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ANALYSIS OF NUMERICAL ERRORS IN LARGE EDDY SIMULATION*

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Abstract. We consider the question of "numerical errors" in large eddy simulation. It is often claimed that straightforward discretization and solution using centered methods of models for large eddy motion can simulate the motion of turbulent flows with complexity independent of the Reynolds number and dependence only on the resolution "δ" of the eddies sought. This report considers this question analytically: Is it possible to prove error estimates for discretizations of *actually used* large eddy models whose error constants depend only on δ but not Re? We consider the most common, simplest, and most mathematically tractable model and the most mathematically clear discretization. In two cases, we prove such an error estimate and try to explain why our technique of proof fails in the most general case. Our analysis aims to assume as little time regularity on the true solution as possible.

Key words. large eddy simulation, Navier–Stokes equations, turbulence, finite element methods

AMS subject classifications. 76F65, 65M60

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1. Introduction.

The laminar or turbulent flow of an incompressible fluid is modeled by solutions (u, p) of the incompressible Navier–Stokes equations:

$$
\begin{array}{rcll}
u_t + u \cdot \nabla u + \nabla p - Re^{-1} \Delta u &=& f & \text{in } \Omega \times (0, T], \\
\nabla \cdot u &=& 0 & \text{in } \Omega \times [0, T], \\
u(x, 0) &=& u_0(x) & \text{in } \Omega, \\
u &=& 0 & \text{on } \Gamma \times [0, T], \\
\displaystyle\int_\Omega p \, dx &=& 0 & \text{in } (0, T].
\end{array}
\tag{1.1}
$$

Here $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded, simply connected domain with polygonal boundary Γ, $u : \Omega \times [0, T] \to \mathbb{R}^d$ is the fluid velocity, $p : \Omega \times (0, T] \to \mathbb{R}$ is the fluid pressure, $f(x, t)$ is the (known) body force, $u_0(x)$ is the initial flow field, and Re is the Reynolds number. Unfortunately, when Re is large the resulting turbulent flow is typically so complex that so-called direct numerical simulation of (u, p) is not practically feasible.

One conjecture of Leray is that "turbulence" in nature is associated with a breakdown of uniqueness of weak solutions to (1.1). It is known that, for example, weak solutions to (1.1) are unique for $d = 3$ and for very small time intervals, e.g., $0 \le t \le O(Re^{-3})$, and, more importantly, over $O(1)$ time intervals $0 \le t \le T$ if

$$\int_0^T \|\nabla u\|^4_{L^2(\Omega)} dt < \infty.$$

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There are numerous generalizations of this basic result [13, 28]. With this in mind, solutions u to (1.1) with $\|\nabla u\|_{L^2(\Omega)} \in L^4(0,T)$ are frequently described as "laminar." Thus, the L^p-regularity in time which can be reasonably assumed is of critical importance.

There are numerous approaches to the simulation of turbulent flows in practical settings. One of the most promising current approaches is large eddy simulation (LES) in which approximations to local spatial averages of u are calculated. A spatial length scale δ is selected. The large eddies are considered to be those of size greater than or equal to $O(\delta)$ and the small eddies are considered to be those of size less than $O(\delta)$. The large eddies are approximated directly while the effects of the small eddies on the large eddies are modeled. In computational turbulence studies using LES it is often reported that *the resulting computational complexity is independent of the Reynolds number* (but dependent on the resolution sought, δ). There has been little or no analytical support for this observation, however. The goal of this report is to begin numerical analysis in support of this claim.

To be more specific, a smooth, nonnegative function $g(x)$ with $g(0) = 1$ and $\int_{\mathbb{R}^d} g\, dx = 1$ is selected and the mollifer $g_\delta(x)$ is defined in the usual way:

$$g_\delta(x) = \delta^{-d} g(x/\delta).$$

One common example is a Gaussian, $g(x) = (6/\pi)^{d/2} \exp(-6x_j x_j)$, where the summation convention is used. The spatial averaging/filtering operation is now defined by convolution:

$$\overline{u}(x,t) = g_\delta * u(x,t), \quad \overline{p} = g_\delta * p, \quad \overline{f} = g_\delta * f, \text{ etc.}$$

In LES, approximations to $(\overline{u}, \overline{p})$ are sought rather than to (u, p). The usual procedure is to first filter the Navier–Stokes equations:

$$\begin{aligned} \overline{u}_t + \nabla \cdot (\overline{u}\, \overline{u}) + \nabla \overline{p} - Re^{-1} \Delta \overline{u} &= \overline{f} + \nabla \cdot \mathbb{T} &&\text{in } \Omega, \\ \nabla \cdot \overline{u} &= 0 &&\text{in } \Omega, \end{aligned}$$

where the "Reynolds stress tensor" $\mathbb{T}$ is

$$\mathbb{T} = \mathbb{T}(\overline{u}, u) = \overline{u}\, \overline{u} - \overline{u\, u}.$$

Closure is addressed by a modeling step in which $\mathbb{T}$ is written in terms of $\overline{u}$. The resulting (closed) space filtered Navier–Stokes equations are solved numerically. In this procedure, there are three essential issues:

1. The "modeling error" committed in approximating $\mathbb{T}$.
2. The "numerical error" in solving the resulting system.
3. Correct boundary conditions for the flow averages.

In this report, we study the numerical error analytically. Since there are many models in LES (see, e.g., [25, 14, 23, 9, 2, 31, 35, 34]) and few analytical studies, we take herein the simplest model commonly in use, presented, for example, in Ferziger and Peric [9, section 9.3].

To describe the model, let $\mathbb{D}(u)$ be the deformation tensor associated with the indicated velocity field by

$$\mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla u^t) = \frac{1}{2}(u_{i,x_j} + u_{j,x_i}).$$

The Reynolds stresses are thought of as a turbulent diffusion process based upon the Boussinesq assumption or eddy viscosity hypothesis that "turbulent fluctuations are dissipative in the mean," [25, 11, 32, 34]. We will accordingly consider a model of the form

$$\nabla \cdot \mathbb{T} \sim \nabla \cdot (\nu_{\text{turb}}(\overline{u}, \delta) \;\; \mathbb{D}(\overline{u})),$$

where $\nu_{\text{turb}} \doteq \nu_{\text{turb}}(\overline{u}, \delta)$ is the so-called turbulent viscosity or eddy viscosity. This turbulent viscosity's determination can be very complex, involving even solutions of accompanying systems of nonlinear partial differential equations. In the simplest case, the turbulent viscosity depends on the mean flow $\overline{u}$ through the magnitude of the deformation of $\overline{u}, \nu_{\text{turb}} = \nu_{\text{turb}}(\mathbb{D}(\overline{u}))$, with a functional dependence. Under the Boussinesq assumption, $\nabla \cdot \mathbb{T}$ should act like a physical viscosity. Following the reasoning of Ladyzhenskaya [29], thermodynamic considerations imply that the Taylor series of $\nu_{\text{turb}}(\mathbb{D})$ should be dominated by odd degree terms. The simplest case is of linear dependence upon $|\mathbb{D}|$:

$$\nu_{\text{turb}} = \nu_{\text{turb}}(|\mathbb{D}(\overline{u})|) = a_0(\delta) + a_1(\delta)|\mathbb{D}(\overline{u})|, \tag{1.2}$$

where $|\mathbb{D}(\overline{u})|$ denotes the Frobenius norm of $\mathbb{D}(\overline{u})$. For specificity and for accord with the most commonly used Smagorinsky [37] model, we take the bulk turbulent viscosity $a_0(\delta) \geq 0$ and $a_1(\delta) = C_s\delta^2$. Other scalings are possible [30] though less tested, as are many other subgridscale models [25, 35]. Here C_s is typically either chosen to be around 0.1 or taken to be a function $C_s = C_s(x, t)$ and extrapolated as in the "dynamic subgridscale model" of Germano et al. [15].

With the model (1.2), the resulting system of equations for the approximations (w, q) to $(\overline{u}, \overline{p})$ is

$$\begin{array}{rcll} w_t + \nabla \cdot (w\; w) + \nabla q - Re^{-1}\Delta w - \nabla \cdot (\nu_{\text{turb}}\mathbb{D}(w)) & = & \overline{f} & \text{in } \Omega \times (0, T], \\ \nabla \cdot w & = & 0 & \text{in } \Omega \times [0, T], \\ w(x, 0) & = & w_0(x) & \text{in } \Omega, \\ \displaystyle\int_\Omega q\; dx & = & 0 & \text{in } [0, T]. \end{array} \tag{1.3}$$

Boundary conditions must be supplied for the large eddies. It is physically clear that large eddies do not adhere to solid walls. (For example, tornadoes and hurricanes move while touching the earth and lose energy as they move.) Therefore, in [14, 26] (see also [34] for the use of similar boundary conditions in a conventional turbulence model), it was proposed that the large eddies should satisfy a no-penetration condition and a slip with friction condition on $\partial\Omega$:

$$\begin{array}{rcll} w \cdot \hat{n} & = & 0 & \text{on } \Gamma, \\ w \cdot \hat{\tau} & = & 0 & \text{on } \Gamma_0, \text{ meas } (\Gamma_0) > 0, \\ \beta w \cdot \hat{\tau} + \vec{t} \cdot \hat{\tau} & = & 0 & \text{on } \Gamma \setminus \Gamma_0, \end{array} \tag{1.4}$$

where $\vec{t}$ is the Cauchy stress vector on Γ (for background information, see Serrin [36]), $\beta = \beta(\delta, Re)$ is the friction coefficient (calculated explicitly in [26]), $\hat{n}$ is the outward unit normal, and $\hat{\tau}$ is an orthonormal system of tangent vectors on each face of Γ. The friction coefficient β can be calculated once a specific filter is chosen [26]. It has the property [26] that no slip conditions are recovered as $\delta \to 0$:

$$\beta(Re,\; \delta) \to \infty \quad \text{as } \delta \to 0.$$

A Dirichlet boundary condition $w = w_{\text{inflow}}$ on Γ_0 is appropriate if Γ_0 is an inflow boundary upon which $\overline{u}$ can be calculated by extending the known, inflow velocity field upstream. We take $w_{\text{inflow}} = 0$ for simplicity.

The Cauchy stress vector $\vec{t}$ includes the action of both the viscous stresses and Reynolds stresses and is given by

$$\vec{t}(w) := \hat{n} \cdot [-q\mathbb{I} + 2Re^{-1}\mathbb{D}(w) + a_0(\delta)\mathbb{D}(w) + C_s\delta^2|\mathbb{D}(w)|\mathbb{D}(w)].$$

Standard properties of convolution operators imply that the flow averages $(\overline{u}, \overline{p})$ are $C^\infty(\Omega)$ in space, have bounded kinetic energy

$$\int_\Omega |\overline{u}|^2 dx \leq \int_\Omega |u|^2 dx \leq C(\Omega, f, u_0),$$

have no solution scales smaller than $O(\delta)$, and converge to u as $\delta \to 0$ [24]. On the other hand, it is *not* obvious, nor has it been proven yet, that solutions (w, q) to the large eddy model approximating $(\overline{u}, \overline{p})$ share *any* of these properties! Nevertheless, the spatial regularity of solutions (w, q) we shall consider to be a *modeling* issue (beyond the scope of this report studying *numerical errors* in LES). The *time regularity* of solutions (w, q) is still an important consideration. For example, we shall show that solutions of this model satisfy

$$\int_0^T \|\nabla w\|_{L^3}^3 dt < \infty$$

uniformly in Re. One goal is thus to assume no greater time regularity than this. The fundamental error analysis of Heywood and Rannacher [22] for the Navier–Stokes equations is based, in part, on a laminar-type assumption $\nabla u \in L^\infty(0, T; L^2(\Omega))$. Weakening this to an assumption of the form $\nabla u \in L^3(0, T; L^3(\Omega))$ (as we seek to do herein) is nontrivial.

2. Preliminaries. This section sets the notation used in the report, describes the function spaces employed, and collects several useful inequalities. The notation used is standard for the most part. The $L^p(\Omega)$ norms, for $p \neq 2$, are explicitly denoted as $\|f\|_{L^p}$. Sobolev spaces $W^{k,p}(\Omega)$ are defined in the usual way [1]. The associated norm is denoted by $\|\cdot\|_{k,p}$. If the domain in question is *not* Ω (e.g., $\Omega \times (0, T)$), then it will be explicitly indicated. If $p = 2$, these norms will be written $\|\cdot\|_k$ for the $W^{k,2}(\Omega)$ norm and $\|\cdot\|_{k,\Gamma}$ for the $W^{k,2}(\Gamma)$ norm and $\|\cdot\|$ and $\|\cdot\|_\Gamma$, respectively, for the $L^2(\Omega)$ and $L^2(\Gamma)$ norms. We suppose the polygonal boundary Γ is composed of faces $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$, where (with some abuse of notation) Γ_0 consists of the face(s) upon which $v = 0$ is strongly imposed.

The spaces associated with the boundary conditions (1.4) are

$$\begin{aligned} X &:= \{v : v \in \left(W^{1,3}(\Omega)\right)^d, v = 0 \text{ on } \Gamma_0 \text{ and } v \cdot \hat{n} = 0 \text{ on } \Gamma_j, j = 1, \ldots, J\}, \\ Q &:= L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \right\}. \end{aligned}$$

The boundary condition in X is defined to hold in the sense of the trace theorem on each Γ_j, and $\hat{n}$ is the outward unit normal to Γ. The $L^2(\Omega)$ and $L^2(\Gamma)$ inner products are denoted by $(\cdot, \cdot)$ and $(\cdot, \cdot)_\Gamma$, respectively.

If $v \in X$, $\mathbb{D}(v)$ denotes the usual deformation tensor, defined in the introduction. The unit vector $\hat{\tau}$ denotes an orthonormal system of tangent vectors on Γ. Whenever

$\hat{\tau}$ occurs, it will be understood that the term is to be summed over the two tangent vectors if $d = 3$; for example,

$$\|v \cdot \hat{\tau}\|^2_{\Gamma_j} \text{ if } d = 3 \text{ means } \left(\|v \cdot \hat{\tau}_1\|^2_{\Gamma_j} + \|v \cdot \hat{\tau}_2\|^2_{\Gamma_j}\right).$$

LEMMA 2.1 (inf-sup condition). *Let* $\tilde{X} := \{v : v \in \left(W^{1,2}(\Omega)\right)^d,\ v = 0$ *on* Γ_0 *and* $v \cdot \hat{n} = 0$ *on* $\Gamma_j, j = 1, \dots, J\}$. *The velocity-pressure spaces* $(\tilde{X}, Q)$ *satisfy the inf-sup condition*

$$\inf_{\lambda \in Q} \sup_{v \in \tilde{X}} \frac{(\lambda, \nabla \cdot v)}{\|\lambda\| \left[\|\mathbb{D}(v)\|^2 + \sum_{j=1}^{J} \|v \cdot \hat{\tau}\|^2_{1/2,\Gamma_j}\right]^{1/2}} \geq C > 0. \tag{2.1}$$

Proof. Since $\|\nabla v\| \geq \|\mathbb{D}(v)\|$, the trace theorem [20] shows that (2.1) is implied by the usual inf-sup condition

$$\inf_{\lambda \in Q} \sup_{v \in \tilde{X} \cap H^1_0(\Omega)^d} \frac{(\lambda, \nabla \cdot v)}{\|\lambda\| \; \|\nabla v\|} \geq C > 0. \qquad \square$$

Lemma 2.1 implies that the space of weakly divergence free functions V,

$$V := \{v \in \tilde{X} : (\lambda, \nabla \cdot v) = 0 \text{ for all } \lambda \in Q\},$$

is a well defined, nontrivial, closed subspace of $\tilde{X}$.

Remark 2.1. Since Γ is *not* C^1, discontinuities in $\hat{\tau}$ and $\hat{n}_j$ have forced modifications in the norms to piecewise definition. For example, $v \cdot \hat{\tau} \notin H^{1/2}(\Gamma)$ for $v \in H^1(\Omega)$ but $v \cdot \hat{\tau} \in H^{1/2}(\Gamma_j), j = 0, \dots, J$.

The conforming finite element method for this problem begins by selecting finite element spaces $X^h \subset X$ and $Q^h \subset Q$, where h denotes as usual a representative mesh width for (X^h, Q^h), satisfying the usual approximation theoretic conditions required of finite element spaces. The condition that $X^h \subset X$ imposes the restriction that $v^h \cdot \hat{n}_{|\Gamma_j} = 0$ for all $v^h \in X^h$. For intricate boundaries, this could possibly be onerous so it is interesting to consider imposing $v^h \cdot \hat{n}_{|\Gamma} = 0$ with penalty or Lagrange multiplier methods, following, e.g., the work in [31]. Nevertheless, there is already considerable computational experience with imposing this condition in finite element methods (see, e.g., [19, 8]), so we shall not focus on the interesting detail of the treatment of corners. Without these additional regularizations in the numerical method, it is useful in the analysis to assume that (X^h, Q^h) satisfies the discrete analogue of (2.1),

$$\inf_{\lambda^h \in Q^h} \sup_{v^h \in X^h} \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \left[\|\mathbb{D}(v^h)\|^2 + \sum_{j=1}^{J} \|v^h \cdot \hat{\tau}\|^2_{1/2,\Gamma_j}\right]^{1/2}} \geq C > 0, \tag{2.2}$$

where $C > 0$ is independent of h. The next lemma shows, in essence, that if the computational mesh follows the boundary and if the velocity space, restricted to no slip boundary conditions, and the pressure space satisfy the usual inf-sup condition, then (2.2) holds.

LEMMA 2.2 (discrete inf-sup condition). *If* (X^h, Q^h) *satisfies*

$$\inf_{\lambda_h \in Q^h} \sup_{v_h \in X^h \cap H^1_0(\Omega)^d} \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \; \|\nabla v^h\|} \geq C_1 > 0,$$

then (2.2) *holds.*

Proof. By trace theorem [20] and the Poincaré–Friedrichs inequality, for any $\lambda_h \neq 0$, $v^h(\neq 0) \in X^h$,

$$\frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \left[\|\mathbb{D}(v^h)\|^2 + \sum_{j=1}^J \|v^h \cdot \hat{\tau}\|^2_{1/2,\Gamma_j}\right]^{1/2}} \geq C \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \ \|v^h\|_1} \geq C \frac{(\lambda^h, \nabla \cdot v^h)}{\|\lambda^h\| \ \|\nabla v^h\|}. \qquad \square$$

Thus, (2.2) will be assumed throughout this report. Under (2.2), the space of discretely divergence free functions

$$V^h := \{v^h \in X^h : (\lambda^h, \nabla \cdot v^h) = 0 \text{ for all } \lambda^h \in Q^h\}$$

is a nontrivial closed subspace of X^h [16, 21].

We shall frequently use Young's inequality in the form

$$ab \leq \frac{\epsilon}{q} a^q + \frac{\epsilon^{-q'/q}}{q'} b^{q'}, \quad 1 < q, q' < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

The generalization of Hölder's inequality

$$\int_\Omega |u| \ |v| \ |w| dx \leq \|u\|_{L^p} \|v\|_{L^q} \|w\|_{L^r}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \quad 1 \leq p, q, r \leq \infty,$$

is also useful. We shall frequently use the Sobolev embedding theorem, often, but not always, in the form that in three dimensions $W^{1,3}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p < \infty$.

The nonlinear form in the subgridscale term, for $v, w \in (W^{1,3}(\Omega))^d$

$$(|\mathbb{D}(w)|\mathbb{D}(w), \mathbb{D}(v)),$$

is of p-Laplacian type (with $p = 3$). Thus, it is strongly monotone and locally Lipschitz continuous in the sense made precise in the following well-known lemma; see, e.g., [30, 7].

LEMMA 2.3 (strong monotonicity and local Lipschitz-continuity). *There are constants $\underline{C}$ and $\overline{C}$ such that for all $u_1, u_2, v \in (W^{1,3}(\Omega))^d$ and $d = 2$ or 3, with $r = \max\{\|\mathbb{D}(u_1)\|_{L^3}, \|\mathbb{D}(u_2)\|_{L^3}\}$,*

$$\begin{aligned}(|\mathbb{D}(u_1)|\mathbb{D}(u_1) - |\mathbb{D}(u_2)|\mathbb{D}(u_2), \mathbb{D}(u_1 - u_2)) &\geq \underline{C}\|\mathbb{D}(u_1 - u_2)\|^3_{L^3}, \\ (|\mathbb{D}(u_1)|\mathbb{D}(u_1) - |\mathbb{D}(u_2)|\mathbb{D}(u_2), \mathbb{D}(v)) &\leq \overline{C} r \|\mathbb{D}(u_1 - u_2)\|_{L^3} \|\mathbb{D}(v)\|_{L^3}.\end{aligned}$$

Korn's inequalities relate L^p norms of the deformation tensor $\mathbb{D}(v)$ to those same norms of the gradient for $1 < p < \infty$ (see Galdi, Heywood, and Rannacher [13], Gobert [17, 18], Temam [39], or Fichera [10]) and fail if $p = 1$.

THEOREM 2.4 (Korn's inequalities). *There is a $C > 0$ such that for $1 < p < \infty$*

$$\|v\|^p_{W^{1,p}} \leq C(\Omega)[\|v\|^p_{L^p} + \|\mathbb{D}(v)\|^p_{L^p}]$$

for all $v \in (W^{1,p}(\Omega))^d$.

Further, if $\gamma(v)$ is a seminorm on $L^p(\Omega)$ which is a norm on the constants, then

$$\|\nabla v\|_{L^p} \leq C(\Omega)[\gamma(v) + \|\mathbb{D}(v)\|_{L^p}]$$

holds for $1 < p < \infty$ and for all $v \in (W^{1,p}(\Omega))^d$.

As a consequence of Korn's inequality it follows that, taking $\gamma(v) = \|v\|_{L^p(\Gamma_0)}$, if $\text{meas}(\Gamma_0) > 0$, then

$$\|\nabla v\|_{L^p}^p \le \|v\|_{1,p}^p \le C_K \|\mathbb{D}(v)\|_{L^p}^p$$

for all $v \in \{v \in W^{1,p}(\Omega)^d : v_{|\Gamma_0} = 0\}$.

We will often use Poincaré's inequality, which holds since $v \cdot \hat{n} = 0$ on Γ (Galdi [12, p. 56]),

$$\|v\| \le C(\Omega)\|\nabla v\| \text{ for all } v \in X.$$

We shall use the Gagliardo–Nirenberg inequality in $W^{1,p}(\Omega) \cap X$. This inequality [1, 33, 13, 6] states that, provided Γ satisfies a weak regularity condition (holding in particular for polygonal domains) and $\text{meas}(\Gamma_0) > 0$ for all $v \in W^{1,p}(\Omega) \cap X, 1 \le q, s \le \infty$,

$$\|v\|_{L^q} \le C\|\nabla v\|_{L^p}^a \|v\|_{L^s}^{1-a} \text{ for all } v \in \left(W^{1,p}(\Omega)\right)^d \cap X,$$

where, for $\Omega \subset \mathbb{R}^3$ (improvable if $\Omega \subset \mathbb{R}^2$), $p \ge 3$, $q \ge s$, $0 \le a < 1$, and

$$a = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{3} - \frac{1}{p} + \frac{1}{s}\right)^{-1}.$$

In particular, note that taking $q = 6$, $p = 3$, and $s = 2$ gives

$$\|v\|_{L^6(\Omega)} \le C\|\nabla v\|_{L^3(\Omega)}^{2/3}\|v\|^{1/3}. \tag{2.3}$$

The following combination of this and Korn's inequality will be useful in section 4.

LEMMA 2.5. *Let* $\text{meas}(\Gamma_0) > 0$ *and* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. *Then,*

$$\|v\|_{L^6} \le C\|v\|^{1/3}\|\mathbb{D}(v)\|_{L^3}^{2/3}, \ C = C(\Omega).$$

Proof. This follows immediately from (2.3) and Korn's inequality. ∎

The following dual norms are defined in an equivalent but slightly nonstandard way: for $\frac{1}{q} + \frac{1}{q'} = 1, 1 < q, q' < \infty$,

$$\|f\|_* := \sup_{v \in X} \frac{(f, v)}{\|\mathbb{D}(v)\|},$$

$$\|f\|_{W^{-1,3/2}} := \sup_{v \in X} \frac{(f, v)}{\|\mathbb{D}(v)\|_{L^3}},$$

$$\|f\|_{W^{-1,q'}(\Omega \times (0,t))} := \sup_{v \in L^q(0,T;X)} \frac{\int_0^t (f, v)dt'}{(\int_0^t \|\mathbb{D}(v)\|_{L^q}^q dt')^{1/q}}.$$

Note that $\|\mathbb{D}(\cdot)\|_{L^3}$ defines a norm in X as a consequence of Poincaré's and Korn's inequality.

3. The finite element formulation. This section develops the finite element method for the LES model. The stability of the model is also studied. In particular, we show w and $w^h \in L^\infty(0, T; L^2(\Omega)) \cap L^3(0, T; H^1(\Omega))$ uniformly in Re. Lastly, the error in an equilibrium projection is considered.

The variational formulation is derived in the usual way by multiplication of (1.3) by $(v, q) \in (X, Q)$ and applying the divergence theorem. The boundary integral terms require careful treatment (following, e.g., [31]) on account of the slip with friction condition on Γ. Let $\alpha \geq 0$ be a constant. The formulation which results is to find $w : [0,T] \to X, q : (0,T] \to Q$ satisfying

$$(3.1)\quad \begin{aligned} &(w_t, v) + \beta(\delta, Re)\sum_{j=1}^{J}(w\cdot\hat\tau, v\cdot\hat\tau)_{\Gamma_j} + \big((2Re^{-1} + a_0(\delta) + C_s\delta^2|\mathbb{D}(w)|)\mathbb{D}(w), \mathbb{D}(v)\big)\\ &+ (w\cdot\nabla w, v) - (q, \nabla\cdot v) + \alpha(\nabla\cdot w, \nabla\cdot v) = (\overline{f}, v) \quad \text{for all } v \in X,\\ &(\lambda, \nabla\cdot w) = 0 \quad \text{for all } \lambda \in Q, \end{aligned}$$

and $w(x,0) = \overline{u}_0(x) \in X$. For compactness, define the nonlinear and trilinear form:

$$\begin{aligned} a(u,w,v) &:= \alpha(\nabla\cdot w, \nabla\cdot v) + \sum_{j=1}^{J}\beta(w\cdot\hat\tau, v\cdot\hat\tau)_{\Gamma_j}\\ &\quad + ((2Re^{-1} + a_0(\delta) + C_s\delta^2|\mathbb{D}(u)|)\mathbb{D}(w), \mathbb{D}(v)),\\ b(u,w,v) &:= \frac{1}{2}(u\cdot\nabla w, v) - \frac{1}{2}(u\cdot\nabla v, w). \end{aligned}$$

It is a simple index calculation to check that for $v \in X, w \in V$ (since such functions have zero normal components on Γ) $(w\cdot\nabla w, v) = b(w,w,v)$. Thus, the variational formulation can be rewritten as follows: find $(w,q) : [0,T] \to (X,Q)$ satisfying $w(x,0) = \overline{u}_0(x)$ and

$$(3.2)\quad (w_t, v) + a(w,w,v) + b(w,w,v) + (\lambda, \nabla\cdot w) - (q, \nabla\cdot v) = (\overline{f}, v)$$

for all $(v,\lambda) \in (X,Q)$.

Using Lemma 2.3, it is easy to prove that the LES model (1.3), (1.4) satisfies the analogue of Leray's inequality for the Navier–Stokes equations.

LEMMA 3.1 (Leray's inequality for the LES model). *A solution of* (3.2) *satisfies*

$$\begin{aligned} \frac{1}{2}\|w(t)\|^2 + \int_0^t \left[\sum_{j=1}^{J}\beta\|w\cdot\hat\tau\|^2_{\Gamma_j} + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w)\|^2 + \underline{C}C_s\delta^2\|\mathbb{D}(w)\|^3_{L^3}\right]dt'\\ \leq \frac{1}{2}\|w(0)\|^2 + \int_0^t(\overline{f}, w)dt'. \end{aligned}$$

Proof. Set $v = w, \lambda = q$ in (3.2) and use Lemma 2.3. ∎

Remark 3.1.

1. Because of the slip with friction boundary conditions (1.4), it is important to choose the formulation of the viscous terms, as in (3.1), (3.2), involving the deformation tensor.

2. Leray's inequality immediately implies stability in various norms (which we will develop) and is the key, first step in proving existence of weak solutions to (1.3), (1.4). This last question is fully investigated (under different boundary conditions) in remarkable papers by Ladyzhenskaya [27], Parés [34], and Du and Gunzburger [7].

LEMMA 3.2. *Let (w,q) be the solution of* (1.3). *Then, there is a constant C independent of Re such that for almost all $t \in (0,T)$ with $0 < T < \infty$*

$$\begin{aligned} \|w_t\|_{W^{-1,3/2}} \leq C\big(&\|w\|^2_{L^3} + \|q\|_{L^{3/2}} + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w)\|_{L^{3/2}}\\ &+ C_s\delta^2\|\mathbb{D}(w)\|^2_{L^3} + \|\overline{f}\|_{W^{-1,3/2}}\big), \end{aligned}$$

$$\|w_t\|^{3/2}_{L^{3/2}(0,T;W^{-1,3/2})} \le C\Big(\|w\|^3_{L^3(0,T;L^3)} + \|q\|^{3/2}_{L^{3/2}(0,T;L^{3/2})} + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w)\|^{3/2}_{L^{3/2}(0,T;L^{3/2})} + C_s\delta^2\|\mathbb{D}(w)\|^3_{L^3(0,T;L^3)} + \|\overline{f}\|^{3/2}_{L^{3/2}(0,T;W^{-1,3/2})}\Big).$$

Proof. From the momentum equation in (1.3) it follows that for almost all $t \in (0,T)$ (alternately, dividing (3.1) by $\|v\|_{W^{1,3}}$ and taking the supremum over v)

$$\|w_t\|_{W^{-1,3/2}} \le \|\nabla\cdot(ww)\|_{W^{-1,3/2}} + \|\nabla q\|_{W^{-1,3/2}} + C_s\delta^2\|\nabla\cdot(|\mathbb{D}(w)|\mathbb{D}(w))\|_{W^{-1,3/2}} + (2Re^{-1} + a_0(\delta))\|\nabla\cdot\mathbb{D}(w)\|_{W^{-1,3/2}} + \|\overline{f}\|_{W^{-1,3/2}}.$$

The definition of the norm, integration by parts, using $v\cdot n = 0$ on Γ for $v \in X$, Hölder's inequality, and Korn's inequality give, e.g.,

$$\|\nabla q\|_{W^{-1,3/2}} = \sup_{v\in X}\frac{\int_\Omega -q(\nabla\cdot v)dx}{\|\mathbb{D}(v)\|_{L^3}} \le \sup_{v\in X}\frac{\|q\|_{L^{3/2}}\|\nabla\cdot v\|_{L^3}}{\|\mathbb{D}(v)\|_{L^3}} \le C\|q\|_{L^{3/2}}.$$

The other terms are estimated in the same way also using $\|ww\|_{L^{3/2}} = \|w\|^2_{L^3}$ and $\||\mathbb{D}(w)|\mathbb{D}(w)\|_{L^{3/2}} = \|\mathbb{D}(w)\|^2_{L^3}$.

The second inequality follows raising both sides to the power $3/2$ and integrating in time. ◻

The continuous-in-time finite element method for (1.3) uses the variational formulation (3.2) as follows. First, velocity-pressure finite element spaces $X^h \subset X \cap (W^{1,3}(\Omega))^d$, $Q^h \subset Q$ satisfying (2.2), and the parameter $\alpha \ge 0$ are selected.

The finite element approximations to (w,q) are maps $(w^h,q^h) : [0,T] \to (X^h,Q^h)$ satisfying

$$(w^h_t, v^h) + a(w^h,w^h,v^h) + b(w^h,w^h,v^h) + (\lambda^h,\nabla\cdot w^h) - (q^h,\nabla\cdot v^h) = (\overline{f},v^h) \tag{3.3}$$

for all $(v^h,\lambda^h) \in (X^h,Q^h)$ where $w^h(x,0) \in X^h$ is an approximation to $w(x,0) = \overline{u}_0$.

It is straightforward to verify that Leray's inequality holds for w^h as well as w.

Lemma 3.3 (Leray's inequality for w^h). *For $\alpha \ge 0$, any solution of* (3.3) *satisfies*

$$\frac{1}{2}\|w^h(t)\|^2 + \int_0^t\Bigg[\beta\sum_{j=1}^J\|w^h\cdot\hat{\tau}\|^2_{\Gamma_j} + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w^h)\|^2 + \alpha\|\nabla\cdot w^h\|^2 + \underline{C}\ C_s\delta^2\|\mathbb{D}(w^h)\|^3_{L^3}\Bigg]dt' \le \frac{1}{2}\|w^h(0)\|^2 + \int_0^t(\overline{f},w^h)dt'.$$

Using various inequalities in the right-hand side, stability bounds for w^h follow from Lemma 3.3.

Proposition 3.4 (stability of w^h). *The solution w^h of* (3.3) *satisfies*

$$\frac{1}{2}\|w^h(t)\|^2 + \int_0^t\Bigg[\sum_{j=1}^J\beta\|w^h\cdot\hat{\tau}\|^2_{\Gamma_j} + (Re^{-1} + a_0(\delta))\|\mathbb{D}(w^h)\|^2 + \alpha\|\nabla\cdot w^h\|^2 + \underline{C}C_s\delta^2\|\mathbb{D}(w^h)\|^3_{L^3}\Bigg]dt' \le \frac{1}{2}\|w^h(0)\|^2 + \frac{Re}{4}\int_0^t\|\overline{f}\|^2_*dt', \tag{3.4}$$

$$\frac{1}{2}\|w^h(t)\|^2 + \int_0^t \left[\sum_{j=1}^J \beta\|w^h\cdot\hat{\tau}\|_{\Gamma_j}^2 + (2Re^{-1}+a_0(\delta))\|\mathbb{D}(w^h)\|^2 + \alpha\|\nabla\cdot w^h\|^2 \right.$$

$$\left. + \frac{2}{3}\underline{C}C_s\delta^2\|\mathbb{D}(w^h)\|_{L^3}^3\right]dt' \le \frac{1}{2}\|w^h(0)\|^2$$

$$+ \frac{2}{3}(\underline{C}C_s)^{-1/2}\delta^{-1}\|\overline{f}\|_{W^{-1,3/2}(\Omega\times(0,t))}^{3/2}, \tag{3.5}$$

$$\|w^h(t)\|^2 + 2\int_0^t e^{t-t'}\left[\sum_{j=1}^J \beta\|w^h\cdot\hat{\tau}\|_{\Gamma_j}^2 + (2Re^{-1}+a_0(\delta))\|\mathbb{D}(w^h)\|^2 + \alpha\|\nabla\cdot w^h\|^2\right.$$

$$\left. + \underline{C}C_s\delta^2\|\mathbb{D}(w^h)\|_{L^3}^3\right]dt' \le e^t\|w^h(0)\|^2 + \int_0^t e^{t-t'}\|\overline{f}\|^2 dt'. \tag{3.6}$$

Proof. Inequality (3.4) follows by applying Young's inequality to Lemma 3.3. The bound (3.5) follows from the definition of the dual norm and $ab \le \frac{\epsilon}{3}a^3 + \frac{2}{3}\epsilon^{-1/2}b^{3/2}$ applied in the same manner.

For (3.6), set $v^h = w^h$ and $\lambda^h = q^h$ in (3.3), use Lemma 2.3, and apply Young's inequality on the right-hand side. This gives

$$\frac{d}{dt}\|w^h\|^2 - \|w^h\|^2 + 2\left[\sum_{j=1}^J \beta\|w^h\cdot\hat{\tau}\|_{\Gamma_j}^2 + (2Re^{-1}+a_0(\delta))\|\mathbb{D}(w^h)\|^2 + \alpha\|\nabla\cdot w^h\|^2\right.$$

$$\left. + \underline{C}C_s\delta^2\|\mathbb{D}(w^h)\|_{L^3}^3\right] \le \|\overline{f}\|^2.$$

Inequality (3.6) now follows by using an integrating factor. ◻

In the analysis of the error in the approximation of the time dependent problem, it is useful to have a clear description of the error in the Stokes projection under slip with friction boundary conditions [31]. It is also necessary that any dependence on Re, δ, and β be made explicit.

Under the discrete inf-sup condition, the Stokes projection $\Pi : (X, Q) \to (X^h, Q^h)$ is defined as follows. Let $\Pi(w, q) = (\tilde{w}, \tilde{q})$, where $(\tilde{w}, \tilde{q})$ satisfies

$$\alpha(\nabla\cdot(w-\tilde{w}), \nabla\cdot v^h) + (2Re^{-1}+a_0(\delta))(\mathbb{D}(w-\tilde{w}), \mathbb{D}(v^h))$$

$$+ \sum_{j=1}^J \beta((w-\tilde{w})\cdot\hat{\tau}, v^h\cdot\hat{\tau})_{\Gamma_j} - (q-\tilde{q}, \nabla\cdot v^h) = 0 \quad \text{for all } v^h \in X^h,$$

$$(\nabla\cdot(w-\tilde{w}), \lambda^h) = 0 \quad \text{for all } \lambda^h \in Q^h.$$

This is equivalent to the following formulation provided $w \in V$ and $v^h \in V^h$. Given (w, q), find $\tilde{w} \in V^h$ satisfying

$$\alpha(\nabla\cdot(w-\tilde{w}), \nabla\cdot v^h) + (2Re^{-1}+a_0(\delta))(\mathbb{D}(w-\tilde{w}), \mathbb{D}(v^h))$$

$$+ \sum_{j=1}^J \beta((w-\tilde{w})\cdot\hat{\tau}, v^h\cdot\hat{\tau})_{\Gamma_j} - (q-\lambda^h, \nabla\cdot v^h) = 0$$

for all $v^h \in V^h$ and $\lambda^h \in Q^h$. Under the discrete inf-sup condition, it is well known that $(\tilde{w}, \tilde{q})$ is a quasi-optimal approximation of (w, q). The dependence of the stability and error constants upon Re and $\beta = \beta(Re, \delta)$ is important to the error analysis. That dependence is described in the next lemma and proposition.

LEMMA 3.5 (stability of the projection $\tilde{w}$). *Let $w \in V$ be given. Then if $\alpha > 0$, $\tilde{w}$ satisfies*

$$\alpha\|\nabla \cdot \tilde{w}\|^2 + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(\tilde{w})\|^2 + \sum_{j=1}^{J} \beta\|\tilde{w} \cdot \hat{\tau}\|^2_{\Gamma_j}$$
$$\leq \alpha^{-1}\|q\|^2 + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w)\|^2 + \sum_{j=1}^{J} \beta\|w \cdot \hat{\tau}\|^2_{\Gamma_j}.$$

If $\alpha = 0$, then

$$\frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\tilde{w})\|^2 + \sum_{j=1}^{J} \beta\|\tilde{w} \cdot \hat{\tau}\|^2_{\Gamma_j}$$
$$\leq 2(2Re^{-1} + a_0(\delta))^{-1}\|q\|^2 + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w)\|^2 + \sum_{j=1}^{J} \beta\|w \cdot \hat{\tau}\|^2_{\Gamma_j}.$$

Proof. Set $v^h = \tilde{w} \in V^h$ in the second formulation of the Stokes projection. This immediately gives

$$\alpha\|\nabla \cdot \tilde{w}\|^2 + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(\tilde{w})\|^2 + \sum_{j=1}^{J} \beta\|\tilde{w} \cdot \hat{\tau}\|^2_{\Gamma_j}$$
$$= (2Re^{-1} + a_0(\delta))(\mathbb{D}(w), \mathbb{D}(\tilde{w})) + \sum_{j=1}^{J} \beta(w \cdot \hat{\tau}, \tilde{w} \cdot \hat{\tau})_{\Gamma_j} + (q - \lambda^h, \nabla \cdot \tilde{w})$$
$$\leq \frac{1}{2}(2Re^{-1} + a_0(\delta))[\|\mathbb{D}(w)\|^2 + \|\mathbb{D}(\tilde{w})\|^2] + \sum_{j=1}^{J} \frac{\beta}{2}[\|w \cdot \hat{\tau}\|^2_{\Gamma_j} + \|\tilde{w} \cdot \hat{\tau}\|^2_{\Gamma_j}]$$
$$+ \frac{\alpha}{2}\|\nabla \cdot \tilde{w}\|^2 + \frac{1}{2\alpha}\|q\|^2,$$

from which the first result follows. If $\alpha = 0$, the term $(q, \nabla \cdot \tilde{w})$ is bounded by noting that $\nabla \cdot \tilde{w} = \text{trace }(\mathbb{D}(\tilde{w}))$ so that

$$(q, \nabla \cdot \tilde{w}) \leq \|q\|\ \|\mathbb{D}(\tilde{w})\| \leq \frac{1}{4}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\tilde{w})\|^2 + (2Re^{-1} + a_0(\delta))^{-1}\|q\|^2. \qquad \square$$

PROPOSITION 3.6. *Suppose the discrete inf-sup condition* (2.2) *holds. Then, $(\tilde{w}, \tilde{q})$ exists uniquely in (X^h, Q^h) and satisfies*

$$\alpha\|\nabla \cdot (w - \tilde{w})\|^2 + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w - \tilde{w})\|^2 + \sum_{j=1}^{J} \beta\|(w - \tilde{w}) \cdot \hat{\tau}\|^2_{\Gamma_j}$$
$$\leq C \inf_{v^h \in X^h, \lambda^h \in Q^h} \Bigg\{ (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w - v^h)\|^2$$
$$+ \sum_{j=1}^{J} \beta\|(w - v^h) \cdot \hat{\tau}\|^2_{\Gamma_j} + \min\{\alpha^{-1}, (2Re^{-1} + a_0(\delta))^{-1}\}\|q - \lambda^h\|^2 \Bigg\}.$$

Proof. The proof follows standard arguments, carefully tracking the dependence of the constants upon Re and β. ☐

Note that the use of least squares penalization of incompressibility allows an error estimate for the Stokes projection whose constants are essentially independent of the Reynolds number in a suitably weighted norm.

4. The convergence theorem. Let us first note that for standard piecewise polynomial finite element spaces it is known that the L^2-projection of a function in $L^p, p \geq 2$, is in L^p itself and the L^2-projection operator is stable in $L^p, p \geq 2$ [5].

Let $e = w - w^h$ and let $\tilde{w}$ denote a stable approximation of w in $V^h \cap (W^{1,3}(\Omega))^d$, for example, the L^2-projection under the conditions of [5]. This stability in $W^{1,p}$ follows for many piecewise polynomial finite element spaces using [5].

The error is decomposed as $e = (w - \tilde{w}) - (w^h - \tilde{w}) = \eta - \phi^h$, where $\eta = w - \tilde{w}$ and $\phi^h = w^h - \tilde{w} \in V^h$. An error equation is obtained by subtracting (3.2) from (3.3) and using the fact that $w \in V$. This gives, for any $v^h \in V^h \cap (W^{1,3}(\Omega))^d$ and $\lambda^h \in Q^h$,

$$\begin{aligned}(e_t, v^h) &+ a(w, w, v^h) - a(w^h, w^h, v^h) \\ &+ b(w, w, v^h) - b(w^h, w^h, v^h) - (q - \lambda^h, \nabla \cdot v^h) = 0.\end{aligned} \tag{4.1}$$

This is rewritten, adding and subtracting terms and setting $v^h = \phi^h$, as follows:

$$\begin{aligned}&(\phi_t^h, \phi^h) + a(w^h, w^h, \phi^h) - a(\tilde{w}, \tilde{w}, \phi^h) = (\eta_t, \phi^h) \\ &\quad + a(w, w, \phi^h) - a(\tilde{w}, \tilde{w}, \phi^h) + b(w, w, \phi^h) - b(w^h, w^h, \phi^h) - (q - \lambda^h, \nabla \cdot \phi^h).\end{aligned} \tag{4.2}$$

The monotonicity lemma (Lemma 2.3) implies that

$$\begin{aligned}&a(w^h, w^h, \phi^h) - a(\tilde{w}, \tilde{w}, \phi^h) \\ &\geq \underline{C} C_s \delta^2 \|\mathbb{D}(\phi^h)\|_{L^3}^3 + \alpha \|\nabla \cdot \phi^h\|^2 + (2Re^{-1} + a_0(\delta)) \|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^J \beta \|\phi^h \cdot \hat{\tau}\|_{\Gamma_j}^2,\end{aligned}$$

and with $r := \max\{\|\mathbb{D}(w)\|_{L^3}, \|\mathbb{D}(\tilde{w})\|_{L^3}\}$

$$\begin{aligned}&a(w, w, \phi^h) - a(\tilde{w}, \tilde{w}, \phi^h) \\ &\leq (2Re^{-1} + a_0(\delta)) \|\mathbb{D}(\phi^h)\| \ \|\mathbb{D}(\eta)\| + \sum_{j=1}^J \beta \|\phi^h \cdot \hat{\tau}\|_{\Gamma_j} \ \|\eta \cdot \hat{\tau}\|_{\Gamma_j} \\ &\quad + C_s \overline{C} \delta^2 r \|\mathbb{D}(\eta)\|_{L^3} \|\mathbb{D}(\phi^h)\|_{L^3} + \alpha \|\nabla \cdot \eta\| \ \|\nabla \cdot \phi^h\|.\end{aligned}$$

Remark 4.1. If $\tilde{w}$ is taken to be the Stokes projection of (w, q) into V^h, then, e.g., the term "$Re^{-1}\|\mathbb{D}(\phi^h)\| \ \|\mathbb{D}(\eta)\|$" on this last right-hand side does not occur.

Inserting these two bounds in (4.2) and using the Cauchy–Schwarz and Young's inequalities gives

$$\begin{aligned}&\frac{1}{2} \ \frac{d}{dt} \|\phi^h\|^2 + \underline{C} C_s \delta^2 \|\mathbb{D}(\phi^h)\|_{L^3}^3 + \alpha \|\nabla \cdot \phi^h\|^2 \\ &\quad + (2Re^{-1} + a_0(\delta)) \|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^J \beta \|\phi^h \cdot \hat{\tau}\|_{\Gamma_j}^2 \\ &\leq |b(w, w, \phi^h) - b(w^h, w^h, \phi^h)| + \frac{\epsilon_1}{3} \delta^2 \|\mathbb{D}(\phi^h)\|_{L^3}^3 + \frac{2}{3} \epsilon_1^{-1/2} \delta^{-1} \|\eta_t\|_{W^{-1,3/2}}^{3/2}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(2Re^{-1}+a_0(\delta))\|\mathbb{D}(\phi^h)\|^2\frac{1}{2}(2Re^{-1}+a_0(\delta))\|\mathbb{D}(\eta)\|^2\\
&+\sum_{j=1}^{J}\left(\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|_{\Gamma_j}^2+\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|_{\Gamma_j}^2\right)+\frac{\epsilon_1}{3}\delta^2\|\mathbb{D}(\phi^h)\|_{L^3}^3\\
&+\frac{2}{3}\epsilon_1^{-1/2}\delta^2\overline{C}^{3/2}C_s^{3/2}r^{3/2}\|\mathbb{D}(\eta)\|_{L^3}^{3/2}+\frac{\alpha}{2}\|\nabla\cdot\phi^h\|^2+\frac{1}{\alpha}\|q-\lambda^h\|^2+\alpha\|\nabla\cdot\eta\|^2.
\end{aligned}
$$

Picking $\epsilon_1=\underline{C}C_s$ and collecting terms gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^h\|^2+\frac{1}{3}\underline{C}C_s\delta^2\|\mathbb{D}(\phi^h)\|_{L^3}^3+\frac{\alpha}{2}\|\nabla\cdot\phi^h\|^2\\
&\quad+\frac{1}{2}(2Re^{-1}+a_0(\delta))\|\mathbb{D}(\phi^h)\|^2+\sum_{j=1}^{J}\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|_{\Gamma_j}^2\\
&\le|b(w,w,\phi^h)-b(w^h,w^h,\phi^h)|+\frac{2}{3}(CC_s)^{-1/2}\delta^{-1}\|\eta_t\|_{W^{-1,3/2}}^{3/2}\\
&\quad+\frac{1}{2}(2Re^{-1}+a_0(\delta))\|\mathbb{D}(\eta)\|^2+\sum_{j=1}^{J}\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|_{\Gamma_j}^2\\
&\quad+\frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2\|\mathbb{D}(\eta)\|_{L^3}^{3/2}+\alpha^{-1}\|q-\lambda^h\|^2+\alpha\|\nabla\cdot\eta\|^2.
\end{aligned}
\tag{4.3}
$$

This is the basic differential inequality for the error. Three cases will be considered, revolving around the treatment of the first term on the right-hand side of (4.3).

Remark 4.2. If $\alpha=0$, an estimate which is uniform in Re can still be obtained by using Korn's inequality and Young's inequality on the term $(q-\lambda^h,\nabla\cdot\phi^h)$ as follows:

$$
\begin{aligned}
(q-\lambda^h,\nabla\cdot\phi^h)&\le\|q-\lambda^h\|_{L^{3/2}}\|\nabla\cdot\phi\|_{L^3}\le C\|\mathbb{D}(\phi^h)\|_{L^3}\|q-\lambda^h\|_{L^{3/2}}\\
&\le\frac{1}{3}\underline{C}C_s\delta^2\|\mathbb{D}(\phi^h)\|_{L^3}^3+C\delta^{-1}\|q-\lambda^h\|_{L^{3/2}}^{3/2}.
\end{aligned}
$$

However, an estimate of the nonlinear convective term which is uniform in Re fails in the case $\alpha=0$; see Remark 4.7.

Consider the convection terms

$$
b(w,w,\phi^h)-b(w^h,w^h,\phi^h)=b(w,\eta-\phi^h,\phi^h)+b(\eta-\phi^h,w^h,\phi^h).
\tag{4.4}
$$

The terms containing η shall be bounded first. Consider $b(w,\eta,\phi^h)$ and $b(\eta,w^h,\phi^h)$. Using the inequalities in section 2 appropriately gives

$$
\begin{aligned}
|b(w,\eta,\phi^h)|&=\left|\frac{1}{2}[(w\cdot\nabla\eta,\phi^h)-(w\cdot\nabla\phi^h,\eta)]\right|\\
&\le\frac{1}{2}\left[\|\phi^h\|\ \|\nabla\eta\|_{L^{s'}}\|w\|_{L^s}+\|\nabla\phi^h\|_{L^3}\|w\|_{L^q}\|\eta\|_{L^{q'}}\right],
\end{aligned}
$$

where $\frac{1}{2}+\frac{1}{s'}+\frac{1}{s}=1$ and $\frac{1}{3}+\frac{1}{q}+\frac{1}{q'}=1$. Picking $s'=3$, $s=6$, $q=2$, and $q'=6$ gives

$$
\begin{aligned}
|b(w,\eta,\phi^h)|&\le\frac{1}{2}\left[\|\phi^h\|\ \|\nabla\eta\|_{L^3}\|w\|_{L^6}+\|\nabla\phi^h\|_{L^3}\|w\|\ \|\eta\|_{L^6}\right]\\
&\le\frac{1}{4}\|\phi^h\|^2\|w\|_{L^6}^2+\frac{1}{4}\|\nabla\eta\|_{L^3}^2+\frac{\epsilon_3}{6}\|\mathbb{D}(\phi^h)\|_{L^3}^3+\frac{C}{3}\epsilon_3^{-1/2}\|w\|^{3/2}\|\eta\|_{L^6}^{3/2}.
\end{aligned}
\tag{4.5}
$$

The term $b(\eta, w^h, \phi^h)$ is similarly bounded as follows:

$$
\begin{aligned}
|b(\eta, w^h, \phi^h)| &= \left|\frac{1}{2}[(\eta \cdot \nabla w^h, \phi^h) - (\eta \cdot \nabla \phi^h, w^h)]\right| \\
&\le \frac{1}{2}\|\nabla w^h\|_{L^3}\|\eta\|_{L^6}\|\phi^h\| + \frac{1}{2}\|\nabla \phi^h\|_{L^3}\|\eta\|_{L^6}\|w^h\| \\
&\le \frac{1}{4}\|\nabla w^h\|_{L^3}^2\|\phi^h\|^2 + \frac{1}{4}\|\eta\|_{L^6}^2 + \frac{\epsilon_3}{6}\|\mathbb{D}(\phi^h)\|_{L^3}^3 + \frac{C}{3}\epsilon_3^{-1/2}\|\eta\|_{L^6}^{3/2}\|w^h\|^{3/2}. \qquad (4.6)
\end{aligned}
$$

Korn's inequality and the stability bounds (3.5) and (3.6) immediately imply that $\mathbb{D}(w^h) \in L^3(0,T;L^3)$ uniformly in Re so that $\|\nabla w^h\|_{L^3}^2 \in L^1(0,T)$, uniformly in Re. The Sobolev imbedding theorem and Korn's inequality also imply $\|w\|_{L^6}^2 \in L^1(0,T)$ uniformly in Re. Thus, these bounds suffice for a later application of Gronwall's inequality.

The first term containing only $\phi^h, b(w, \phi^h, \phi^h)$, is zero due to skew symmetry. Thus, there only remains the term $b(\phi^h, w^h, \phi^h)$. Estimating the term $b(\phi^h, w^h, \phi^h)$ is the essential, core difficulty in obtaining an error bound which is uniform in Re. There are only a few natural ways to bound this using Hölder's inequality and the Sobolev embedding theorem. There are two cases in which the analysis is successful:

(i) $a_0(\delta) \neq 0$ and $\nabla w \in L^3(0,T;L^3(\Omega))$,

(ii) $a_0(\delta) = 0$ and ∇w very regular, $\nabla w \in L^2(0,T;L^\infty(\Omega))$.

There is one important case in which the analysis fails:

(iii) $a_0(\delta) = 0$ and $\nabla w \in L^3(0,T;L^3(\Omega))$.

To highlight subsequent analysis and, hopefully, spur further study, we shall first present the case (iii) and explain the failure of the analysis.

Remark 4.3. On the condition $a_0(\delta) > 0$ in part (i), if $a_1(\delta) > 0$ and $a_0(\delta) \ge 0$, then it is known that the difference between two weak solutions of (1.3) can be bounded (nonuniformly in Re) by the change in the problem data [7, 29, 34]. These bounds imply uniqueness over $O(1)$ time intervals. On the other hand, if $a_1(\delta) \equiv 0$ and $a_0(\delta) > 0$, weak solutions are then only known to be unique over very small time intervals $0 \le t \le T^*(\delta)$, where (loosely speaking) $T^*(\delta) \sim (a_0(\delta) + Re^{-1})^3$.

4.1. The case $\nabla w \in L^3(0,T;L^3(\Omega))$ and $a_0(\delta) = 0$. If we assume only that $\nabla w \in L^3(0,T;L^3(\Omega))$, there is no need to add and subtract terms since a priori bounds on $\|\nabla w^h\|_{L^3(0,T;L^3)}$ have been proven which are uniform in Re. Thus, we can use Hölder's inequality to write

$$
\begin{aligned}
|b(\phi^h, w^h, \phi^h)| &= \left|\frac{1}{2}(\phi^h \cdot \nabla w^h, \phi^h) - \frac{1}{2}(\phi^h \cdot \nabla \phi^h, w^h)\right| \\
&\le \frac{1}{2}\|\nabla w^h\|_{L^3}\|\phi^h\|_{L^s}\|\phi^h\|_{L^{s'}} + \frac{1}{2}\|\nabla \phi^h\|_{L^3}\|\phi^h\|\ \|w^h\|_{L^6},
\end{aligned}
$$

where $\frac{1}{3} + \frac{1}{s'} + \frac{1}{s} = 1$ and $1 \le s', s \le \infty$. Thus, picking $s' = 2,\ s = 6$, using the embedding $W^{1,3}(\Omega) \to L^6(\Omega)$ and Poincaré's inequality gives

$$
\begin{aligned}
|b(\phi^h, w^h, \phi^h)| &\le \frac{C(\Omega)}{2}\|\nabla w^h\|_{L^3}\|\phi^h\|\ \|\phi^h\|_{1,3} + \frac{C(\Omega)}{2}\|\nabla \phi^h\|_{L^3}\|\phi^h\|\ \|w^h\|_{1,3} \\
&\le \frac{\epsilon}{6}\|\phi^h\|_{1,3}^3 + C\epsilon^{-1/2}\|\nabla w^h\|_{L^3}^{3/2}\|\phi^h\|^{3/2} + \frac{\epsilon}{6}\|\mathbb{D}(\phi^h)\|_{L^3}^3. \qquad (4.7)
\end{aligned}
$$

Remark 4.4. Using Lemma 2.5 instead of the embedding of $W^{1,3} \to L^6$ changes the critical exponent on $\|\phi^h\|$ "3/2" to 12/7 in the first term of (4.7) but not the final conclusion.

Combining (4.5), (4.6), (4.7) with $\epsilon_3 = \epsilon$ gives an initial bound on the convection term's difference:

$$
\begin{aligned}
&|b(w, w, \phi^h) - b(w^h, w^h, \phi^h)| \\
&\leq \left[\frac{1}{4}\|\nabla\eta\|_{L^3}^2 + \frac{1}{4}\|\eta\|_{L^6}^2 + C\epsilon^{-1/2}\left(\|w\|^{3/2} + \|w^h\|^{3/2}\right)\|\eta\|_{L^6}^{3/2}\right] \\
&+ \frac{2\epsilon}{3}\|\nabla\phi^h\|_{L^3}^3 + C\epsilon^{-1/2}\|\nabla w^h\|_{L^3}^{3/2}\|\phi^h\|^{3/2} + \left[\frac{1}{4}\|w\|_{L^6}^2 + \frac{1}{4}\|\nabla w^h\|_{L^3}^2\right]\|\phi^h\|^2.
\end{aligned} \tag{4.8}
$$

Inserting (4.8) into (4.3), applying Korn's inequality, and collecting terms gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^h\|^2 + \left(\frac{1}{3}\underline{C}C_s\delta^2 - \frac{2\epsilon}{3}\right)\|\mathbb{D}(\phi^h)\|_{L^3}^3 + \frac{\alpha}{2}\|\nabla\cdot\phi^h\|^2 \\
&\quad + \frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\leq \Bigg[\frac{2}{3}(CC_s)^{-1/2}\delta^{-1}\|\eta_t\|_{W^{-1,3/2}}^{3/2} + \frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\eta)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\quad + \frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2\|\mathbb{D}(\eta)\|_{L^3}^{3/2} + \alpha^{-1}\|q - \lambda^h\|^2 + \alpha\|\nabla\cdot\eta\|^2 + \frac{1}{4}\|\nabla\eta\|_{L^3}^2 \\
&\quad + C\epsilon^{-1/2}\|w\|^{3/2}\|\eta\|_{L^6}^{3/2} + \frac{1}{4}\|\eta\|_{L^6}^2 + C\epsilon^{-1/2}\|\eta\|_{L^6}^{3/2}\|w^h\|^{3/2}\Bigg] \\
&\quad + C\epsilon^{-1/2}\|\nabla w^h\|_{L^3}^{3/2}\|\phi^h\|^{3/2} + \left[\frac{1}{4}\|w\|_{L^6}^2 + \frac{1}{4}\|\nabla w^h\|_{L^3}^2\right]\|\phi^h\|^2.
\end{aligned}
$$

Thus, pick ϵ such that

$$\frac{2\epsilon}{3} = \frac{1}{6}\underline{C}C_s\delta^2,$$

i.e., $\epsilon = O(\delta^2)$. This gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^h\|^2 + \frac{1}{6}\underline{C}C_s\delta^2\|\mathbb{D}(\phi^h)\|_{L^3}^3 + \frac{\alpha}{2}\|\nabla\cdot\phi^h\|^2 + Re^{-1}\|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\leq \Bigg[\frac{2}{3}(CC_s)^{-1/2}\delta^{-1}\|\eta_t\|_{W^{-1,3/2}}^{3/2} + Re^{-1}\|\mathbb{D}(\eta)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\quad + \frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2\|\mathbb{D}(\eta)\|_{L^3}^{3/2} + \alpha^{-1}\|q - \lambda^h\|^2 + \alpha\|\nabla\cdot\eta\|^2 + \frac{1}{4}\|\nabla\eta\|_{L^3}^2 \\
&\quad + \frac{C}{\delta}\left(\|w\|^{3/2} + \|w^h\|^{3/2}\right)\|\eta\|_{L^6}^{3/2} + \frac{1}{4}\|\eta\|_{L^6}^2\Bigg] \\
&\quad + \left[C\delta^{-1}\|\nabla w^h\|_{L^3}^{3/2}\right]\|\phi^h\|^{3/2} + \left[\frac{1}{4}\|w\|_{L^6}^2 + \frac{1}{4}\|\nabla w^h\|_{L^3}^2\right]\|\phi^h\|^2.
\end{aligned}
$$

Consider the bracketed terms on this right-hand side. The first is approximation theoretic; the second is an L^1 function multiplying $\|\phi^h(t)\|^{3/2}$; the third is an L^1

function multiplying $\|\phi^h(t)\|^2$. Let $y(t) := \|\phi^h(t)\|^2$. This inequality may then be written as

$$\frac{d}{dt}y(t) + (\text{nonnegative terms}) \le C(t)h^\gamma + a(t)y(t) + b(t)\delta^{-1}y^{3/4}(t),$$

where $a(t), b(t) \in L^1(0,T)$.

The final step would normally be to apply Gronwall's inequality to deduce $y(t) = \frac{1}{2}\|\phi^h(t)\|^2$ to be bounded by its initial values and approximation theoretic terms. Unfortunately, the term $y^{3/4}$ is not Lipschitz, so the argument fails at this last step.

Tracing the inequalities backward, the problem term arises from the steps used to bound $b(\phi^h, w^h, \phi^h)$ to obtain *Re* independence. The error analysis in the successful cases (i) and (ii) centers therefore on alternate bounds for this term. We shall first consider case (i).

Remark 4.5. If the estimate in (4.7) is improved as noted in Remark 4.3, the term $y(t)^{3/4}$ is changed to $y(t)^{6/7}$ but the final conclusion still holds.

4.2. The case $\nabla w \in L^3(0,T;L^3(\Omega))$ and $a_0(\delta) > 0$. The main result of this section is the following theorem.

THEOREM 4.1. *Assume $\alpha > 0$ and $a_0(\delta) > 0$. Let*

$$a(t) = \frac{1}{4}\|w\|_{L^6}^2 + \frac{1}{4}\|\nabla w^h\|_{L^3}^2 + \frac{C}{a_0(\delta)}\|\nabla w^h\|_{L^3}^2 + Ca_0(\delta)^{-1/2}\alpha^{-3/2}\|\mathbb{D}(w^h)\|_{L^3}^3.$$

Then, there is a $C_1 = C_1(\delta)$, independent of Re and h, such that

$$\|a(t)\|_{L^1(0,T)} \le C_1(\delta).$$

Further, there is a $C_2 = C_2(\delta)$, independent of Re and h, such that

$$\frac{C}{\delta}\left(\|w\|^{3/2} + \|w^h\|^{3/2}\right) \le C_2(\delta).$$

Then, the error $w - w^h$ satisfies for $T > 0$

$$\begin{aligned}
&\|w - w^h\|_{L^\infty(0,T;L^2)}^2 + \delta^2\|\mathbb{D}(w - w^h)\|_{L^3(0,T;L^3)}^3 + \alpha\|\nabla\cdot(w - w^h)\|_{L^2(0,T;L^2)}^2 \\
&\quad + \left(Re^{-1} + Ca_0(\delta)\right)\|\mathbb{D}(w - w^h)\|_{L^2(0,T;L^2)}^2 + \sum_{j=1}^J \beta\|(w - w^h)\cdot\hat{\tau}\|_{L^2(0,T;L^2(\Gamma_j))}^2 \\
&\le C\exp(C_1(\delta))\|(w - w^h)(x,0)\|^2 + C\inf_{\tilde{w}\in V^h\cap(W^{1,3}(\Omega))^d,\lambda^h\in Q^h}\mathcal{F}(w - \tilde{w}, q - \lambda^h, \delta)
\end{aligned}$$

with

$$\begin{aligned}
&\mathcal{F}(w - \tilde{w}, r - q^h, \delta) \\
&= \|w - \tilde{w}\|_{L^\infty(0,T;L^2)}^2 + \delta^2\|\mathbb{D}(w - \tilde{w})\|_{L^3(0,T;L^3)}^3 \\
&\quad + \exp(C_1(\delta))\Bigg[\|(w - \tilde{w})(x,0)\|^2 + \delta^{-1}\|(w - \tilde{w})_t\|_{L^{3/2}(0,T;W^{-1,3/2})}^{3/2} \\
&\quad + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(w - \tilde{w})\|_{L^2(0,T;L^2)}^2 + \sum_{j=1}^J \beta\|(w - \tilde{w})\cdot\hat{\tau}\|_{L^2(0,T;L^2(\Gamma_j))}^2 \\
&\quad + C(\delta)\|\mathbb{D}(w - \tilde{w})\|_{L^3(0,T;L^3)}^{3/2} + \alpha^{-1}\|q - \lambda^h\|_{L^2(0,T;L^2)}^2 + \alpha\|\nabla\cdot(w - \tilde{w})\|_{L^2(0,T;L^2)}^2 \\
&\quad + \|\nabla(w - \tilde{w})\|_{L^2(0,T;L^3)}^2 + \|w - \tilde{w}\|_{L^2(0,T;L^6)}^2 + C_2(\delta)\|w - \tilde{w}\|_{L^{3/2}(0,T;L^6)}^{3/2}\Bigg].
\end{aligned}$$

Proof. This analysis follows the previous discussion closely except for the treatment of the $b(\phi^h, w^h, \phi^h)$ term and the final application of Gronwall's inequality.

Consider, therefore, $b(\phi^h, w^h, \phi^h)$. Integration by parts and using the fact that $\phi^h \cdot \hat{n} = 0$ on Γ give

$$\begin{aligned} b(\phi^h, w^h, \phi^h) &= \frac{1}{2}(\phi^h \cdot \nabla w^h, \phi^h) - \frac{1}{2}(\phi^h \cdot \nabla \phi^h, w^h) \\ &= (\phi^h \cdot \nabla w^h, \phi^h) + \frac{1}{2}(\nabla \cdot \phi^h, \phi^h \cdot w^h) \\ &\leq \|\nabla w^h\|_{L^3} \|\phi^h\|_{L^3}^2 + \frac{1}{2}|(\nabla \cdot \phi^h, \phi^h \cdot w^h)|. \end{aligned} \tag{4.9}$$

Using the embedding $H^{1/2} \hookrightarrow L^3$ in $d = 2, 3$ and Young's inequality give

$$\left|b(\phi^h, w^h, \phi^h)\right| \leq \frac{\epsilon_1}{2}\|\mathbb{D}(\phi^h)\|^2 + \frac{C}{2\epsilon_1}\|\nabla w^h\|_{L^3}^2 \|\phi^h\|^2 + \frac{1}{2}|(\nabla \cdot \phi^h, \phi^h \cdot w^h)|. \tag{4.10}$$

Consider now the last term on the above right-hand side. By Hölder's inequality, we obtain

$$|(\nabla \cdot \phi^h, \phi^h \cdot w^h)| \leq \|\nabla \cdot \phi^h\| \ \|\phi^h\|_{L^{r'}} \|w^h\|_{L^s},$$

where $\frac{1}{r'} + \frac{1}{s} = \frac{1}{2}$. Thus,

$$|(\nabla \cdot \phi^h, w^h \cdot \phi^h)| \leq \frac{\alpha}{4}\|\nabla \cdot \phi^h\|^2 + \alpha^{-1}\|\phi^h\|_{L^{r'}}^2 \|w^h\|_{L^s}^2. \tag{4.11}$$

The Sobolev embedding theorem implies that for any $s, 1 \leq s < \infty$ in two or three dimensions, $W^{1,3}(\Omega) \hookrightarrow L^s(\Omega)$. Thus,

$$\|w^h\|_{L^s}^2 \leq C(s, \Omega) \ \|w^h\|_{W^{1,3}(\Omega)}^2 \leq C(s, \Omega)\|\mathbb{D}(w^h)\|_{L^3}^2.$$

This implies that for any $r' > 2$

$$|(\nabla \cdot \phi^h, w^h \cdot \phi^h)| \leq \frac{\alpha}{4}\|\nabla \cdot \phi^h\|^2 + C(r', \Omega)\alpha^{-1}\|\phi^h\|_{L^{r'}}^2 \|\mathbb{D}(w^h)\|_{L^3}^2.$$

Consider the last term on the above right-hand side. The Sobolev embedding theorem also implies

$$\|\phi^h\|_{L^{r'}} \leq C(r', \Omega)\|\phi^h\|_{W^{t,2}(\Omega)} \text{ for } t \geq \frac{3}{2} - \frac{3}{r'}.$$

(The final result is not improved by applying here instead the Gagliardo–Nirenberg inequality.) As $r' \to 2, t \to 0$ in this inequality. Thus, picking $r' = r'(t) > 2$ close enough to 2 implies that, using an embedding inequality and Korn's inequality,

$$\|\phi^h\|_{L^{r'}}^2 \leq C(t, \Omega)\|\phi^h\|_t^2 \leq C(t, \Omega)\|\phi^h\|^{2(1-t)} \ \|\mathbb{D}(\phi^h)\|^{2t}$$

for any $t > 0$. Thus, for these values of r' and s

$$\frac{1}{\alpha}\|\phi^h\|_{L^{r'}}^2 \|w^h\|_{L^s}^2 \leq \frac{C}{\alpha}\|\mathbb{D}(\phi^h)\|^{2t}\|\phi^h\|^{2(1-t)}\|\mathbb{D}(w^h)\|_{L^3}^2$$

for any $t > 0$. For conjugate exponents $q = 3$ and $q' = \frac{3}{2}$ in Young's inequality, we then have

$$\frac{1}{\alpha}\|\phi^h\|_{L^{r'}}^2 \|w^h\|_{L^s}^2 \leq \frac{\epsilon}{3}\|\mathbb{D}(\phi^h)\|^{6t} + \epsilon^{-1/2}\alpha^{-3/2}C\|\phi^h\|^{3(1-t)}\|\mathbb{D}(w^h)\|_{L^3}^3.$$

Picking $t = \frac{1}{3} > 0$ gives for these values of r' and s

$$\frac{1}{\alpha}\|\phi^h\|^2_{L^{r'}}\|w^h\|^2_{L^s} \le \frac{\epsilon}{3}\|\mathbb{D}(\phi^h)\|^2 + C(r', s, t, \Omega)\epsilon^{-1/2}\alpha^{-3/2}\|\phi^h\|^2\|\mathbb{D}(w^h)\|^3_{L^3}.$$

Using this bound, (4.10) and (4.11) finally give

$$\begin{aligned}|b(\phi^h, w^h, \phi^h)| \le{}& \frac{\epsilon_1}{2}\|\mathbb{D}(\phi^h)\|^2 + \frac{C}{2\epsilon_1}\|\nabla w^h\|^2_{L^3}\|\phi^h\|^2\\ &+ \frac{\alpha}{8}\|\nabla\cdot\phi^h\|^2 + \frac{\epsilon_2}{6}\|\mathbb{D}(\phi^h)\|^2 + C\epsilon_2^{-1/2}\alpha^{-3/2}\|\mathbb{D}(w^h)\|^3_{L^3}\|\phi^h\|^2.\end{aligned}$$

Remark 4.6. It appears on first consideration that this last term $(\nabla\cdot\phi^h, w^h\cdot\phi^h)$ can be agreeably bounded more directly and easily by

$$\begin{aligned}|(\nabla\cdot\phi^h, w^h\cdot\phi^h)| &\le C\|\nabla\cdot\phi^h\|\ \|\nabla w^h\|\ \|\phi^h\|^{1/2}\ \|\nabla\phi^h\|^{1/2}\\ &\le C\|\nabla\phi^h\|^{3/2}\|\phi^h\|^{1/2}\|\nabla w^h\| \le \epsilon\|\nabla\phi^h\|^2 + C(\epsilon)\|\nabla w^h\|^4\|\phi^h\|^2.\end{aligned}$$

This bound, while certainly true, is not sufficient because of the condition that inevitably arises from using it that w^h or $w \in L^4(0,T;H^1(\Omega))$. The extra work in the bound we use reduces the time regularity requirements arising from this term to $w^h \in L^3(0,T;W^{1,3}(\Omega))$ (which is bounded uniformly in *Re* by problem data in section 3).

Substituting this bound for $b(\phi^h, w^h, \phi^h)$ in the derivation of the upper estimate (4.8) for the difference of the convection terms gives

$$\begin{aligned}&|b(w,w,\phi^h) - b(w^h,w^h,\phi^h)|\\ &\le \left[\frac{1}{4}\|\nabla\eta\|^2_{L^3} + \frac{C}{3}\epsilon_3^{-1/2}\|w\|^{3/2}\|\eta\|^{3/2}_{L^6} + \frac{1}{4}\|\eta\|^2_{L^6} + \frac{C}{3}\epsilon_3^{-1/2}\|\eta\|^{3/2}_{L^6}\|w^h\|^{3/2}\right]\\ &\quad + \left[\frac{\epsilon_3}{3}\|\mathbb{D}(\phi^h)\|^3_{L^3} + \frac{\epsilon_1}{2}\|\mathbb{D}(\phi^h)\|^2 + \frac{\alpha}{8}\|\nabla\cdot\phi^h\|^2 + \frac{\epsilon_2}{6}\|\mathbb{D}(\phi^h)\|^2\right]\\ &\quad + \left[\frac{1}{4}\|w\|^2_{L^6} + \frac{1}{4}\|\nabla w^h\|^2_{L^3} + \frac{C}{2\epsilon_1}\|\nabla w^h\|^2_{L^3} + C\epsilon_2^{-1/2}\alpha^{-3/2}\|\mathbb{D}(w^h)\|^3_{L^3}\right]\|\phi^h\|^2.\end{aligned} \tag{4.12}$$

To proceed further, (4.12) is inserted in the right-hand side of (4.3). This yields the differential inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\phi^h\|^2 + \left(\frac{1}{3}\underline{C}C_s\delta^2 - \frac{\epsilon_3}{3}\right)\|\mathbb{D}(\phi^h)\|^3_{L^3} + \frac{3}{8}\alpha\|\nabla\cdot\phi^h\|^2\\ &\quad + \left(\frac{1}{2}(2Re^{-1} + a_0(\delta)) - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{6}\right)\|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|^2_{\Gamma_j}\\ &\le \left[\frac{2}{3}(\underline{C}C_s)^{-1/2}\delta^{-1}\|\eta_t\|^{3/2}_{W^{-1,3/2}} + \frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\eta)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|^2_{\Gamma_j}\right.\\ &\quad + \frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2\|\mathbb{D}(\eta)\|^{3/2}_{L^3} + \alpha^{-1}\|q-\lambda^h\|^2 + \alpha\|\nabla\cdot\eta\|^2 + \frac{1}{4}\|\nabla\eta\|^2_{L^3}\\ &\quad \left. + \frac{C}{3}\epsilon_3^{-1/2}\left(\|w\|^{3/2} + \|w^h\|^{3/2}\right)\|\eta\|^{3/2}_{L^6} + \frac{1}{4}\|\eta\|^2_{L^6}\right]\\ &\quad + \left[\frac{1}{4}\|w\|^2_{L^6} + \frac{1}{4}\|\nabla w^h\|^2_{L^3} + \frac{C}{\epsilon_1}\|\nabla w^h\|^2_{L^3} + C\epsilon_2^{-1/2}\alpha^{-3/2}\|\mathbb{D}(w^h)\|^3_{L^3}\right]\|\phi^h\|^2.\end{aligned} \tag{4.13}$$

Pick $\epsilon_3 = C\underline{C}C_s\delta^2, C < 1/3$, $\epsilon_1 = a_0(\delta)/3$, and $\epsilon_2 = a_0(\delta)$. These choices simplify (4.13) to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^h\|^2 + C\underline{C}C_s\delta^2\|\mathbb{D}(\phi^h)\|_{L^3}^3 + \frac{3}{8}\alpha\|\nabla\cdot\phi^h\|^2 \\
&\quad + \left(Re^{-1} + \frac{a_0(\delta)}{6}\right)|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\leq \Bigg[\frac{2}{3}(CC_s)^{-1/2}\delta^{-1}\|\eta_t\|_{W^{-1,3/2}}^{3/2} + \frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\eta)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\quad + \frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2\|\mathbb{D}(\eta)\|_{L^3}^{3/2} + \alpha^{-1}\|q-\lambda^h\|^2 + \alpha\|\nabla\cdot\eta\|^2 \\
&\quad + \frac{1}{4}\|\nabla\eta\|_{L^3}^2 + \frac{C}{\delta}(\|w\|^{3/2} + \|w^h\|^{3/2})\|\eta\|_{L^6}^{3/2} + \frac{1}{4}\|\eta\|_{L^6}^2\Bigg] \\
&\quad + \left[\frac{1}{4}\|w\|_{L^6}^2 + \frac{1}{4}\|\nabla w^h\|_{L^3}^2 + \frac{C}{a_0(\delta)}\|\nabla w^h\|_{L^3}^2 + \frac{C}{a_0(\delta)^{1/2}\alpha^{3/2}}\|\mathbb{D}(w^h)\|_{L^3}^3\right]\|\phi^h\|^2.
\end{aligned} \tag{4.14}$$

Before applying Gronwall's inequality, let us first verify that it will indeed give us an error bound that is uniform in the Reynolds number by considering the coefficients on the right-hand side of (4.14).

First, note that $r \leq C\|D(w)\|_{L^3}$. By the stability estimates $\|w\| \in L^\infty(0,T)$ and $\|w^h\| \in L^\infty(0,T)$ uniformly in Re. Thus,

$$\frac{C}{\delta}\|w\|^{3/2}\|\eta\|_{L^6}^{3/2} + \frac{C}{\delta}\|\eta\|_{L^6}^{3/2}\|w^h\|^{3/2} \leq \frac{C}{\delta}(\|w\|^{3/2} + \|w^h\|^{3/2})\|\eta\|_{L^6}^{3/2} \leq C_2(\delta)\|\eta\|_{L^6}^{3/2}.$$

Consider the (critical) bracketed coefficient of the last term on the right-hand side. We must show this coefficient is in $L^1(0,T)$ uniformly in Re. Indeed, by the stability estimates and the Sobolev imbedding $\|w\|_{L^6}$, $\|\mathbb{D}(w^h)\|_{L^3}$, $\|\mathbb{D}(w)\|_{L^3} \in L^3(0,T)$ uniformly in Re. Since $T < \infty$, $L^3(0,T) \subset L^2(0,T)$, and thus the first factor of the last term is in $L^1(0,T)$ uniformly in Re.

Hiding all constants in generic C's, Gronwall's lemma now implies for almost all $t \in [0,T]$ that

$$\begin{aligned}
&\|\phi^h(x,t)\|^2 + \delta^2\|\mathbb{D}(\phi^h)\|_{L^3(0,t;L^3)}^3 + \alpha\|\nabla\cdot\phi^h\|_{L^2(0,t;L^2)}^2 \\
&\quad + \left(Re^{-1} + Ca_0(\delta)\right)\|\mathbb{D}(\phi^h)\|_{L^2(0,t;L^2)}^2 + \sum_{j=1}^{J}\beta\|\phi^h\cdot\hat{\tau}\|_{L^2(0,t;L^2(\Gamma_j))}^2 \\
&\leq C\exp\left(\|a(t)\|_{L^1(0,t)}\right)\|\phi^h(x,0)\|^2 \\
&\quad + C\exp\left(\|a(t)\|_{L^1(0,t)}\right)\Bigg[\delta^{-1}\|\eta_t\|_{L^{3/2}(0,T;W^{-1,3/2})}^{3/2} + (2Re^{-1} + a_0(\delta))\|\mathbb{D}(\eta)\|_{L^2(0,t;L^2)}^2 \\
&\quad + \sum_{j=1}^{J}\beta\|\eta\cdot\hat{\tau}\|_{L^2(0,t;L^2(\Gamma_j))}^2 + \delta^2\int_0^t\|\mathbb{D}(w)\|_{L^3}^{3/2}\|\mathbb{D}(\eta)\|_{L^3}^{3/2}dt' \\
&\quad + \alpha^{-1}\|q-\lambda^h\|_{L^2(0,t;L^2)}^2 + \alpha\|\nabla\cdot\eta\|_{L^2(0,t;L^2)}^2 + \|\nabla\eta\|_{L^2(0,t;L^3)}^2 + \|\eta\|_{L^2(0,t;L^6)}^2 \\
&\quad + \int_0^t\frac{1}{\delta}(\|w\|^{3/2} + \|w^h\|^{3/2})\|\eta\|_{L^6}^{3/2}dt'\Bigg].
\end{aligned}$$

Note that by the Cauchy–Schwarz inequality in $L^2(0,t)$, $t \in [0,T]$, and the stability estimates

$$\int_0^t \|\mathbb{D}(w)\|_{L^3}^{3/2} \|\mathbb{D}(\eta)\|_{L^3}^{3/2} dt' \le \|\mathbb{D}(w)\|_{L^3(0,t;L^3)}^{3/2} \|\mathbb{D}(\eta)\|_{L^3(0,t;L^3)}^{3/2} \le C(\delta) \|\mathbb{D}(\eta)\|_{L^3(0,t;L^3)}^{3/2}.$$

Now, the essential supremum of $t \in [0,T]$ is applied on both sides of the inequality. As $w - w^h = \eta - \phi^h$, the triangle inequality completes the proof of Theorem 4.1. ∎

Remark 4.7. On the condition $\alpha > 0$, the least squares control of $\nabla \cdot u$ seems to be essential to get an estimate uniform in *Re*. Consider (4.9) in the proof of Theorem 4.1. There are two important nonlinear terms in the error equation corresponding loosely to convection and reaction. The reaction term is controlled by the subgrid model. The convection term can be converted into a reaction-like term. It is controllable provided that $\nabla \cdot \phi^h$ is controllable, which $\alpha > 0$ accomplishes.

Another promising approach is to use a variational formulation, such as SUPG developed by Brooks and Hughes [3], which will control the convection term directly. We note that both SUPG and least squares control of $\nabla \cdot u$ are consistent: they work on the error and do not change the solution.

4.3. The case $\boldsymbol{\nabla w \in L^2(0,T;L^\infty(\Omega))}$ and $\boldsymbol{a_0(\delta) \ge 0}$. We now consider the case of smoother w, i.e.,

$$w \in L^2(0,T;W^{1,\infty}(\Omega)) \text{ uniformly in } Re,$$

allowing for the case $a_0(\delta) \equiv 0$. This case is primarily of interest because many tests involve "academic" flow fields given in closed form (as in section 5). These are typically smooth and bounded. In this case Theorem 4.2 gives an error estimate with constants independent of *Re* (but depending on δ and α). It is noteworthy in this estimate that *multiplicative constants* depend on δ but the *rate constant* in the (inevitable) exponential term takes the form

$$\exp(C_3(w)), \quad C_3 = C_3(\|w\|_{L^2(0,T;W^{1,\infty}(\Omega))}),$$

with no *explicit* dependence on δ.

THEOREM 4.2. *Suppose $a_0(\delta) \ge 0$, $\alpha > 0$, and $w \in L^2(0,T;W^{1,\infty}(\Omega))$ uniformly in Re. Let*

$$a(t) := \frac{3}{4} + \|\nabla w\|_{L^\infty} + \left(\frac{1}{4} + \frac{1}{4\alpha}\right) \|w\|_{L^\infty}^2 + \frac{1}{2}\|\nabla w\|_{L^\infty}^2;$$

then there is a $C_3 = C_3(w)$ such that

$$\|a(t)\|_{L^1(0,T)} \le C_3(w).$$

Let $C_4 = C_4(\delta)$ be such that

$$\|\mathbb{D}(w^h)\|_{L^3(0,T;L^3)} \le C_4(\delta).$$

Then, the error $w - w^h$ satisfies

$$\begin{aligned} &\|w - w^h\|_{L^\infty(0,T;L^2)}^2 + \delta^2 \|\mathbb{D}(w - w^h)\|_{L^3(0,T;L^3)}^3 + \alpha \|\nabla \cdot (w - w^h)\|_{L^2(0,T;L^2)}^2 \\ &\quad + \left(Re^{-1} + Ca_0(\delta)\right) \|\mathbb{D}(w - w^h)\|_{L^2(0,T;L^2)}^2 + \sum_{j=1}^J \beta \|(w - w^h) \cdot \hat{\tau}\|_{L^2(0,T;L^2(\Gamma_j))}^2 \\ &\le C \exp(C_3(w)) \|(w - w^h)(x,0)\|^2 + C \inf_{\tilde{w} \in V^h \cap (W^{1,3}(\Omega))^d, \lambda^h \in Q^h} \mathcal{F}(w - \tilde{w}, q - \lambda^h, \delta) \end{aligned}$$

with

$$
\begin{aligned}
&\mathcal{F}(w-\tilde{w}, r-q^h, \delta)\\
&= \|w-\tilde{w}\|^2_{L^\infty(0,T;L^2)} + \delta^2 \|\mathbb{D}(w-\tilde{w})\|^3_{L^3(0,T;L^3)}\\
&\quad + \exp(C_3(w)) \Bigg[\|(w-\tilde{w})(x,0)\|^2 + \delta^{-1} \|(w-\tilde{w})_t\|^{3/2}_{L^{3/2}(0,T;W^{-1,3/2})}\\
&\quad + (2Re^{-1} + a_0(\delta)) \|\mathbb{D}(w-\tilde{w})\|^2_{L^2(0,T;L^2)} + \sum_{j=1}^{J} \beta \|(w-\tilde{w})\cdot\hat{\tau}\|^2_{L^2(0,T;L^2(\Gamma_j))}\\
&\quad + C(\delta) \|\mathbb{D}(w-\tilde{w})\|^{3/2}_{L^3(0,T;L^3)} + \alpha^{-1} \|q-\lambda^h\|^2_{L^2(0,T;L^2)}\\
&\quad + \left(\frac{1}{4}+\alpha\right) \|\nabla\cdot(w-\tilde{w})\|^2_{L^2(0,T;L^2)} + \|w-\tilde{w}\|^2_{L^2(0,T;L^2)}\\
&\quad + C_4(\delta) \left(\|\mathbb{D}(w-\tilde{w})\|^2_{L^{18/5}(0,T;L^3)} + \|w-\tilde{w}\|^2_{L^6(0,T;L^6)} \right) \Bigg].
\end{aligned}
$$

Proof. In this case, the difference in the nonlinear terms is decomposed a bit differently as

$$
\begin{aligned}
|b(w,w,\phi^h) - b(w^h,w^h,\phi^h)| &= |b(\eta-\phi^h, w, \phi^h) + b(w^h, \eta-\phi^h, \phi^h)|\\
&= |b(\eta, w, \phi^h) - b(\phi^h, w, \phi^h) + b(w^h, \eta, \phi^h)|.
\end{aligned} \tag{4.15}
$$

Consider the individual terms on the right-hand side of (4.15):

$$
\begin{aligned}
|b(\eta, w, \phi^h)| &= \left| \frac{1}{2}(\eta\cdot\nabla w, \phi^h) - \frac{1}{2}(\eta\cdot\nabla\phi^h, w) \right|\\
&= \left| (\eta\cdot\nabla w, \phi^h) + \frac{1}{2}(\nabla\cdot\eta, \phi^h\cdot w) \right|\\
&\le \frac{1}{2}\|\eta\|^2 + \frac{1}{2}\|\nabla w\|^2_{L^\infty}\|\phi^h\|^2 + \frac{1}{4}\|\nabla\cdot\eta\|^2 + \frac{1}{4}\|w\|^2_{L^\infty}\|\phi^h\|^2,\\
|b(\phi^h, w, \phi^h)| &= \left| (\phi^h\cdot\nabla w, \phi^h) + \frac{1}{2}(\nabla\cdot\phi^h, w\cdot\phi^h) \right|\\
&\le \|\nabla w\|_{L^\infty}\|\phi^h\|^2 + \frac{\alpha}{4}\|\nabla\cdot\phi^h\|^2 + \frac{1}{4\alpha}\|w\|^2_{L^\infty}\|\phi^h\|^2,\\
|b(w^h, \eta, \phi^h)| &= \left| (w^h\cdot\nabla\eta, \phi^h) + \frac{1}{2}(\nabla\cdot w^h, \eta\cdot\phi^h) \right|\\
&\le \|w^h\|_{L^6}\|\nabla\eta\|_{L^3}\|\phi^h\| + \frac{1}{2}\|\nabla\cdot w^h\|_{L^3}\|\eta\|_{L^6}\|\phi^h\|\\
&\le C\left(\|w^h\|^2_{L^6}\|\nabla\eta\|^2_{L^3} + \|\mathbb{D}(w^h)\|^2_{L^3}\|\eta\|^2_{L^6} \right) + \frac{3}{4}\|\phi^h\|^2.
\end{aligned}
$$

Combining these three estimates gives

$$
\begin{aligned}
&|b(w,w,\phi^h) - b(w^h,w^h,\phi^h)|\\
&\le \frac{1}{2}\|\eta\|^2 + \frac{1}{4}\|\nabla\cdot\eta\|^2 + C\|w^h\|^2_{L^6}\|\nabla\eta\|^2_{L^3} + C\|\mathbb{D}(w^h)\|^2_{L^3}\|\eta\|^2_{L^6} + \frac{\alpha}{4}\|\nabla\cdot\phi^h\|^2\\
&+ \left(\frac{3}{4} + \frac{1}{2}\|\nabla w\|^2_{L^\infty} + \frac{1}{4}\|w\|^2_{L^\infty} + \|\nabla w\|_{L^\infty} + \frac{1}{4\alpha}\|w\|^2_{L^\infty} \right)\|\phi^h\|^2.
\end{aligned} \tag{4.16}
$$

The term $\|w^h\|_{L^6}$ is bounded using the Gagliardo–Nirenberg inequality (Lemma 2.5)

$$\|w^h\|_{L^6}^2 \leq C\|w^h\|^{2/3}\|\mathbb{D}(w^h)\|_{L^3}^{4/3}.$$

Since $\|w^h\|$ is bounded uniformly in ν and h by (3.5) or (3.6), it follows that

$$\|w^h\|_{L^6}^2 \leq C\|\mathbb{D}(w^h)\|_{L^3}^{4/3}.$$

This bound, together with (4.16), is now inserted in the right-hand side of (4.3) giving

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^h\|^2 + \frac{1}{3}\underline{C}C_s\delta^2\|\mathbb{D}(\phi^h)\|_{L^3}^3 + \frac{\alpha}{2}\|\nabla\cdot\phi^h\|^2 \\
&\quad + \frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\phi^h)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\phi^h\cdot\hat{\tau}\|_{\Gamma_j}^2 \\
&\leq \left[\frac{2}{3}(CC_s)^{-1/2}\delta^{-1}\|\eta_t\|_{W^{-1,3/2}}^{3/2} + \frac{1}{2}(2Re^{-1} + a_0(\delta))\|\mathbb{D}(\eta)\|^2 + \sum_{j=1}^{J}\frac{\beta}{2}\|\eta\cdot\hat{\tau}\|_{\Gamma_j}^2\right. \\
&\quad + \frac{2}{3}\underline{C}^{-1/2}C_s\overline{C}^{3/2}r^{3/2}\delta^2\|\mathbb{D}(\eta)\|_{L^3}^{3/2} + \alpha^{-1}\|q-\lambda^h\|^2 + \alpha\|\nabla\cdot\eta\|^2 + \frac{1}{2}\|\eta\|^2 \\
&\quad \left.+ \frac{1}{4}\|\nabla\cdot\eta\|^2 + C\|\mathbb{D}(w^h)\|_{L^3}^{4/3}\|\nabla\eta\|_{L^3}^2 + C\|\mathbb{D}(w^h)\|_{L^3}^2\|\eta\|_{L^6}^2\right] + \left[\frac{\alpha}{4}\|\nabla\cdot\phi^h\|^2\right] \\
&\quad + \left(\frac{3}{4} + \|\nabla w\|_{L^\infty} + \left(\frac{1}{4} + \frac{1}{4\alpha}\right)\|w\|_{L^\infty}^2 + \frac{1}{2}\|\nabla w\|_{L^\infty}^2\right)\|\phi^h\|^2.
\end{aligned}$$

To apply Gronwall's inequality we need

$$\frac{3}{4} + \|\nabla w\|_{L^\infty} + \left(\frac{1}{4} + \frac{1}{4\alpha}\right)\|w\|_{L^\infty}^2 + \frac{1}{2}\|\nabla w\|_{L^\infty}^2 \in L^1(0,T),$$

in other words $w \in L^2(0,T;W^{1,\infty}(\Omega))$. The term on the right-hand side of this inequality containing $r^{3/2}$ is treated as in the proof of Theorem 4.1. In the final result of Gronwall's lemma, we must also verify that the resulting terms containing $\|\mathbb{D}(w^h)\|_{L^3}$ are bounded uniformly in Re. To this end, apply Hölder's inequality

$$\int_0^T \|\mathbb{D}(w^h)\|_{L^3}^{4/3}\|\mathbb{D}(\eta)\|_{L^3}^2 dt \leq \|\mathbb{D}(w^h)\|_{L^{4q/3}(0,T;L^3)}^{4/3}\|\mathbb{D}(\eta)\|_{L^{2q'}(0,T;L^3)}^2,$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. From the stability estimates, we clearly must take q such that $4q/3 \leq 3$. Accordingly, take $q = \frac{9}{4}, q' = \frac{9}{5}$. This gives

$$\begin{aligned}
\int_0^T \|\mathbb{D}(w^h)\|_{L^3}^{4/3}\|\mathbb{D}(\eta)\|_{L^3}^2 dt &\leq C\|\mathbb{D}(w^h)\|_{L^3(0,T;L^3)}^{4/3}\|\mathbb{D}(\eta)\|_{L^{18/5}(0,T;L^3)}^2 \\
&\leq CC_4(\delta)\|\mathbb{D}(\eta)\|_{L^{18/5}(0,T;L^3)}^2.
\end{aligned}$$

Similarly, for q and q' conjugate exponents take $q = \frac{3}{2}$, $q' = 3$,

$$\begin{aligned}
\int_0^T \|\mathbb{D}(w^h)\|_{L^3}^2\|\eta\|_{L^6}^2 dt &\leq \|\mathbb{D}(w^h)\|_{L^{2q}(0,T;L^3)}^2\|\eta\|_{L^{2q'}(0,T;L^6)}^2 \\
&\leq \|\mathbb{D}(w^h)\|_{L^3(0,T;L^3)}^2\|\eta\|_{L^6(0,T;L^6)}^2 \leq C_4(\delta)\|\eta\|_{L^6(0,T;L^6)}^2.
\end{aligned}$$

The stated error estimate now follows from Gronwall's inequality and the triangle inequality as in the proof of Theorem 4.1. $\square$

5. A numerical example. To give a numerical illustration several decisions must be made, mainly whether to work on an "academic" flow problem with a known exact solution or to work on a more realistic flow problem containing the accompanying uncertainties. Since our aim is to illustrate a convergence theorem, we have chosen the former. (To assess a model or study the limitations of an algorithm, we would naturally have chosen the latter.) Accordingly, we have selected the vortex decay problem of Chorin [4], used also by others, e.g., Tafti [38]. The domain is $\Omega = (0,1)^2$ and we choose

$$\begin{aligned} w_1 &= -\cos(n\pi x)\sin(n\pi y)\exp(-2n^2\pi^2 t/\tau), \\ w_2 &= \sin(n\pi x)\cos(n\pi y)\exp(-2n^2\pi^2 t/\tau), \\ q &= -\tfrac{1}{4}(\cos(2n\pi x)+\cos(2n\pi y))\exp(-4n^2\pi^2 t/\tau). \end{aligned} \tag{5.1}$$

For the relaxation time $\tau = Re$ this is a solution of the Navier–Stokes equation consisting of an array of opposite signed vortices which decay as $t \to \infty$. The right-hand side f, initial condition, and nonhomogeneous Dirichlet boundary conditions are chosen so that (w_1, w_2, q) is the closed form solution of (1.3).

Since we are studying convergence as $h \to 0$ for δ fixed and Re varying we have accordingly chosen a 4×4 array of the vortices (so $n = 4$) and

$$\begin{aligned} \tau &= 1000, \\ \text{final time } T &= 8, \\ \text{eddy scale } \delta &= 0.1, \\ \text{Smagorinski constant } C_s &= 0.05, \\ a_0(\delta) &= 0. \end{aligned}$$

It is significant that $\delta = 0.1 \le \frac{1}{4} = \frac{1}{n}$ so that the vortices are larger than $O(\delta)$ and hence should be "visible" to the model.

The fractional—step θ—scheme with an equal distant time step $\Delta t_n = 0.001$ is used as discretization in time. The time discretization error should be kept small by using this very small time step. In space, the Q_2/P_1^{disc} and the Q_3/P_2^{disc} finite element discretizations are applied; see Table 1 for the number of degrees of freedom for different mesh sizes. The unit square was divided into an $h \times h$ mesh with $h = 1/2$ on level 0. Both the Smagorinsky subgridscale model and the convection term are treated implicitly. The viscous term is treated not as $(\nabla w^h, \nabla v^h)$ but as using the deformation tensor formulation, $(\mathbb{D}(w^h), \mathbb{D}(v^h))$, as analyzed herein. The least squares constant α is chosen to be zero and we used the convective form of the nonlinear convection term. The nonlinear system in each time step is solved up to a Euclidean norm of the residual vector less than 10^{-10}.

The numbers of degrees of freedom in space are certainly not extremely large. However, their importance is only relative to the Reynolds number, ranging from 10^2

TABLE 1
Mesh widths and degrees of freedom in space.

	Q_2/P_1^{disc}			Q_3/P_2^{disc}		
Mesh width	Velocity	Pressure	Total	Velocity	Pressure	Total
1/4	-	-	-	338	96	434
1/8	578	192	770	1 250	384	1 634
1/16	2 178	768	2 946	4 802	1 536	6 338
1/32	8 450	3 072	11 522	18 818	6 144	24 962
1/64	33 282	12 288	45 570	-	-	-
1/128	132 098	49 152	181 250	-	-	-

Table 2
Q_2/P_1^{disc} *finite element discretization,* $\|e\|_{L^\infty(0,T;L^2)}$.

$Re \setminus h$	1/8	1/16	1/32	1/64	1/128
10^2	2.20176e-2	2.76780e-3	3.47796e-4	4.35185e-5	5.43988e-6
10^3	3.19389e-2	3.50372e-3	4.81015e-4	4.86864e-5	5.50381e-6
10^4	5.97051e-2	7.01100e-3	1.00294e-3	1.39466e-4	1.44707e-5
10^5	7.67057e-2	7.73782e-3	1.09801e-3	1.62252e-4	1.92555e-5
10^6	7.86394e-2	7.81755e-3	1.10830e-3	1.64891e-4	1.98666e-5
10^7	7.88349e-2	7.82560e-3	1.10934e-3	1.65161e-4	1.99290e-5
10^8	7.88545e-2	7.82641e-3	1.10945e-3	1.65188e-4	1.99373e-5
10^9	7.88564e-2	7.82649e-3	1.10946e-3	1.65190e-4	1.99379e-5
10^{10}	7.88566e-2	7.82650e-3	1.10946e-3	1.65191e-4	1.99380e-5

Table 3
Q_3/P_2^{disc} *finite element discretization,* $\|e\|_{L^\infty(0,T,L^2)}$.

$Re \setminus h$	1/4	1/8	1/16	1/32
10^2	3.09237e-2	2.14568e-3	1.39746e-4	8.96881e-6
10^3	7.61050e-2	2.53153e-3	1.40003e-4	8.85819e-6
10^4	1.09160e-1	2.79077e-3	1.43102e-4	8.81959e-6
10^5	1.13716e-1	2.82963e-3	1.43815e-4	8.81915e-6
10^6	1.14186e-1	2.83371e-3	1.43896e-4	8.81835e-6
10^7	1.14234e-1	2.83412e-3	1.43904e-4	8.81929e-6
10^8	1.14238e-1	2.83416e-3	1.43905e-4	8.81853e-6
10^9	1.14239e-1	2.83417e-3	1.43905e-4	8.81961e-6
10^{10}	1.14239e-1	2.83417e-3	1.43905e-4	8.81754e-6

to 10^{10}, and the resolution sought, $\delta = 0.1$. Again, LES is focused on situations in which the number of degrees of freedom is small relative to Re. Thus, the chosen values of h and Re seem appropriate.

Tables 2 and 3 present the $L^\infty(0,T;L^2)$ norm of the error for both discretizations in space. Note that the behavior is exactly as anticipated by the theory: the error in this norm is clearly independent of Re.

Tables 4 and 5 present the errors in $L^2(0,T;H^1)$. These errors are not predicted to be in general uniform in Re. But in the particular example which we have chosen, one can observe uniformity in Re.

6. Conclusions. Reynolds number dependence in finite element error analysis arises in three basic places: multiplicative error constants (Re), time scale constants ($\exp(C(Re)T)$), and time regularity assumptions on the true solution (needed even to prove continuous dependence on the initial data) which might fail for turbulent flows. In the error analysis of a large eddy model all three sources must be addressed. The idea of our error analysis herein for the Smagorinsky model has been that the greater spatial regularity of the large eddies must be used to compensate for the reduced time regularity of the underlying turbulent flow. The execution of this idea is necessarily technical since it entails using, in so far as possible, L^3 bounds (the natural norm arising from the model) for the nonlinear error terms. For different models, this same idea can be possibly applied; its execution will vary with the particular features of the model.

The error equation contains nonlinear terms resembling both convection and reaction. Our analysis suggests that uniformity in Re can be accomplished by the control of both effects. The second is controlled by the subgrid model while the first seems to need a correctly adapted numerical method; see Remark 4.7.

TABLE 4
Q_2/P_1^{disc} *finite element discretization,* $\|\mathbb{D}(e)\|_{L^2(0,T,L^2)}$.

$Re \setminus h$	1/8	1/16	1/32	1/64	1/128
10^2	1.24827	3.13720e-1	7.84736e-2	1.96114e-2	4.90234e-3
10^3	1.56935	3.60470e-1	8.42787e-2	2.00913e-2	4.93406e-3
10^4	2.35100	4.66554e-1	1.05387e-1	2.34301e-2	5.28506e-3
10^5	2.68127	4.98844e-1	1.14609e-1	2.61063e-2	5.79700e-3
10^6	2.72037	5.02793e-1	1.15920e-1	2.66473e-2	5.96091e-3
10^7	2.72434	5.03197e-1	1.16058e-1	2.67093e-2	5.98435e-3
10^8	2.72474	5.03237e-1	1.16072e-1	2.67156e-2	5.98686e-3
10^9	2.72478	5.03241e-1	1.16073e-1	2.67162e-2	5.98711e-3
10^{10}	2.72478	5.03242e-1	1.16073e-1	2.67163e-2	5.98714e-3

TABLE 5
Q_3/P_2^{disc} *finite element discretization,* $\|\mathbb{D}(e)\|_{L^2(0,T,L^2)}$.

$Re \setminus h$	1/4	1/8	1/16	1/32
10^2	1.15300	1.65587e-1	2.07562e-2	2.60050e-3
10^3	2.87920	1.93565e-1	2.15627e-2	2.62325e-3
10^4	4.79996	2.24195e-1	2.27512e-2	2.67749e-3
10^5	5.07949	2.31037e-1	2.31277e-2	2.71302e-3
10^6	5.10843	2.31820e-1	2.31768e-2	2.72525e-3
10^7	5.11134	2.31899e-1	2.31819e-2	2.72679e-3
10^8	5.11163	2.31907e-1	2.31824e-2	2.72674e-3
10^9	5.11166	2.31908e-1	2.31825e-2	2.72698e-3
10^{10}	5.11166	2.31908e-1	2.31825e-2	2.72759e-3

We note that replacing multipliers like $\exp(C(Re)T)$ in the error estimate by $\exp(C(\delta)T)$ establishes that a LES will be valid over a much longer time interval than a DNS, although still not over $0 < t \leq \infty$. It would certainly be interesting to know which flow statistics could be accurately approximated over $0 < t \leq \infty$, but this requires a different analysis.

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