbol{z}_h)\big)+b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{z}-\boldsymbol{z}_h) \\
&-b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{z}-\boldsymbol{z}_h)+\big(s-s_h, \nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\big)\big]+b(\boldsymbol{u}_h-\boldsymbol{u}_H, \boldsymbol{u}_*-\boldsymbol{u}_h, \boldsymbol{z}_h) \\
&-b(\boldsymbol{u}_h-\boldsymbol{u}_*, \boldsymbol{u}_H-\boldsymbol{u}_*, \boldsymbol{z}_h)+b\big(\boldsymbol{u}_*-\boldsymbol{u}_H, \boldsymbol{u}_*-\boldsymbol{u}_H, R_V(\boldsymbol{z}_h, s_h)-\boldsymbol{z}_h\big).
\end{aligned}
$$

Now, each term can be estimated as in Propositions 2 and 4. □

Table 1
A posteriori error estimates and bounds for the asymptotic behaviour of the additional terms

Algorithm	$\|\|\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|\|\|$	$\\|\boldsymbol{u}-\boldsymbol{u}_h\\|_0$
1	η_1	η_2
2	$\eta_1+\mathrm{O}(H^{2k+1-\varepsilon})$	$\eta_2+\mathrm{O}(H^{2k+1})$
3	$\eta_1+\mathrm{O}(H^{k+1-\varepsilon})$	$\eta_2+\mathrm{O}(H^{k+1})$
4	$\eta_1+\mathrm{O}(H^{k+1-\varepsilon})$	$\eta_2+\mathrm{O}(H^{k+2-\varepsilon})$
5	$\eta_1+\mathrm{O}(H^{2k+1-\varepsilon})$	$\eta_2+\mathrm{O}(H^{2k+2-\varepsilon})$

Assuming (5), all additional terms in the estimate (39) behaves asymptotically like $\mathrm{O}(H^{2k+1-\varepsilon})$. To prove this, use the a priori error estimates (6) for $\boldsymbol{u}-\boldsymbol{u}_H$ and (21), (22), (37), to estimate $\boldsymbol{u}-\boldsymbol{u}_h$ and $\boldsymbol{u}-\boldsymbol{u}_*$. An analogous estimate of the first, third and fifth additional term in (40) shows that these terms behave like $\mathrm{O}(H^{2k+2-\varepsilon})$. However, for the second and the fourth extra term, we obtain only $\mathrm{O}(H^{2k+1})$. But with the assumption $h=\mathrm{O}(H^{1+(1-\varepsilon)/k})$, we get also for these terms $\mathrm{O}(H^{2k+2-\varepsilon})$ with the help of (21), (22), and (37). Thus, to obtain a $\mathrm{O}(H^{2k+2-\varepsilon})$ behaviour of the extra terms in (40), $\mathcal{T}_h$ needs to be sufficiently fine but not optimal.

9. Summary

We have derived residual based a posteriori error estimates for several two-level algorithms to compute discrete approximations of the solution of the Navier–Stokes equations. Additional terms arise in the estimates due to an approximate Galerkin orthogonality of these algorithms. The estimates and the asymptotic behaviour of the additional terms are summarized in Table 1, where $\eta_1=\eta_1((\boldsymbol{u}_h, p_h))$ and $\eta_2=\eta_2((\boldsymbol{u}_h, p_h))$ are defined in (16) and (19), respectively, ε in (9), and k in (5).

The asymptotic behaviour of the additional terms in the error estimates of Algorithms 2–5 has been studied under the assumption $h<H$ (except the L^2-error estimate in Algorithm 5, where $h=\mathrm{O}(H^{1+(1-\varepsilon)/k})$ has been used). For optimally scaled meshes, the bounds of the asymptotic behaviour of the extra terms are in general not of higher order than the order of convergence of the discrete solution. Thus, the extra terms have to be computed in practice to guarantee an asymptotically correct a posteriori error estimate. Table 1 shows that the asymptotic behaviour of the additional terms in all a posteriori error estimates decreases by k powers of H if the Newton step is replaced by the Oseen step. The additional correction step improves the asymptotic order of convergence of the extra terms in the a posteriori error estimates for the L^2-norm of the velocity by the factor $H^{1-\varepsilon}$ and it does not influence the asymptotic behaviour of the extra terms in the a posteriori error estimates in the $|||\cdot|||$-norm.

Acknowledgements

The author wishes to acknowledge Prof. W. Layton for fruitful discussions on the subject of this paper.

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